

Bhabha scattering and a special pencil of K3 surfaces

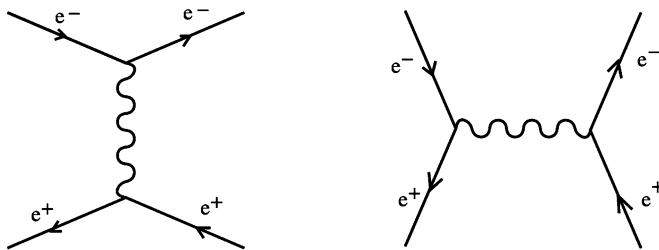
DINO FESTI AND DUCO VAN STRATEN

We study a pencil of K3 surfaces that appeared in the 2-loop diagrams in Bhabha scattering. By analysing in detail the Picard lattice of the general and special members of the pencil, we identify the pencil with the celebrated Apéry–Fermi pencil, that was related to Apéry’s proof of the irrationality of $\zeta(3)$ through the work of F. Beukers, C. Peters and J. Stienstra. The same pencil appears miraculously in different and seemingly unrelated physical contexts.

Introduction

The electron-positron scattering process $e^+ + e^- \rightarrow e^+ + e^-$ is called *Bhabha scattering*, after Homi J. Bhabha, who calculated the differential cross-section to lowest order in 1935, [4]. This calculation can be found in almost any textbook on quantum field theory and is now routinely relegated to the exercise sheets of the courses on the subject.

At lowest order, there are two Feynman diagrams contributing to the amplitude of this process, called the scattering and the annihilation diagram.



Scattering and annihilation diagrams.

In terms of the *Mandelstam variables*

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2$$

the answer obtained by Bhabha can be written in the form

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s} \left\{ \frac{1}{s^2} \left[st + \frac{s^2}{2} + (t - 2m^2)^2 \right] + \frac{1}{st} [(s+t)^2 - 4m^4] + \frac{1}{t^2} \left[st + \frac{t^2}{2} + (s - 2m^2)^2 \right] \right\}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant and m is the mass of the electron (cf. [7, pp. 2, 3]).

The three terms in the formula have an interesting interpretation in terms of the two Feynman diagrams. As the determination of the cross section involves the square modulus of the amplitude, the first and last term in the formula describe the contributions of scattering and annihilation, whereas the middle term represents the interference (or ‘exchange’) between the two virtual processes.

According to the rules of Quantum Electrodynamics (QED), the complete amplitude for Bhabha scattering appears as a sum of integrals that correspond to certain Feynman diagrams. The next order of approximation depends on the virtual processes described by the ten different 1-loop diagrams and at 2-loops there are many diagrams to consider.

The corresponding integrals in fact diverge and need to be regularised. One of the most powerful regularisation schemes is *dimensional regularisation*. All integrals are taken in D spacetime dimensions, then developed as a Laurent series in $\epsilon = \frac{4-D}{2}$ and regularised by subtracting the polar part. It was found in [6], [7] that all coefficients in the ϵ -expansion of 1-loop Feynman integrals can be represented in terms of so-called *harmonic polylogarithms*, a class of functions introduced in [22], evaluated at arguments that are *rational expressions* in variables x, y related to the Mandelstam variables

$$\begin{aligned} -s &= m^2 \frac{(1-x)^2}{x}, & x &= \frac{\sqrt{4m^2 - s} - i\sqrt{s}}{\sqrt{4m^2 - s} + i\sqrt{s}}, \\ -t &= m^2 \frac{(1-y)^2}{y}, & y &= \frac{\sqrt{4m^2 - t} - i\sqrt{t}}{\sqrt{4m^2 - t} + i\sqrt{t}}. \end{aligned}$$

In [19] the 2-loop master integrals were reconsidered and expressed in terms of Chen iterated integrals: the irrationality

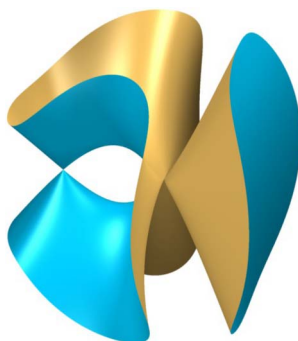
$$Q = \sqrt{\frac{(x+y)(1+xy)}{x+y-4xy+x^2y+xy^2}}$$

was needed as argument in the logarithmic forms of the differential equation associated to one special master integral. The same irrationality appears in [9], where the same 2-loop master integrals are reconsidered, but this time are expressed in terms of elliptic polylogarithms. The question arises if one can ‘undo’ the square root by a rational substitution of the form

$$x = x(s, t), \quad y = y(s, t).$$

More precisely, if we introduce a further variable z and put $z = (x + y)/Q$, we obtain, after squaring and clearing denominators, a quartic surface in the complex affine space in $\mathbb{A}_{\mathbb{C}}^3$ with equation

(1)
$$Q: z^2(1 + xy) = (x + y)(x + y - 4xy + x^2y + xy^2).$$



The quartic Q .

Remark 0.1. The quartic Q has eight singular points, all of type A_k for some k , two of which are visible in the picture above; six more singular points lie in the plane at infinity. More precisely, the singular points have the following coordinates.

Coordinates	Singularity
$(0 : 0 : 0 : 1)$	A_3
$(0 : 1 : 1 : 0), (0 : 1 : -1 : 0), (1 : 0 : 1 : 0), (1 : 0 : -1 : 0)$	A_2
$(1 : 1 : 0 : 1), (0 : 0 : 1 : 0), (1 : -1 : 0 : 0)$	A_1

As this quartic has only simple singularities (see remark above), it is birational to a K3 surface, which implies that no rational parameterisation is possible, see also [10].

Using coordinate transformations, this surface defined by the irrationality Q can be transformed in many different forms. In his talk (June 2, 2015)

at the MITP conference *Amplitudes, Motives and beyond*, Johannes Henn mentioned the particular nice surface with the equation

$$0 = 1 + \frac{x^2}{1-x^2} + \frac{y^2}{1-y^2} + \frac{z^2}{1-z^2},$$

which after clearing of denominators leads to the equation

$$B: 0 = f(x, y, z) := 1 - (x^2y^2 + y^2z^2 + z^2x^2) + 2x^2y^2z^2.$$

He posed the question if this surface is rational, i.e., if there exist non-trivial rational functions $x(s, t), y(s, t), z(s, t)$ that satisfy the above equation identically, thus providing a rational parametrisation of the surface B . The answer to this question is no. In this paper we will compute the geometric Picard lattice of a family of K3 surfaces containing B and we will show that this surface is in fact a well-known beauty, as stated in the following theorem.

Theorem. *The surface B is birational to a K3 surface with Picard number equal to 20. Its geometric Picard lattice is isomorphic to*

$$U \oplus E_8(-1)^2 \oplus \langle -4 \rangle \oplus \langle -2 \rangle.$$

It is not hard to see that the 8 points $(x, y, z) = (\pm 1, \pm 1, \pm 1)$ are ordinary double points and make up all singularities of the surface B in affine space, but more complicated singularities are present at infinity. By the reciprocal transformation

$$u = \frac{1}{x}, \quad v = \frac{1}{y}, \quad w = \frac{1}{z}$$

the surface is mapped to a surface R

$$R: 0 = 1 + \frac{1}{u^2-1} + \frac{1}{v^2-1} + \frac{1}{w^2-1}$$

which after clearing denominators produces a slightly simpler equation

$$R: 0 = u^2v^2w^2 - u^2 - v^2 - w^2 + 2$$

for the surface.



The surface B and its reciprocal R .

1. The pencil and representation as double sextic

The above surface R can be seen as a member of a nice symmetric (cf. Remark 1.1) one-parameter family of surfaces with equation

$$\mathcal{R} : 0 = 1 + s + \frac{1}{u^2 - 1} + \frac{1}{v^2 - 1} + \frac{1}{w^2 - 1} = 0 \subset \mathbb{A}^1 \times \mathbb{A}^3,$$

where we consider s as parameter. We write

$$\pi : \mathcal{R} \rightarrow \mathbb{A}^1, \quad (u, v, w, s) \mapsto s$$

for the projection and denote by $\mathcal{R}_s = \pi^{-1}(s)$ the fibre over s , so $R = \mathcal{R}_0$. Clearing denominators we can write the equation for $\mathcal{R}_s \subset \mathbb{A}^3$ as:

$$u^2v^2w^2 - u^2 - v^2 - w^2 + 2 + s(u^2 - 1)(v^2 - 1)(w^2 - 1) = 0.$$

Remark 1.1. From its defining equation, one can easily see that \mathcal{R}_s admits (at least) 48 automorphisms: those coming from changing the sign of and permuting the coordinates. More precisely, these automorphisms generate a subgroup of $\text{Aut}(\mathcal{R}_s)$ isomorphic to

$$S_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3,$$

where the action of S_3 on $(\mathbb{Z}/2\mathbb{Z})^3$ is given by permuting the coordinates.

In order to study the geometry of these surfaces, we will brutally break the symmetry and use this equation to express w^2 in terms of u, v (and s), obtaining

$$w^2 = \frac{u^2 + v^2 - 2 + s(u^2v^2 - u^2 - v^2 + 1)}{u^2v^2 - 1 + s(u^2v^2 - u^2 - v^2 + 1)}.$$

Using the birational map $(u, v, w) \mapsto (1/u, 1/v, w) =: (x, y, w)$ the threefold \mathcal{R} is seen to be birationally equivalent to the threefold given by

$$w^2 = \frac{f_1}{f_0}$$

with

$$f_i(x, y) := (1 - x^2y^2) + (s - i)(x^2 - 1)(y^2 - 1), \quad i = 0, 1.$$

Using the map $(x, y, w) \mapsto (x, y, wf_1(x, y))$, we see that \mathcal{R} is birationally equivalent to the threefold defined by the equation

$$w^2 = f_1 \cdot f_0.$$

This represents our threefold as a double cover, ramified over the union of two s -dependent quartic curves. Let us write $F_i(x, y, z)$ for the homogenization in z of the polynomial f_i , for $i = 0, 1$. The quartics in \mathbb{P}^2 with homogeneous coordinates (x, y, z) defined by F_1 and F_2 intersect in the points $(\pm 1 : \pm 1 : 1)$ with multiplicity 2 and in the points $(1 : 0 : 0), (0 : 1 : 0)$ with multiplicity 4. The last two points form also the singular locus of both curves.

The Cremona transformation γ on the base points

$$(1 : 0 : 0), \quad (0 : 1 : 0), \quad (1 : 1 : 1)$$

transforms these singular quartics into non-singular cubics B_i defined by the vanishing of the polynomials

$$G_i(x, y, z) := (x^2 + y^2)z - 2xy(x + y) + (s + 1 - i)(2x - z)(2y - z)z, \quad i = 0, 1.$$

We let $\mathbb{P} := \mathbb{P}(1, 1, 1, 3)$ be the weighted projective space on the variables x, y, z of weight 1 and a variable w of weight 3 and let \mathcal{D} to be the threefold in $\mathbb{A}^1 \times \mathbb{P}$ defined by the equation

$$(2) \quad \mathcal{D} : w^2 = G_0 \cdot G_1.$$

The fibre over s of the map $\mathcal{D} \rightarrow \mathbb{A}^1$ will be denoted by \mathcal{D}_s .

So we have shown the following result.

Proposition 1.2. *The family \mathcal{R} is birationally equivalent to the family \mathcal{D} .*

2. Singularities of the generic fibre

We first consider the generic fibre \mathcal{D}_η , i.e. the fibre over the generic point $\eta \in \mathbb{A}^1$. The surface \mathcal{D}_η is the double cover of \mathbb{P}^2 branched over the union of the two smooth cubic curves B_0, B_1 . As a consequence, the branch curve $B := B_0 \cup B_1$ and hence the surface \mathcal{R}_η , has singularities of type A_{2n-1} at a point where B_0 and B_1 have intersection multiplicity n . A direct computation learns that, for generic s , the curves B_0 and B_1 intersect in exactly four points $P_i, i = 1, 2, 3, 4$.

Point	P_1	P_2	P_3	P_4
Coordinates	$(1 : 0 : 0)$	$(0 : 1 : 0)$	$(1 : 1 : 2)$	$(-1 : 1 : 0)$
Multiplicity	3	3	2	1
Singularity	A_5	A_5	A_3	A_1

So the surface \mathcal{R}_η has exactly 4 singular points $Q_i, i = 1, 2, 3, 4$ corresponding to the singular points $P_i, i = 1, 2, 3, 4$ of the branch curve B . All singularities are of type A_k , hence we have the following corollary.

Corollary 2.1. *The minimal resolution of \mathcal{D}_η is a K3 surface.*

As is well-known (see e.g. [1]), the minimal resolution

$$\pi : S \longrightarrow \mathcal{D}_\eta$$

is obtained by replacing a singular point of type A_k by a chain of k projective lines. For example, to resolve the singularity of the branch curve B at P_1 , one needs three blow-ups and hence we get three exceptional divisors $D_{1,2}, D_{1,1}, D_{1,0}$, intersecting in a chain. The strict transform of the branch locus intersects the exceptional divisor coming from the last transform, say $D_{1,0}$, in two distinct points and does not intersect the other two. By Riemann–Hurwitz, the double cover of $D_{1,2}$ and $D_{1,1}$ is an unramified double cover of \mathbb{P}^1 , i.e., two copies of \mathbb{P}^1 ; the double cover of $D_{1,0}$ is a double cover of \mathbb{P}^1 ramified above two points, i.e., it is isomorphic to \mathbb{P}^1 . Let $E_{1,i}$ and $E_{1,-i}$ be the two components of the double cover of $D_{1,i}$, for $i = 1, 2$, and let $E_{1,0}$ the cover of $D_{1,0}$. As $D_{1,1}$ intersect $D_{1,0}$ and $D_{1,2}$, we have that $E_{1,1}$ intersect $E_{1,0}$ and $E_{1,\pm 2}$. As the name of the components are only defined up to a switch of the cover, we define $E_{1,2}$ as the component of the double cover $D_{1,2}$ intersecting $E_{1,1}$. It follows that $E_{1,-1}$ intersects $E_{1,-2}$ and $E_{1,0}$. Hence the divisors $E_{1,-2}, E_{1,-1}, E_{1,0}, E_{1,1}, E_{1,2}$ give rise to an A_5 -configuration. We will use the analogous notations for the exceptional divisors lying above the

other points Q_i , $i = 2, 3, 4$. So we have exceptional divisors $E_{i,j}$ on S lying above the point Q_i ; the divisor $E_{i,0}$ will always denote the middle one, the divisors $E_{i,\pm 1}$ will be those intersecting E_0 , for $i = 1, 2, 3$, the divisor $E_{i,\pm 2}$ will denote the divisors intersecting $E_{i,\pm 1}$, for $i = 1, 2$.

3. Some divisors on the generic fibre

It is our aim to determine the Picard lattice of the smooth model S of the generic fibre \mathcal{D}_η . Not all the divisors we present here are defined over $K = \mathbb{Q}(s)$, some of them are defined only over an algebraic extension of degree 2. The divisors we present are of three types:

- hyperplane section,
- exceptional divisors coming from the resolution of the singularities of \mathcal{D}_η ,
- pullback on S of components of the pullback on \mathcal{D}_η of plane lines that are tritangent to the branch locus B .

We will denote the hyperplane class by H . As to the exceptional divisors, we have seen that the resolution of the singularities of \mathcal{D}_η gives rise to $5 + 5 + 3 + 1 = 14$ linearly independent divisors $E_{i,j}$.

Further divisors on S can be obtained from the pullback of lines of \mathbb{P}^2 that intersect the branch locus B with even multiplicity everywhere. These pullbacks are themselves linearly equivalent to the hyperplane section, but they are reducible and their components are linearly independent to the fifteen divisors listed so far.

Let \overline{K} be a fixed algebraic closure of $K = \mathbb{Q}(s)$, and let α be a square root of $s^2 - s$ inside \overline{K} . We put $L := K(\alpha)$ and consider the following lines in \mathbb{P}_L^2 .

$$\begin{aligned} \ell_1: z = 0, & & \ell_2: z = 2x, & & \ell_3: z = 2y, & & \ell_4: z = x + y, \\ \ell_5: z = \frac{x}{s+\alpha}, & & \ell_6: z = \frac{y}{s+\alpha}, & & \ell_7: z = \frac{x}{s-\alpha}, & & \ell_8: z = \frac{y}{s-\alpha}. \end{aligned}$$

Lemma 3.1. *Let ℓ be any of the lines defined above, that is, $\ell = \ell_i$ for some $i \in \{1, \dots, 8\}$. Then ℓ intersects B with even multiplicity everywhere.*

Proof. Notice that in the equation of any of the lines ℓ_i , one can express z in terms of x or y . Then, in order to prove the statement, it is enough to substitute z with the expression in terms of the other variables in the equation of B , and check that in this way one obtains a polynomial that is a square in $\overline{L}[x, y]$. \square

Proposition 3.2. *Let ℓ_i be one of the lines defined above, and let $D_i := \pi^{-1}(\ell_i)$ denote the pullback of ℓ_i on \mathcal{D}_η . Then D_i splits into two components, L_i and L'_i , both isomorphic to ℓ_i .*

Proof. It is obvious in view of Lemma 3.1. For more details see, for example, [15, Proposition 1.2.26]. □

For every line ℓ_i we explicitly fix a component of the pullback on \mathcal{D}_η , by writing down the explicit equations.

$$\begin{aligned}
 L_1: z = 0, \quad w = 2xy(x + y) & \quad L_2: z = 2x, \quad w = 2x^2(x - y) \\
 L_3: z = 2y, \quad w = 2y^2(y - x) & \quad L_4: z = x + y, \quad w = \alpha(x - y)^2(x + y) \\
 L_5: z = \frac{x}{s + \alpha}, \quad w = xy(x - (\alpha + s)y)/s \\
 L_6: z = \frac{y}{s + \alpha}, \quad w = xy(y - (\alpha + s)x)/s \\
 L_7: z = \frac{x}{s - \alpha}, \quad w = xy(x + (\alpha - s)y)/s \\
 L_8: z = \frac{y}{s - \alpha}, \quad w = xy(y + (\alpha - s)x)/s
 \end{aligned}$$

4. Intersection numbers

These eight divisors, together with the fifteen aforementioned ones, give us a list of twenty-three divisors. The next step is to compute the intersection numbers of these divisors.

Lemma 4.1. *We have the following values for the intersection numbers of H :*

- (1) $H^2 = 2$;
- (2) $H \cdot E_{i,j} = 0$, for every i, j ;
- (3) $H \cdot L_i = 1$, for every i .

Proof. (1) By definition, H is the pullback of a line in the plane. Two lines in the projective plane always meet in one point. As \mathcal{D}_η is a double cover of the plane, the pullbacks of the lines will meet in two points. As S is birational to \mathcal{D}_η and we considered two general lines, the statement holds on S as well. Hence, using the fact that lines in the plane are all linearly equivalent, we conclude that $H^2 = 2$.

(2) One can choose a line in \mathbb{P}^2 not passing through P_i , for $i = 1, \dots, 4$. The statement follows.

(3) Two lines on the plane always meet in one point, hence the pullback of ℓ_i meets H in two points. But this pullback splits into two isomorphic components, L_i and L'_i , each of them intersecting H in one point. □

Lemma 4.2. *The intersection numbers of the exceptional divisors are as follows:*

- (1) $E_{i,j}^2 = -2$;
- (2) $E_{i,j} \cdot E_{i,j'} = 1$ (for $j \neq j'$) if and only if $|j - j'| = 1$, $E_{i,j} \cdot E_{i,j'} = 0$ otherwise;
- (3) $E_{i,j} \cdot E_{i',j'} = 0$, for every $i \neq i'$.

Proof. (1) By construction, every $E_{i,j}$ has genus 0. Since S is a K3 surface, its canonical divisor K is trivial. Then the result follows by the adjunction formula (cf. [18, Proposition V.1.5]).

- (2) Follows directly from the labeling of the components $E_{i,j}$.
- (3) If $i \neq i'$, it means that the divisors lie above distinct points of \mathcal{D}_η , and therefore they do not intersect. □

Lemma 4.3. *We have the following intersection matrix for L_1, \dots, L_8 .*

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2 \end{pmatrix}$$

Proof. By Proposition 3.2, L_i is a genus 0 curve on the K3 surface S we have $L_i^2 = -2$ by the adjunction formula. The intersection numbers $L_i \cdot L_j$ for $i \neq j$ can be explicitly computed on \mathcal{D}_η (see the accompanying MAGMA code [16]), using their defining equations. Also, we already know that if ℓ_i and ℓ_j meet in any of the points P_1, \dots, P_4 , then $L_i \cdot L_j = 0$. □

The intersection numbers $L_i \cdot E_{j,k}$ are slightly more difficult to determine, as the exceptional divisors $E_{j,k}$ only live on S . The following partial determination of these intersections will be sufficient for our purposes.

Lemma 4.4. *The following statements hold.*

- (1) *Each of the divisors L_1, L_7, L_8 intersect either $E_{1,2}$ or $E_{1,-2}$; they do not intersect the other exceptional divisors above Q_1 .*
- (2) *Each of the divisors L_1, L_5, L_6 intersect either $E_{2,2}$ or $E_{2,-2}$; they do not intersect the other exceptional divisors above Q_2 .*

- (3) L_2 intersects either $E_{2,2}$ or $E_{2,-2}$; it does not intersect the other exceptional divisors above Q_2 ; it does intersect either $E_{3,1}$ or $E_{3,-1}$; it does not intersect $E_{3,0}$ nor any exceptional divisor above Q_1 .
- (4) L_3 intersects either $E_{1,2}$ or $E_{1,-2}$; it does not intersect the other exceptional divisors above Q_1 ; it does intersect either $E_{3,1}$ or $E_{3,-1}$; it does not intersect $E_{3,0}$ nor any exceptional divisor above Q_2 .
- (5) L_4 intersect $E_{3,0}$ and $E_{4,0}$; it does not intersect any other exceptional divisor.
- (6) $E_{4,0}$ intersects L_1 and L_4 ; it does not intersect any other of the L_i 's.
- (7) L_1, L_5, \dots, L_8 do not intersect any exceptional divisor above Q_3 .
- (8) L_5, L_6 do not intersect any exceptional divisor above Q_1 .
- (9) L_7, L_8 do not intersect any exceptional divisor above Q_2 .

Proof. The statements can easily be proven by case by case analysis. As an example, we provide the argument for the first statement. The others are analogous.

The lines ℓ_1, ℓ_7, ℓ_8 intersect at the point P_1 . This implies that L_1, L_2, L_8 intersect the exceptional divisor lying above Q_1 . As none of these lines is tangent to any of the irreducible components of B in P_1 , we have that they get separated from the branch locus already after the first blow up, i.e., they intersect $D_{1,2}$. The statement follows from the definition of $E_{1,\pm 2}$. \square

5. The geometric Picard lattice of S

We let $\bar{S} = \bar{K} \otimes_K S$ and denote by Λ the sublattice of $\text{Pic } \bar{S}$ generated by the divisors $H, L_i, E_{j,k}$.

Proposition 5.1. *The lattice Λ has rank 19, signature $(1, 18)$ and its discriminant group is isomorphic to $\mathbb{Z}/12\mathbb{Z}$.*

Proof. Lemma 4.4 determines some of the intersection numbers $L_i \cdot E_{j,k}$, but not all of them. In fact, the intersection numbers concerning the exceptional divisors above Q_1 and Q_2 are only determined up to the double cover involution. The same holds for the intersection numbers $L_3 \cdot E_{3,\pm 1}$ and $L_2 \cdot E_{3,\pm 1}$. Using the fact that we only defined these exceptional divisors up to the double cover involution, we can fix some of the intersection numbers.

We define $E_{1,2}$ as the extremal exceptional divisor above Q_1 intersecting L_1 . It follows that $L_1 \cdot E_{1,2} = 1$ and $L_1 \cdot E_{1,-2} = 0$. Analogously, we define $E_{2,2}$ as the extremal exceptional divisor above Q_2 intersecting L_1 . It follows that $L_1 \cdot E_{2,2} = 1$ and $L_1 \cdot E_{2,-2} = 0$. Finally, we define $E_{3,1}$ as the extremal exceptional divisor above Q_3 intersecting L_2 . It hence follows that $L_2 \cdot E_{3,1} = 1$ and $L_1 \cdot E_{3,-1} = 0$.

These definitions leave fourteen intersection numbers pairwise undetermined. The values of these intersection numbers can only vary between 1 and 0, and hence it is possible to run through all the possible values. The Picard lattice of a K3 surface injects into the $H^{1,1}$ of the surface (in fact, Lefschetz (1,1)-theorem tells us that if X is a K3 surface over \mathbb{C} , then $\text{Pic } X \cong H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$, cf. [20, 1.3.3]). As the $H^{1,1}$ of a K3 surface is 20-dimensional, we only look for those combinations of intersection numbers giving a Gram matrix of rank less than or equal to 20.

Among all the combinations in $\{0, 1\}^{\times 7}$, four satisfy this condition. A computation shows that these four combinations all give rise to a lattice as in the statement of the result. \square

Corollary 5.2. *The lattice Λ is isometric to the lattice*

$$U \oplus E_8(-1)^{\oplus 2} \oplus \langle -12 \rangle.$$

Proof. The lattice in the statement has the same rank, signature, discriminant group, and discriminant form as Λ . Then the result follows from [21, Corollary 1.13.3]. \square

We now will show that Λ is in fact the full Picard lattice.

Lemma 5.3. *The family \mathcal{D} is not isotrivial.*

Proof. We will see in the next section that the fibers above $s = 1$ and $s = -1$ have non-isometric geometric Picard lattices (cf. subsections 6.1 and 6.3), and therefore the fibers themselves are not isomorphic as K3 surfaces. \square

Corollary 5.4. $\rho(\overline{S}) \leq 19$.

Proof. As the family \mathcal{D} is a 1-dimensional non-isotrivial family. Then, recalling that S is birational to the generic fiber \mathcal{D}_η of the family, the statement follows for example from [13, Corollary 3.2]. \square

Proposition 5.5. *The surface S has geometric Picard rank 19.*

Proof. From Corollary 5.4 we know that the geometric Picard number is at most 19; Proposition 5.1 tells us that Λ is a sublattice of $\text{Pic}(\overline{S})$ of rank 19. Hence the rank of $\text{Pic}(\overline{S})$ is exactly 19. \square

Theorem 5.6. *The lattice Λ is the whole geometric Picard lattice $\text{Pic}(\overline{S})$.*

Proof. In this proof, for sake of brevity, let P denote the geometric Picard lattice of S . Assume by contradiction that Λ is not the whole geometric

Picard lattice P of S ; from Proposition 5.5 the Picard lattice of S (here simply denoted by P) has rank 19, and therefore Λ is a proper finite-index sublattice of P . As Λ has discriminant 12 (cf. 5.1), it follows that P has discriminant 3. So P is an even lattice of rank 19 and discriminant -3 , getting a contradiction as even lattices of odd rank have even discriminant. This shows that Λ is the whole Picard lattice. \square

Corollary 5.7. *The transcendental lattice $T(S)$ of S is isometric to the lattice*

$$U \oplus \langle 12 \rangle.$$

Proof. It follows from [21, Corollary 1.13.3], noticing that the lattice in the statement satisfies the hypothesis. Note that we know the form on the discriminant group A_T as it is minus the form on $A_{NS(X)}$, which we know by Theorem 5.6. \square

6. Special fibers

For special values of s , the singularities of the branch curve B change. We will compute the geometric Picard lattice in these particular cases, obtaining examples of singular K3 surfaces with non-isometric Picard lattice. This implies that these fibers are not isomorphic, proving Lemma 5.3.

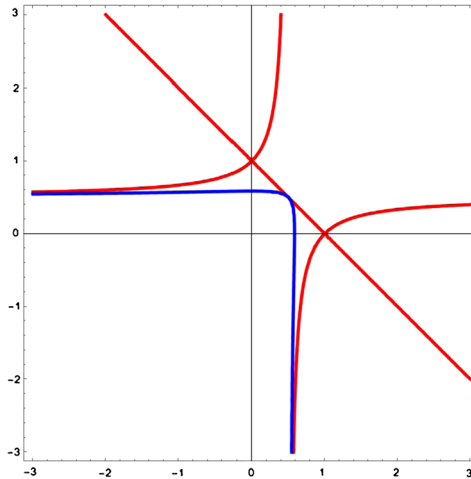
The computations performed and the arguments used in these special cases are completely analogous to the general case, and therefore they will be omitted in this paper. They are nevertheless available in the accompanying MAGMA code (cf. [16]). Notice that as these surfaces turn out to have Picard lattice with maximal rank, no particular upper bound on the rank is needed.

6.1. The case $s = 1$

The surface \mathcal{D}_1 in the weighted projective space $\mathbb{P}(1, 1, 1, 3)$ is a double cover of \mathbb{P}^2 ramified along a sextic which has exactly seven singular points:

$$\begin{aligned} Q_1 &= (1 : 0 : 0 : 0), Q_2 = (0 : 1 : 0 : 0), \\ Q_3 &= (1 : 1 : 2 : 0), Q_4 = (-1 : 1 : 0 : 0), \\ Q_5 &= (1 : 0 : 1 : 0), Q_6 = (0 : 1 : 1 : 0), \\ Q_7 &= (1 : 1 : 1 : 0). \end{aligned}$$

The points Q_1 and Q_2 are singularities of A_5 -type; the point Q_3 is of A_3 -type; the points Q_4, \dots, Q_7 are of A_1 -type.



The curve B for $s = 1$.

Consider the following five lines of \mathbb{P}^2 .

$$l_1: z = 0, \quad l_2: z = 2x, \quad l_3: z = 2y, \quad l_4: z = x, \quad l_5: z = y.$$

All these lines intersect the branch locus of \mathcal{D}_1 with even multiplicity everywhere. Let $S_1 \rightarrow \mathcal{D}_1$ be the resolution of the singularities of \mathcal{D}_1 . Proceeding as in the previous sections, we find that the (geometric) Picard lattice of S_1 is isometric to the lattice

$$U \oplus E_8(-1)^{\oplus 2} \oplus \langle -4 \rangle \oplus \langle -2 \rangle.$$

From this it also follows that the transcendental lattice $T(S_1)$ is isometric to the lattice $\langle 2 \rangle \oplus \langle 4 \rangle$ (as it is the only positive definite binary quadratic form with discriminant 8).

Remark 6.1. As the discriminant of $\text{Pic } S_1$ is -8 , we know by Elkies and Schütt (cf. [26, Section 10]) that the geometric Picard lattice of S_1 is realised over \mathbb{Q} , i.e., $\text{Pic } S_1 = \text{Pic } \overline{S_1}$. This can also be directly observed by noting that all the divisors used are defined over \mathbb{Q} .

6.2. The case $s = 0$

The surface S_0 is isomorphic to S_1 . To see this, it is enough to notice that the branch locus of S_0 is obtained by reflecting the branch locus of S_1 with respect to the line $x + y - z = 0$. Then the two surfaces have the same Picard

lattice. It follows that the geometric Picard lattice of S_0 is isometric to the lattice

$$U \oplus E_8(-1)^{\oplus 2} \oplus \langle -4 \rangle \oplus \langle -2 \rangle.$$

Notice that the surface S_0 is the fiber of our family corresponding to the surface Q (cf. (1)) we started with. As it is a singular K3 surface, its automorphism group is infinite and therefore it has more than just 48 automorphisms (cf. Remark 1.1). In [24, Section 10.2], Shimada provides a finite set of generators for the automorphisms group, consisting of the 48 symmetries and six extra automorphisms coming from reflection with respect to the walls of the ample chamber which do not correspond to -2 -vectors.

6.3. The case $s = -1$

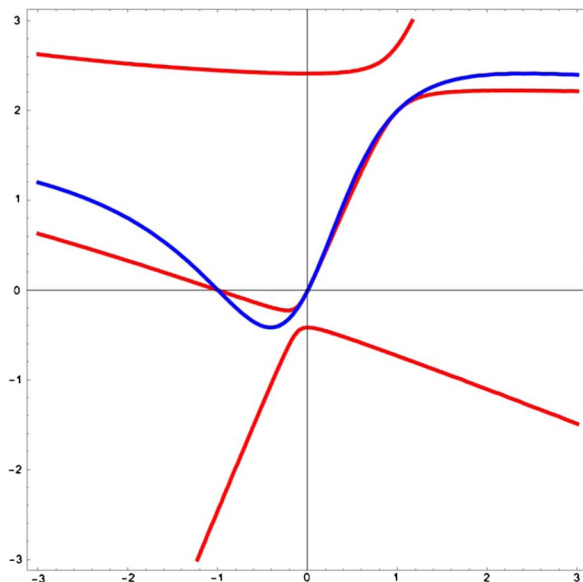
The surface X_{-1} has exactly five singular points,

$$Q_1 = (1 : 0 : 0 : 0), Q_2 = (0 : 1 : 0 : 0),$$

$$Q_3 = (1 : 1 : 2 : 0), Q_4 = (-1 : 1 : 0 : 0),$$

$$Q_5 = (0 : 0 : 1 : 0).$$

The points Q_1 and Q_2 are singularities of A_5 -type; the point Q_3 is of A_3 -type; the points Q_4, Q_5 are of A_1 -type.



The curve B for $s = -1$.

In order to generate a sublattice of the Picard lattice we use the exceptional divisors lying above the singular points, the hyperplane sections and the divisors L_1, \dots, L_8 defined before, with α a square root of 2. Using the same strategy as before we have that the geometric Picard lattice of X_{-1} is isometric to the lattice

$$U \oplus E_8(-1)^{\oplus 2} \oplus \langle -12 \rangle \oplus \langle -2 \rangle.$$

Reasoning as in Subsection 6.1, we show that $T(S_{-1})$ is isometric to $\langle 2 \rangle \oplus \langle 12 \rangle$. In [25, Theorem 1.2(1)], Shimada proves that the automorphism group of the surface is generated by 15 involutions.

6.4. The case $s = 2$

Using the same argument as in Subsection 6.2, one sees that S_2 is isomorphic to S_{-1} .

7. Transformation to the Apéry–Fermi-family

We have seen that

$$\mathcal{R}: 0 = 1 + s + \frac{1}{u^2 - 1} + \frac{1}{v^2 - 1} + \frac{1}{w^2 - 1}$$

is birationally a pencil of K3 surfaces with $U \oplus \langle 12 \rangle$ as transcendental lattice.

We will now explain how this family is related to a famous piece of mathematics that started with Apéry's spectacular proof of the irrationality of $\zeta(3)$, in which the Apéry numbers

$$A_0 = 1, A_1 = 5, A_2 = 73, A_4 = 1445, \dots, A_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

played an important role. In the paper [3], F. Beukers and C. Peters showed that the generating function

$$\Phi(\lambda) := \sum_{k=0}^{\infty} A_n \lambda^n$$

of the Apéry numbers is a period function of a certain pencil $\mathcal{A} \rightarrow \mathbb{P}^1$ of K3 surfaces. The differential operator

$$\theta^3 - \lambda(2\theta + 1)(17\theta^2 + 17\theta + 5) + \lambda^2(\theta + 1)^3,$$

of which $\Phi(\lambda)$ is the unique solution, holomorphic around 0, was identified as the Picard–Fuchs operator of the pencil, which has

$$0, \quad 17 + 12\sqrt{2}, \quad 17 - 12\sqrt{2}, \quad \infty$$

as singularities.

Somewhat later, C. Peters and J. Stienstra [23] studied the level sets of the Laurent polynomial

$$F := x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z},$$

which turned up in the theory of *Fermi surfaces*, [17]. These are affine parts of K3 surfaces, and the pencil

$$\mathcal{Z} : x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} = \xi + \frac{1}{\xi}$$

over the ξ -line has

$$\theta^3 - \xi^2(\theta + 1)(17\theta^2 + 34\theta + 20) + \xi^4(\theta + 2)^3, \quad \theta = \xi \frac{d}{d\xi}$$

as Picard–Fuchs operator, and

$$0, \quad \pm 3 \pm \sqrt{2}, \quad \infty$$

as singularities. This operator is just the pullback of the Apéry operator via the substitution $\lambda = \xi^2$. In fact, the family $\mathcal{A} \rightarrow \mathbb{P}^1$ can be identified as the *quotient* of $\mathcal{Z} \rightarrow \mathbb{P}^1$ by the map induced by

$$(x, y, z, \xi) \mapsto (-x, -y, -z, -\xi)$$

and thus provides a simpler and more symmetric description of the Apéry–Fermi pencil $\mathcal{A} \rightarrow \mathbb{P}^1$ of [3]. Obviously, the Apéry–Fermi family itself is a

pullback from the family

$$\mathcal{F} : x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} = t$$

over the t -line. Note that this family still has a symmetry, as the fibre over t is isomorphic to the fibre over $-t$ via the map $(x, y, z) \rightarrow (-x, -y, -z)$. As a result, the Picard–Fuchs operator of \mathcal{F} has a symmetry and is the pullback via the substitution $\mu = 1/t^2$ of the operator

$$\theta^3 - 2\mu(2\theta + 1)(10\theta^2 + 10\theta + 3) + \mu^2(2\theta + 1)(\theta + 1)(2\theta + 3), \quad \theta := \mu \frac{d}{d\mu},$$

with singularities at $0, 1/4, 1/36, \infty$. So this is a close relative of the Apéry operator; the holomorphic solution $\Psi(\mu)$ around 0 expands as follows:

$$\Psi(\mu) = \sum_{n=0}^{\infty} b_n \mu^n = 1 + 6t + 90t^2 + 1860t^3 + 44730t^4 + \dots$$

$$b_n := \binom{2n}{n} a_n, \quad a_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}.$$

The differential equation and the numbers b_n appeared in [14] in certain models of conductivity in crystals. The number b_n counts the number of paths of length $2n$ in the standard cubical lattice \mathbb{Z}^3 with the Euclidean distance. The series

$$\sum_{n=0}^{\infty} a_n \mu^n$$

is the period of a rational elliptic surface with singular fibres I_6, I_3, I_2, I_1 and appears in [28]. It can be realised by the family of cubics

$$(x + y)(y + z)(z + x) + \mu xyz = 0.$$

In their paper [23], Peters and Stienstra also determined the Picard lattice of the general fibre of the Apéry–Fermi family and found them to be

$$U \oplus (-E_8)^{\oplus 2} \oplus \langle -12 \rangle$$

which is exactly the lattice we found for our family of K3 surfaces! By the global Torelli theorem for K3 surfaces, we suspect the two families to be related. One can use the determination of the Picard lattice for the special

members of Section 6 and compare them with the Picard lattices of for the special members of the Apéry–Fermi family determined in [23]. This suggests the identification

$$t = \pm(2 - 4s)$$

of the fibres of the families \mathcal{R} and \mathcal{F} . And indeed, if we put

$$G(x, y, z) := \frac{1}{x^2 - 1} + \frac{1}{y^2 - 1} + \frac{1}{z^2 - 1}$$

then the substitution

$$x \mapsto \frac{1+x}{1-x}, \quad y \mapsto \frac{1+y}{1-y}, \quad z \mapsto \frac{1+z}{1-z}$$

converts G into F , up to a factor and a constant. To be precise, one has the following

Remarkable identity

$$4G\left(\frac{1+x}{1-x}, \frac{1+y}{1-y}, \frac{1+z}{1-z}\right) = F(x, y, z) - 6.$$

Hence, the above substitution maps the pencil of surfaces

$$0 = 1 + s + \frac{1}{x^2 - 1} + \frac{1}{y^2 - 1} + \frac{1}{z^2 - 1}$$

to the pencil of surfaces

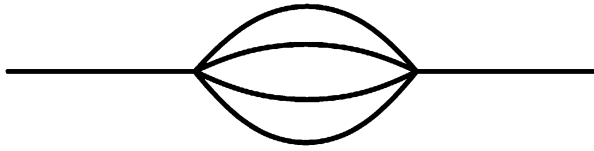
$$x + 1/x + y + 1/y + z + 1/z = 2 - 4s.$$

So we see that our family \mathcal{R} is, up to a linear transformation in the parameter, the Apéry–Fermi family \mathcal{F} ! There is much more to be said about the rich algebraic geometry around this pencil. Note that by removing denominators we obtain the family of quartics given by

$$x^2yz + yz + xy^2z + xz + xyz^2 + xy - (2 - 4s)xyz = 0.$$

We refer to [2] for a recent contribution. In that paper, Bertin and Lecacheux prove (among other things) that the generic member of the family admits 27 elliptic fibrations, and they also describe the singular fibers of all of them.

We remark that the Apéry–Fermi family and its Picard–Fuchs operator also play a role in the analysis of the equal mass scalar Feynmann diagram shown below.



Banana graph.

From the extensive literature on the subject, we mention [30], [29], [5], [8].

Another interesting occurrence of the family of K3 surfaces with the same Picard lattice is the family of *Hessian quartics* of the family of cubic surfaces

$$C_a: 0 = x_0^3 + x_1^3 + x_2^3 + x_3^3 - 3x_0^2(x_1 + x_2 + x_3)$$

described in [12, Proposition 5.7.(4)]. This gives yet another realisation of the pencil as a family of quartics in \mathbb{P}^3 .

We conclude with the remark that the appearance of the irrationality Q from the introduction is to be expected from the symmetry considerations. In fact, as already stated in [19], it is natural to introduce a variable z related to the third Mandelstam variable u by the relation

$$-u = m^2 \frac{(1-z)^2}{z}.$$

Then the well-known relation

$$s + t + u = 4m^2$$

between the Mandelstam variables translates directly into the algebraic relation

$$\frac{(1-x)^2}{x} + \frac{(1-y)^2}{y} + \frac{(1-z)^2}{z} + 4 = 0,$$

which upon expansion is just the equation

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} = 2.$$

By the above *Remarkable Identity*, this surface is the transform of the surface $R = \mathcal{R}_0$.

On this surface there is a symmetric collection of divisors and the 2-loop amplitudes appearing in Bhabha scattering seem to be special solutions of a natural differential system on this K3 surface, rather than on the affine

x, y -plane, and a theory of harmonic polylogarithms should properly be formulated as living on the above K3 surface.

So we see the ‘same’ pencil of K3 surfaces appearing over and over again, with a multitude of links to various physical contexts. A question posed by J. Stienstra [27] is if there are more direct links between these physical occurrences that would provide *a priori* explanations for the miracles.

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