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Komplexe Analysis

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Zusammenfassung. ...

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Introduction by the Organisers

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Workshop: Komplexe Analysis

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Abstracts

Arctic Computation of Monodromy

Duco van Straten
(joint work with K. Jung)

We consider a Lefschetz fibration $f : \mathcal{X} \longrightarrow \mathbb{C}$ with critical set $\Sigma \subset \mathcal{X}$ and set of critical values $D := f(\Sigma) := \{p_1, p_2, \ldots, p_N\} \subset \mathbb{C}$. The fibre over each p_i has a single A_1-singularity. By choosing a base point $b \in \mathbb{C}$ and paths connecting b the critical values in the usual way we get a natural basis (up to sign) in the relative group $H_n(\mathcal{X}, X_b)$ consisting of *Lefschetz thimbles*, the traces of the cycles $\delta_i \in H_n(X)$ that vanish at p_i, see [6]. The Picard-Lefschetz formula implies that the monodromy representation $\rho : \pi_1(\mathbb{C} \setminus D, b) \longrightarrow Aut(H_n(\mathcal{X}, X_b))$ is determined by the intersection numbers $(\delta_i \cdot \delta_j)$, which are conveniently summarized in the Stokes-Seifert matrix S, see for example [1].

We are interested in the case where $\mathcal{X} = (\mathbb{C}^*)^n$ and $f \in \mathbb{C}[x_0^{\pm}, x_1^{\pm}, \ldots, x_n^{\pm}]$ a Laurent-polynomial with Newton-diagram Δ. It follows from the Bernstein-Koushnirenko formula that the number of critical points N is equal to $n!Vol(\Delta)$.

The paper [7] provides a beautiful way of seeing the vanishing cycles and their intersection for the case $n = 2$. In this case $X := X_b$ is a Riemann surface of genus $g :=$ number of internal lattice points of Δ with $r :=$ number of lattice points on the boundary of Δ punctures. The exact sequence

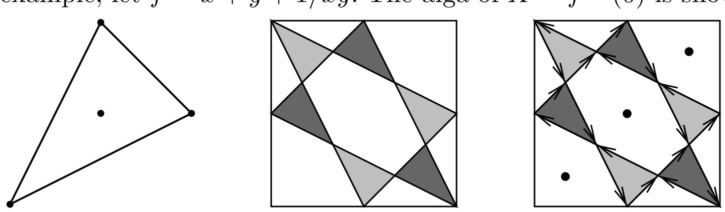
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(\mathcal{X}) & \longrightarrow & H_2(\mathcal{X}, X) & \longrightarrow & H_2(X) & \longrightarrow & H_1(\mathcal{X}) & \longrightarrow & 0 \\ & & =\downarrow & & =\downarrow & & =\downarrow & & =\downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^N & \longrightarrow & \mathbb{Z}^{2g+r-1} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 \end{array}$$

leads to the relation $N = 2g + r$, which is equivalent to Pick's formula.

The *amoeba* of a set $X \subset (\mathbb{C}^*)^2$ is defined as the image under the map $(\mathbb{C}^*)^2 \longrightarrow \mathbb{R}^2, (x, y) \mapsto (\log |x|, \log |y|)$ and is well studied, see e.g. [4], [8], [9]. In the *Harnack case* the amoeba has g holes and r *tentacles* that approach its spine, which is a tropical curve that lies in the tropical plane $\mathbb{R}^2$. The map from X to its amoeba is then two-to-one.

The *alga* of a set $X \subset (\mathbb{C}^*)^2$ is the image under the map $(\mathbb{C}^*)^2 \longrightarrow \mathbb{R}^2, (x, y) \mapsto (arg(x), arg(y)) \in \mathbb{T}$. Here $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^2$ is the real 2-torus of arguments, which we call the *arctic or argtic plane*, which complements to the usual tropical plane.

As an example, let $f = x + y + 1/xy$. The alga of $X = f^{-1}(0)$ is shown below



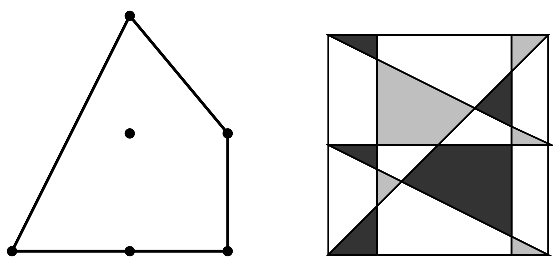
Newton diagram of $f = x + y + 1/xy$ and alga of $X = f^{-1}(0)$.

The bounding straight lines are circles on $\mathbb{T}$ and are the images of the asymptotic boundary circles that go 'around' the tentacles, and are orthogonal to the boundary edges of Δ. In the example, the map to its alga is generically one-to-one, and from the alga one can reconstruct a model of the Riemann surface X, very much like the construction of a Seifert surface of a link. The map f has $(1,1)$, (ω,ω),(ω^2,ω^2) ($\omega = \exp(2\pi i/3)$) as critical points, which are mapped by the alga map into the three 'free' regions complementary to the alga. Moreover, if we move the base point along a straight line towards one of the critical values, one can observe that the alga changes its shape and closes up the corresponding hole. From this we see that the cycles that vanish along these paths are exactly the cycles on the abstract Riemann surface that surround these free regions. From this one sees that in the example the vanishing cycles intersect pairwise in three points.

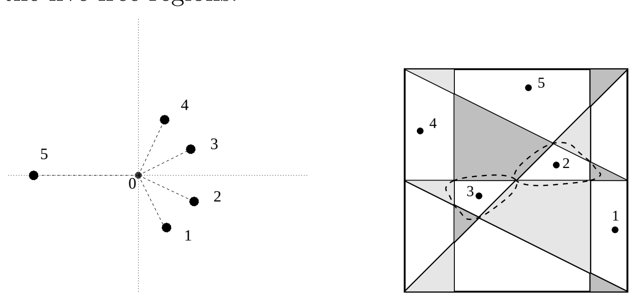
The beautiful observation of [7] is that this state of affairs holds in much greater generality, although the precise limits of applicability are not clear yet.

One starts by drawing arctic lines, orthogonal to the sides of Δ (as many as the integral length of that side). These lines come with an orientation in the direction of the outward normals to Δ. One avoids triple intersections of these lines using appropriate parallel shifts. We bi-color, according to orientation, the regions that are enclosed by oriented polygons. (The dual graph is a bipartite graph on the torus and leads to what is called a *brane tiling* in the physics literature.)

In this way one can obtain, almost without calculation, the monodromy for many Newton-diagrams, including the 16 reflexive polytopes in dimension two, [5]. We give one example.



The alga of $-x+y+1/xy+1/y+x/y=0$ is close to the above picture, and the five critical points, ordered according to increasing argument, land under the alga projection in the five free regions.



The five critical values and $(\delta_2 \cdot \delta_3) = \pm 1$

From this one reads off all intersection numbers and obtains the following Stokes-Seifert matrix

$$S = \begin{pmatrix} 1 & -1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which leads to the monodromy $h := S^{-t} \cdot S$ around infinity that has all eigenvalues equal to 1, and a single Jordan-block of size two, as it should be.

Mirror symmetry relates these families of Laurent polynomials to certain 3-dimensional toric singularities Y, which geometrically are cones over toric surfaces in their anti-canonical embedding. The geometry of the vanishing cycles is then related to exceptional collections of line bundles on these surfaces, [3]. The first example $f = x + y + 1/xy$ corresponds to $Y = \mathbb{C}^3/(\mathbb{Z}/3)$, which is the cone over $\mathbb{P}^2$ in its anti-canonical embedding. For general $\mathbb{C}^3/A$ with A a finite abelian subgroup of $SL_3(\mathbb{C})$, corresponds to the case that Δ a triangle, see [10].

Literatur

[1] V. Arnold, S. Gusein-Zade, A. Varchenko, *Singularities of differentiable maps. Volume II: Monodromy and asymptotics of integrals* Monographs in Mathematics, 83. Boston, MA etc.: Birkhuser Verlag, (1988).

[2] C. van Enckevort, D. van Straten, *Monodromy calculatons of fourth order equations of Calabi-Yau type*, `math.AG/0412539`

[3] A. Hanany, C. Herzog, D. Vegh, *Brane Tilings and exceptional collections*, `hepth/0602041`

[4] M. Gross, *Amoebas of complex curves and tropical curves.* [CA] Guest, Martin (ed.) et al., UK-Japan winter school 2004–Geometry and analysis towards quantum theory. Lecture notes from the school, University of Durham, Durham, UK, January 6-9, 2004. Yokohama: Keio University, Department of Mathematics. Seminar on Mathematical Sciences 30, 24-36 (2004).

[5] K. Jung, D. van Straten, *Arctic Monodromy Calculation for reflexive Polytopes in Dimension two.*, in preparation.

[6] S. Lefschetz, *L'analysis situs et la gomtrie algbrique. (Collection de monographies sur la théorie des fonctions.)*, Paris: Gauthier-Villars,(1924).

[7] B. Feng, Y-H. He, K. Kennaway, C. Vafa, *Dimer Models from Mirror Symmetry and Quivering Amoebæ*, `hep-th/0511287`

[8] G. Mikhalkin, *Amoebas of algebraic varieties and tropical geometry.* Donaldson, Simon (ed.) et al., Different faces of geometry. New York, NY: Kluwer Academic/Plenum Publishers. International Mathematical Series (New York) 3, 257-300 (2004).

[9] M. Passare, H. Rulgård, *Amoebas, Monge-Ampre measures, and triangulations of the Newton polytope.*, Duke Math. J. 121, No.3, 481-507 (2004).

[10] K. Ueda, M. Yamazaki, *A Note on Brane Tilings and McKay Quivers*, `math.AG/0605780`.

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