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Deformation of singular lagrangian subvarieties

Christian Sevenheck · Duco van Straten

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Abstract. We investigate deformations of lagrangian manifolds with singularities. We introduce a complex similar to the *de Rham*-complex whose cohomology calculates deformation spaces. This cohomology turns out to be constructible in many cases. Examples of singular lagrangian varieties are presented and deformations are calculated explicitly.

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1. Introduction

In this paper, we develop some ideas of a deformation theory of singular lagrangian subvarieties. Lagrangian submanifolds are quite fundamental objects, so in a sense it is natural to extend the study of them to a larger class of objects which are allowed to have singularities. This has been done by Arnold, Givental and others ([Giv88]). However, not much is known on the behavior of lagrangian singularities under deformations. The aim of this article is to describe the spaces of infinitesimal deformations and obstructions of a lagrangian subvariety and to perform calculations for some concrete examples. It turns out that the lagrangian property of a space has a strong influence on its deformations, e.g., there are examples of spaces X with $\dim(T^1_X) = \infty$, which have nevertheless a versal deformation space for the lagrangian deformations.

In the sequel, we will consider the following situation: Let M be a $2n$-dimensional symplectic manifold over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ (that is, a C^∞ or complex analytic manifold of real resp. complex dimension $2n$ endowed with a closed, non-degenerated 2-form ω, holomorphic in the second case) and L a reduced analytic subspace of dimension n, given by an involutive ideal sheaf $\mathcal{I}$, i.e. an ideal sheaf satisfying $\{\mathcal{I}, \mathcal{I}\} \subset \mathcal{I}$ where $\{\ ,\ \}$ denotes the Poisson bracket corresponding to ω. This condition ensures that L is a lagrangian submanifold in a neighborhood of each of its smooth points. A lagrangian deformation of L will be a deformation in the usual sense (a flat family $L_S \to S$) with the additional condition that all

C. Sevenheck, D. van Straten
FB Mathematik, Johannes-Gutenberg-Universität Mainz, 55099 Mainz, Germany
(e-mail: `sevenheck@mathematik.uni-mainz.de`; `straten@mathematik.uni-mainz.de`)

fibers are lagrangian subvarieties of M. More precisely, we will call a diagram

$$\begin{tikzcd}
 & M \arrow[rr, hook] \arrow[ddl, bend left] & & M \times S \arrow[ddl, bend left] \\
L \arrow[ur, hook] \arrow[rr] \arrow[d] & & L_S \arrow[ur, hook] \arrow[d] & \\
\{*\} \arrow[rr] & & S &
\end{tikzcd}$$

a lagrangian deformation of L iff $L_S \to S$ is flat and $\{\mathcal{I}_S, \mathcal{I}_S\}_S \subset \mathcal{I}_S$. Here $\mathcal{I}_S$ is the ideal sheaf defining L_S in $M \times S$ and $\{\ ,\ \}_S$ is the Poisson structure defined on $M \times S$ by the form $\omega_S := p^*\omega$, $p : M \times S \to M$ being the canonical projection. This definition can be formalized using the language of deformation functors (see [Sev99] and [Sch68]). The more formal approach yields the definition of morphisms of deformations, in particular, two deformations $L_S \subset M \times S$ and $L_T \subset M \times T$ are called equivalent iff there is an isomorphism $F : M \times S \to M \times T$ satisfying $F^*\omega_T = \omega_S$ (i.e., inducing a family of symplectomorphism parameterized by S), mapping L_S to L_T and specializing to the identity over the zero fibre. In case that M is simply connected (and Stein for $\mathbb{K} = \mathbb{C}$), F is induced by a family of time 1 maps of hamiltonian vector field, see lemma 3.

The tangent space to the functor of lagrangian deformations of L (that is, the space of lagrangian deformations of L over $Spec(\mathbb{K}[\epsilon])$ up to those induced by hamiltonian vector fields of the ambient manifold) will be denoted by LT^1_L. However, we will focus our attention to the local case mainly, that is, we will study the *sheaf* $\mathcal{LT}^1_L$ of lagrangian deformations of L.

2. The complex $\mathcal{C}^\bullet$

We start with a slightly more general situation: Let $\mathcal{I} \subset \mathcal{O}_M$ be an involutive ideal sheaf, $\mathcal{O}_L$ the structure sheaf of the subvariety L described by $\mathcal{I}$ and denote by $\mathcal{L} := \mathcal{I}/\mathcal{I}^2$ the conormal sheaf. The formula $\{\mathcal{I}^i, \mathcal{I}^j\} \subset \mathcal{I}^{i+j-1}$, which can be easily verified, shows that there are well-defined operations

$$\begin{array}{rcl} \mathcal{L} \times \mathcal{O}_L & \longrightarrow & \mathcal{O}_L \\ (g, f) & \longmapsto & \{g, f\} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{L} \times \mathcal{L} & \longrightarrow & \mathcal{L} \\ (g_1, g_2) & \longmapsto & \{g_1, g_2\} \end{array}$$

The bracket on $\mathcal{L}$ is antisymmetric and satisfies the Jacobi identity, thus making $\mathcal{L}$ into a (infinite dimensional) Lie algebra over $\mathbb{K}$. The first operation acts as a derivation on $\mathcal{O}_L$, i.e. satisfies $\{g, f \cdot h\} = \{g, f\}h + f\{g, h\}$ for $g \in \mathcal{L}$ and $f, h \in \mathcal{O}_L$. This implies that it can be rewritten as

$$\mathcal{L} \to \mathcal{D}er(\mathcal{O}_L, \mathcal{O}_L) = \Theta_L$$

and one checks that this is a morphisms of Lie-algebras. In such a situation $\mathcal{L}$ is called a *Lie algebroid* (see [Mac87] and [Käl98]).

Definition 1. *Let $\mathcal{C}_L^p$ be the following $\mathcal{O}_L$-module*

$$\mathcal{C}_L^p := \mathcal{H}om_{\mathcal{O}_L}\left(\bigwedge^p \mathcal{L}, \mathcal{O}_L\right)$$

and define a differential:

$$\begin{aligned}&(\delta(\phi))\left(h_1 \wedge \ldots \wedge h_{p+1}\right) := \\ &\textstyle\sum_{i=1}^{p+1} (-1)^i \left\{h_i, \phi\left(h_1 \wedge \ldots \wedge \widehat{h_i} \wedge \ldots h_{p+1}\right)\right\} \\ &+ \sum_{1\le i<j\le p+1} (-1)^{i+j-1} \phi\left(\left\{h_i, h_j\right\} \wedge h_1 \wedge \ldots \wedge \widehat{h_i} \wedge \ldots \wedge \widehat{h_j} \wedge \ldots \wedge h_{p+1}\right)\end{aligned}$$

It is a straightforward computation to check that $\delta \circ \delta = 0$, so we get a complex. Following [Mac87], it is called the standard complex for the Lie algebroid $\mathcal{L}$. Remark that $\mathcal{C}^0 = \mathcal{O}_L$ and $\mathcal{C}^1 = \mathcal{H}om_{\mathcal{O}_L}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_L) =: \mathcal{N}_L$, the normal sheaf of $\mathcal{I}$ in $\mathcal{O}_M$. For the definition of δ, the fact that $\mathcal{I}$ is involutive is essential: the second term would not make sense otherwise.
We may define a product on the complex $(\mathcal{C}^\bullet, \delta)$:

$$\begin{aligned}\mathcal{C}^p \times \mathcal{C}^q &\longrightarrow \mathcal{C}^{p+q}\\ (\Phi, \Psi) &\longmapsto \Phi \wedge \Psi\end{aligned}$$

with

$$(\Phi \wedge \Psi)(f_1 \wedge \ldots \wedge f_{p+q}) = \sum_{\substack{I \coprod J=\{1,\ldots,n\}\\ i_1<\ldots<i_p\\ j_1<\ldots<j_q}} sgn(I, J) \cdot \Phi(f_{i_1} \wedge \ldots \wedge f_{i_p}) \cdot \Psi(f_{j_1} \wedge \ldots \wedge f_{j_q})$$

The sign is defined as

$$sgn(I, J) := sgn\begin{pmatrix}1, \ldots\ldots\ldots\ldots, p+q\\ i_1, \ldots, i_p, j_1, \ldots, j_q\end{pmatrix}$$

Proposition 1. *Let $\Phi \in \mathcal{C}^p$, $\Psi \in \mathcal{C}^q$ et $\Gamma \in \mathcal{C}^r$. Then we have*

1. $\Phi \wedge \Psi = (-1)^{deg(\Phi)\cdot deg(\Psi)} \cdot \Psi \wedge \Phi$
2. $(\Phi \wedge \Psi) \wedge \Gamma = \Phi \wedge (\Psi \wedge \Gamma)$
3. $\delta(\Phi \wedge \Psi) = \delta(\Phi) \wedge \Psi + (-1)^{deg(\Phi)} \cdot \Phi \wedge \delta(\Psi)$

Proof. The first two points are trivial, while the third has to be checked by an explicit calculation.

Note that the last proposition says that $(\mathcal{C}_L^\bullet, \delta, \wedge)$ is a *differential graded algebra*, furthermore, we have $\mathcal{C}_L^0 = \mathcal{O}_L = \Omega_L^0$. As one might hope, there is indeed a tight connection between $\Omega_L^\bullet$ and $\mathcal{C}_L^\bullet$.

Proposition 2. *Suppose that L is lagrangian. Then there exists a morphism $J : \Omega^1_L \to \mathcal{C}^1_L$ which is an isomorphism outside the singular locus of L.*

Proof. On a symplectic manifold, there is a canonical isomorphism β between vector fields and one forms, given by $\beta(V) := i_V\omega$. On the other hand, for each analytic subspace $L \subset M$ we have two exact sequences, dual to each other, namely, the conormal and the normal sequence, thus, there is the following diagram:

$$\begin{tikzcd} & \mathcal{L} \arrow[r] & \Omega^1_M \otimes \mathcal{O}_L \arrow[r] \arrow[d, "\alpha:=\beta^{-1}"] & \Omega^1_L \arrow[r] & 0 & \\ 0 \arrow[r] & \Theta_L \arrow[r] & \Theta_M \otimes \mathcal{O}_L \arrow[r] & \mathcal{N}_L \arrow[r] & \mathcal{T}^1_L \arrow[r] & 0 \end{tikzcd}$$

Now the fundamental fact is that this diagram can be completed: the morphism $\mathcal{L} \to \Theta_L$ from above commutes with α, so we have

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d, "\alpha'"'] & \Omega^1_M \otimes \mathcal{O}_L \arrow[d, "\alpha"] \\ \Theta_L \arrow[r] & \Theta_M \otimes \mathcal{O}_L \end{tikzcd} \tag{1}$$

Note that the image of an element $g \in \mathcal{L}$ under α' is just the hamiltonian vector field H_g. The morphisms $J : \Omega^1_L \to \mathcal{C}^1_L = \mathcal{N}_L$ we are looking for can now be defined as the map induced by α, explicitly

$$J(df) = (g \mapsto \{f, g\})$$

To see that J is an isomorphism near a smooth point of L it will be sufficient to prove this for the map α' (because at smooth points x we have $\mathcal{T}^1_{(L,x)} = 0$ and the map $\mathcal{L}_x \to \Omega^1_{(L,x)} \otimes \mathcal{O}_{L,x}$ is injective). So assume the sheaves $\mathcal{L}$, Ω^1_L, and Θ_L to be defined in a neighborhood of a smooth point which means that they all become locally free. $\mathcal{L}$ then has to be identified with the conormal bundle. To prove that α' is an isomorphism, we will construct an inverse. First note that, by the fact that L is coisotropic, the morphism $\beta : \Theta_{M|L} \to \Omega^1_{M|L}$ actually sends an element of Θ_L to a form vanishing on all vectors tangent to L. So the restriction of β to Θ_L defines a morphism $\beta' : \Theta_L \to \mathcal{L}$. The situation is as follows:

$$\begin{tikzcd} 0 \arrow[r] & \Theta_L \arrow[r] \arrow[d, "\beta'"] & \Theta_M \otimes \mathcal{O}_L \arrow[r] \arrow[d, "\beta"] & \mathcal{N}_L \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{L} \arrow[r] & \Omega^1_M \otimes \mathcal{O}_L \arrow[r] & \Omega^1_L \arrow[r] & 0 \end{tikzcd}$$

One sees that β' is injective. On the other hand, from the fact that L is lagrangian we see that $\dim(\mathcal{L}) = \dim(\Theta_L)$. So β' is an isomorphism and the inverse of α'.

Remark. J can equally be described as the composition of the canonical morphism

$$\Omega^1_L \longrightarrow (\Omega^1_L)^{**} = (\Theta^1_L)^*$$

with the dual of the morphism α'. But the above diagram makes the explicit description of J using the Poisson bracket more transparent.

Corollary 1. *The morphism $J : \Omega^1_L \to \mathcal{C}^1_L$ can be extended to a morphism of DGA's*

$$J : (\Omega^\bullet_L, d, \wedge) \longrightarrow (\mathcal{C}^\bullet_L, \delta, \wedge)$$

which is an isomorphism at smooth points of L.

Proof. Set

$$J(\omega_1 \wedge \ldots \wedge \omega_p) := J(\omega_1) \wedge \ldots \wedge J(\omega_p)$$

where $\omega_i \in \Omega^1_L$. Then it is immediate that J is an isomorphism on L_{reg}. To prove that $J \circ d = \delta \circ J$, it suffices to check this in the lowest degrees, that is, we have to show that the diagram

$$\begin{tikzcd} \Omega^0_L \arrow[r, "d"] \arrow[d] & \Omega^1_L \arrow[d, "J"] \\ \mathcal{C}^0_L \arrow[r, "\delta"] & \mathcal{C}^1_L \end{tikzcd}$$

commutes. This follows directly from $\Omega^0_L = \mathcal{C}^0_L = \mathcal{O}_L$.

In the last section, we use the following elementary fact.

Lemma 1. *The kernel of J is the complex $\mathcal{T}ors(\Omega^\bullet_L)$ consisting of the torsion subsheaves of Ω^p_L.*

Proof. We have $\mathcal{T}ors(\Omega^\bullet_L) \subset \mathcal{K}er(J)$ as $\mathcal{C}^\bullet_L$ is torsion free. On the other hand, the kernel is supported on the singular locus of L, so it must be a torsion sheaf, hence $\mathcal{K}er(J) \subset \mathcal{T}ors(\Omega^\bullet_L)$.

Remark. Although the definition of the modules $\mathcal{C}^p_L$ involves the ideal $\mathcal{I}$, the following lemma shows that they are intrinsic at least in some special cases.

Lemma 2. *Let $\mathbb{K} = \mathbb{C}$ and suppose L to be Cohen-Macaulay and regular in codimension one. Then there is an isomorphism*

$$(\Omega^p_L)^{**} \xrightarrow{\cong} \mathcal{C}^p_L$$

where for an $\mathcal{O}_L$-module $\mathcal{F}$, $\mathcal{F}^*$ denotes $\mathcal{H}om_{\mathcal{O}_L}(\mathcal{F}, \mathcal{O}_L)$.

Proof. We will make use of the following fact: Let $\mathcal{F}$ be an $\mathcal{O}_L$-module of type $\mathcal{G}^*$, then $\mathcal{F}$ is reflexive, i.e. $\mathcal{F}^{**} = \mathcal{F}$. The morphism $(\Omega_L^p)^{**} \to \mathcal{C}_L^p$ we are looking for is obtained by dualizing twice the morphism $J : \Omega_L^p \to \mathcal{C}_L^p$, this yields $J^{**} : (\Omega_L^p)^{**} \to (\mathcal{C}_L^p)^{**} = \mathcal{C}_L^p$ as $\mathcal{C}_L^p$ is of type $\mathcal{H}om(-, \mathcal{O}_L)$. Clearly, J^{**} is an isomorphism on the regular locus. We have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow (\Omega_L^p)^{**} \xrightarrow{J^{**}} \mathcal{C}_L^p \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{K}$ and $\mathcal{G}$ are the kernel resp. cokernel sheaves of the map J^{**}. This sequence can be split

$$\begin{gathered} 0 \longrightarrow \mathcal{K} \longrightarrow (\Omega_L^p)^{**} \longrightarrow \mathcal{H} \longrightarrow 0 \\ 0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{C}_L^p \longrightarrow \mathcal{G} \longrightarrow 0 \end{gathered}$$

with $\mathcal{H} = \mathcal{I}m(J^{**})$. Applying $\mathcal{H}om_{\mathcal{O}_L}(-, \mathcal{O}_L)$ yields

$$\begin{gathered} 0 \longrightarrow \mathcal{H}^* \longrightarrow ((\Omega_L^p)^{**})^* \longrightarrow \mathcal{K}^* \\ 0 \longrightarrow \mathcal{G}^* \longrightarrow (\mathcal{C}_L^p)^* \longrightarrow \mathcal{H}^* \longrightarrow \mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_L) \end{gathered}$$

Now we use the lemma of Ischebeck (see [Mat89]): Given a local ring R, two R-modules M and N with $k = \dim(M)$ and $r = depth(N)$, then for all $p < r - k$, the modules $Ext^p(M, N)$ vanish. It follows that $\mathcal{K}^* = \mathcal{G}^* = \mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_L) = 0$, so we have $((\Omega_L^p)^{**})^* = (\mathcal{C}_L^p)^*$. Then obviously $((\Omega_L^1)^{**})^{**} = (\mathcal{C}_L^1)^{**}$ and by the argument above $(\Omega_L^1)^{**} = \mathcal{C}_L^1$ so the map J^{**} is an isomorphism.

3. Deformations

Recall that the space of infinitesimal embedded deformations of an analytic algebra R, given as $R = S/I$ where S is the ring of convergent power series, is equal to the normal module of I in S, i.e. $Hom_R(I/I^2, R)$. Dividing out trivial deformations gives the space T_R^1, defined by the sequence

$$0 \to Hom_R(\Omega_R^1, R) \to Hom_S(\Omega_S^1, S) \widehat{\otimes} R$$

$$\to Hom_R(I/I^2, R) \to T_R^1 \to 0$$

On the other hand, the deformations of a manifold X over $Spec(\mathbb{K}[\epsilon]/(\epsilon^2))$ are parameterized by $H^1(X, \Theta_X)$. The cotangent complex is a tool to handle these two special cases in an integrated manner: infinitesimal deformations of an analytic space L are in bijection with $\mathbb{H}^1(\mathbb{L}_X)$. It seems that the complex $\mathcal{C}_L^\bullet$ has to be seen as a first approximation to an equivalent for the cotangent complex in the lagrangian context. The theorems 1 and 2 give the precise meaning of this statement.

Theorem 1. *The cohomology sheaves of $\mathcal{C}_L^\bullet$ have the following interpretations.*

1. $\mathcal{H}^0(\mathcal{C}_L^\bullet) = \mathbb{K}_L$.
2. $\mathcal{H}^1(\mathcal{C}_L^\bullet) = \mathcal{LT}_L^1$.

The proof of the following preliminary lemma can be found in [Ban94].

Lemma 3. *Let U be a symplectic manifold and suppose that $H^1(U, \mathbb{K}) = 0$ (and that U is Stein for $\mathbb{K} = \mathbb{C}$). Then the Lie algebra of the symplectomorphism group of U is exactly the Lie algebra of Hamiltonian vector fields on U.*

Proof. For an open set $U \subset M$ let us choose sections $f_1, \dots, f_k$ generating $\mathcal{I}(U)$ (In what follows, when we speak about a sheaf $\mathcal{F}$, we mean its sections over U). $\mathcal{H}^0(\mathcal{C}_L^\bullet)$ equals $\mathcal{K}er(\delta : \mathcal{O}_L \to \mathcal{C}_L^1)$. Take an element f of $\mathcal{K}er(\delta)$. Then $\{f, g\} \in \mathcal{I}$ for all $g \in \mathcal{I}$. If f is not a constant, then the ideal $(\mathcal{I}, f)$ is strictly larger than $\mathcal{I}$, not the whole ring and still involutive. This is a contradiction to the fact that L is lagrangian, which means that $\mathcal{I}$ is maximal under all involutive ideals. So the kernel must be the constant sheaf.

To prove that $\mathcal{H}^1(\mathcal{C}_L^\bullet) = \mathcal{LT}_L^1$, two things have to be checked: As $\mathcal{C}_L^1 = \mathcal{N}_L$, we must first identify the elements of $\mathcal{K}er(\delta^1 : \mathcal{C}_L^1 \to \mathcal{C}_L^2)$ with the **lagrangian** deformations. Then we have to show that the image of $\delta^0 : \mathcal{O}_L \to \mathcal{C}_L^1$ are the trivial deformations. But this is easy, because for $f \in \mathcal{O}_L$, $\delta(f)$ acts as H_f, thus inducing a trivial deformation. Furthermore, of all deformations coming from vector fields on M, only those induced by hamiltonian vector fields are trivial in the lagrangian sense (this follows from the preceding lemma, as U can assumed to be Stein and contractible).

Take an element $\Phi \in \mathcal{K}er(\delta^1)$, which means that

$$\phi(\{g, h\}) - \{g, \phi(h)\} - \{\phi(g), h\} = 0$$

for all $f, g \in \mathcal{I}/\mathcal{I}^2$. Then Φ corresponds to the deformation given by

$$\widetilde{\mathcal{I}} = (f_1 + \epsilon\phi(f_1), \ \dots \ , f_k + \epsilon\phi(f_k))$$

The ideal $\widetilde{\mathcal{I}}$ is involutive iff for any two elements $f + \epsilon\phi(f)$, $g + \epsilon\phi(g)$, we have $\{f + \epsilon\phi(f), g + \epsilon\phi(g)\} \in \widetilde{\mathcal{I}}$, which is equivalent to

$$F := \{f, g\} + \epsilon\left(\{f, \phi(g)\} + \{\phi(f), g\}\right) \in \widetilde{\mathcal{I}}$$

Consider $G := \{f, g\} + \epsilon\phi(\{f, g\})$, which is an element of $\widetilde{\mathcal{I}}$, so the condition $F \in \widetilde{\mathcal{I}}$ is equivalent to $F - G \in \widetilde{\mathcal{I}}$, that is

$$\{f, \phi(g)\} + \{\phi(f), g\} - \phi(\{f, g\}) \in \mathcal{I}$$

This means exactly that $\phi \in \mathcal{K}er(\delta^1)$.

The following theorem shows that we can extract some information concerning the obstruction theory of L from the second cohomology of the complex $\mathcal{C}_L^\bullet$.

Theorem 2. *Chose for a given deformation $\Phi \in \mathcal{C}^1_L$ elements $g_i \in \mathcal{O}_M$ such that the class of g_i modulo I equals $\Phi(f_i)$. Denote by $ob_{f_i \wedge f_j}$ the class of the element $\{g_i, g_j\}$ in $\mathcal{O}_L$. Then we have the following: If there exists a map $ob : \mathcal{C}^1_L \to \mathcal{C}^2_L$ such that $ob(\Phi)(f_i \wedge f_j) = ob_{f_i \wedge f_j}$ then*

- $\delta\left(\mathcal{I}m(ob)\right) = 0$ *and* $ob\left(\mathcal{I}m(\delta : \mathcal{O}_L \to \mathcal{C}^1_L)\right) = 0$*, so ob defines a map*
$$ob : \mathcal{H}^1(\mathcal{C}^\bullet_L) \longrightarrow \mathcal{H}^2(\mathcal{C}^\bullet_L)$$
- $ob(\Phi) = 0 \in \mathcal{H}^2(\mathcal{C}^\bullet_L)$ *iff there exits a (not necessarily flat) deformation over $Spec(\mathbb{K}[\epsilon]/\epsilon^3)$ whose fibers are lagrangian subvarieties inducing the given deformation over $Spec(\mathbb{K}[\epsilon]/\epsilon^2)$.*

Proof. The first statement can be verified by a direct calculation which uses several times the Jacobi identity. So we suppose that there is a map $ob : \mathcal{H}^1(\mathcal{C}^\bullet_L) \to \mathcal{H}^2(\mathcal{C}^\bullet_L)$. Let $\Phi \in \mathcal{H}^1(\mathcal{C}^\bullet_L)$ be an element of $\mathcal{K}er(ob)$. This condition is equivalent to the existence of $\Psi \in \mathcal{H}^1(\mathcal{C}^\bullet_L)$ with $ob(\Phi) = \delta(\Psi)$, i.e.

$$\{\Phi(f), \Phi(g)\} = \Psi\left(\{f, g\}\right) - \{f, \Psi(g)\} - \{\Psi(f), g\} \qquad \forall f, g \in \mathcal{L}$$

But this means that the following ideal is involutive.

$$J = (f_1 + \epsilon\Phi(f_1) + \epsilon^2\Psi(f_1), \ldots, f_k + \epsilon\Phi(f_k) + \epsilon^2\Psi(f_k))$$

proving that the given lagrangian deformation can be lifted to third order.

Remark. Due to the non-linearity of the Poisson bracket, it is not clear whether the elements $ob_{f_i \wedge f_j}$ always extend to a map $ob : \mathcal{C}^1_L \to \mathcal{C}^2_L$. Furthermore, $\mathcal{H}^2(\mathcal{C}^\bullet_L)$ does not contain any information on whether a given $\Phi \in \mathcal{H}^1(\mathcal{C}^\bullet_L)$ can be lifted as a *flat* deformation. We see that $\mathcal{H}^2(\mathcal{C}^\bullet_L)$ is not the right obstruction space for the deformation problem under consideration. This makes precise what was meant by saying that the complex $\mathcal{C}^\bullet_L$ is a first approximation of the object we are looking for: Hopefully, there is a modified version of this complex whose cohomology gives, in complete analogy with the cotangent complex, the spaces T^1 and T^2 for lagrangian deformations as defined in the introduction, i.e., for *flat lagrangian* deformations. On the other hand, it is perhaps not even necessary to impose flatness as the involutivity condition implies that the dimension of the fibers cannot drop, see also [Mat].

Corollary 2. *There is an exact sequence*

$$0 \to H^1(L, \mathbb{K}_L) \to \mathbb{H}^1(\mathcal{C}^\bullet_L) \to H^0(L, \mathcal{LT}^1_L) \to H^2(L, \mathbb{K}_L) \to \mathbb{H}^2(\mathcal{C}^\bullet_L)$$

Furthermore, there are two special cases:

- *Let L be a contractible space. Then $\mathbb{H}^1(\mathcal{C}^\bullet_L) = H^0(L, \mathcal{LT}^1_L)$ and in fact: $LT^1_L = H^0(L, \mathcal{LT}^1_L)$.*

– *Let L be Stein and smooth. Then it follows that $\mathbb{H}^1(\mathcal{C}_L^\bullet) = H^1(L, \mathbb{K}_L)$ and the space of global deformations is indeed $LT_L^1 = H^1(L, \mathbb{K}_L)$.*

Proof. The existence of the exact sequence is immediate and the assertion for the case that L is contractible is just the definition of the sheaf $\mathcal{LT}_L^1$. In the second case, note that the space of embedded flat deformations is $H^0(L, \mathcal{N}_L)$, where $\mathcal{N}_L$ is the normal bundle of L in M. As L is smooth, this happens to be $H^0(L, \Omega_L^1)$, so each infinitesimal flat deformation corresponds to globally defined one-form on L. It is closed iff the deformation is lagrangian and the subspace of exact one-forms are deformations induced by hamiltonian vector fields (which were called *isodrastic deformations* in [Wei90]), these are the trivial ones. L is assumed to be a Stein manifold, in this case the first *de Rham*-cohomology group is exactly $H^1(L, \mathbb{K}_L)$.

By analogy with the cotangent complex, the following generalization is probably true.

Conjecture 1. The space of infinitesimal lagrangian deformations of a complex space L which is a lagrangian subvariety of a symplectic manifold (M, ω) is given by

$$LT_L^1 = \mathbb{H}^1(\mathcal{C}_L^\bullet)$$

4. Finiteness of the cohomology

In this section we will study a class of lagrangian subvarieties admitting a special stratification.

Definition 2. *Let $L \subset M$ be as above. Denote by $edim(p)$ the embedding dimension of a point $p \in L$, that is*

$$edim(p) := dim_{\mathbb{K}}(\mathbf{m}_p/\mathbf{m}_p^2)$$

where $\mathbf{m}_p$ is the maximal ideal in the local ring $\mathcal{O}_{(L,p)}$. Let S_k^L be the following set

$$S_k^L := \{p \in L \mid edim(p) = 2n - k\} \subset L$$

for $k \in \{0, \dots, n\}$. Then we will say that L satisfies "Condition P" iff the following inequality holds for all k.

$$\dim(S_k^L) \leq k$$

The goal of this section is to prove that for lagrangian spaces L satisfying "Condition P", the cohomology of $\mathcal{C}_L^\bullet$ is finite dimensional (see theorem 3). The following lemma explains the meaning of this condition in somewhat more geometric terms.

Lemma 4. *Let $p \in S_k^L \subset L$ with $k > 0$. Then the germ (L, p) can be decomposed into a product*

$$(L, p) = (L', p') \times (\mathbb{K}, 0)$$

This decomposition is compatible with the decomposition of the ambient symplectic space

$$(\mathbb{K}^{2n}, 0) = (\mathbb{K}^{2n-2}, 0) \times (\mathbb{K}^2, 0)$$

by symplectic reduction. Therefore, (L', p') is a germ of a lagrangian variety in the symplectic space $\mathbb{K}^{2n-2}$. Furthermore, we have $p' \in S_{k-1}^{L'}$.

Proof. Let $x_1, \dots, x_{2n}$ be coordinates of M centered at p. Then the fact that $edim(p) < 2n$ implies that there are coefficients $\alpha_i \in \mathcal{O}_{L,p}$ such that the following equation holds in $\mathcal{O}_{L,p}$

$$\sum_{i=1}^{2n} \alpha_i x_i + h = 0$$

where h is an element of $\mathcal{O}_{L,p}$ vanishing at second order. So we have an element in the ideal describing (L, p) whose derivative does not vanish. Then (L, p) is fibred by the hamiltonian flow of this function. Explicitly, we can make an analytic change of coordinates, such that $\alpha_1 = 1$, $\alpha_i = 0$ for all $i > 1$ and $h = 0$. Then the ideal of (L, p) is of the form $(x_1, f_1, \dots, f_m)$ for some functions f_i which are independent of the variable x_{n+1}.

This result implies that whenever a stratum S_k^L is non-empty then there are k independent hamiltonian vector fields defined in a neighborhood of a point $p \in S_k^L$ which are tangent to S_k^L. Thus, the dimension of this stratum must be at least k. So "condition P" can be restated by saying that either $\dim(S_k^L) = k$ or $S_k^L = \emptyset$.

The preceding lemma can be used to show that there are germs of singular spaces which do not admit any lagrangian embedding.

Corollary 3. *Let $n > 1$ and $(X, 0) \subset (\mathbb{K}^{n+1}, 0)$ be an isolated hypersurface singularity. Then there does not exist any lagrangian embedding $(X, 0) \hookrightarrow (\mathbb{K}^{2n}, 0)$.*

Proof. Suppose that a lagrangian embedding exists. The embedding dimension of the germ $(X, 0)$ is $n + 1 < 2n$, so by the previous lemma there is a decomposition $(X, 0) = (Y, 0) \times (\mathbb{K}^{n-1}, 0)$ showing that $(X, 0)$ has non-isolated singularities.

Note that the two preceding results can be found in [Giv88]. We will now state and prove the main theorem of this section. We restrict to the complex case for simplicity.

Theorem 3. *Suppose "Condition P" to be satisfied for a lagrangian subvariety $L \subset M$, where M is holomorphic symplectic. Then all $\mathcal{H}^i(\mathcal{C}_L^\bullet)$ are constructible sheaves of finite dimensional $\mathbb{C}$-vector spaces with respect to the stratification given by the S_k^L.*

According to lemma 4, the set of points of the variety L can be divided into two classes, those with maximal embedding dimension (these are the "bad points") and those (with $edim(p) < 2n$) at which L is decomposable. "Condition P" implies that the bad points are isolated. The proof of the theorem consists of two parts: First, we will show that the cohomology sheaves are locally constant on the strata S_k^L. This is an immediate consequence of the following lemma. Then it suffices to show that all stalks of $\mathcal{H}^p(\mathcal{C}_L^\bullet)$ are finite-dimensional.

Lemma 5 (Propagation of deformations)**.** *Let*

$$(L, 0) \subset (\mathbb{C}^{2n}, 0)$$

be a germ of a lagrangian subvariety which can be decomposed, i.e., there is a germ $(L', 0)$ *(which is lagrangian in* $(\mathbb{C}^{2n-2}, 0)$*) such that* $(L, 0) = (L', 0) \times (\mathbb{C}, 0)$*. Denote by* $\pi : L \to L'$ *the projection. Then there is a quasi-isomorphism of sheaf complexes*

$$j : \pi^{-1}\mathcal{C}_{L'}^\bullet \to \mathcal{C}_L^\bullet$$

Proof. The proof is a direct calculation with the complex $\mathcal{C}^\bullet$ for the spaces L' and L . We choose local Darboux coordinates $(p_2, \dots, p_n, q_2, \dots, q_n)$ on $\mathbb{C}^{2n-2}$ and $(p_1, \dots, p_n, q_1, \dots, q_n)$ on $\mathbb{C}^{2n}$ such that L' and L are given by ideals

$$\begin{aligned} I' &= (f_1, \dots, f_m) \subset \mathbb{C}\{p_2, \dots, p_n, q_2, \dots, q_n\} \\ I &= (f_1, \dots, f_m, p_1) \subset \mathbb{C}\{p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n\} \end{aligned}$$

This implies in particular that

$$I/I^2 = \left(I'/I'^2 \otimes_{\mathcal{O}_{L',0}} \mathcal{O}_{L,0}\right) \oplus \mathcal{O}_{L,0}$$

Let $\mathcal{L}_0 = I/I^2$ and $\mathcal{L}_0' = I'/I'^2$ be germs of the conormal sheaves. It follows that

$$\begin{aligned} \mathcal{C}_{L,0}^p = &\, Hom_{\mathcal{O}_{L,0}}\left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \textstyle\bigwedge^p \mathcal{L}_0', \mathcal{O}_{L,0}\right) \\ &\oplus Hom_{\mathcal{O}_{L,0}}\left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \textstyle\bigwedge^{p-1} \mathcal{L}_0', \mathcal{O}_{L,0}\right) \end{aligned}$$

Now we have to describe the differential on $\mathcal{C}_{L,0}^\bullet$ in terms of the differential on $\mathcal{C}_{L',0}^\bullet$. Elements of

$$\begin{aligned} &Hom_{\mathcal{O}_{L,0}}\left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \textstyle\bigwedge^p \mathcal{L}_0', \mathcal{O}_{L,0}\right) \\ &= Hom_{\mathcal{O}_{L',0}}\left(\textstyle\bigwedge^p \mathcal{L}_0', \mathcal{O}_{L',0}\right) \otimes_{\mathcal{O}_{L',0}} \mathcal{O}_{L,0} \end{aligned}$$

are power series in q_1 with coefficients in $\mathcal{C}_{L',0}^\bullet$ (this is because $\mathcal{O}_{L,0} \cong \mathcal{O}_{L',0}\{q_1\}$). So let $\Phi = \sum_{i=0}^\infty \Phi_i q_1^i$ with $\Phi_i \in \mathcal{C}_{L',0}^p$ and $\Psi = \sum_{i=0}^\infty \Psi_i q_1^i$ with $\Psi_i \in \mathcal{C}_{L',0}^{p-1}$. Then a direct calculation shows that

$$\begin{array}{rccc} \delta : & \mathcal{C}_{L,0}^p & \longrightarrow & \mathcal{C}_{L,0}^{p+1} \\ & \sum\limits_{i=0}^\infty (\Phi_i, \Psi_i)\, q_1^i & \mapsto & \sum\limits_{i=0}^\infty \left(\delta\Phi_i, \delta\Psi_i + (-1)^{p+1}(i+1)\Phi_{i+1}\right) q_1^i \end{array}$$

We define the morphism j to be the inclusion

$$\begin{aligned} Hom_{\mathcal{O}_{L',0}}\left(\bigwedge\nolimits^p \mathcal{L}'_0, \mathcal{O}_{L',0}\right) \hookrightarrow\ & Hom_{\mathcal{O}_{L,0}}\left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \bigwedge\nolimits^p \mathcal{L}'_0, \mathcal{O}_{L,0}\right) \\ \oplus\ & Hom_{\mathcal{O}_{L,0}}\left(\mathcal{O}_{L,0} \otimes_{\mathcal{O}_{L',0}} \bigwedge\nolimits^{p-1} \mathcal{L}'_0, \mathcal{O}_{L,0}\right) \end{aligned}$$

$$\Phi \longmapsto (\Phi, 0) \cdot q_1^0$$

It remains to show that the cokernel of this inclusion is acyclic. Then it follows immediately from the long exact cohomology sequence that j is a quasi-isomorphism. So let Γ be an element outside of the image of j such that $\delta(\Gamma) = 0$, that is:

$$\Gamma = \sum_{i=1}^{\infty} (\Phi_i, \Psi_i) q_1^i + (0, \Psi_0)$$

with $\delta\Phi_i = 0$ and $\delta\Psi_i = (-1)^p (i+1)\Phi_{i+1}$ for all $i \in \{0, 1, \ldots\}$. But then Γ vanishes in the cohomology because it can be written as $\Gamma = \delta\Lambda$ with

$$\Lambda := \sum_{i=1}^{\infty} \left(\frac{(-1)^p \Psi_{i-1}}{i}, 0\right) q_1^i \in \mathcal{C}_L^{p-1}$$

Corollary 4. *We have isomorphisms of sheaves*

$$\pi^{-1}\mathcal{H}^i(\mathcal{C}^\bullet_{L'}) \cong \mathcal{H}^i(\mathcal{C}^\bullet_L)$$

Proof. This is obvious since π^{-1} is an exact functor.

Let $p \in S_k^L$ be a point at which L is decomposable, i.e. $k > 0$. By induction, we find a neighborhood $U \subset L$ of p such there is an analytic isomorphism $h : U \xrightarrow{\cong} Z \times B_\epsilon^k$, where Z is lagrangian in $\mathbb{C}^{2(n-k)}$, $B_\epsilon := \{z \in \mathbb{C} \mid |z| < \epsilon\}$ and each $q \in U \cap S_l^L$ corresponds via h to a point $(q', b) \in Z \times B_\epsilon^k$ with $q' \in S_{l-k}^Z$. In particular, the image of $U \cap S_k^L$ under h is $(\{pt\}, B(\epsilon)^k)$, so by the last corollary, $\mathcal{H}^p(\mathcal{C}^\bullet_L)$ is constant on $U \cap S_k^L$.

It remains to show that the stalks of the cohomology are finite-dimensional. Again by corollary 4, this is done once we have shown it for points with maximal embedding dimension.

Our proof relies on a method which was used several times in similar situations, see e.g. [vS87]. In this paper, a morphism of complex spaces $f : X \to S$ and a sheaf complex $\mathcal{K}^\bullet$ on X is considered. Then, under certain circumstances, the relative hypercohomology $\mathbb{R}^p f_* \mathcal{K}^\bullet$ is $\mathcal{O}_S$-coherent. The result we will use is contained in the following statement, which we quote from [vS87].

Theorem 4. *Let a germ $f : (Y, 0) \to (T, 0)$ of complex spaces be given, where T is smooth and one-dimensional. Suppose Y and T embedded in some $\mathbb{C}^N$ and in $\mathbb{C}$, respectively. Choose a so called **standard representative** $f : X \to S$, i.e., a morphism representing the given germ such that:*

1. $X := (B_\epsilon \cap Y) \cap f^{-1}(D_\eta)$
2. $S := T \cap D_\eta$

for an open ϵ-ball $B_\epsilon \subset \mathbb{C}^N$ and an open η-disc $D_\eta \subset \mathbb{C}$. For small ϵ and η the space X will be Stein and contractible. Let $(\mathcal{K}^\bullet, d)$ be a sheaf complex on X with the following properties

1. all $\mathcal{K}^p$ are $\mathcal{O}_X$-coherent
2. the differentials $d : \mathcal{K}^p \to \mathcal{K}^{p+1}$ are $f^{-1}\mathcal{O}_S$-linear
3. there is a neighborhood U of $\overline{\partial X} := \overline{\partial B_\epsilon \cap Y \cap f^{-1}(D_\eta)}$ in $\mathbb{C}^N$ and a vector field ϑ of class C^∞ on U such that
  - *ϑ is transversal to ∂B_ϵ*
  - *the flow of ϑ respects X and the fibers of f.*
  - *the restriction of the cohomology sheaves $\mathcal{H}^p(\mathcal{K}^\bullet)$ to the integral curves of ϑ are locally constant sheaves.*

Then the sheaves $\mathbb{R}^p f_* \mathcal{K}^\bullet$ are $\mathcal{O}_S$-coherent.

In our situation, we take for $(Y, 0)$ the germ $(L, p) \subset (M, p) \cong (\mathbb{C}^{2n}, p)$ of a lagrangian variety satisfying “Condition P”, where $edim(p) = 2n$. The morphism f is simply the constant mapping to a point, so that most of the conditions of the preceding theorem are trivially fulfilled (existence of a standard representative, linearity of the differential etc.). What we have to do is to construct a vector field ϑ with the required properties. We first choose a representative $V := L \cap B_\epsilon$ of (L, p) such that p is the only point in V with embedding dimension equal to $2n$ (this is possible due to “Condition P”) and such that the intersection of all strata S_k^L with ∂B_ϵ is transversal.

Lemma 6. *There is a C^∞-vector field ϑ on a neighborhood U of ∂B_ϵ in $\mathbb{C}^{2n}$ such that $\mathcal{H}^p(\mathcal{C}_L^\bullet)$ is* ***transversally constant*** *with respect to U and ϑ, i.e., it satisfies the third condition in the last theorem.*

Proof. We will first construct such a field locally around a point $q \in V$ different from p. Consider thus $q \in V \cap S_k^L$ with $k > 0$. It follows from lemma 4 that in a neighborhood W of q in $\mathbb{C}^{2n}$ there exist k linear independent holomorphic hamiltonian vector fields $\eta_1, \ldots, \eta_k$ on W which are tangent to the stratum S_L^k. We know from “Condition P” that $\dim(S_k^L) = k$. Therefore, the hamiltonian fields span the holomorphic tangent space of S_k^L at q. The real dimension of a stratum S_L^k is $2k$. Consider the holomorphic fields $\eta_1, \ldots, \eta_k, i\eta_1, \ldots, i\eta_k$. They span the real tangent space at q. The intersection of ∂B_ϵ and S_k^L is transversal, so there is a linear combination η of the $2k$ vector fields which is transversal to ∂B_ϵ and tangent to S_k^L. Then the cohomology sheaves are constant on the integral curves of η, as the integral curves are contained in S_k^L.

The next task is to glue the locally defined C^∞-fields. For this purpose we set $\widetilde{U} := (V \setminus \{p\})^\circ$. The last lemma yields a covering $\widetilde{U}_i$ of $\widetilde{U}$ and vector fields ϑ_i

defined in a neighborhood U_i of $\widetilde{U}_i$ in $\mathbb{C}^{2n}$. Put $U = \bigcup_i U_i$ and chose a partition of unity of U subordinate to this covering. This produces a vector field ϑ defined on U such that the cohomology sheaves are transversally constant with respect to ϑ and U.

Now we are able to apply theorem 4. It follows that the (absolute) hypercohomology groups $\mathbb{H}^p(\mathcal{C}_L^\bullet)$ are finite-dimensional. But we already know that the cohomology sheaves $\mathcal{H}^p(\mathcal{C}_L^\bullet)$ are locally constant on the strata S_k^L. Therefore the germ $\mathcal{H}^p(\mathcal{C}_L^\bullet)_p$ for a point p with $edim(p) = 2n$ must also be finite dimensional. This finally proves our theorem.

Remark. Although most of the known examples of lagrangian singularities (in particular those which we will consider in the next section) satisfy "Condition P", it is easy to construct spaces where points with maximal embedding dimension are not isolated. The following example is taken from [Giv88]:

Consider a non-quasihomogenous plane irreducible curve singularity $(C, 0) \subset (\mathbb{C}^2, 0)$ which is Lagrangian with respect to the symplectic structure $dp \wedge dq$. Let $\nu : (\widetilde{C}, 0) \to (C, 0)$ be the normalization of this curve. Then we can consider the *generating function* of C, i.e., a continuous function $F : (C, 0) \to (\mathbb{C}, 0)$ which is holomorphic on C_{reg} and satisfies

$$dF_{|C_{\text{reg}}} = \alpha_{|C_{\text{reg}}}$$

where α is the Liouville form pdq. As $(C, 0)$ was not homogeneous, F is not holomorphic on all of C (for a discussion of this fact, see [Her], Chap. 7). Therefore, the image $(\Lambda, 0)$ of the mapping

$$\begin{aligned} (\nu, F) : (\widetilde{C}, 0) &\longrightarrow (\mathbb{C}^3, 0) \\ p &\longmapsto (\nu(p), (F \circ \nu)(p)) \end{aligned}$$

is a legendrian space curve singularity, i.e. the contact form $dz - pdq$ vanishes on Λ_{reg} and $(\Lambda, 0)$ has embedding dimension 3. Now for any germ of a contact manifold $(K, 0)$ of dimension $2n - 1$ we can equip the direct product $(M, 0) = (K, 0) \times (\mathbb{C}^*, p)$ with a symplectic structure (which is called symplectization of $(K, 0)$ in [Giv88]): in our example, taking $K = \mathbb{C}^3$ with coordinates (p, q, z), we have local coordinates (p, q, z, t) on $(M, 0)$ and

$$\omega = d\,(t(dz - pdq))$$

There is a natural projection $\pi : (M, 0) \to (K, 0)$ and the preimage $L := \pi^{-1}(\Lambda)$ is a lagrangian subspace of $(M, 0)$. Obviously, at all points $(0, q) \in L$ we have $edim_{(0,q)} L = 4$. Therefore, $(L, 0)$ does not satisfy "Condition P". Probably, there are examples of this type where the cohomology of $\mathcal{C}_{L,0}^\bullet$ (and in particular the module $\mathcal{LT}_{L,0}^1$) is not finite over $\mathbb{C}$. However, as these spaces are non-quasi-homogenous, a direct calculation of the cohomology of the complex $\mathcal{C}^\bullet$ is very difficult (see the next section).

Remark. By the *Riemann-Hilbert-correspondence* ([Bjö93]), the complex $\mathcal{C}_L^\bullet$, viewed as an object of $\mathcal{D}_c^b(\mathbb{C}_M)$ (the derived category of constructible sheaves of $\mathbb{C}$-vector spaces on M) corresponds via the *de Rham*-functor to a unique complex of coherent $\mathcal{D}_M$-modules with regular holonomic cohomology supported on L (i.e., an object of $\mathcal{D}^b_{\text{r.h.}}(\mu_L(\mathcal{D}_M))$).

Lemma 7. *The complex $\mathcal{C}_L^\bullet$ satisfies the first perversity condition, that is, the following inequality holds.*

$$\dim supp(\mathcal{H}^i(\mathcal{C}_L^\bullet)) \leq n - i$$

Proof. Let $p \in S_k^L$. Then $(L, p) = (L', p') \times (\mathbb{C}^k, 0)$ and $\mathcal{H}^i(\mathcal{C}_L^\bullet)_p = \mathcal{H}^i(\mathcal{C}_{L'}^\bullet)_{p'}$. But $\dim(L') = n - k$, so $\mathcal{H}^i(\mathcal{C}_{L'}^\bullet)_{p'} = 0$ for all $i > n - k$. This means that for fixed i, $\mathcal{H}^i(\mathcal{C}_L^\bullet)_p = 0$ for $p \in S_k^L$ for all $k > n - i$. So $\mathcal{H}^i(\mathcal{C}_L^\bullet)$ is supported on the strata S_k^L for $k \leq n - i$. By "Condition P" they are of dimension less or equal $n - i$.

The second perversity condition means that

$$\dim supp(\mathcal{H}_V^i(\mathcal{C}_L^\bullet)) \leq \dim(V)$$

for any irreducible subspace $V \subset L$ and any $i \in \{0, \dots, n - \dim(V)\}$. Here $\mathcal{H}_V^i(\mathcal{C}_L^\bullet)$ is the i-th local cohomology sheaf with respect to V of $\mathcal{C}^\bullet$. It is not known whether this condition is always satisfied by a variety L with constructible complex $\mathcal{C}_L^\bullet$. Whenever this is the case, the $\mathcal{H}^i$'s are the *de Rham*-cohomology modules of a *single* $\mathcal{D}_M$-module supported on L. This suggest that the complex $\mathcal{C}_L^\bullet$ is related to $\mathcal{D}$-module theory. Some more evidence for this conjecture comes from the following consideration: Every complex manifold X is lagrangian in its cotangent bundle T^*X. Consider *Spencer's* complex, which is a resolution of $\mathcal{O}_X$ as a $\mathcal{D}_X$-module, explicitly:

$$\begin{aligned} Sp(\mathcal{O}_X)^\bullet : \qquad & \ldots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X^{p+1} \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X^p \to \ldots \\ & \to \mathcal{D}_X \to \mathcal{O}_X \to 0 \end{aligned}$$

The *de Rham*-complex of a $\mathcal{D}_X$-module $\mathcal{M}$ is obtained as

$$DR(M) := \mathcal{H}om_{\mathcal{D}_X}(Sp(\mathcal{O}_X)^\bullet, \mathcal{M})$$

If we define a generalized version of the complex $\mathcal{C}_L^\bullet$ as

$$\mathcal{C}_L^p(\mathcal{M}) := \mathcal{H}om_{\mathcal{O}_L}\left(\bigwedge^p \mathcal{L}, \mathcal{M}\right)$$

for some module $\mathcal{M}$ over the Lie algebroid $\mathcal{L}$, then $\mathcal{C}_X^p(\mathcal{M})$ is exactly the *de Rham*-complex of the $\mathcal{D}_X$-Module $\mathcal{M}$.

5. Examples and results

In this section we will describe some of the basic examples of singular lagrangian submanifolds, in particular those for which results on their deformation spaces are available. We start with the simplest case, a plane curve C in $\mathbb{K}^2$, given as the zero set of a mapping $f : \mathbb{K}^2 \to \mathbb{K}$. Such a curve C is obviously lagrangian. In this case the complex $\mathcal{C}_C^\bullet$ simplifies to

$$\begin{array}{rcl} \mathcal{C}_C^0 = \mathcal{O}_C & \xrightarrow{\delta} & \mathcal{C}_C^1 = \mathcal{H}om_{\mathcal{O}_C}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C) = \mathcal{H}om_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C) = \mathcal{O}_C \\ h & \longmapsto & \{h, f\} \end{array}$$

It follows immediately that $\mathcal{H}^2(\mathcal{C}_C^\bullet) = 0$, while

$$\mathcal{LT}_C^1 = \mathcal{H}^1(\mathcal{C}_C^\bullet) = \mathcal{C}oker(\delta)$$

This sheaf is supported on the singular points of the curve, let x_0 be such a point. Then we have

$$\mathcal{LT}_{C,x_0}^1 = \frac{\mathcal{O}_{C,x_0}}{\{\{h, f\} \mid h \in \mathcal{O}_{C,x_0}\}}$$

Now the following equalities hold

$$\begin{aligned} \frac{\mathcal{O}_{C,x_0}}{\left\{\{h, f\} | \, h \in \mathcal{O}_{C,x_0}\right\}} &= \frac{\Omega^2_{\mathbb{K}^2,x_0}}{\left\{f\Omega^2_{\mathbb{K}^2,x_0} + \{df \wedge dh | h \in \mathcal{O}_{C,x_0}\}\right\}} \\ &= \frac{\Omega^2_{\mathbb{K}^2,x_0}}{\left\{f\Omega^2_{\mathbb{K}^2,x_0} + df \wedge d\Omega^0_{\mathbb{K}^2,x_0}\right\}} \end{aligned}$$

because $\mathcal{O}_{C,x_0} \cong \Omega^2_{\mathbb{K}^2,x_0}/(f\Omega^2_{\mathbb{K}^2,x_0})$ and the Poisson bracket of two functions h and g corresponds under the isomorphism $\mathcal{O}_{\mathbb{K}^2,x_0} \cong \Omega_{\mathbb{K}^2,x_0}$ to the 2-form $dh \wedge dg$. But it is known (see [Mal74]) that the dimension of the last quotient equals μ, the Milnor number of the plane curve singularity (C, x_0). So the result is:

$$\mathcal{LT}_C^1 = \prod_{x_0 \in Sing(C)} \mathbb{K}^{\mu(C,x_0)}$$

This is remarkable because the usual T_C^1 has dimension τ (the Tjurina number) which is in general smaller than μ. The difference corresponds to the space of symplectic structures on $\mathbb{K}^2$ relative to which C is lagrangian (see [Giv88] and [Her], Theorem 7.2b).

Applying lemma 5, we see that the dimension of $\mathcal{LT}^1$ for a surface singularity which is a curve germ, crossed with a smooth factor is also equal to the Milnor number of this curve. This result can also be obtained by a direct calculation, e.g., for a cuspidal edge given in four-space (with coordinates A, B, C, D and symplectic form $dA \wedge dC + dB \wedge dD$) by the two equations $A, B^2 - C^3$, we get $LT^1 = \mathbb{K}^2$ and $LT^2 = 0$.

We will proceed with further examples of lagrangian surfaces in $\mathbb{K}^4$, which satisfy "Condition P" of theorem 3. So there are three strata: one point with embedding dimension four (supposed to be the origin), the singular locus away from this point and the regular locus. In order to simplify the calculation of the cohomology of $\mathcal{C}^\bullet$, we will suppose that our varieties are strongly quasi-homogeneous in the sense of [CJNMM96], that is, one can choose local coordinates of the ambient space around each point of L such that the defining equations become weighted homogeneous with positive weights. In this case, the *de Rham*-complex is a resolution of the constant sheaf as one can see by considering the decomposition of the modules Ω_L^p into eigenspaces of the Lie-derivative.

Lemma 8. *Let $L \subset M$ be a strongly quasi-homogeneous lagrangian subvariety. Consider the map $J : (\Omega_L^\bullet, d, \wedge) \to (\mathcal{C}_L^\bullet, \delta, \wedge)$ of DGA's from corollary 1. Denote by $\widetilde{\Omega}_L^\bullet$ the subcomplex $\mathcal{I}m(J)$ in $\mathcal{C}_L^\bullet$. Then $\widetilde{\Omega}_L^\bullet$ is a resolution of $\mathbb{K}_L$.*

Proof. By the long exact cohomology sequence, it suffices to prove that the complex $\mathcal{K}er(J)$ is acyclic. This can be done in exactly the same way as for $\Omega_L^\bullet$ provided that the inner derivative i_E (E being the quasi-homogeneous Euler vector field) maps $\mathcal{K}er(J) \cap \Omega_L^p$ into $\mathcal{K}er(J) \cap \Omega_L^{p-1}$. But this follows from lemma 1 because if ω is a torsion element than the same holds for $i_E\omega$.

Corollary 5. *Denote by $\mathcal{G}_L^\bullet$ the cokernel of the map J. Then there is an exact sequence of complexes of $\mathcal{O}_L$-modules*

$$0 \longrightarrow \widetilde{\Omega}_L^\bullet \longrightarrow \mathcal{C}_L^\bullet \longrightarrow \mathcal{G}_L^\bullet \longrightarrow 0$$

and the associated long exact sequence gives

$$\mathcal{H}^i(\mathcal{C}_L^\bullet) = \mathcal{H}^i(\mathcal{G}_L^\bullet)$$

for all $i \geq 0$. In particular, if L is of dimension two, then we get

$$\mathcal{H}^1(\mathcal{C}_L^\bullet) = \mathcal{K}er(\delta : \mathcal{G}_L^1 \to \mathcal{G}_L^2)$$
$$\mathcal{H}^2(\mathcal{C}_L^\bullet) = \mathcal{C}oker(\delta : \mathcal{G}_L^1 \to \mathcal{G}_L^2)$$

We can thus calculate $\mathcal{LT}_L^1$ and $\mathcal{LT}_L^2$ by computing the induced morphism $\delta : \mathcal{G}_L^1 \to \mathcal{G}_L^2$. As J is an isomorphism at smooth points, the sheaves $\mathcal{G}_L^i$ are supported on the singular locus of L, which is of dimension one. In a neighborhood of all of its regular points q (points with embedding dimension three), the germ is decomposable and the dimension of $\mathcal{H}^i(\mathcal{C}_L^\bullet)_q$ is given by lemma 5. So we are only interested in the one special point with maximal embedding dimension. We now choose a quasihomogenous (with respect to the given grading) element $p \in \mathcal{O}_L$ which is finite when restricted to the support of $\mathcal{G}_L^i$, note that although this is set-theoretically equal to the singular locus of L, it may have embedded components. We will suppose that p maps the origin in $\mathbb{K}^4$ to the origin in $\mathbb{K}$. Consider the sheaves $p_*\mathcal{G}^1_L$ and $p_*\mathcal{G}^2_L$, these are modules over $\mathcal{O}_{\mathbb{K}}$. Denote by $\widetilde{E}$ resp. $\widetilde{F}$ the modules of section of $p_*\mathcal{G}^1_L$ resp. $p_*\mathcal{G}^2_L$ in a small neighborhood of the origin. Then they can be decomposed into torsion and torsion free parts, the former being supported on the origin while the latter is free over $\mathbb{K}\{t\}$. In practice, this is done as follows: As $\mathcal{G}^1_L$ and $\mathcal{G}^2_L$ are graded modules over $\mathcal{O}_L$ and the map $\delta : \mathcal{G}^1_L \to \mathcal{G}^2_L$ is homogeneous, we consider the decomposition of these modules into homogeneous parts. The map p is finite, so the torsion submodules of $\widetilde{E}$ and $\widetilde{F}$ corresponds to homogeneous parts of $\mathcal{G}^1_L$ and $\mathcal{G}^2_L$ in a **finite** number of degrees. This yields a decomposition of $\widetilde{E}$ and $\widetilde{F}$ into $\widetilde{E} = \widehat{E} \oplus E$ and $\widetilde{F} = \widehat{F} \oplus F$ such that $\widehat{E}$ and $\widehat{F}$ are supported on the origin, while E and F are free. We first calculate the rank of these modules.

Lemma 9. *The rank of E and F is the Milnor number μ of the transversal curve singularity, i.e. the germ $(L', 0)$ such that $(L, p) = (L', 0) \times (\mathbb{K}, 0)$ for all $p \in Sing(L) \setminus 0$.*

Proof. We will determine the rank of $(\mathcal{G}^1_L)_p$ and $(\mathcal{G}^2_L)_p$ at a decomposable point p. This is an explicit calculation involving the definition of the complex $\mathcal{C}^\bullet_L$ and the map $J : \Omega^\bullet_L \to \mathcal{C}^\bullet_L$. So suppose that (L, p) is a decomposable germ. We choose coordinates $(x, y, s, t) \in \mathbb{K}^4$ (with symplectic form $\omega = dx \wedge dy + ds \wedge dt$) around p such that L is given as the zero locus of s and a function f depending only on x and y. Denote the ideal generated by these two functions by I and by R the stalk of $\mathcal{O}_L$ at the point p. Then we can identify I/I^2 with R^2, so $Hom_R(I/I^2, R)$ is free on the two generators n_1 and n_2, where

$$\begin{array}{l} n_1(f) = 1 \; n_1(s) = 0 \\ n_2(f) = 0 \; n_2(s) = 1 \end{array}$$

while $Hom_R(I/I^2 \wedge I/I^2, R)$ is just R, generated by the homomorphism sending $f \wedge s$ to 1 in R. The complex $\mathcal{C}^\bullet$ at the point p then reads:

$$\begin{array}{ccccc} R & \longrightarrow & R n_1 \oplus R n_2 & \longrightarrow & R \\ h & \longmapsto & (\{h, f\}, \{h, s\}) & & \\ & & (p, q) & \longmapsto & \{p, s\} + \{f, q\} \end{array}$$

where the pair $(p, q) \in R^2 = Hom_R(I/I^2, R)$ denotes the homomorphism sending $f \in I/I^2$ to $p \in R$ and $s \in I/I^2$ to $q \in R$.

Now we have to investigate the modules of differential forms on L at x. In general

$$\Omega^p_R = \Omega^p_S / \left(I \Omega^p_S + dI \wedge \Omega^{p-1}_S \right)$$

where S is the ring $\mathbb{K}\{x, y, s, t\}$. This leads to

$$\Omega^1_R = M_1 \oplus M_2$$
$$\Omega^2_R = M_3 \oplus M_4$$

where we have used the following abbreviations:

$$M_1 = \frac{R\,dx \oplus R\,dy}{R\,df}$$
$$M_2 = R\,dt$$
$$M_3 = \frac{R\,dx \wedge dy}{R\,df \wedge dx \oplus R\,df \wedge dy}$$
$$M_4 = \frac{R\,dx \wedge dt \oplus R\,dy \wedge dt}{R\,df \wedge dt}$$

$J : \Omega^\bullet_L \to \mathcal{C}^\bullet_L$ can be described as

$$\begin{aligned} J : M_1 &\longrightarrow R\,n_1 \oplus R\,n_2 \\ dx &\longmapsto (\{x, f\}, \{x, s\}) = (\partial_y f, 0) \\ dy &\longmapsto (\{y, f\}, \{y, s\}) = (-\partial_x f, 0) \\ J : M_2 &\longrightarrow R\,n_1 \oplus R\,n_2 \\ dt &\longmapsto (\{t, f\}, \{t, s\}) = (0, 1) \end{aligned}$$

$$\begin{aligned} J : M_3 &\longrightarrow R \\ dx \wedge dy &\longmapsto J(dx) \wedge J(dy) = 0 \\ J : M_4 &\longrightarrow R \\ dx \wedge dt &\longmapsto J(dx) \wedge J(dt) = \partial_y f \\ dy \wedge dt &\longmapsto J(dx) \wedge J(dt) = -\partial_x f \end{aligned}$$

E and F are the cokernels of the maps $J : M_1 \oplus M_2 \to R\,n_1 \oplus R\,n_2$ and $J : M_3 \oplus M_4 \to R$, respectively. We see that E is a quotient of $R\,n_1$ and F a quotient of $R\,(n_1 \wedge n_2)$. Using the form $dx \wedge dy$, we can identify the modules $R\,n_1$ and $R\,(n_1 \wedge n_2)$ with $\left(\Omega^2_{\mathbb{K}^2,0}/f\Omega^2_{\mathbb{K}^2,0}\right) \widehat{\otimes} \mathbb{K}\{t\}$. Under this identification, the modules E and F are isomorphic to $\left({}''H/(f \cdot {}''H)\right) \widehat{\otimes} \mathbb{K}\{t\}$, where ${}''H := \Omega^2_{\mathbb{K}^2,0}/df \wedge d\Omega^0_{\mathbb{K}^2,0}$ is the Brieskorn lattice of the function $f \in \mathbb{K}\{x, y\}$. The Brieskorn lattice is a rank μ $\mathbb{K}\{f\}$-module, so that E and F are of rank μ over $\mathbb{K}\{t\}$.

The lemma shows that the operator δ defines an (E, F)-connection in the sense of [Mal74]. Denote $\delta_{|E}$ by D for short. Then D is a first-order differential operator $D : \mathcal{O}^\mu_{\mathbb{K}} \to \mathcal{O}^\mu_{\mathbb{K}}$ which respects the grading. So it is of the form

$$D = t\partial_t \mathbb{1} + A$$

where A is a constant $\mu \times \mu$-matrix. We can thus calculate the cohomology of the operator δ in two steps: $\widehat{E}$ and $\widehat{F}$ are of finite dimension, so the kernel and cokernel of $\delta_{|\widehat{E}}$ can be computed explicitly. Secondly, the cohomology of $\delta_{|E}$ can be deduced from the solutions of the differential system given by D. All explicit calculations have been done using *Macaulay2* (see [GS]).

The first interesting example we are going to study is the so called "open swallowtail". For details of its definition, see [Giv82] and [Giv83]. Consider the space of polynomials in one variable of degree $d := 2k + 1$ with fixed leading coefficient and sum of roots equal to zero, that is, the space

$$\mathcal{P}_{2k+1} = \left\{x^{2k+1} + A_2 x^{2k-1} + \ldots + A_{2k+1} x^0\right\} \cong \mathbb{K}^{2k}$$

which comes equipped with the following symplectic structure

$$\omega = \sum_{i=2}^{k+1} (2k + 1 - i)!\,(i - 2)! \cdot (-1)^i dA_i \wedge dA_{2k+3-i}$$

We will write Σ_k for the subspace consisting of those polynomials which have a root of multiplicity greater than k. This space is obviously of dimension k and it can be shown that the form ω vanishes on its regular locus. So we have a lagrangian subvariety in the space $\mathcal{P}_{2k+1}$, which is called open swallowtail. To get a more concrete impression of how it looks like, we will describe the easiest examples. For $k = 1$, $\Sigma_1 \subset \mathcal{P}_3$ is just the ordinary cusp in the plane, this case has already been discussed above. For $k = 2$, we obtain a surface in the four-dimensional space (see the conceptual figure 1)

$$\mathcal{P}_5 = \left\{x^5 + Ax^3 + Bx^2 + Cx + D \mid (A, B, C, D) \in \mathbb{K}^4\right\}$$

(the symplectic form is $\omega = 3d\,A \wedge d\,D + d\,C \wedge d\,B$) consisting of those polynomials f with a root of multiplicity at least three. Such a f can be written as $f = (x - a)^3(x^2 + 3ax + b)$, so there is a normalization of Σ_2 given by

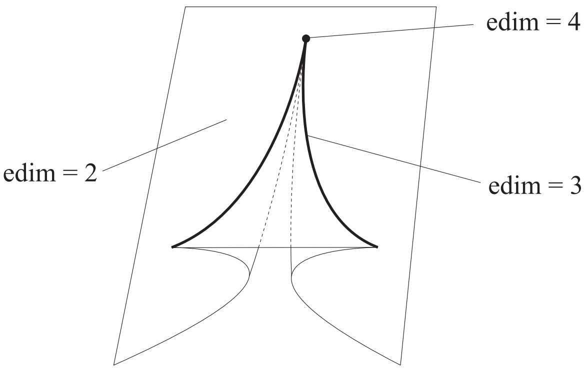

Fig. 1. The open swallowtail $\Sigma_2 \subset \mathbb{K}^4$

$$\begin{aligned} n : \mathbb{K}^2 &\longrightarrow \mathcal{P}_5 = \mathbb{K}^4 \\ (a, b) &\longmapsto (b - 6a^2, 8a^3 - 3ab, 3a^2b - 3a^4, -a^3b) \end{aligned}$$

Note that the singular locus of Σ_2 is a again a cusp as well as the transversal curve singularity.

The space Σ_2 is our main example, we will describe in some detail how to apply the general results in this case. Using elimination theory, we can calculate the defining equations of Σ_2 in $\mathbb{K}^4$. It turns out that the swallowtail is a determinantal variety given by the minors of the matrix

$$\begin{pmatrix} 9D & 9B^2 - 32AC \\ 3C & -5AB + 125D \\ -9B & 45A^2 - 100C \end{pmatrix}$$

The ideal which defines Σ_2 is generated by the following three polynomials

$$\begin{aligned} f_1 &= -27B^2C + 96AC^2 - 45ABD + 1125D^2, \\ f_2 &= 81B^3 - 288ABC + 405A^2D - 900CD \\ f_3 &= -45AB^2 + 135A^2C - 300C^2 + 1125BD \end{aligned}$$

So Σ_2 is not a complete intersection but nevertheless Cohen-Macaulay by the Hilbert-Burch theorem. We list the commutators $\{f_i, f_j\}$ (for $1 \leq i < j \leq 3$) with respect to the given set of generators (this is a direct proof that $\Sigma_2 \subset \mathbb{K}^4$ is involutive):

$$\begin{aligned} \{f_1, f_2\} &= -576Af_1 + 81Bf_2 - 96Cf_3 \\ \{f_1, f_3\} &= 15Af_2 - 12Bf_3 \\ \{f_2, f_3\} &= -900f_1 + 18Af_3 \end{aligned}$$

Σ_2 is quasi-homogeneous with the weights $(2, 3, 4, 5)$ for the variables A, B, C, D, respectively. We can thus apply the machinery developed above to obtain that $\left(\mathcal{LT}^1_{\Sigma_2}\right)_0 = 0$, while $\left(\mathcal{LT}^2_{\Sigma_2}\right)_0 = \mathbb{K}$. The operator D is in this case

$$t\partial_t \mathbb{1} + \begin{pmatrix} 11/40 & -245/2 & 0 & 0 \\ 33/4000 & 109/40 & 0 & 0 \\ 0 & 0 & 49/15 & -59/27 \\ 0 & 0 & 51/100 & 11/15 \end{pmatrix}$$

For $\mathbb{K} = \mathbb{C}$, the residue of the meromorphic connection on the trivial bundle $\mathcal{O}^{\mu}_{\mathbb{C}}$ given by the operator ∂_t has the following eigenvalues (spectral numbers) which determine the monodromy of the locally constant sheaf $\mathcal{LT}^1_{|Sing(\Sigma_2)\setminus 0}$

$$-\frac{8}{10}, -\frac{13}{10}, -\frac{22}{10}, -\frac{27}{10}$$

The second large class of examples are the conormal spaces. Given any submanifold Y of an n-dimensional manifold X, the total space of the conormal bundle $T^*_Y X$ is always a lagrangian submanifold of T^*X. More generally, if Y is an analytic subspace, we can take the closure of the space of conormals to all smooth points of Y. The result (which is called conormal space of Y in X) is still lagrangian, but may have singularities. Examples of such spaces are provided by holonomic $\mathcal{D}_X$-modules. Their characteristic varieties are always a finite union of conormal spaces. Obviously, these spaces are conical in the fibers of T^*X. If Y is a plane curve $C \subset X = \mathbb{K}^2$, then the conormal space $T^*_C\mathbb{K}^2$ will be a surface in $\mathbb{K}^4$. Here the results are as follows.

equation of C	**LT^1**	**LT^2**	**spectrum (multiplicity, if $\neq 1$)**
$y^2 - x^5$	0	0	$-\frac{4}{5}, -\frac{16}{5}$
$y^3 - x^7$	0	0	$-\frac{37}{7}, -\frac{61}{7}, -\frac{69}{7}, -\frac{85}{7}, -\frac{93}{7}, -\frac{117}{7}$
$y^5 - x^7$	0	0	$-\frac{116}{7}, -\frac{132}{7}, -\frac{148}{7}, -\frac{164}{7},$
$y^3 - x^6$	$\mathbb{K}$	$\mathbb{K}$	$-\frac{7}{2}, -\frac{10}{2}^{(2)}, -\frac{13}{2}$
$xy(x+y)(x-y)(x-2y)$	$\mathbb{K}^2$	$\mathbb{K}^2$	$-$

In the last example, there is only an isolated singularity, so the modules $\mathcal{G}^1_L$ and $\mathcal{G}^2_L$ are artinien.

Finally, there is a third class of singular lagrangian subvarieties, these are *completely integrable hamiltonian systems*. Such a system is given in the $2n$-dimensional phase space by n Poisson-commuting functions. The ideal formed by them then obviously satisfies the involutivity condition. If, additionally, the common zero set of these function is a complete intersection, then it will be lagrangian in our sense. The lagrangian deformation space of such a system is at least n-dimensional (deforming by a constant is flat and the ideal stays involutive).

To get the equations of some interesting examples, we will proceed as follows. Choose coordinates (p_1, q_1, p_2, q_2) of $\mathbb{K}^4$ and set $z_1 = p_1 + iq_1$ and $z_2 = p_2 + iq_2$ (This can obviously be done only in the real case, but for any real lagrangian singularity, we might consider its complexification, which is given by the same equations). We can now express functions on $\mathbb{K}^4$ in the variables $z_1, z_2, \overline{z_1}, \overline{z_2}$, and the Poisson bracket becomes

$$\{f, g\} = 2i\left(\partial_{\overline{z}_1} f \cdot \partial_{z_1} g - \partial_{\overline{z}_1} g \cdot \partial_{z_1} f + \partial_{\overline{z}_2} f \cdot \partial_{z_2} g - \partial_{\overline{z}_2} g \cdot \partial_{z_2} f\right)$$

We want to find functions f, g such that $\{f, g\} = 0$. Set, for example $f = \lambda z_1\overline{z_1} + \mu z_2\overline{z_2}$ and let us look for a $g = z_1^\alpha \overline{z_1}^\beta z_2^\gamma \overline{z_2}^\delta$ for some parameters $\lambda, \mu, \alpha, \beta, \gamma, \delta \in \mathbb{N}$. It can be easily verified that the commuting condition transforms to

$$\lambda(\alpha - \beta) - \mu(\gamma - \delta) = 0$$

The function f is real, so we might take f and $Re(g)$ to get a completely integrable system. The following table shows results for some resonance (**r**) coefficients λ, μ and exponents (**e**) α, β, γ, δ.

r	**e**	**LT**^{1}	**LT**^{2}	**spectrum (multiplicity)**
$1, 0$	$0, 0, 1, 1$	$\mathbb{K}^2$	$\mathbb{K}$	$-3^{(4)}$
$1, 2$	$0, 2, 1, 0$	$\mathbb{K}^3$	$\mathbb{K}^2$	$-\frac{2}{2}^{(2)}, -\frac{3}{2}^{(2)}, -\frac{4}{2}^{(2)}, -\frac{5}{2}^{(2)}, -\frac{6}{2}^{(2)}$
$1, 3$	$3, 0, 0, 1$	$\mathbb{K}^4$	$\mathbb{K}^3$	$-\frac{3}{3}^{(2)}, -\frac{5}{3}^{(2)}, -\frac{7}{3}^{(4)}, -\frac{9}{3}^{(4)}, -\frac{11}{3}^{(4)}, -\frac{13}{3}^{(2)}, -\frac{15}{3}^{(2)}$
$1, 4$	$4, 0, 0, 1$	$\mathbb{K}^5$	$\mathbb{K}^4$	$-\frac{4}{4}^{(2)}, -\frac{7}{4}^{(2)}, -\frac{9}{4}^{(2)}, -\frac{10}{4}^{(2)}, -\frac{12}{4}^{(2)}, -\frac{13}{4}^{(2)},$ $-\frac{14}{4}^{(2)}, -\frac{15}{4}^{(2)}, -\frac{16}{4}^{(2)}, -\frac{17}{4}^{(2)}, -\frac{18}{4}^{(2)}, -\frac{19}{4}^{(2)},$ $-\frac{20}{4}^{(2)}, -\frac{22}{4}^{(2)}, -\frac{23}{4}^{(2)}, -\frac{25}{4}^{(2)}, -\frac{28}{4}^{(2)}$

Remark. It seems that the spectral numbers have a symmetry property, but this has not yet been proved. They look similar to the spectrum of an isolated hypersurface singularity (spectral numbers of the Brieskorn lattice). One might speculate that there is a mixed Hodge structure related to this theory and that the eigenvalues share further properties with the singularity spectrum, e.g. the semi-continuity under deformations.

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