A Visual Introduction to Cubic Surfaces Using the Computer Software Spicy

Duco van Straten and Oliver Labs

University of Mainz, Algebraic Geometry Group, Mainz, Germany

Abstract. At the end of the 19^{th} century geometers like Clebsch, Klein and Rodenberg constructed plaster models in order to get a visual impression of their surfaces, which are so beautiful from an abstract point of view. But these were static visualizations. Using the computer program Spicy[1], which was written by the second author, one can now draw algebraic curves and surfaces depending on parameters interactively.

Using this software and Coble's explicit equations for the cubic surface that arises as the blowing-up of the projective plane in six points it was possible for the first time to visualize how some of the 27 lines upon the cubic surface coalesce when the surface develops a double point.

When the user drags one of the six points, the equation and a raytraced image of the cubic surface are computed using external programs. As the whole process takes less than half a second, one nearly gets the impression of a continuously changing surface.

1 Introduction

A cubic surface in the real projective space $\mathbf{P}^3 := \mathbf{P}^3(\mathbf{R})$ is the vanishing set of a homogenous cubic polynomial in $\mathbf{P}^3$, i.e. it consists of all $(x : y : z : w) \in \mathbf{P}^3$, such that

$$a_0x^3 + a_1x^2y + \cdots + a_{19}w^3 = 0,$$

where $a_i \in \mathbf{R}, i = 0, 1, \ldots, 19$.

The intensive study of cubic surfaces started in 1849, when the British mathematicians Salmon and Cayley published the results of their correspondence on the number of straight lines on a smooth cubic surface ([1], [19]). In a letter, Cayley told Salmon that there could only exist a finite number of such lines – and Salmon found this number to be 27 (allowing complex lines[2]).

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1 Spicy – Space and Plane Interactive Constructive and Algebraic Geometry – is a dynamic constructive geometry software, that uses external programs like Singular ([12]) and Surf ([9]) to visualize algebraic curves and surfaces.

2 In fact, Clebsch constructed the famous Diagonal Surface in [5] and showed that it contained 27 real lines.

Shortly after that Steiner wrote a short but fruitful article ([21]) which became the basis for a purely geometrical treatment of cubic surfaces. In that paper he formulated many theorems, but for most of them he did not even indicate the idea of the proof. Many of these proofs were supplied ten years later independently by Cremona ([7]) and Sturm ([23]).

Other important contributions were made by Cayley, Schläfli, Klein, Rodenberg and Clebsch[3] – the latter gave, e.g., the explicit equation (in terms of determinants) of a covariant of order 9, which intersects the cubic surface exactly in the 27 lines (cf. appendix). This proves both the existence and the number of lines on smooth cubic surfaces.

After all these results had been established, the mathematicians of the 19^{th} century started to build plaster models of cubic surfaces in order to get a visual impression of these objects, which are so beautiful from an abstract point of view.
The objective of the computer program SPICY is to allow mathematicians of the 21^{st} century not only to make static models or movies of surfaces and curves, but also to manipulate them interactively.

For this purpose, another theorem of Clebsch ([4]) is important:

Theorem 1 *Every smooth cubic surface can be represented in the plane using 4 plane cubic curves through six points and vice versa.*[4]

As we will not only visualize cubic surfaces, but also the straight lines on them, we need a notation for these 27 lines. Over the last 150 years, several notations have been established, but for our needs, the one introduced by Schläfli in 1858 seems to be the most convenient one. In his article on the classification of cubic surfaces with respect to the number of real lines and triple tangent planes on them ([20]), Schläfli discovered a very interesting property of the 27 lines:

Notation 1 On every smooth cubic surface one can always choose two sets of six lines, say $a_1, \dots, a_6$ and $b_1, \dots, b_6$, such that the incidence diagram is as shown in fig. 1.
Let P be the plane in $\mathbf{P}^3$ spanned by two intersecting lines a_i, b_j for some $i \neq j$. As any plane, P intersects the cubic surface in a cubic curve. This curve is reducible; it consists of the two lines a_i, b_j and a third line c_{ij}. Repeating this process for any $i \neq j$, we get the remaining $\binom{6}{2} = 15$ lines c_{ij} and have thus established a notation for all the 27 lines, which is called *Schläfli's double-six notation.* The configuration of these 27 lines was a subject of intense investigation in the 19^{th} century. For a more modern approach we

[3] See [15] and [22] for a more complete list of references.

[4] For some of those surfaces, we must allow complex coordinates for some points. In this article, we will start from a representation in the real plane, so we will not be able to get all the smooth cubic surfaces, but only those on which all the lines are real.

	a_1	a_2	a_3	a_4	a_5	a_6
b_1		×	×	×	×	×
b_2	×		×	×	×	×
b_3	×	×		×	×	×
b_4	×	×	×		×	×
b_5	×	×	×	×		×
b_6	×	×	×	×	×	

Fig. 1. Incidence diagram of a double-six. An × indicates, that the two lines intersect.

refer to [17, sections 6.1.2, 6.1.3] and [11], where the 27 lines are seen as lines of the unique finite generalized quadrangle with parameters $(4, 2)$.

In our application the initial choice of the $2 \cdot 6$ lines a_i, b_i, $i = 1, \ldots, 6$, will arise in a natural way in the next section – e.g. the lines $a_1, \ldots, a_6$ will correspond to the six points in the plane.

2 Blowing-Up the Plane in Six Points

We will first construct the blowing-up of the affine plane $\mathbf{A}^2 := \mathbf{A}^2(\mathbf{R})$ at the origin $O = (0, 0)$.

Definition 1 Let (x_1, x_2) be the affine coordinates of $\mathbf{A}^2$ and let $(y_1 : y_2)$ be the projective coordinates of $\mathbf{P}^1$. We define the *blowing-up of* $\mathbf{A}^2$ *at the point* O to be the closed subset $\widetilde{\mathbf{A}^2}$ of $\mathbf{A}^2 \times \mathbf{P}^1$ defined by the equation $x_1 y_2 = x_2 y_1$. We have a natural morphism $\pi : \widetilde{\mathbf{A}^2} \longrightarrow \mathbf{A}^2$, obtained by

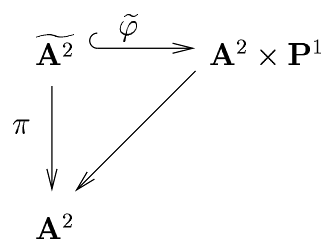

Fig. 2. Blowing-up the affine plane $\mathbf{A}^2$ in a point.

restricting the projection of $\mathbf{A}^2 \times \mathbf{P}^1$ to the first factor.

As we want to understand the blowing-up visually, we give the following properties (cf. [14, p. 28]):

Fact 1 *With the notations of the previous definition, we have:*

- *The restriction of* π *to the set* $\widetilde{\mathbf{A}^2}\backslash\pi^{-1}(O)$ *is bijective.*
- $\pi^{-1}(O) \cong \mathbf{P}^1$.
- *The points of* $\pi^{-1}(O)$ *are in* $1-1$ *correspondence with the set of lines through* O *in* $\mathbf{A}^2$ *(fig. 3).*

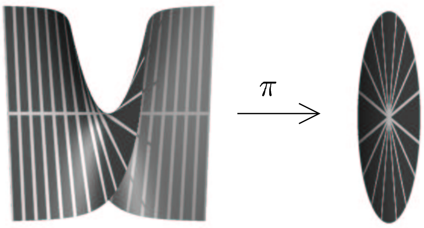

Fig. 3. Blowing-up the plane in one point. This was visualized using SPICY, but we will not explain that here; the construction is similar to the one presented in section 3.3.

Definition 2 If $C \subset \mathbf{A}^2$ is a curve in $\mathbf{A}^2$, we call

$$\widetilde{C} := \overline{\pi|^{-1}_{\mathbf{A}^2\backslash\{O\}}(C\backslash\{O\})}$$

the *strict transform* of C.

All the notions introduced in this section can be generalized (cf. [14, 163-171] or [8]), so that we can talk of the blowing-up of the projective plane $\mathbf{P}^2 := \mathbf{P}^2(\mathbf{R})$ in six points and the strict transform of a curve under this blowing-up.

This enables us to formulate the following well-known facts (for proofs, cf. [14, p. 400, 401] and [15, p. 30]):

Fact 2 *Let* $S := \{P_1, \ldots, P_6\} \subset \mathbf{P}^2$ *be a set of six points in the plane, such that no three are collinear and not all six are on a common conic. Then the blowing-up* $\widetilde{\mathbf{P}^2}$ *of the projective plane* $\mathbf{P}^2$ *in* S *can be embedded as a smooth cubic surface in projective three-space* $\mathbf{P}^3$.

Denoting by $\widetilde{\varphi} : \widetilde{\mathbf{P}^2} \hookrightarrow \mathbf{P}^3$ this embedding and by $\pi : \widetilde{\mathbf{P}^2} \hookrightarrow \mathbf{P}^2$ the projection as in fig. 2, we have:

Fact 3 *Let* $S := \{P_1, \ldots, P_6\} \subset \mathbf{P}^2$ *be a set of six points in the plane, such that no three are collinear and not all six are on a common conic. Let* $Q_i \subset \mathbf{P}^2$ *denote the unique conic through the five points* $\{P_1, \ldots, P_6\}\backslash\{P_i\}$ *for* $i = 1, 2, \ldots, 6$ *and let* $l_{ij} \subset \mathbf{P}^2$ *denote the straight line through the points* P_i *and* P_j *for* $i, j = 1, 2, \ldots, 6$, $i \neq j$. *Then the 27 lines lying on the cubic surface are as follows:*

- *The* 6 *exceptional lines over the* 6 *base points* P_i*:*

$$a_i := \widetilde{\varphi}(\pi^{-1}(P_i)) \subset \mathbf{P}^3,\ i = 1, 2, \ldots, 6.$$

- *The* 6 *strict transforms of the* 6 *plane conics* Q_i*:*

$$b_i := \widetilde{\varphi}(\widetilde{Q}_i) \subset \mathbf{P}^3,\ i = 1, 2, \ldots, 6.$$

- *The* 15 *strict transforms of the* $15 = \binom{6}{2}$ *lines* l_{ij} *joining the* P_i*:*

$$c_{ij} := \widetilde{\varphi}(\widetilde{l}_{ij}) \subset \mathbf{P}^3,\ i, j = 1, 2, \ldots, 6,\ i \neq j.$$

Furthermore, the lines a_i *and* b_i*,* $i = 1, \ldots, 6$*, intersect according to the diagram in fig. 1 (i.e. they form a double six) and the* c_{ij}*,* $i \neq j$*, are the remaining* 15 *lines in Schläfli's notation.*

3 Visualizing Cubic Surfaces using SPICY

Using the results presented in the previous section, we can study some special point configurations and some properties of the corresponding cubic surfaces and the 27 lines on them.

First, we will see how the general situation of blowing-up the plane in six points can be visualized using SPICY and then we will focus on some interesting cases, where the use of this interactive software enlightens the situation.

3.1 SPICY

The core of the computer software SPICY ([16]) is a constructive geometry program designed both for visualizing geometrical facts interactively on a computer and for including them in publications. Its main features are:

- Connection to external software like the computer algebra system SINGULAR ([12]) and the visualization software SURF ([9]), which enables the user to include algebraic curves and surfaces in dynamic constructions.
- Comfortable graphical user-interface (cf. fig. 4) for interactive constructions using the computer-mouse including macro-recording, animation, etc.
- High quality export to `.fig`-format (and in combination with external software like XFIG or FIG2DEV export to many other formats, like `.eps`, `.pstex`, etc.).

3.2 Eckardt Points

The first situation we discuss concerns a very special kind of smooth points on a cubic surface:

Definition 3 An *Eckardt Point* on a cubic surface is a smooth point, where three of its straight lines meet.

On a general cubic surface there is no Eckardt Point, but when starting from the plane representation, the following is easy to see:

Proposition 1 *Let $S := \{P_1, \ldots, P_6\} \subset \mathbf{P}^2$ be a set of six points, such that no three are collinear and not all six are on a common conic. Denote by l_{ij} the 15 lines through P_i and P_j for $i \neq j$ and by $Q_i \subset \mathbf{P}^2$ the unique conic through the five points $\{P_1, \ldots, P_6\} \backslash \{P_i\}$ for $i = 1, 2, \ldots, 6$. Then:*

- *If three of the lines l_{ij} meet in a point $P \in \mathbf{P}^2 \backslash \{P_1, \ldots, P_6\}$, then the corresponding lines $\widetilde{l_{ij}}$ on the cubic surface meet in an Eckardt Point (fig. 4).*
- *If l_{ij} touches the conic Q_i in P_j for some $i, j \in \{1, \ldots, 6\}$, $i \neq j$, then the corresponding lines $a_j := (\pi^{-1} \circ \widetilde{\varphi})(P_j)$, $b_i := \widetilde{Q_i}$ and $c_{ij} := \widetilde{l_{ij}}$ on the cubic surface meet in an Eckardt Point.*

Proof: Recall the properties of blowing-up given in fact 1. Then both parts are easy to see:

- The application $\pi^{-1} \circ \widetilde{\varphi}$ is bijective on $\mathbf{P}^2 \backslash S$.
- Both Q_i and l_{ij} have the same tangent direction in P_j, so the corresponding lines $b_i := \widetilde{Q_i}$ and $c_{ij} := \widetilde{l_{ij}}$ meet the line $a_j := (\pi^{-1} \circ \widetilde{\varphi})(P_j)$ in the same point. □

We will describe in the following how we can visualize the first of these situations using SPICY.

The Computer Algebra Part. First, we need to know from an abstract point of view, what we want to visualize. So we implement a function, say `cubicSurfaceFor6PointsWithLines`, for the computer algebra program SINGULAR, that takes the projective coordinates of the six points as an input and returns the equations of the surface and the 27 lines on it in a format that the visualization software SURF understands. This implies, in particular, that we have to choose a plane at infinity in order to get an equation in affine three-space. This function is part of the SINGULAR library `spicy.lib`, which can be downloaded from [16] or [22].

Example 1 *The output for the Clebsch Diagonal Surface (without showing the lines on it) could be, e.g.* `x^3+y^3+z^3+1-(x+y+z+1)^3`.

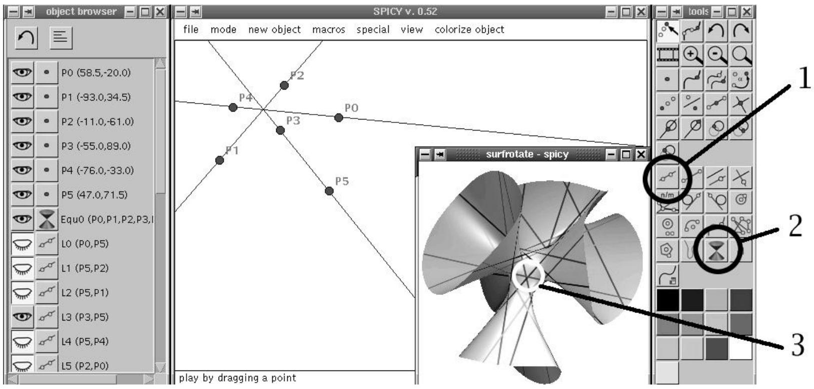

Fig. 4. A screen shot of the SPICY user interface showing three lines, that meet in a point and the corresponding cubic surface, which contains an Eckardt Point (3). Buttons 1 and 2 are used to draw the lines and the surface, respectively.

In fact, the method we use there does not calculate the blowing-up by means of computer algebra. Coble explains in [6] how the equations can be calculated from the six points simply by computing some products and sums of 3×3 determinants (cf. appendix). The detailed study of this (cf. [15]) enables us to calculate the equations defining each of the 27 lines a_i, b_i, c_{ij} separately[5]. Thus, using SPICY we can not only visualize all the 27 lines at a time, but also just some selected ones.

The SPICY **Part.** We start the software by typing `spicy` on the command line[6]. The working area will show a visualization of Poncelet's theorem, which was historically the first problem constructed using SPICY. So, to clean up, we select `file -> new construction` from the menu.

To satisfy the condition of the first case of proposition 1, we need three lines that meet in a point (in fig. 4, the Eckardt Point is marked by circle 3). To construct them, we first press button 1 (marked in fig. 4) in SPICY's *tools window* to switch to the *infinite line through two points mode* and then click twice on the working area to define these two points. After that, the program switches automatically back to the *default mode*, which allows us to drag the points and thus move the lines they define.
Repeating this process three times, we get three lines in the plane. By drag-

[5] The SINGULAR functions related to Coble's equations are packed in the library `coble.lib`, which can also be downloaded from [16] or [22].

[6] For instructions on how to download and install the software, cf. [16]. As SPICY is implemented in Java, it runs on most systems. The external software SINGULAR ([12]) is available for most systems, too. Unfortunately, the visualization software SURF, which is also developed at the university of Mainz, is only available for Linux or Unix at the moment.

ging the points, we can adjust them, so that all the three lines meet in a point (approximately[7]) as in fig. 4.

It remains to define the surface. We press button 2 (marked in fig. 4) to switch to the *surface mode*. To tell SPICY, that the calculation of the surface is based on the six points, we click on each of them once and then we press the mouse button again somewhere on the working area together with the `control` key held down.
A window shows up, where we can enter the command that tells SPICY how to calculate the surface. In our case, this will be
`{cubicSurfaceFor6PointsWithLines:#0.p#,#1.p#,#2.p#,#3.p#,#4.p#,#5.p#}`,
where `cubicSurfaceFor6PointsWithLines` is the name of the SINGULAR function, that we discussed in the previous subsection and where `#0.p#` is a place-holder for the projective coordinates of the 0^{th} point that we selected earlier etc.
Once we have done this, SPICY opens a separate window that shows the cubic surface, which is the blowing-up of the six points (fig. 4). If we do not see three lines intersecting in a point yet, we can rotate the surface just by dragging the mouse on the image.

Playing with Cubic Surfaces. We could have visualized this just by using SINGULAR and SURF – without SPICY. But here, SPICY's main feature comes into play: We can drag the points and accordingly the software recomputes the image of the cubic surface. As the whole process of calculating the new equations and the new image (fig. 5) only takes less than half a second, the user nearly gets the impression of a continuously changing surface while dragging points.

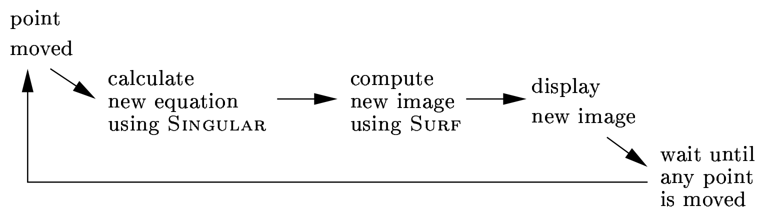

Fig. 5. The whole process of recomputing the image of the cubic surface only takes less than half a second.

This could not be done using other existing dynamic geometry software like CINDERELLA or GEONEXT, because these programs can not perform groebner basis computations or visualization of algebraic surfaces. On the other hand, standard software like MATHEMATICA and MAPLE do not have

[7] To make them meet exactly, we may enter the coordinates of the points explicitly; cf. the tutorials on [16].

any dynamic geometry features and visualization of algebraic surfaces takes much longer and is less exact than with SURF. This is the reason why we combined our interactive geometry software SPICY with the computer algebra software SINGULAR and the visualization software SURF.

3.3 Double Points

While playing with the cubic surfaces as explained in the previous section, one could discover the following: If three of the six points are collinear or if all the six points are on a common conic, then the blowing-up is no longer smooth, but it is a cubic surface in projective three-space with singularities[8]. More exactly, using the notations of fact 3, we have in the case of six points on an irreducible conic (cf. [15] for a discussion of this and the other cases[9]):

Fact 4 *Let* $S := \{P_1, \ldots, P_6\} \subset \mathbf{P}^2$ *be a set of six points in the projective plane, where no three are collinear. Then: If all the six points are on a conic (this conic is irreducible, because no three points are collinear), then the corresponding cubic surface contains an ordinary double point* Q. *More exactly, with the notations introduced in proposition 1, we have:*

$$Q = \widetilde{Q_1} = \widetilde{Q_2} = \cdots = \widetilde{Q_6}.$$

Furthermore, if $P_i', i \in \{1, 2, \ldots, 6\}$ *is a point, such that* $S_i' := S \backslash P_i \cup P_i'$ *is a set of six points, where no three are collinear and not all the six points are on a common conic, denote by* $\widetilde{Q_i'}$ *the strict transform of* Q_i *under the blowing-up of the projective plane* $\mathbf{P}^2$ *in* S_i'. *Then the straight lines*

$$a_i \text{ and } b_i' := \lim_{P_i' \to P_i} \widetilde{Q_i'}, \ i = 1, \ldots, 6,$$

coincide and all these six distinct lines meet in Q *(fig. 6).*

3.4 Cutting the Surface by a Plane

Now we describe how we can use SPICY to learn more about this surface. We cut the surface by a plane in three-space and show the resulting curve in the plane (fig. 7).

To do so, we must define a plane P in three-space and the curve in the plane $\mathbf{P}^2$ cut out of the cubic surface by P. Given the equation of a surface and a plane, it is easy to write a SINGULAR script, that computes their intersection. For the equation of the plane, we choose $x + (8 \cdot r - 4)$, where

8 Already Clebsch knew that the projective plane blown up in six points, which lie on a common plane conic, is a cubic surface with an ordinary double point ([4]).

9 If three points are collinear, the corresponding cubic surface has an A_1 singularity and if six points are on a common reducible conic, the corresponding cubic surface has an A_2 singularity.

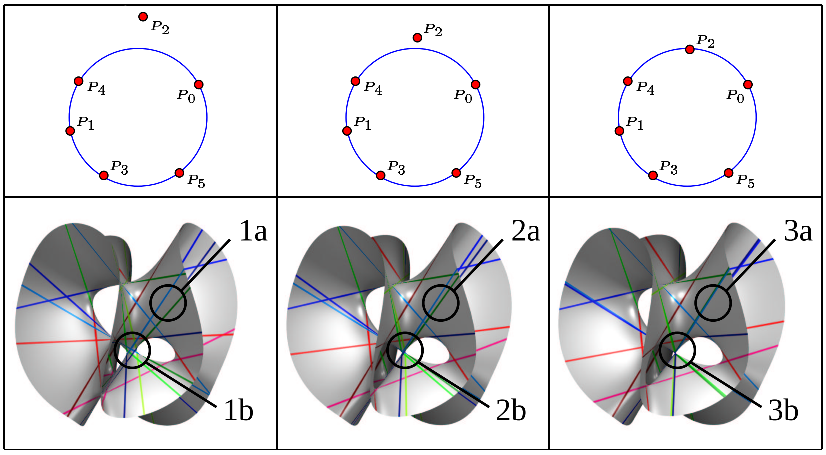

Fig. 6. The blowing-up of the projective plane in six points, such that all six are on a common conic, is a cubic surface with an ordinary double point. Note the changing of the lines, when we drag the point P_2. When P_2 lies on the conic through the other five points, $2 \cdot 6$ lines meet in the double point (1b – 3b) and six pairs of two lines coincide (1a – 3a).

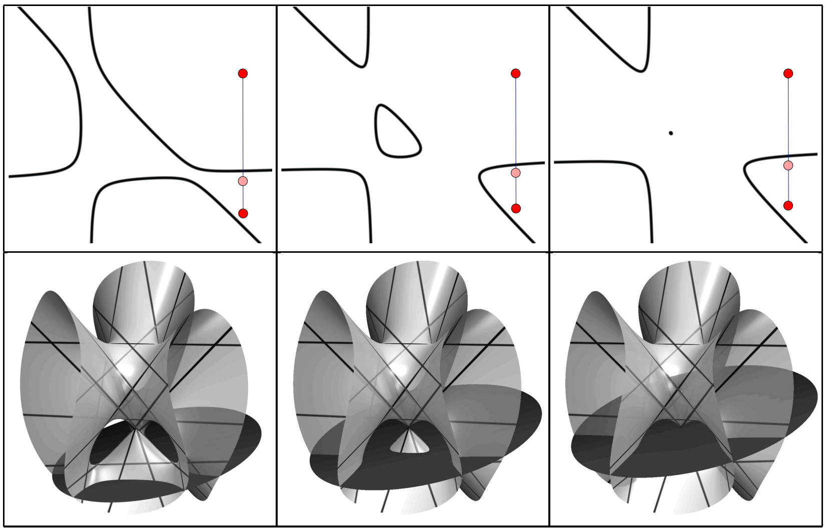

Fig. 7. Cutting the surface by a plane. The equation of the plane is just $x+(8\cdot r-4)$, where r is defined as the ratio of the length of the upper part of the segment and the length of the whole segment. So, while dragging the point on the segment, the plane passes through the surface and SPICY shows both the plane and the surface in the 3d-view and the curve cut out by the plane in the 2d-view.

$r \in [0, 1]$ is a parameter that we would like to change by dragging some point of our construction. We can do this as follows: We draw a segment and a point on this segment (fig. 7, upper part). Then we define r as the ratio of the length of the upper part of the segment and the length of the whole segment. We hide the six points and the lines between them in the SPICY construction using SPICY's so-called *object browser* (the left window in fig. 4), so that we only see the curve and the parameter segment. This visualization can now be exported in high quality to `.fig` format in order to use it for a publication.

4 Appendix

4.1 Clebsch's Explicit Equation for the Covariant of Order 9 that Meets the Cubic Surface in the 27 Lines

In [3] Clebsch gives an explicit equation for the covariant F of order 9 that meets a smooth cubic surface $u(x_1, x_2, x_3, x_4) = 0$ in projective three-space in its 27 lines in terms of three determinants. Its existence proves both the existence and the number of lines on a smooth cubic surface:

$$F = \Theta - 4\Delta T,$$

where Θ, Δ and T are defined as follows.

- $\Delta := det(u_{ij}) := det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$, the *hessian* of u.

For the other two determinants we write $\Delta_p := \frac{\partial \Delta}{\partial x_p}$ and accordingly $\Delta_{pq} := \frac{\partial^2 \Delta}{\partial x_p \partial x_q}$. Furthermore, we denote by U_{ij} the entries of the *Cramer Matrix* of Δ, i.e. $U_{ij} := (-1)^{i+j} u^{ij}$, where we denote by u^{ij} the determinant of the submatrix of Δ, where the i^{th} row and the j^{th} column are removed. With this we can define:

- $\Theta := \sum U_{pq} \Delta_p \Delta_q := \sum_p \sum_q U_{pq} \Delta_p \Delta_q,$
- $T := \sum U_{pq} \Delta_{pq} := \sum_p \sum_q U_{pq} \Delta_{pq}.$

4.2 Coble's Explicit Parametrization for the Cubic Surface and the 27 Lines on it.

Let $S := \{P_1, \ldots, P_6\} \subset \mathbf{P}^2$ be a set of six points in the plane, such that no three are collinear and not all the six points are on a common conic. For such a configuration of six points, Coble ([6]) gives explicit equations for the blowing-up of the projective plane $\mathbf{P}^2$ in S and the 45 so-called *triple tangent planes* cutting out the 27 straight lines of this cubic surface. They can be calculated as follows.

Denoting by $(P_{ix} : P_{iy} : P_{iz})$ the coordinates of $P_i \in \mathbf{P}^2$, $i = 1, 2, \ldots, 6$, we write for $i, j, k, l, m, n \in \{1, 2, \ldots, 6\}$:

$$(ijk) := det \begin{pmatrix} P_{ix} & P_{jx} & P_{kx} \\ P_{iy} & P_{jy} & P_{ky} \\ P_{iz} & P_{jz} & P_{kz} \end{pmatrix}$$

and

$$(ij, kl, mn) := (ijm)(kln) - (ijn)(klm).$$

We can now define six coefficients $\overline{x_0}, \overline{x_1}, \ldots, \overline{x_5} \in \mathbf{R}$:

$$\begin{aligned}
6\,\overline{x_0} &= +(15,24,36)+(14,35,26)+(12,43,56)+(23,45,16)+(13,52,46),\\
6\,\overline{x_1} &= -(15,24,36)+(25,34,16)+(13,54,26)+(12,35,46)+(14,23,56),\\
6\,\overline{x_2} &= -(14,35,26)-(25,34,16)+(15,32,46)+(13,24,56)+(12,45,36),\\
6\,\overline{x_3} &= -(12,43,56)-(13,54,26)-(15,32,46)+(14,52,36)+(24,35,16),\\
6\,\overline{x_4} &= -(23,45,16)-(12,35,46)-(13,24,56)-(14,52,36)+(15,34,26),\\
6\,\overline{x_5} &= -(13,52,46)-(14,23,56)-(12,45,36)-(24,35,16)-(15,34,26).
\end{aligned}$$

With these notations, Coble ([6]) gives the explicit equation of the blowing-up of the projective plane in the six points $P_1, P_2, \ldots, P_6$ as an equation in projective five-space $\mathbf{P}^5$:

$$\begin{aligned}
x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 &= 0\,, \text{ where}\\
x_0 + x_1 + x_2 + x_3 + x_4 + x_5 &= 0\,, \text{ and}\\
\overline{x_0}x_0 + \overline{x_1}x_1 + \overline{x_2}x_2 + \overline{x_3}x_3 + \overline{x_4}x_4 + \overline{x_5}x_5 &= 0\,.
\end{aligned}$$

Via the two linear equations this surface can be embedded in projective three-space $\mathbf{P}^3$, because for our set of six points S (no three points are collinear, not all six are on a conic) there are always two indices $i \neq j$, such that $\overline{x_i} \neq \overline{x_j}$.

Furthermore, the 15 triple tangent planes which meet the cubic surface in three lines c_{ab}, c_{cd}, c_{ef}, $\{a, b, c, d, e, f\} = \{1, 2, \ldots, 6\}$ each are:

$$E_{ij} := x_i + x_j = 0,\ i, j = 1, 2, \ldots, 6,\ i \neq j.$$

There are 30 other triple tangent planes. These planes E^{ij} meet the cubic surface in the three lines a_i, b_j and c_{ij}, $i, j \in \{1, 2, \ldots, 6\}$, $i \neq j$. They are linear combinations of the 15 planes given above, e.g.

$$E^{12} = (531)(461)(x_0 + x_3) - (341)(561)(x_1 + x_4) = 0.$$

[6] discusses this in more detail and [15] lists all the 45 triple tangent planes and 27 lines. Here we only give the set of 9 triple tangent planes, that we used to cut out all the 27 lines for the visualizations in this article:

$$E_{01}, E_{02}, E_{05}, E^{12}, E^{23}, E^{34}, E^{45}, E^{56}, E^{16}.$$

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