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A note on the number of periodic orbits near a resonant equilibrium point

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Abstract. In this paper we derive a formula for the number $M(n, d)$ of *complex* nonlinear normal modes in the $(n+1)$ degrees of freedom system with Hamiltonian $H_2 + \tilde{H}$, where H_2 describes a system of $(n+1)$ uncoupled oscillators with the *same* frequency (the so-called $(1:1:1:\ldots:1)$ resonance) and $\tilde{H}$ is a generic homogeneous perturbation of degree $2d$ which Poisson commutes with H_2. The formula is:

$$\sum_{n\geq 0} M(n, d)T^n = (1-T)^{-3/2}[1-(2d-1)^2T]^{-1/2}.$$

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1. Introduction

Consider a mechanical system consisting of n uncoupled harmonic oscillators with angular frequencies $\omega_1, \omega_2, \ldots, \omega_n$. It can be described by the Hamiltonian

$$H_2 : \mathbb{R}^{2n} \rightarrow \mathbb{R} \qquad (p, q) \mapsto \sum_{i=1}^{n} \omega_i(p_i^2 + q_i^2)/2. \tag{1}$$

When these frequencies ω_i are *rationally independent* then there are precisely n periodic orbits on an energy level set. They lie in the (p_i, q_i) planes and have periods $2\pi/\omega_i$. All the other orbits densely fill out tori of dimension $\geqslant 2$. In the other extreme, we say that the system described by the Hamiltonian (1) is *fully resonant* when all the ω_i are integral multiples of some fundamental frequency ω_0. In such a system *all* orbits will be periodic and the period of the general orbit will be $T_0 := 2\pi/\omega_0$.

In the neighbourhood of an equilibrium point 0 of a general Hamiltonian system, one can expand the Hamiltonian function H in a power series:

$$\begin{aligned} H &= H_2 + H_3 + H_4 + \ldots \\ &= H_2 + \tilde{H} \end{aligned} \tag{2}$$

where H_k is homogeneous of degree k in (p, q).

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So near 0 one can consider H as a 'small' perturbation of H_2. Now, we have seen that when H_2 is fully resonant, then all orbits are periodic with period T_0. However, for a 'generic' perturbation $\tilde{H}$ one expects to find on an energy level set $H = E$ only a *finite* number of periodic solutions with period near the unperturbed period T_0, forming 'families' when one varies E in a neighbourhood of 0. In the following we shall refer to such (families of) periodic orbits as non-linear normal modes.

The following problem naturally arises. How many non-linear normal modes are there in the neighbourhood of an equilibrium point and what is their structure?

Our course, the answer might, and does, depend on the precise nature of the perturbation $\tilde{H}$.

In the last few decades a number of important results of a general nature have been obtained. In the first place there is a theorem due to Weinstein [15], stating that if H_2 is definite, then there are *at least* n non-linear normal modes. This lower bound is obtained by topological considerations. It turns out, however, that in many cases with H_2 *resonant* the actual number is much bigger. Secondly, there is the realisation by Moser [14] and others that the problem of finding the normal modes can be reduced to a *finite-dimensional* problem. We now briefly sketch one such set up, where one reduces it to the problem of finding critical points of a function on $\mathbb{R}^{2n}$. (For more details and proofs see [3, 8, 13, 14].) For simplicity we take H to be of the form (2) with H_2 fully resonant.

Let $\mathscr{L} := C^1(S^1, \mathbb{R}^{2n})$ be the space of C^1-loops in $\mathbb{R}^{2n}$. ($S^1 = \mathbb{R}/2\pi\mathbb{Z}$). On $\mathscr{L}$ consider the function

$$S_T : \mathscr{L} \to \mathbb{R} \qquad \gamma \to \int \gamma^* \alpha - T(1/2\pi) \int_0^{2\pi} H \circ \gamma \, \mathrm{d}\vartheta$$

where $\alpha := \sum_{i=1}^n p_i \, \mathrm{d}q_i$. There is an obvious S^1 action on the space $\mathscr{L}$ induced by rotation, and the function S_T is invariant under this action.

Then one can verify the following facts about S_T.

(i) $\gamma \in \mathscr{L}$ is a critical point of S_T iff $\gamma(S^1) \supset \mathbb{R}^{2n}$ is a periodic orbit for H with period T.

(ii) $\mathrm{d}S_T(0) = 0$ for all T. The second derivative of S at 0 is the quadratic mapping $\mathrm{d}^2 S_T(0) : \mathscr{L} \to \mathbb{R}$ given by

$$\mathrm{d}^2 S_T(0)(u) = \int u^* \alpha - T(1/2\pi) \int_0^{2\pi} H_2 \circ u \, \mathrm{d}\vartheta.$$

(iii) So, by (i) and (ii), if T is not in $\mathbb{Z}T_0$, then $\mathrm{d}^2 S_T(0)$ is non-degenerate and if $T = T_0$, then the null space $V \subset \mathscr{L}$ of $\mathrm{d}^2 S_T(0)$ can be identified with the solutions of the H_2 system, which is isomorphic to $\mathbb{R}^{2n}$.

(iv) One can invoke here the *splitting lemma with parameters* (see [6]), which states that S_T in a neighbourhood of $0 \times T_0$ is right equivalent to $G_T(v) + Q(z)$, where $v \in V$ and $z \in Z$, a complement to V in $\mathscr{L}$ and Q is a non-degenerate quadratic form. The function $G_T : V \to \mathbb{R}$ is invariant under the natural S^1 action on $V = \mathbb{R}^{2n}$, and has the property that $\mathrm{d}^2 G_T(0) = 0$ (see also [13]).

(v) Hence the critical points of G_T are in one-to-one correspondence with the periodic orbits of H with period T. We call the set of equations $\mathrm{d}G_T(x) = 0$ the *periodicity equations*. The geometrically distinct periodic orbits are in one-to-one correspondence with the S^1 orbits of solutions of the periodicity equation. Alternatively, one can consider G_T as a function on the orbit space $\mathbb{R}^{2n}/S^1$ and the periodic orbits as critical points of this function on orbit space.

The problem now is: how does one determine this function G_T? Recall that by a

The problem now is: how does one determine this function G_T? Recall that by a sequence of symplectic coordinate transformations we can bring H into Birkhoff normal form to order k (see [8]) (we say 'H is BNF(k)'), meaning that $H = H_2 + H_3 + \ldots + H_k + R$ with $\{H_2, H_i\} = 0$, $i = 2, 3, \ldots, k$. When $\{H_2, \tilde{H}\} = 0$ we simply say that H is in Birkhoff normal form ('H is BNF'). Then one has the following:

(vi)

$$H \text{ is BNF}(k) \Rightarrow G_T = T_0 H_2 - TH + \mathrm{O}(k+1)$$

$$H \text{ is BNF} \Rightarrow G_T = T_0 H_2 - TH = T(\lambda H_2 - \tilde{H})$$

where $\lambda = T_0/T - 1$. In this last case the periodicity equations are $\lambda \, \mathrm{d}H_2(x) = \mathrm{d}\tilde{H}(x)$. Hence, by considering λ as a Lagrange multiplier we see the following.

(vii) For a system in Birkhoff normal form, the nonlinear normal modes on a given 'energy level set' $H_2 = E$ correspond exactly to the critical points of the perturbation $\tilde{H}$ on this energy level set.

In the case that G_T is finitely determined in the class of S^1-invariant functions (and in the case that $\{H_2, \tilde{H}\} = 0$, $\tilde{H}$ polynomial), the periodicity equations $\mathrm{d}G_T = 0$ are equivalent to a set of polynomial equations. Therefore, in these cases it makes sense to look for *complex solutions* of the periodicity equations. (Alternatively, one could redo the above analysis replacing S^1 by $\mathbb{C} - \{0\}$, $\mathbb{R}^{2n}$ by $\mathbb{C}^{2n}$, etc.) The obvious advantage of looking for these 'complex nonlinear normal modes' is that their number will be less dependent on the detailed coefficients entering in the perturbation $\tilde{H}$. Furthermore, this number of complex nonlinear normal modes is clearly an *upper bound* for the number of real ones.

The rest of this paper is devoted to the computation of this number for the $(1:1:1:\ldots:1)$ resonance. To be more precise, we assume the following.

(I) H_2 is of the form (1) with all ω_i equal to some ω_0.

(II) The system is in Birkhoff normal form, i.e. $\{H_2, \tilde{H}\} = 0$.

(III) $\tilde{H}$ is homogeneous of degree k and generic in some appropriate sense. (It follows from (I) and (II) that $k = 2d$ for some integer d; cf lemma 2.1.)

To satisfy the curiosity of the reader we list, in table 1, this number of complex normal modes for a small number of degrees of freedom n and order of perturbation d. (For general formulae see §6.)

The organisation of the paper is as follows. In §2 we reformulate the problem of counting the nonlinear normal modes on a given energy level set via the orbit space as counting critical points of a map $\pi: U \to \mathbb{C}$. To see that π is a C^∞-fibre bundle outside a finite set of critical values we need some control of π at infinity. We do this by describing a compactification of π to a map $\Pi: X \to \mathbb{P}^1$ in §3. In §4 we then use the complex version of Morse theory, called Lefschetz theory, to relate the critical points of π to Euler characteristics of some explicit projective varieties. In §5 we compute these Euler characteristics by computing the Chern classes of their tangent bundles. This then leads to explicit formulae which are given in §6, where we also discuss applications and possible generalisations.

Table 1. The number of normal modes for n degrees of freedom at dth order of perturbation.

$2d$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
2	1	2	3	4	5
4	1	6	39	284	2205
6	1	14	255	5260	114 605

2. The orbit space for the (1 : 1 : . . . : 1) resonance

Consider the Hamiltonian system of $(n+1)$ uncoupled harmonic oscillators with the same frequency. After a (complex) symplectic coordinate change and time rescaling this system is described by the Hamiltonian

$$H_2 : \mathbb{C}^{2(n+1)} \to \mathbb{C}$$
$$(x_0, \ldots, x_n, y_0, \ldots, y_n) \mapsto \sum_{i=0}^{n} x_i y_i.$$

This Hamiltonian generates a $\mathbb{C}^*$ action which is given by

$$\mathbb{C}^* \times \mathbb{C}^{2(n+1)} \to \mathbb{C}^{2(n+1)} \qquad (\lambda, (x_i, y_j)) \mapsto (\lambda x_i, \lambda^{-1} y_j).$$

The orbit space Y is the space whose points correspond to the closed orbits of the above $\mathbb{C}^*$ action. It has a natural structure of an analytic space.

Lemma 2.1. The orbit space Y is isomorphic to the affine cone $\subset \mathbb{C}^m$, $m = (n+1)^2$, over the Segre embedding (cf examples 2.10 and 2.14 on p 13 of [4]):

$$\sigma : \mathbb{P}^n \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{m-1}$$
$$(x_0 : x_1 : \ldots : x_n ; y_0 : y_1 : \ldots : y_n) \mapsto (x_0 y_0 : x_1 y_0 : \ldots : x_i y_j : \ldots : x_n y_n).$$

In particular, Y has a unique singular point at the origin.

Proof. This is immediate, because the ring of polynomial functions which are invariant under the $\mathbb{C}^*$ action is generated by the $\Pi_{ij} := x_i y_j$. These functions define a mapping from $\mathbb{C}^{2(n+1)}$ to $\mathbb{C}^m$ which is constant on the orbits of the $\mathbb{C}^*$ action and separates the closed ones. Hence the image in $\mathbb{C}^m$ can be identified with the orbit space Y. □

Note that the dimension of Y is $2n+1$, but in $\mathbb{C}^m$ it is defined by the vanishing of the $\binom{n+1}{2}^2$ 2×2 minors of the matrix $\Pi = (\Pi_{ij})$. Hence, for $n \geqslant 2$, this number is much bigger than the codimension of Y in $\mathbb{C}^m$, i.e. Y is not a complete intersection.

Next we study the energy levels E_ε defined by $H_2 = \varepsilon$ inside Y. Note that the function $H_2 = \sum_{i=0}^{n} \Pi_{ii}$ is *linear* on the ambient space $\mathbb{C}^m$ of Y. Let $\rho : Y - \{0\} \to \mathbb{P}^n \times \mathbb{P}^n$ be the radial projection map.

Lemma 2.2.

(i) The space $\mathscr{k} := \{z \in \mathbb{P}^n \times \mathbb{P}^n \mid H_2(z) = 0\}$ is a *smooth* hyperplane section of $\mathbb{P}^n \times \mathbb{P}^n$.

(ii) For each $\varepsilon \neq 0$ the radial projection map ρ gives an isomorphism

$$\rho : E_\varepsilon \subset Y \to U := \mathbb{P}^n \times \mathbb{P}^n - \mathscr{k}.$$

Proof. Statement (i) follows from the fact that $\{H_2 = 0\}$ is transverse to $\mathbb{P}^n \times \mathbb{P}^n$, as can be checked by a direct computation which we omit. For statement (ii) see also figure 1: the energy level set E_ε is the intersection in $\mathbb{C}^m$ of Y and the hyperplane $H_2 = \varepsilon$. For $\varepsilon \neq 0$ this hyperplane does not pass through 0, the singular point of Y, so there is a unique line between a point of E_ε, $\varepsilon \neq 0$, and 0. This sets up an injective map from E_ε to $\mathbb{P}^n \times \mathbb{P}^n$ which clearly has as image the set U. □

Figure 1 is a schematic representation of the state of affairs in lemma 2.1 and lemma 2.2. Note the two rulings of $\mathbb{P}^n \times \mathbb{P}^n$.

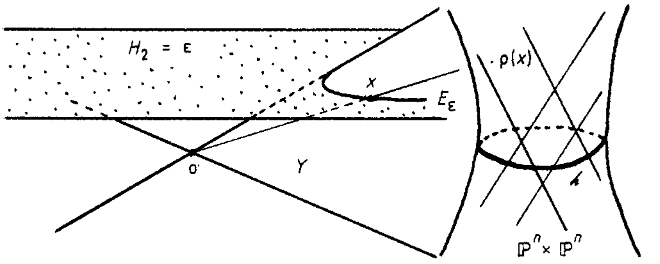

Figure 1. The orbit space Y and its associated structure.

Let V_d be the vector space of *homogeneous* polynomials of degree d in the invariants Π_{ij} and let $\tilde{H} \in V_d$. We consider the perturbed Hamiltonian system with Hamiltonian $H = H_2 + \tilde{H}$. We are interested in the complex nonlinear normal modes of this system, which lie on a given energy level set E_ε, $\varepsilon \neq 0$. By the result quoted in the introduction, these periodic orbits correspond to the critical points of the perturbation $\tilde{H}$ on the level sets of H_2. We translate this into the following lemma.

Lemma 2.3. Let $\pi: U \to \mathbb{C}$ be given by $z \mapsto \tilde{H}(y)/(H_2(y)^d)$ where $y \in \rho^{-1}(z)$. Then the following are equivalent:

(i) $y \in Y - \{0\}$ is a complex nonlinear normal mode for the system with Hamiltonian $H = H_2 + \tilde{H}$

(ii) the point $z = \rho(y)$ is a critical point for the map π.

Proof. Note first that the map π is indeed well defined because $\tilde{H}$ and $(H_2)^d$ are homogeneous of degree d. By property (vii) of §1 y is such a periodic orbit if and only if the function $\tilde{H}$ has a critical point on the energy level set E_ε, i.e. if the differentials of the functions H_2 and $\tilde{H}$ are linearly dependent on y. By differentiation of the map π this is seen to be equivalent to $\rho(y) = z$ being a critical point for π (as one could expect from lemma 2.2). $\square$

The usual way to study critical points of a real function on a smooth manifold is to use Morse theory (see [9]). Here we will use the complex analogue of Morse theory, called Lefschetz theory, to count the number of critical points of the map $\pi: U \to \mathbb{C}$. As the space U is not compact, we need some control over the behaviour of π 'at infinity' to conclude that π is a smooth fibration over the complement of a finite bad set $\subset \mathbb{C}$. This can be done in a nice way by compactifying the map π.

3. Compactification of the map π

As a first step we can extend the map π to a map

$$\pi_e: \mathbb{P}^n \times \mathbb{P}^n - D \to \mathbb{P}^1$$
$$z \to (\tilde{H}(z) : H_2(z)^d)$$

where $D := \{z \in \mathbb{P}^n \times \mathbb{P}^n \mid \tilde{H}(z) = H_2(z) = 0\}$ is the intersection of the hyperplane section $\mathscr{h}$ with the hypersurface $\tilde{H} = 0$ of degree d.

Lemma 3.1. The set W_0 defined by

$W_0 := \{\tilde{H} \in V_d \mid$ the hypersurface $\tilde{H} = 0$ in $\mathbb{P}^{m-1}$ intersects $\mathscr{k}$ transversely$\}$

is Zariski open dense in V_d.

Proof. We can use the d-tuple embedding $v_d : \mathbb{P}^{m-1} \to \mathbb{P}^N$, $N = \binom{m-1+d}{d} - 1$, whose components are the monomials of degree d in m variables (see example 2.12 on p 13 of [4]). The linear hyperplanes in $\mathbb{P}^N$ correspond to the degree-d hypersurfaces in $\mathbb{P}^{m-1}$. Then Bertini's theorem (the forerunner of Sard's theorem, see theorem 8.18 on p 179 of [4]) states that a general hyperplane intersects $v_d(\mathscr{k})$ transversely, i.e. $\mathscr{k}$ and $\tilde{H} = 0$ intersect transversely. □

From now on we assume that $\tilde{H} \in W_0$. So in that case the space D is smooth and around a point $p \in D$ we can find local coordinates $v_1, v_2, \ldots, v_{2n}$ on $\mathbb{P}^n \times \mathbb{P}^n$ such that one has

$$D = \{v_1 = v_2 = 0\} \qquad H_2 = v_1 \qquad \tilde{H} = v_2.$$

When we restrict the map π_e to the (punctured) (v_1, v_2) plane it is given in these coordinates by $(v_1, v_2) \mapsto (v_2 : v_1^d) \in \mathbb{P}^1$. Hence the fibre of this map over the point $(\lambda : \mu)$ is a curve in the (v_1, v_2) plane with equation $\lambda v_1^d - \mu v_2 = 0$. When we vary the point $(\lambda : \mu) \in \mathbb{P}^1$ we cover the whole plane.

Fibres of π_e for two different values of d are shown in figure 2.

We see that the problem of extending π_e to a map from the whole of $\mathbb{P}^n \times \mathbb{P}^n$ to $\mathbb{P}^1$ is that the closures of these curves all contain the origin, so that point 'does not know where to go'. One can overcome this difficulty by *blowing up*. The blowing up of a space X along a subspace Y results in a space Z which is the same as X, except that Y is replaced by the projectified normal bundle $\mathbb{P}(N_Y)$ of Y in X (see pp 28–30 and 163–171 of [4]). There is a map $bl : Z \to X$, the blowing down map, which contracts $\mathbb{P}(N_Y)$ to Y. The strict transform of a subvariety $W \subset X$ is defined to be the closure of $bl^{-1}(W \backslash Y)$ in Z. For example, blowing up the (v_1, v_2) plane at the origin has the effect of replacing this point by the projective line of directions through the origin. The blow-up space Z in this case is a manifold that can be covered by two copies of $\mathbb{C}^2$. In coordinates for these charts the blow-down map is the following:

$$\begin{array}{lll} \text{Copy I} & bl : \mathbb{C}^2 \to \mathbb{C}^2;\ (w_1, w_2) \mapsto (w_1 w_2, w_1) & \multirow{2}{*}{$\left.\right\} = (v_1, v_2).$} \\ \text{Copy II} & \qquad (u_1, u_2) \mapsto (u_1, u_1 u_2) & \end{array}$$

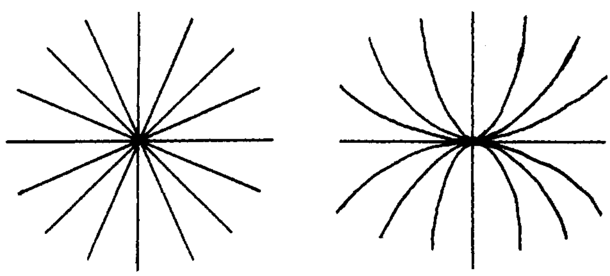

Figure 2. Fibres of π_e around a point of D for $d = 1$ (left) and $d = 2$ (right).

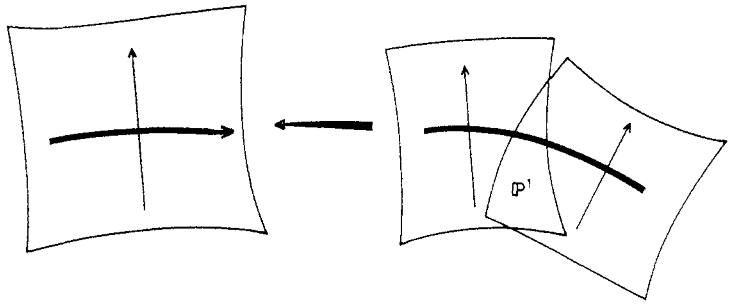

Figure 3. Blowing up $\mathbb{C}^2$ in a point.

(In fact, the blow-up space Z is obtained by gluing these two copies of $\mathbb{C}^2$ via these maps, and turns out to be isomorphic to the cotangent bundle of $\mathbb{P}^1$.) The $\mathbb{P}^1$ that is blown down is given on the left of figure 3 by $w_1 = 0$, and on the right by $u_1 = 0$.

One can pull back the family of curves $\lambda\ v_1^d - \mu v_2 = 0$ to the space obtained by blowing up the origin. The resulting curves in figure 4 are given by the following equations:

for $d = 1$ $\qquad \lambda w_1^{d-1} w_2^d - \mu = 0;$

for $d = 2$ $\qquad \lambda u_1^{d-1} - \mu u_2 = 0.$

We see that after one blow up we get for $d = 2$ the same picture as before the blow up, but with d replaced by $d - 1$. If we blow up the family of curves with $d = 1$, they become separated on the blow up. The nice thing about blowing up is that one can *repeat* the process. As soon as the curves become separated, one can extend the map. So after blowing up d times at a point, the map

$$\mathbb{C}^2 - \{0\} \to \mathbb{P}^1 \qquad (v_1, v_2) \mapsto (v_2 : v_1^d)$$

can be extended to a map $\tilde{\mathbb{C}}^2 \to \mathbb{P}^1$.

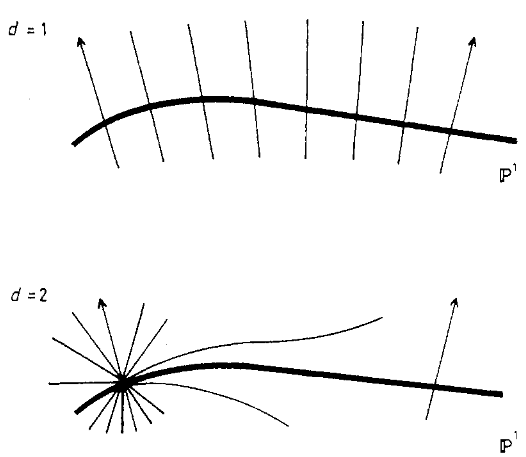

Figure 4. Blowing up the fibres of figure 2.

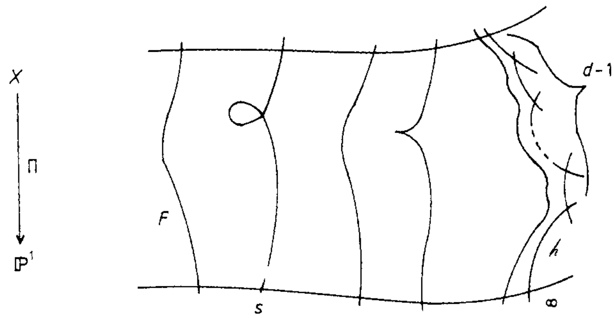

Figure 5. Fibres of the map Π.

Remember that the above map was a representation of what happened with the map $\pi_e: \mathbb{P}^n \times \mathbb{P}^n - D \to \mathbb{P}^1$ in a two-dimensional slice transverse to a point of D. The above arguments then provide the following proposition.

Proposition 3.2. Let $\tilde{H} \in W_0$. The map $\pi_e: \mathbb{P}^n \times \mathbb{P}^n - D \to \mathbb{P}^1$ can be extended to a map $\Pi: X \to \mathbb{P}^1$. The space X is obtained from $\mathbb{P}^n \times \mathbb{P}^n$ by blowing up d times along a space that is isomorphic to D. The fibre $F_\infty := \Pi^{-1}(\infty)$, $\infty = (1:0)$, consists of (the strict transform of) the hyperplane $\hbar$, together with a chain of $(d-1)$ $\mathbb{P}^1$ bundles over (spaces isomorphic to) D, which intersect each other in copies of D. The $\mathbb{P}^1$ bundle over D introduced in the last blow up intersects all fibres transversely. □

The schematic picture in figure 5 may clarify the situation.

4. Topology of the map Π

In §3 we compactified the map $\pi: U \to \mathbb{C}$ to some map $\Pi: X \to \mathbb{P}^1$. As in ordinary Morse theory, there is a relation between the topology of X and the critical points of the map Π. By (1.4), these critical points correspond to the complex non-linear normal modes we want to count.

Lemma 4.1. Let $\tilde{H} \in W_0$. Then there is a finite set $S = S' \cup \{\infty\} \subset \mathbb{P}^1$ such that:

(i) $\Pi: X - \Pi^{-1}(S) \to \mathbb{P}^1 - S$ is a C^∞ fibre bundle with fibre F diffeomorphic to a smooth intersection of $\mathbb{P}^n \times \mathbb{P}^n$ and a degree d hypersurface in $\mathbb{P}^{m-1}$

(ii) $\pi: U - \pi^{-1}(S') \to \mathbb{C} - S'$ is a C^∞ fibre bundle with fibre diffeomorphic to $F - D$, where F is as above.

Proof. The set S of non-regular values of Π is finite, by Bertini–Sard. Because X is compact, it follows from the Ehresmann fibration theorem that Π is a C^∞ fibre bundle away from S. The fibre over the point t is equal to (the strict transform of) the intersection of $\mathbb{P}^n \times \mathbb{P}^n$ with the degree d hypersurface $\tilde{H} - tH_2^d = 0$. This proves (i), and (ii) follows immediately from (i). □

By the fibration property (if $f: X \to Y$ is a fibre bundle with fibre F, then $\chi(X) = \chi(Y)\chi(F)$; see for example, p 182 of [2]) we find

$$\chi(X - \Pi^{-1}(S)) = \chi(F)\chi(\mathbb{P}^1 - S). \tag{4.1}$$

Next we look at the special fibres F_s, $s \in S$. These fibres will be singular in general. If the critical points on F_s are all *isolated*, then it is not so difficult to relate the topology of F_s to F. In fact one has in general

$$\chi(F_s) = \chi(F) + (-1)^{\dim_{\mathbb{C}}(X)}\mu_s \tag{4.2}$$

where $\mu_s := \sum_{x \in F_s} \mu_x$ and μ_x is the so-called Milnor number of the germ of the map Π around $x \in X$.

Intermezzo 4.2. The Milnor number μ is a number that is attached to a germ $f:(\mathbb{C}^P, 0) \to (\mathbb{C}, 0)$ with an isolated critical point. It is the most important invariant of f and can be defined as follows:

$$\mu := \dim_{\mathbb{C}}(\mathcal{O}/J(f))$$

where $\mathcal{O} := \mathbb{C}\{x_1, x_2, \ldots, x_p\}$ is the ring of convergent power series near the origin in $\mathbb{C}^P$ and $J(f) := (\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_p)$ is the Jacobian ideal in $\mathcal{O}$, generated by the partial derivatives of f. This number μ determines the relation between the topology of $f = 0$ and $f = t$, $t \neq 0$. To be precise, one can find a representative of f on a space of the form $B(\varepsilon, \eta) := \{x \in \mathbb{C}^P \mid \|x\| \leq \varepsilon,\ |f(x)| \leq \eta\}$ such that:

(i) the fibres $F_t := f^{-1}(t)$ are transverse to $\{x \in \mathbb{C}^P \mid \|x\| = \varepsilon\}$;

(ii) the fibre F_0 is contractible.

Then one has the following fundamental result of Milnor (see [10]).

(i) $f: B(\varepsilon, \eta) - F_0 \to \{t \in \mathbb{C} \mid |t| \leq \eta,\ t \neq 0\}$ is a C^∞ locally trivial fibre bundle with fibre F.

(ii) The restriction $f: \{x \in \mathbb{C}^P \mid \|x\| = \varepsilon,\ |f(x)| \leq \eta\} \to \{t \in \mathbb{C} \mid |t| \leq \eta\}$ is a trivial fibre bundle with fibre ∂F.

(iii) The fibre F has the homotopy type of a bouquet of μ spheres of dimension $(p-1)$.

For more details we refer to the standard works on this subject, like [1, 10]. An immediate consequence of the above local structure is (4.2) mentioned above. We say that f has a *non-degenerate* critical point if its Milnor number μ is equal to 1. This is the same as saying that f has a Morse point at 0, i.e. after a coordinate change one has $f = x_1^2 + x_2^2 + \ldots + x_p^2$.

Lemma 4.3. There is a Zariski open dense subset $W_2 \subset W_0 \subset V_d$ such that for $\tilde{H} \in W_2$ the map $\Pi: X - \Pi^{-1}(\infty) \to \mathbb{P}^1 - \{\infty\}$ has only non-degenerate critical points (with distinct Π values).

Proof. Let $Z \subset \mathbb{P}^N$ be the image of the d-tuple embedding of $\mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{m-1}$ (cf lemma 3.1). The hypersurfaces $H_2^d = 0$ and $\tilde{H} = 0$ can be considered as linear hyperplanes in $\mathbb{P}^N$, hence as elements of the dual space $(\mathbb{P}^N)^*$. So the family of hypersurfaces $s\tilde{H} - tH_2^d = 0$ for $(s:t) \in \mathbb{P}^1$ corresponds to a line L in this $(\mathbb{P}^N)^*$. We let $\mathbb{P}$ be the projective space of all lines L through the point $a := [H_2^d = 0] \in (\mathbb{P}^N)^*$ and let $p: (\mathbb{P}^N)^* - \{a\} \to \mathbb{P}$ be the radial projection map. Consider the set $\Gamma := \{(x, H) \in Z \times (\mathbb{P}^N)^* \mid H \cap Z \text{ is singular at the point } x\}$. It is easy to see that dim $\Gamma = N - 1$. (cf pp 179–80 of [4]). By a local calculation one can show that the set Δ, the projection of Γ in $(\mathbb{P}^N)^*$, also has dimension $N - 1$ and that a smooth point of Δ exactly corresponds to $H \cap Z$ having precisely one ordinary double point. The radial projection map induces a map $p:\Delta - \{a\} \to \mathbb{P}$ between spaces of the same dimension $N-1$. The complement of the ramification locus of this map now is a Zariski open dense set. Points of this set correspond to lines $L \subset (\mathbb{P}^N)^*$ of the form $s\tilde{H} - tH_2^d = 0$ which intersect Δ transversely (away from a), i.e. to choices of $\tilde{H} \in W_0$ such that the corresponding map $\Pi: X - \Pi^{-1}(\infty) \to \mathbb{P}^1 - \{\infty\}$ has only non-degenerate critical points (with distinct values). (We omit some further details, which are standard anyway.) $\square$

Let $W_1 \subset W_0$ be the set of $\tilde{H} \in W_0$ such that the associated map $\Pi: X - \Pi^{-1}(\infty) \to \mathbb{P}^1 - \{\infty\}$ has only isolated critical points. Because W_1 contains W_2, it is also (Zariski) dense in V_d.

Definition 4.4. For a $\tilde{H} \in W_1$ we define a number M by

$$M := \sum_{s \in S'} \mu_s.$$

So M is the number of critical points of the map $\pi: U \to \mathbb{C}$, counted with multiplicity equal to the Milnor number. Note that for $\tilde{H} \in W_2$ all multiplicities are equal to 1.

Theorem 4.5. Let $\tilde{H} \in W_1$. Then one has

$$M = \chi(\mathbb{P}^n \times \mathbb{P}^n) - \chi(\mathscr{k}) - \chi(F) + \chi(D).$$

Proof. Choose small discs $B(s) \subset \mathbb{P}^1$ around the points $s \in S$ and let $B := \bigcup_{s \in S} B(s)$. Now we have by Mayer–Vietoris the following formula:

$$\chi(X) = \chi(X - \Pi^{-1}(S)) + \chi(\Pi^{-1}(B)) - \chi((X - \Pi^{-1}(S)) \cap \Pi^{-1}(B)).$$

But $(X - \Pi^{-1}(S)) \cap \Pi^{-1}(B) = \Pi^{-1}(B - S)$. As Π is a fibration over the union of the punctured discs $B - S$, and the Euler characteristic of the circle is zero, we see that the last term of the above formula is actually zero. Furthermore, the set B contracts to S, so $\Pi^{-1}(B)$ contracts to $\bigcup_{s \in S} F_s$. By proposition 3.2 we can compute the Euler characteristic of X: each time D is replaced by its projectivised normal bundle, the Euler characteristic increases by χ(bundle)-$\chi(D)$; the bundle is a $\mathbb{P}^1$ bundle over D, so by the fibration property the first term is $2\chi(D)$. Thus

$$\chi(X) = \chi(\mathbb{P}^n \times \mathbb{P}^n) + d\chi(D) \qquad \chi(F_\infty) = \chi(\mathscr{k}) + (d-1)\chi(D)$$

where the second result is obtained by a similar argument.

Using this in combination with (4.1) and (4.2) then gives, after some rearrangements, the formula of the theorem. (Alternatively, one can use part (ii) of Lemma 4.1 and do the above calculation immediately for the original map $\pi: U \to \mathbb{C}$.) $\square$

The result in theorem 4.5 relates M, which, by lemma 2.3 and definition 4.4, is equal to the number of complex nonlinear normal modes of the Hamiltonian system $H_2 + \tilde{H}$ for $\tilde{H} \in W_2$, to Euler characteristics of some explicit spaces. In §5 we compute these numbers using Chern classes.

5. Euler characteristics

Every topological space X has cohomology groups $H^k(X) := H^k(X, \mathbb{Z})$, which should be thought of as the group of codimension k cycles. The intersection product

$H^k(X) \otimes H^m(X) \to H^{k+m}(X)$ makes $H^*(X) := \bigoplus_{k \geq 0} H^k(x)$ into a ring. When X is a smooth compact connected manifold of dimension n, then there is a natural isomorphism $\deg : H^n(X) \to \mathbb{Z}$. A map $f: X \to Y$ between smooth compact manifolds of dimension n or respectively m induces two maps:

(I) $f^*: H^k(Y) \to H^k(X)$ pull back of cycles

(II) $f_*: H^k(X) \to H^{m-n+k}(Y)$ Gysin map

which are related to each other by the projection formula:

$$f_*(f^*xy) = xf_*y.$$

If f is an inclusion map, then $[X] := f_*(1_X) \in H^{m-n}(Y)$ is called the cohomology class of X in Y. For all these facts we refer to any standard text book on algebraic topology like [2, 7].

Furthermore, every complex vector bundle E on X carries Chern classes $c_i(E) \in H^{2i}(X)$ which measure the non-triviality of E. The total Chern class is defined as $c(E) = \sum c_i(E) \in H^*(X)$. Two important properties of the Chern class are:

(i) if $f: X \to Y$ is a map and E a vector bundle on Y then $c(f^*E) = f^*c(E)$

(ii) if $0 \to E \to F \to G \to 0$ is a short exact sequence of vector bundles, then $c(F) = c(E)c(G)$.

When X is a complex manifold, then its tangent bundle T_X has a complex structure and one puts $c(X) := c(T_X)$. The topological Euler characteristic $\chi(X)$ is related to $c(X)$ by $\chi(X) = \deg(c_n(X))$, where $n = \dim_{\mathbb{C}} X$. As an example, consider the space $\mathbb{P}^n$. Then $H^*(\mathbb{P}^n) = \mathbb{Z}[x]/(x^{n+1})$, where $x \in H^2(\mathbb{P}^n)$ is the class of a linear hyperplane. The total Chern class of $\mathbb{P}^n$ is given by $c(\mathbb{P}^n) = (1+x)^{n+1}$, and indeed $n+1 = \chi(\mathbb{P}^n) = \deg(c(\mathbb{P}^n))$. (For all these facts on Chern classes we refer to [5, 11] or any other textbook on the subject.)

Similarly, $H^*(\mathbb{P}^n \times \mathbb{P}^n) = \mathbb{Z}[x, y]/(x^{n+1}, y^{n+1})$, as readily follows from the Künneth formula, and $c(\mathbb{P}^n \times \mathbb{P}^n) = (1+x)^{n+1}(1+y)^{n+1}$.

Consider now our Segre embedding of $\mathbb{P}^n \times \mathbb{P}^n$ in $\mathbb{P}^m$, $m = (n+1)^2 - 1$. Because each of the factors $\mathbb{P}^n$ is embedded linearly, the cohomology class of the linear hyperplane section $\hbar$ is $x + y$. Let X_d be the smooth intersection in $\mathbb{P}^m$ of $\mathbb{P}^n \times \mathbb{P}^n$ and a hypersurface of degree d and let $i: X_d \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n$ be the inclusion map. The tangent bundle of X_d sits in an exact sequence

$$0 \to T_{X_d} \to i^* T_{\mathbb{P}^n \times \mathbb{P}^n} \to N \to 0$$

where N is the normal bundle of X_d in $\mathbb{P}^n \times \mathbb{P}^n$.

Because X_d is a degree-d hyperplane section of $\mathbb{P}^n \times \mathbb{P}^n$ one has $c(N) = i^*(1 + d(x+y))$. Using the projection formula and the multiplicative property of Chern classes over short exact sequences we find

$$i_*c(X_d) = [d(x+y)/(1+d(x+y))](1+x)^{n+1}(1+y)^{n+1} \in \mathbb{Z}[x, y]/(x^{n+1}, y^{n+1}). \quad (5.1)$$

Similarly, if $X_{d,e}$ is the smooth intersection of $\mathbb{P}^n \times \mathbb{P}^n$ and hypersurfaces of degree d and e in $\mathbb{P}^m$ we get

$$i_*c(X_{d,e}) = [de(x+y)^2/(1+d(x+y))(1+e(x+y))](1+x)^{n+1}(1+y)^{n+1}. \quad (5.2)$$

Corollary 5.1. Let $\tilde{H} \in W_1$. Then one has

$M =$ coefficient of $x^n y^n$ in $[(1+(x+y))(1+d(x+y))]^{-1}(1+x)^{n+1}(1+y)^{n+1}$.

In particular, M depends only on n and d and we put $M = M(n, d)$.

Proof. The result in theorem 4.5 expresses M in Euler characteristics of the spaces $\mathbb{P}^n \times \mathbb{P}^n$, $\mathcal{k}$, F and D. But $\mathcal{k} = X_1$, $F = X_d$ and $D = X_{1,d}$, so we can use (5.1) and (5.2) and take top Chern classes. After some rearrangements we then find the above formula. □

The fact that this final formula is simpler than the intermediate steps suggests that there might be a simpler argument for obtaining the result of corollary 5.1.

6. Final results

Corollary 5.1 gives a formula for the number $M = M(n, d)$ of complex nonlinear normal modes in the $(n+1)$ degrees of freedom system with Hamiltonian $H_2 + \tilde{H}$, where H_2 is the $(1:1:1:\ldots:1)$ resonance and $\tilde{H}$ is a generic homogeneous perturbation of degree $2d$ which commutes with H_2. We now will study these numbers $M(n, d)$ a little closer.

Theorem 6.1. The numbers $M(n, d)$ have the following properties.

(i) $M(n, d)$ is a polynomial of degree $2n$ in d whose leading coefficient is $\binom{2n}{n} d^{2n}$.

(ii) The generating function $F(T) := \sum_{n \geq 0} M(n, d) T^n$ is given by

$$F(T) = (1 - T)^{-3/2}[1 - (2d-1)^2]^{-1/2}.$$

(iii) Define polynomials a_m in d by $a_m := M(m-1, d)/m$, $m \geq 1$. Then the a_m satisfy the following recursion relation:

$$a_1 = 1$$

$$a_{2k} = 1 - 2r(a_1 a_{2k-1} + a_2 a_{2k-2} + \ldots + a_k a_k)$$

$$a_{2k+1} = 1 - 2r(a_1 a_{2k} + a_2 a_{2k-1} + \ldots + a_k a_{k+1})$$

where $r := d(1-d)$.

Proof. Statement (i) is obtained immediately from corollary 5.1. For statement (ii) note that

coefficient of $x^n y^n$ in $A(1+x)^{n+1}(1+y)^{n+1} = \operatorname{Res}_{\{0\}}(A[(1+x^{-1})(1+y^{-1})]^{n+1})$.

(In our case $A = [(1+\sigma)(1+d\sigma)]^{-1}$, $\sigma = x + y$.)

So the generating function can be written as

$$F(T) = \sum_{n \geq 0} M(n, d) T^n = T^{-1} \operatorname{Res}_{\{0\}}\left(A \sum_{m \geq 0} [(1+x^{-1})(1+y^{-1})T]^m\right)$$

$$= T^{-1}(1/2\pi \mathrm{i})^2 \iint \{Axy[xy - (1+x)(1+y)T]\}^{-1} \,\mathrm{d}x\, \mathrm{d}y$$

where the integration is over (something homologous to) a small torus $|x| = \varepsilon$, $|y| = \varepsilon$. (I thank F Beukers for showing me this trick.) This double integral can be computed in two steps. First integrate out the x, using the residue theorem (around $x = T(1+y)/[y(1-T) - T]$). Then we are left with the y integral, which has as denominator a quadratic polynomial in y (with coefficients depending on T). Using the residue theorem again we find the above formula for $F(T)$.

Statement (iii) is obtained by observing that the function $G(T) := (2r)^{-1}[1 + 4rT/(1-T)]^{-1/2}$ has as derivative the function $F(T)$; hence the coefficient of T^m in $G(T)$ is precisely a_m. The recursion formula then follows by comparing coefficients of

$$G(T)^2 = \Big(\sum_{n\geq 0} a_m T^m\Big)^2 = (2r)^{-2} + r^{-1}(T + T^2 + T^3 + \ldots). \quad \square$$

Corollary 6.2.

(i) $\lim_{n\to\infty} (M(n, d)/(2d-1)^{2n}) = 1$;

(ii) the polynomials a_m have integral coefficients and in fact $M(n, d)$ is an odd multiple of $(n+1)$.

Proof. Statement (i) follows from part (ii) of theorem 6.1 by looking at the radius of convergence of the power series $F(T)$. Statement (ii) follows from part (iii) of theorem 6.1 by induction. $\square$

We now give a list of the first five a_i polynomials, computed via the recursion formula in part (iii) of theorem 6.1:

$$\begin{aligned}
a_1 &= 1\\
a_2 &= 1 - d + d^2\\
a_3 &= 1 - 2d + 4d^2 - 4d^3 + 2d^4\\
a_4 &= 1 - 3d + 9d^2 - 17d^3 + 21d^4 - 15d^5 + 5d^6\\
a_5 &= 1 - 4d + 16d^2 - 44d^3 + 86d^4 - 116d^5 + 104d^6 - 56d^7 + 14d^8.
\end{aligned} \tag{6.1}$$

Remark 6.3. The number of complex nonlinear normal modes tends to be rather big, as the list in the introduction already indicates. It is important to find how many of these can be real, because those are the only ones which correspond to physically relevant periodic orbits. Unfortunately I do not have any results in this direction, but I believe that for an open set of coefficients for $\tilde{H}$ all $M(n, d)$ complex orbits actually are *real*. For one and two degrees of freedom it is easily checked to be the case. A very interesting case is that of the (1:1:1) resonance with a quartic perturbation commuting with H_2 (so $n = 2$, $d = 2$). Here we find 39 complex nonlinear normal modes. In [12, 13] Montaldi *et al* study Hamiltonian systems with symmetry near an equilibrium point. Usually the group action will force all frequencies ω_i to be equal. Furthermore, from symmetry considerations alone one can deduce the existence of a number of nonlinear normal modes. Montaldi *et al* discuss a model of the methane molecule, which has as an essential subsystem such a (1:1:1) resonance with a quartic perturbation (which has tetrahedral symmetry). From symmetry considerations it follows that there are always at least 27 nonlinear normal modes, but for special choices of the perturbation (an open set in coefficient space) there can be 12 more, so indeed 39 in total. This example actually was the initial motivation for this paper.

Remark 6.4. One can imagine several generalisations of the above results. In the first place one can ask what happens for arbitrary perturbations $\tilde{H}$ instead of homogeneous ones. One would expect that if the lowest-order term of $\tilde{H}$ is generic, then the number of complex nonlinear normal modes is not changed by the higher-order terms. Secondly, one can ask what happens for the other resonances. Part of the constructions of this paper can be carried out in this more general context: the orbit space of a general resonance is isomorphic to a weighted homogeneous cone over a product of two weighted projective spaces. For a perturbation which is quasi-homogeneous in the appropriate sense one can imitate the construction of the map $\Pi : X \rightarrow \mathbb{P}^1$, but now X is no longer a smooth space, which complicates the Chern class computation. Hopefully these ideas will be substantiated in a future publication.

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