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Extendability of holomorphic differential forms near isolated hypersurface singularities

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Introduction

Let $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an holomorphic function with an isolated singularity at the origin. Let $X = \{z \in \mathbb{C}^{n+1} \mid |z| < \varepsilon \text{ and } |f(z)| < \eta\}$ for $0 < \eta \ll \varepsilon$, ε sufficiently small, and let V be the set of zeroes of f on X. Then V is a contractible Stein space and $U = V - \{0\}$ is smooth.

A holomorphic form ω on U is called *of first kind* if there exists a resolution $\pi: \tilde{V} \to V$ of the singularity $(V, 0)$ such that $\pi^*(\omega)$ extends holomorphically to $\tilde{V}$. A result of Greuel ([3], Proposition 2.3) implies that for $p \leqq n - 2$ every holomorphic p-form on U is of first kind on V. In fact this result holds for arbitrary isolated singularities (Theorem (1.3)). An application of this is a proof of the following (easy) case of a conjecture of Zariski and Lipman [16]: If $(V, 0)$ is an isolated singularity of dimension at least 3 and $\Theta_{V,0} := \mathrm{Hom}_{\mathcal{O}_{V,0}}(\Omega^1_{V,0}, \mathcal{O}_{V,0})$ is a free $\mathcal{O}_{V,0}$-module, then $(V, 0)$ is in fact smooth. The crucial case of $\dim V = 2$ remains open, however.

The remaining cases are n-forms and $(n-1)$-forms. From now on we take $n \geqq 2$. Concerning n-forms one has the invariant

$$p_g = \dim \left\{ \begin{matrix} \text{holomorphic} \\ n\text{-forms on } U \end{matrix} \right\} \Big/ \left\{ \begin{matrix} n\text{-forms of} \\ \text{first kind} \end{matrix} \right\}$$

which is equal to the geometric genus of $(V, 0)$. It counts the number of adjunction conditions imposed by the singularity. See [6] for a detailed discussion of this invariant.

Our main attention goes to the invariant

$$q = \dim \left\{ \begin{matrix} \text{holomorphic} \\ (n-1)\text{-forms on } U \end{matrix} \right\} \Big/ \left\{ \begin{matrix} (n-1)\text{-forms} \\ \text{of first kind.} \end{matrix} \right\}.$$

It has been studied by Yau [14] and Wahl [13].

Our main result indicates how to compute q (and p_g) for isolated hypersurface singularities. Our formula uses the Gauss-Manin system of f, see [9, 12]. As an application of the formula we give an example of a deformation of a function of three variables with constant Milnor number, depending on two

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parameters s_1, s_2 such that

$$q = 1 \quad \text{if} \quad s_1 \neq 0,$$
$$q = 0 \quad \text{if} \quad s_1 = 0, \quad s_2 \neq 0,$$
$$q = 2 \quad \text{if} \quad s_1 = s_2 = 0.$$

As a consequence, the invariant q is not a semicontinuous function on the stratum with constant Milnor number. The example is

$$f_{s_1,s_2}(x, y, z) = x^7/7 + y^3/3 + z^3/3 - s_1x^5y - s_2x^4yz.$$

We have also obtained a similar formula for an invariant q' closely related to q. This enables one to compute the invariants α, β, γ which have been considered by Wahl [13]. It is hoped that their study gives deeper insight to the moduli problem for isolated hypersurface singularities.

§ 1. Extendability of forms of low degree

Let V be an n-dimensional complex space with singular locus V_{sing} and let $U = V - V_{\text{sing}}$.

(1.1) Proposition. *For a holomorphic p-form ω on U the following conditions are equivalent:*

(i) *for each C^∞ map $\gamma: \Delta^p \to V$ the integral $\int_\gamma \omega$ exists;*

(ii) *there exists a complex manifold $\tilde{V}$ and a proper holomorphic map $\pi: \tilde{V} \to V$ such that $\pi: \tilde{V} - \pi^{-1}(V_{\text{sing}}) \to U$ is biholomorphic and $\pi^*(\omega)$ extends to a holomorphic p-form on the whole of $\tilde{V}$;*

(iii) *for every pair $(\tilde{V}, \pi)$ as in* (ii) *the form $\pi^*(\omega)$ extends holomorphically.*

Proof. See [6] for the case of n-forms. The general case is similar.

(1.2) Definition. We call a holomorphic p-form on U *of first kind on* V if it satisfies the equivalent conditions of the preceding proposition.

(1.3) Theorem. *Let V be a complex space with isolated singular locus, $U = V - V_{\text{sing}}$, $n = \dim(V) \geqq 2$. Let $p \leqq n - 2$. Then every holomorphic p-form on U is of first kind on V.*

Proof. Without loss of generality we may assume that V is a contractible Stein space with only one singular point x. We choose a resolution $\pi: \tilde{V} \to V$ such that $\pi^{-1}(x) = D_1 \cup \cdots \cup D_k$ is a union of smooth divisors on V with normal crossings. Then we have the vanishing theorem

$$H^q(\tilde{V}, I_D\Omega^p_{\tilde{V}}(\log D)) = 0 \quad \text{for} \quad p + q > n$$

([11], Theorem 2b). Here $\Omega^{\bullet}_{\tilde{V}}(\log D)$ is the logarithmic De Rham complex on $\tilde{V}$ and I_D is the ideal sheaf of the divisor D. By duality we have

$$H^1_D\big(\tilde{V}, \Omega^p_{\tilde{V}}(\log D)\big) = 0 \quad \text{for} \quad p < n-1 .$$

$\big(H^i(\tilde{V}, \mathcal{F})^{\wedge}$ is dual to $H^{n-i}_D(\tilde{V}, \mathcal{F}^* \otimes \omega_{\tilde{V}})$ for F locally free on $\tilde{V}$ and $\big(I_D\Omega^p_{\tilde{V}}(\log D)\big)^* \otimes \omega_{\tilde{V}} \cong \Omega^{n-p}_{\tilde{V}}(\log D)\big)$.

So if $p < n-1$, every holomorphic p-form on U extends to $\tilde{V}$ as a form with logarithmic poles along D.

By [10], § 1 the spaces $H^p(D)$ and $H^p(U)$ carry mixed Hodge structures such that

$$F^p H^p(D, \mathbb{C}) = H^0\big(D, \Omega^p_{\tilde{V}}/I_D\Omega^p_{\tilde{V}}(\log D)\big)$$

and

$$F^p H^p(U, \mathbb{C}) = H^0\big(D, \Omega^p_{\tilde{V}}(\log D)/I_D\Omega^p_{\tilde{V}}(\log D)\big) .$$

The natural map $H^p(D) \cong H^p(V) \to H^p(U)$ is a morphism of mixed Hodge structures, hence it is strictly compatible with the Hodge filtrations. For $p < n$ this map is surjective (ibid. (1.11)) so we may conclude that the natural map

$$\varrho\colon H^0\big(D, \Omega^p_{\tilde{V}}/I_D\Omega^p_{\tilde{V}}(\log D)\big) \to H^0\big(D, \Omega^p_{\tilde{V}}(\log D)/I_D\Omega^p_{\tilde{V}}(\log D)\big)$$

is also surjective for $p < n$. From the exact sequence

$$0 \to \Omega^p_{\tilde{V}} \to \Omega^p_{\tilde{V}}(\log D) \to \Omega^p_{\tilde{V}}(\log D)/\Omega^p_{\tilde{V}} \to 0$$

we obtain the connecting homomorphism

$$\delta\colon H^0\big(\tilde{V}, \Omega^p_{\tilde{V}}(\log D)/\Omega^p_{\tilde{V}}\big) \to H^1(\tilde{V}, \Omega^p_{\tilde{V}}) .$$

If we compose this map with the natural map

$$H^1(\tilde{V}, \Omega^p_{\tilde{V}}) \to H^1\big(\tilde{V}, \Omega^p_{\tilde{V}}/I_D\Omega^p_{\tilde{V}}(\log D)\big)$$

we also get a connecting homomorphism for a suitable sequence, which is injective because ϱ is surjective. Hence δ is injective too. We conclude that $H^0(\tilde{V}, \Omega^p_{\tilde{V}}) \xrightarrow{\sim} H^0\big(V, \Omega^p_{\tilde{V}}(\log D)\big)$ for $p < n$. Hence every form on $\tilde{V}$ with only logarithmic poles along D is already holomorphic. □

(1.4) Corollary: *Let V be a contractible Stein space with one singular point x and $U = V - \{x\}$. Let $\pi\colon \tilde{V} \to V$ be a resolution. Then the map $d\colon H^0(\Omega^{n-1}_U)/H^0(\Omega^{n-1}_{\tilde{V}}) \to H^0(\Omega^n_U)/H^0\big(\Omega^n_{\tilde{V}}(\log D)\big)$ induced by differentiation is injective.*

Proof. We have $H^0(\Omega^{n-1}_U)/H^0(\Omega^{n-1}_{\tilde{V}}) = H^0(\Omega^{n-1}_U)/H^0\big(\Omega^{n-1}_{\tilde{V}}(\log D)\big)$ and by [5] the differentiation map

$$H^1_D\big(\tilde{V}, \Omega^{n-1}_{\tilde{V}}(\log D)\big) \xrightarrow{\beta} H^1_D\big(V, \Omega^n_{\tilde{V}}(\log D)\big)$$

is injective. In the commutative diagram

$$\begin{array}{ccc} H^0(\Omega_U^{n-1})/H^0(\Omega_{\tilde V}^{n-1}) & \xrightarrow{d} & H^0(\Omega_U^n)/H^0(\Omega_{\tilde V}^n(\log D)) \\ \downarrow{\scriptstyle\alpha} & & \downarrow \\ H_D^1(\Omega_{\tilde V}^{n-1}(\log D)) & \xrightarrow{\beta} & H_D^1(\Omega_{\tilde V}^n(\log D)) \end{array}$$

the maps α and β are injective, hence d is injective too. □

(1.5) *Definition.* For arbitrary isolated singularities (V, x) we define the *irregularity* q and the *geometric genus* p_g as in the introduction.

From Corollary (1.4) we obtain the inequality $q \leqq p_g - h^{n-1}(\mathcal{O}_D)$. In particular $q = 0$ holds for rational singularities.

(1.6) We now prove the special case of the Zariski-Lipman conjecture mentioned in the introduction. Assume that $n > 2$ and that Θ_V is free. Take a basis $\vartheta_1, \ldots, \vartheta_n$ of sections. Let $(\tilde V, D) \xrightarrow{\pi} (V, x)$ be a good resolution with $\pi_* \Theta_{\tilde V} = \Theta_V$ (this exists by a result of Hironaka [15]). The vector fields ϑ_i lift to vector fields $\tilde\vartheta_i$ on $\tilde V$ which are tangent to D as V has an isolated singularity at x. Outside D, we have holomorphic 1-forms ω_j, $j = 1, \ldots, n$ with $\langle \tilde\vartheta_i, \omega_j\rangle = \delta_{ij}$. By Theorem (1.3) the ω_j are holomorphic on the whole of $\tilde V$. For $P \in D_{\mathrm{reg}}$ the vectors $\tilde\vartheta_i(P)$ must be linearly dependent contradicting the fact that $\langle \tilde\vartheta_i, \omega_j\rangle = \delta_{ij}$.

In the surface case this argument shows that the freeness of Θ_V implies that $q > 0$.

§ 2. The Gauss-Manin system

Let $f: X \to S$ be a good representative of a holomorphic function germ $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 as in the introduction, with $S = \{z \in \mathbb{C} \mid |z| < \eta\}$. Let D be an indeterminate and let $\Omega_X^p[D] = \bigoplus_{k=0}^{\infty} \Omega_X^p \cdot D^k$. Then $\Omega_X^{\cdot}[D]$ becomes a complex of sheaves by the differentiation

$$d(\omega \cdot D^k) = d\omega \cdot D^k - df \wedge \omega \cdot D^{k+1}.$$

(2.1) *Definition.* The Gauss-Manin system of f is the $\mathcal{D}_S$-module

$$\mathcal{H}_X = f_* \Omega_X^{n+1}[D]/d(f_* \Omega_X^n[D]).$$

The operator ∂_t acts on $\mathcal{H}_X$ via $\omega \cdot D^k \mapsto \omega \cdot D^{k+1}$ and t acts as $\omega \cdot D^k \mapsto f\omega \cdot D^k - k\omega \cdot D^{k-1}$. These formulas become clear if one uses the identification $\omega \cdot D^k \to [k!\, \omega/(f - t)^{k+1}]$ of $\Omega_X^{\cdot}[D]$ with a complex of meromorphic differential forms on $X \times S$ with poles along the graph of f modulo forms without poles; see [9], § 3.

The Gauss-Manin system $\mathcal{H}_X$ is a regular holonomic $\mathcal{D}_S$-module on which the operator ∂_t is invertible. It contains the sheaf

$$H'' = f_* \Omega_X^{n+1}/df \wedge d(f_* \Omega_X^{n-1})$$

(the lattice of Brieskorn) as a free $\mathcal{O}_S$-submodule.

The *Hodge filtration* $F^{\cdot}$ on $\mathcal{H}_X$ is given by

$$F^p\mathcal{H}_X = 0 \text{ for } p > n, \quad F^p\mathcal{H}_X = \partial_t^{n-p}H'' \text{ for } p \leqq n.$$

The *V-filtration* on the stalk $\mathcal{H}_{X,0}$ is defined as follows. Let $C^a \subset \mathcal{H}_{X,0}$, $C^a = \bigcup_{k \geqq 1} \operatorname{Ker}(t\,\partial_t - a)^k$. We let V_a (resp. $V_{>a}$) be the $\mathcal{O}_{S,0}$-submodule of $\mathcal{H}_{X,0}$ generated by all C^b with $b \in \mathbb{Q}$ and $b \geqq a$ (resp. $b > a$). (Observe that the monodromy is quasi-unipotent so $C^a = 0$ for $a \notin \mathbb{Q}$). We will use two results which describe the V-filtration in a different way for special cases. The first one is due to A. Varchenko. To formulate this, let $\pi: \tilde{X} \to X$ be a good embedded resolution of f, i.e. $\tilde{X}$ is a complex manifold, π is a proper holomorphic map such that $(f\pi)^{-1}(0)$ is a divisor with normal crossings on $\tilde{X}$ and π maps $\tilde{X} - \pi^{-1}(0)$ biholomorphically to $X - \{0\}$. Let $E_1, \ldots, E_k$ be the irreducible components of $\pi^{-1}(0)$. Each divisor E_i determines a valuation v_{E_i} on the spaces of holomorphic functions and holomorphic $(n+1)$-forms on X. For a holomorphic $(n+1)$-form ω on X we define its *geometrical weight* (w.r.t. π)

$$g(\omega) = \min_i \left\{\left(v_{E_i}(\omega) + 1\right)/v_{E_i}(f)\right\}$$

and we let

$$\alpha(\omega) = \max\{a \in \mathbb{Q} \mid [\omega] \in V_a\}$$

where $[\omega]$ denotes the image of ω in $H_0'' \subset \mathcal{H}_{X,0}$.

(2.2) Theorem. *For any holomorphic* $(n+1)$*-form on* X

$$g(\omega) \leqq \alpha(\omega) + 1 \text{ and if } g(\omega) \leqq 1 \text{ then } g(\omega) = \alpha(\omega) + 1.$$

Proof. See [12], Theorem 4.3.1.

Corollary: *If* $-1 < a \leqq 0$ *then*

$$H_0'' \cap V_a = \{[\omega] \in H_0'' \mid \omega \in H^0(X, \Omega_X^{n+1}) \text{ and } g(\omega) \geqq a + 1\};$$

$$H_0'' \cap V_{>a} = \{[\omega] \in H_0'' \mid \omega \in H^0(X, \Omega_X^{n+1}) \text{ and } g(\omega) > a + 1\}$$

for any good embedded resolution of f.

The second result we want to mention expresses the V-filtration for "nondegenerate functions" in terms of their Newton diagram. This is due to M. Saito [8]. For $f \in \mathbb{C}\{z_0, \ldots, z_n\}$ write $f = \Sigma_\nu a_\nu z^\nu$ where ν runs over all $(n+1)$-tuples of non-negative integers. Let $\operatorname{supp}(f) = \{\nu \mid a_\nu \neq 0\} \subset \mathbb{R}_+^{n+1}$ and let $\Gamma(f)$ be the convex hull of $\operatorname{supp}(f) + \mathbb{R}_+^{n+1}$. Let $\Gamma_+(f)$ be the union of all compact faces of $\Gamma(f)$. For σ a face of $\Gamma_+(f)$ we let $f_\sigma = \sum_{\nu \in \sigma} a_\nu z^\nu \in A_\nu =$ the subalgebra of $\mathbb{C}[z_0, \ldots, z_n]$ generated by all monomials z^ν with ν in the closed cone with vertex 0 on σ. We call f *nondegenerate* if for all such faces the ideal in A_σ generated by all $z_i \partial f_\sigma/\partial z_i$ has finite codimension.

Assume that f is nondegenerate and that $\Gamma_+(f)$ contains a point of all the coordinate axes in $\mathbb{R}^{n+1}$. Then the region bounded by the coordinate hyperplanes and $\Gamma_+(f)$ has finite volume. Define for $a > 0$

$$V'_a H''_0 = \{x \in H''_0 \mid \exists \text{ a holomorphic function } h \text{ on } X \text{ with } x = [h \cdot (dz_0/z_0) \wedge \cdots \wedge (dz_n/z_n)] \text{ and } \operatorname{supp}(h) \subset a\Gamma(f)\}.$$

(2.3) Theorem. *For any nondegenerate function as above we have*

$$V'_a H''_0 = V_{a+1} \cap H''_0, \text{ for any } a \in \mathbf{Q},\ a > 0.$$

Let us return to the Hodge filtration. Because $V_a = C^a + V_{>a}$ for all a, we have $C^a = V_a / V_{>a}$. We define

$$F^p C^a = \text{image of } F^p \cap V_a \text{ in } C^a$$

to be the Hodge filtration on C^a. It follows from the results in [9] and [12] that $C = \bigoplus_{-1<a\leq 0} C^a$ carries a mixed Hodge structure with Hodge filtration F given by $F^pC = \bigoplus_{-1<a\leq 0} F^pC^a$ and such that the nilpotent endomorphism N of C which is $-2\pi i(t\,\partial_t - a)$ on C^a is a morphism of mixed Hodge structures of type $(-1, -1)$. Because such morphisms are always strictly compatible with the Hodge filtration, we have in particular that

(2.4) Lemma. $N(C) \cap F^pC = N(F^{p+1}C)$ *for all* p.

For future use we formulate an application:

(2.5) Corollary. *Let ω be a holomorphic $(n+1)$-form on X such that $f\omega = df \wedge \eta$ for some holomorphic n-form η on X. Then*

$$[\omega] \in V_0 \text{ if and only if } [d\eta - \omega] \in V_{>0}.$$

Proof. We have $\partial_t t[\omega] = \partial_t[f\omega] = \partial_t[df \wedge \eta] = [d\eta]$ so

$$t\,\partial_t[\omega] = \partial_t t[\omega] - [\omega] = [d\eta - \omega].$$

Suppose that $[\omega] \in V_0$. Because $t\partial_t V_0 \subset V_0$ we have $[d\eta - \omega] \in V_0$. Let x and y denote the images of $[\omega]$ and $[d\eta - \omega]$ in C^0. Then $y = t\,\partial_t x \in NC^0 \cap F^nC^0 = N(F^{n+1}C^0) = 0$ because $F^{n+1} = 0$. Hence $y = 0$ so $[d\eta - \omega] \in V_{>0}$.

Conversely, because the map $t\,\partial_t\colon V_{>0} \to V_{>0}$ is invertible, if $[d\eta - \omega] \in V_{>0}$ there exists $z \in V_{>0}$ with $t\,\partial_t z = [d\eta - \omega]$. Then $[\omega] - z \in \ker(t\,\partial_t) \subset C^0$ so $[\omega] \in C^0 + V_{>0} = V_0$. □

§ 3. The geometric genus

In this section we prove a formula for the geometric genus of an isolated hypersurface singularity due to M. Saito [7]. In the next section we will use this proof to derive a formula for the irregularity as well.

(3.1) Theorem. *Let $n \geq 2$ and let $(V, 0) \subset (C^{n+1}, 0)$ be an isolated hypersurface singularity defined by a holomorphic function germ f. Then*

$$p_g(V, 0) = \dim H''_0 / H''_0 \cap V_{>0}.$$

Proof. Let $f\colon X \to S$ be a good representative for the germ f as in § 2. Let $\pi\colon \tilde{X} \to X$ be a good embedded resolution of f. Write $\tilde{V}$ for the strict transform of the singular fibre V of f and $\pi^{-1}(0) = E_1 \cup \cdots \cup E_k$ so $\tilde{V} \cup E_1 \cup \cdots \cup E_k$ is a divisor with normal crossings on X. Let $W = X - \{0\} = \tilde{X} - E$ and $U = V - \{0\}$. The exact sequence of sheaves

$$0 \to \Omega^{n+1}_{\tilde{X}} \to \Omega^{n+1}_{\tilde{X}}(\log \tilde{V}) \to \Omega^n_{\tilde{V}} \to 0$$

where the last morphism is the residue map gives the diagram

$$\begin{array}{ccccccccc} 0 \to & H^0(\tilde{X}, \Omega^{n+1}_{\tilde{X}}) & \to & H^0(\tilde{X}, \Omega^{n+1}_{\tilde{X}}(\log \tilde{V})) & \xrightarrow{\mathrm{res}_{\tilde{V}}} & H^0(\tilde{V}, \Omega^n_{\tilde{V}}) & \to 0 \\ & \varrho \downarrow \cong & & \downarrow & & \downarrow & \\ 0 \to & H^0(W, \Omega^{n+1}_W) & \to & H^0(W, \Omega^{n+1}_W(\log U)) & \xrightarrow{\mathrm{res}_U} & H^0(U, \Omega^n_U) & \to 0 \end{array}$$

in which the vertical mappings are the restrictions of sections to W. It is clear that the diagram is commutative. Because $H^1(\tilde{X}, \Omega^{n+1}_{\tilde{X}}) = 0$ (by Grauert-Riemenschneider) the map $\mathrm{res}_{\tilde{V}}$ is surjective. The map res_U is also surjective because $H^1(W, \Omega^{n+1}_W) = 0$ (here we use that $n \geqq 2$). The vertical mappings are clearly injective. Surjectivity of the map ϱ follows from the fact that the map $H^0(X, \Omega^{n+1}_X) \to H^0(W, \Omega^{n+1}_W)$ is already surjective. The same argument shows that each element of $H^0(W, \Omega^{n+1}_W(\log U))$ is of the form ω/f for some holomorphic $(n+1)$-form ω on X. It is easy to see that the map $H^0(X, \Omega^{n+1}_X) \to H^0(U, \Omega^n_U)/H^0(\tilde{V}, \Omega^n_{\tilde{V}})$, $\omega \mapsto$ the class of $\mathrm{res}_U(\omega/f)$ modulo forms of first kind, factors via H_0'', so we obtain a surjective mapping

$$\varphi\colon H_0'' \to H^0(U, \Omega^n_U)/H^0(\tilde{V}, \Omega^n_{\tilde{V}}).$$

Moreover

$$\begin{aligned} [\omega] \in \ker(\varphi) &\Leftrightarrow \mathrm{res}_U(\omega/f) \in H^0(\tilde{V}, \Omega^n_{\tilde{V}}) \\ &\Leftrightarrow \pi^*(\omega/f) \text{ extends to a section of } \Omega^{n+1}_{\tilde{X}}(\log \tilde{V}) \\ &\Leftrightarrow g(\omega) > 1 \Leftrightarrow [\omega] \in V_{>0}. \quad \square \end{aligned}$$

§ 4. The irregularity

We keep the notations of the preceding section.

(4.1) Theorem. *Let $n \geqq 2$ and let $(V, 0) \subset (C^{n+1}, 0)$ be an isolated hypersurface singularity, defined by a holomorphic function germ f. Let $K = \{[\omega] \in H_0'' \mid f\omega = df \wedge \eta$ for some holomorphic n-form η on $X\}$. Then the irregularity of $(V, 0)$ is given by*

$$q(V, 0) = \dim(K/K \cap V_0).$$

Proof. Take the notations of the proof of Theorem (3.1). Let $\tilde{K} = \{\omega \in H^0(X, \Omega^{n+1}_X) \mid [\omega] \in K\}$. For $\omega \in \tilde{K}$ choose η with $f\omega = df \wedge \eta$. Then

$$d(\eta/f) = d\eta/f - df \wedge \eta/f^2 = (d\eta - \omega)/f$$

so η/f and $d(\eta/f)$ both have a first order pole along U. In particular η/f is a section of $\Omega^n_W(\log U)$ and $\operatorname{res}_U(\eta/f)$ is a well-defined section of Ω^{n-1}_U. If an n-form η' also satisfies $f\omega = df \wedge \eta'$ then $df \wedge (\eta - \eta') = 0$ so $\eta - \eta' = df \wedge \zeta$ for some $(n-1)$-form ζ on X. Then $\operatorname{res}_U(\eta/f) = \operatorname{res}_U(\eta'/f) + \zeta|_U$ so the class of $\operatorname{res}_U(\eta/f)$ modulo forms of first kind on V depends only on ω. We denote it by $\tilde{\psi}(\omega)$. If ω is itself devisible by df then $\tilde{\psi}(\omega) = 0$ so we obtain a mapping $\psi: K \to H^0(U, \Omega^{n-1}_U)/H^0(\tilde{V}, \Omega^{n-1}_{\tilde{V}})$. We first show that ψ is surjective. As $n \geqq 2$ we have $H^1(W, \Omega^n_W) = 0$ so for all $x \in H^0(U, \Omega^{n-1}_U)$ there exist $y \in H^0\big(W, \Omega^n_W(\log U)\big)$ such that $x = \operatorname{res}_U(y)$. Then the $(n+1)$-form $\omega = df \wedge y$ is holomorphic on W and hence on X. Moreover $\omega \in K$ because $f\omega = df \wedge \eta$ where $\eta = fy$ is holomorphic on X. Then $x = \psi([\omega])$.

To determine Ker (ψ) we observe that Ker (ψ) = Ker $(d \circ \psi)$ where

$$d: H^0(U, \Omega^{n-1}_U)/H^0(\tilde{V}, \Omega^{n-1}_{\tilde{V}}) \to H^0(U, \Omega^n_U)/H^0(\tilde{V}, \Omega^n_{\tilde{V}})$$

is the differentiation map which is injective by Corollary (1.4). For $\omega \in \tilde{K}$, $f\omega = df \wedge \eta$, we have

$$d\tilde{\psi}(\omega) = d \operatorname{res}_U(\eta/f) = \operatorname{res}_U\big(d(\eta/f)\big) = \operatorname{res}_U\big((d\eta - \omega)/f\big)$$

hence, if φ is the mapping of the proof of Theorem (3.1), then for $[\omega] \in K$ one gets $[\omega] \in \operatorname{Ker}(\psi) \Leftrightarrow [d\eta - \omega] \in \operatorname{Ker}(\varphi) \Leftrightarrow [d\eta - \omega] \in V_{>0} \Leftrightarrow [\omega] \in V_0$ by Corollary (2.5). So Ker $(\psi) = K \cap V_0$ and the theorem follows. $\square$

§ 5. Example

Let $f(x, y, z) = x^7/7 + y^3/3 + z^3/3 - s_1x^5y - s_2x^4yz$. Here $s_1, s_2 \in \mathbb{C}$. Then f is semi-quasihomogeneous with weights (1/7, 1/3, 1/3). We will compute q for all values of the parameters s_1, s_2.

First observe that the space $K \subset H_0''$ contains $\partial_t^{-1}H_0'' = H_0' = df \wedge \Omega^n_{X,0}$, $df \wedge d(\Omega^{n-1}_{X,0})$ (see [9], § 3). Hence it is convenient to pass to the quotient $Q^f = H_0''/H_0'$ which is a $\mathbb{C}$-vectorspace of dimension μ, the Milnor number of f. The space K/H_0' is just the kernel of multiplication by f in Q^f. We let V also denote the induced filtration on Q^f. We can actually compute the V-filtration on our Q^f by the result (2.3) of M. Saito. Let $\omega_0 = dx \wedge dy \wedge dz$. Then a basis for Q^f is given by the forms $x^iy^jz^k\omega_0$ with $i \in \{0, 1, 2, 3, 4, 5\}$, $j \in \{0, 1\}$ and $k \in \{0, 1\}$. We have $\alpha(x^iy^jz^k\omega_0) = (3i + 7j + 7k - 4)/21$ for these forms. For every $\alpha \in \mathbb{Q}$, $V_\alpha Q^f$ is generated by those basis elements for which this number is at least α.

Multiplication by $21f$ in Q^f is easily seen to be the same as multiplication by $s_1x^5y + 5s_2x^4yz$ (use the Euler relation). It is clear that this maps V_α to $V_{>1+\alpha}$ for every α. As $Q^f = V_{-4/21}Q^f$, the image is contained in $V_{19/21}Q^f$ and because $V_{>25/21}Q^f = 0$, its kernel contains $V_{4/21}Q^f$. So if

$$A = Q^f/V_{4/21}Q^f, \qquad B = V_{19/21}Q^f$$

and $P: A \to B$ is the operator induced by multiplication with $s_1x^5y + 5s_2x^4yz$, then $q = \dim(\operatorname{Ker} P/\operatorname{Ker} P \cap V_0A)$ where V_0A is the image of V_0Q^f in A.

A basis for A is ω_0, $x\omega_0$, $x^2\omega_0$, $y\omega_0$, $z\omega_0$ and for B: $x^5y\omega_0$, $x^5z\omega_0$, $x^3yz\omega_0$, $x^4yz\omega_0$, $x^5yz\omega_0$. With respect to these bases, the operator P is given by the matrix

$$\begin{pmatrix} s_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 5s_2 & 0 & 0 & 0 & 0 \\ 0 & 5s_2 & 0 & 0 & s_1 \end{pmatrix}$$

Hence Ker $(P) = A$ for $s_1 = s_2 = 0$ and else it is generated by $(s_1x - 5s_2z)\,\omega_0$, $x^2\omega_0$ and $y\omega_0$. Moreover V_0A is generated by $x^2\omega_0$, $y\omega_0$ and $z\omega_0$. We conclude that

$$\begin{cases} q = 0 \text{ if } s_1 = 0, \quad s_2 \neq 0, \\ q = 1 \text{ if } s_1 \neq 0, \\ q = 2 \text{ if } s_1 = s_2 = 0. \end{cases}$$

§ 6. Related invariants

We take the notations of § 3. If ω is a holomorphic $(n-1)$-form on U which is of first kind on V, then $\pi^*(\omega)$ is a section of $\Omega_{\tilde{V}}^{n-1}$. We let $D_i = E_i \cap V$ for $i = 1, \ldots, k$ and define

$$q'(V, 0) = \dim \left\{\begin{matrix}\text{holomorphic} \\ (n-1)\text{-forms on } U\end{matrix}\right\} \Big/ \left\{\begin{matrix}\text{forms of first kind which} \\ \text{restrict to 0 on each } D_i\end{matrix}\right\}$$

$$= \dim H^0(U, \Omega_U^{n-1})/H^0\big(V, I_D\Omega_{\tilde{V}}^{n-1}(\log D)\big)$$

where $D = D_1 \cup \cdots \cup D_k$.

The fact that q' does not depend on the choice of the resolution $\pi\colon \tilde{V} \to V$ follows from its relation with the filtered De Rham complex of V (see [1, 11]). One has

$$\pi_* I_D\Omega_{\tilde{V}}^{n-1}(\log D) = \mathcal{H}^{n-1}\mathrm{Gr}_F^{n-1}\mathcal{K}^{\cdot}_{V.O} \quad ([11], \text{Cor. } (3.4)).$$

(6.1) Theorem. *With notations as in Theorem* (4.1) *we have*

$$q'(V, 0) = \dim\,(K/K \cap V_{>0}).$$

Proof. Let $W = X - \{0\}$. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^0\big(\tilde{X}, I_E\Omega_{\tilde{X}}^n(\log E)\big) & \to & H^0\big(\tilde{X}, I_E\Omega_{\tilde{X}}^n(\log \tilde{V} + E)\big) & \to & H^0\big(\tilde{V}, I_D\Omega_{\tilde{V}}^{n-1}(\log D)\big) & \to 0 \\ & \downarrow\cong & & \downarrow & & \downarrow & \\ 0 \to & H^0(W, \Omega_W^n) & \longrightarrow & H^0\big(W, \Omega_W^n(\log U)\big) & \longrightarrow & H^0(U, \Omega_U^{n-1}) & \longrightarrow 0. \end{array}$$

The bottom row is exact because $H^1(W, \Omega_W^n) = 0$ (we take $n \geqq 2$ again) and exactness of the top row follows from $H^1\big(\tilde{X}, I_E\Omega_{\tilde{X}}^n(\log E)\big) = 0$. To explain this, observe that for any singularity (Y, Σ) and any resolution $p: \tilde{Y} \to Y$ with exceptional divisor A the sheaves $R^i p_* I_A \Omega^j_{\tilde{Y}}(\log A) = \mathcal{H}^{i+j}(\mathrm{Gr}^j_F \mathcal{K}^{\cdot}_{Y,\Sigma})$ ([11], Cor. (3.4)) are invariants of (Y, Σ) which do not depend on the resolution. Because X is smooth they vanish for $i \neq 0$ so $R^1\pi_* I_E \Omega^n_{\tilde{X}}(\log E) = 0$. Finally the left vertical map is again an isomorphism because $H^0(\tilde{X}, I_E \Omega^n_{\tilde{X}}(\log E)) \cong H^0(X, \Omega^n_X)$ as X is smooth.

We conclude from this diagram that we have

$$q' = \dim H^0(W, \Omega^n_W(\log U))/H^0(\tilde{X}, I_E \Omega^n_{\tilde{X}}(\log V + E)).$$

Let $\omega \in H^0(X, \Omega^{n+1}_X)$ such that $f\omega = df \wedge \eta$ i.e. $[\omega] \in K$.

Claim: $\alpha([\omega]) > 0$ if and only if $\pi^*(\eta/f)$ is a section of $I_E \Omega^n_{\tilde{X}}(\log \tilde{V} + E)$. It is clear that the theorem follows from this.

So suppose that $\pi^*(\eta/f) \in H^0(\tilde{X}, I_E \Omega^n_{\tilde{X}}(\log \tilde{V} + E))$. Then $\pi^*(\omega/f) = df/f \wedge \pi^*(\eta/f) \in H^0(\tilde{X}, I_E \Omega^{n+1}_{\tilde{X}}(\log \tilde{V} + E)) = H^0(\tilde{X}, \Omega^{n+1}_{\tilde{X}}(\log \tilde{V}))$ hence $\alpha(\omega) > 0$ by (2.2).

Conversely, let $\alpha(\omega) > 0$. Then by Theorem (4.1) $\mathrm{Res}_U(\eta/f)$ is of first kind. Now recall that $H^0(\tilde{V}, \Omega^{n-1}_{\tilde{V}}) = H^0(\tilde{V}, \Omega^{n-1}_{\tilde{V}}(\log D))$ (see the proof of Theorem (1.3)). We have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^0(\tilde{X}, \Omega^n_{\tilde{X}}(\log E)) & \to & H^0(\tilde{X}, \Omega^n_{\tilde{X}}(\log \tilde{V} + E)) & \to & H^0(\tilde{V}, \Omega^{n-1}_{\tilde{V}}(\log D)) & \to 0 \\ & \downarrow\cong & & \uparrow & & \uparrow & \\ 0 \to & H^0(W, \Omega^n_W) & \longrightarrow & H^0(W, \Omega^n_W(\log U)) & \longrightarrow & H^0(U, \Omega^{n-1}_U) & \longrightarrow 0. \end{array}$$

This time the top row is exact because $H^1(\tilde{X}, I_E \Omega^n_{\tilde{X}}(\log E)) = 0$ as before and $H^1(E, \Omega^n_{\tilde{X}}(\log E) \otimes \mathcal{O}_E) = \mathrm{Gr}^n_F H^{n+1}_{\{0\}}(X, \mathbb{C}) = 0$ again because X is smooth (see [5]). So the fact that $\mathrm{Res}_U(\eta/f)$ is of first kind implies that η/f is a section of $\Omega^n_{\tilde{X}}(\log \tilde{V} + E)$. To prove that it is in fact a section of $I_E \Omega^n_{\tilde{X}}(\log \tilde{V} + E)$ we first show that it has zero residue along each E_i and then that its restriction to each E_i is zero. We will use the fact that global logarithmic forms on E_i are zero if their cohomology class is zero (see [2], Cor. (3.2.13) (ii)). Let E_i be a component of E and let C_i be the intersection of E_i with the remaining components of $E \cup \tilde{V}$. Let γ be an $(n-1)$-cycle on $E_i - C_i$. We show that $\int_\gamma R(\eta/f) = 0$ where $R = \mathrm{Res}_{E_i} \circ \pi^*$.

Let $T_\varepsilon(\gamma)$ be the ε-tube over γ in $\tilde{X} - \tilde{V}$, which is an $(n+1)$-chain with boundary $\tau_\varepsilon(\gamma) = \partial T_\varepsilon(\gamma)$. Because the map $H_n(X_\varepsilon) \to H_n(X - V)$ is surjective (where $X_\varepsilon = f^{-1}(\varepsilon)$), there exists an $(n+1)$-chain Γ_ε on $X - V$ with $\partial\Gamma_\varepsilon = \tau_\varepsilon(\gamma) - \alpha_\varepsilon$ with α_ε an n-cycle on X_ε. Because the inclusion $E - D \hookrightarrow \tilde{X} - \tilde{V}$ is a homotopy equivalence, we have $H_i(\tilde{X} - \tilde{V}, E - D) = 0$ for all i. This means that, when Z_i and B_i denote i-cycles and i-boundaries respectively:

$$Z_i(E - D) + B_i(\tilde{X} - \tilde{V}) = Z_i(X - V) \qquad (1)_i$$

$$Z_i(E - D) \cap B_i(\tilde{X} - \tilde{V}) = B_i(E - D). \qquad (2)_i$$

So there exist $\beta_\varepsilon \in Z_n(E - D)$ and $\Delta_\varepsilon \in C_{n+1}(X - V)$ with $\alpha_\varepsilon = \beta_\varepsilon + \partial\Delta_\varepsilon$ by $(1)_n$. Let $Z = T_\varepsilon(\gamma) - \Gamma_\varepsilon - \Delta_\varepsilon$. Then $\partial Z = \beta_\varepsilon$. By $(2)_n$ there exists $H \in C_{n+1}(E - D)$ with $\partial H = \beta_\varepsilon$. Thus $Z - H \in Z_{n+1}(\tilde{X} - \tilde{V})$ so $Z - H = G + \partial K$ for some $G \in Z_{n+1}(E - D)$, $K \in C_{n+2}(\tilde{X} - \tilde{V})$. So, putting $F = G + H$ we see that Z is homologous to $F \in C_{n+1}(E - D)$.

By the residue formula ([4], Prop. 8.16(b)) we have

$$\int_\gamma R(\eta/f) = \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{\tau_\varepsilon(\gamma)} (\eta/f) = \frac{1}{2\pi i} \lim_{\varepsilon\to 0} \left[\int_{\alpha_\varepsilon} (\eta/f) + \int_{\partial\Gamma_\varepsilon} (\eta/f) \right].$$

Now it follows from $\alpha(\omega) > 0$ that

$$\lim_{\varepsilon\to 0} \int_{\alpha_\varepsilon} (\eta/f) = \lim_{\varepsilon\to 0} \int_{\alpha_\varepsilon} (\omega/df) = 0$$

(see [12], § 1.2). Moreover

$$\int_{\partial\Gamma_\varepsilon} (\eta/f) = \int_{\Gamma_\varepsilon} d(\eta/f) = \int_{T_\varepsilon(\gamma)} d(\eta/f) - \int_{\Delta_\varepsilon} d(\eta/f) + \int_Z d(\eta/f)$$

and

$$\lim_{\varepsilon\to 0} \int_{T_\varepsilon(\gamma)} d(\eta/f) = \lim_{\varepsilon\to 0} \int_{\Delta_\varepsilon} d(\eta/f) = 0$$

because the form $d(\eta/f)$ is regular on the whole of $X - V$ and we are integrating over smaller and smaller chains. Finally

$$\int_Z d(\eta/f) = \int_F d(\eta/f) = 0$$

because $d(\eta/f)$ is closed and its restriction to E has to be zero, as E is an n-dimensional complex space and $d(\eta/f)$ is a holomorphic $(n + 1)$-form. We conclude that $R(\eta/f) = 0$. Hence η/f is a section of $\Omega^n_{\tilde{X}}(\log V)$. Its restriction to E_i is then a section of $\Omega^{n-1}_{E_i}(\log D_i)$ so to prove that this restriction is zero we check that $\int_\gamma \eta/f = 0$ for all n-cycles γ on $E_i - C_i$.

So let γ be an n-cycle on $E_i - C_i$ and let e_i be the multiplicity of E_i in the fibre of $f \circ \pi$ over 0. Then there exists an open neighborhood Y of the support of γ in $\tilde{X}$ such that for $\varepsilon > 0$ small enough, the intersection $X_\varepsilon \cap Y$ is an e_i-fold unramified covering of $(E_i - D_i) \cap Y$. If $\gamma_i \in H_n(X_\varepsilon)$ is the inverse image of γ then $e_i\gamma$ is the limit of the cycles γ_ε for $\varepsilon \to 0$. As a consequence

$$\int_\gamma \eta/f = \frac{1}{e_i} \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} \eta/f = 0$$

again because $\alpha(\omega) > 0$. This completes the proof. □

Let us consider the case $n = 2$. In [13] the following invariants are considered:

$\alpha = \dim H^0(\tilde{V}, \Omega^2_{\tilde{V}})/dH^0(\tilde{V}, I_D\Omega^1_{\tilde{V}}(\log D))$;

$\beta = \dim \operatorname{Coker}[H^0(\tilde{V}, \Omega^1_{\tilde{V}}) \to H^0(\tilde{D}, \Omega^1_{\tilde{D}})]$;

$\gamma = \operatorname{rank}[H^1_D(\tilde{V}, \Omega^1_{\tilde{V}}) \to H^1(\tilde{V}, \Omega^1_{\tilde{V}})] - k$ where k is the number of irreducible components of D.

Let $g = \dim H^0(\tilde{D}, \Omega^1_{\tilde{D}}) = \sum_{i=1}^{k} g_i$ where g_i is the genus of D_i and let b denote the number of cycles in the dual graph of D. Finally let $\tau = \dim T^1_{V,0}$. Then one has the formulas

$$\mu - \tau = b + 2(\alpha + \beta) + \gamma \quad \text{([13], Theorem (2.7))};$$

$$q = p_g - g - b - \alpha - \beta - \gamma \quad \text{(ibid. Theorem (1.9))}.$$

(6.2) Proposition. *We have* $q' = q + g - \beta$.

Proof. Use that $\Omega^1_{\tilde{D}} = \Omega^1_{\tilde{V}}/I_D\Omega^1_{\tilde{V}}(\log D)$ to obtain the sequence

$$0 \to H^0(\tilde{V}, \Omega^1_{\tilde{V}})/H^0(\tilde{V}, I_D\Omega^1_{\tilde{V}}(\log D)) \to H^0(\tilde{D}, \Omega^1_{\tilde{D}}) \to \mathbb{C}^\beta \to 0$$

in which the first term has dimension $q' - q$ and the second has dimension g. □

If $\bar{K} = K/H_0' \subset Q^f$ then $\tau = \dim \bar{K}$. The invariants p_g, g and b are constant under deformations with constant Milnor number whereas $\alpha, \beta, \gamma, \tau$ may jump. To obtain certain semicontinuous linear combinations of these invariants (restricted to the stratum with constant Milnor number) observe that $\dim \bar{K}$ is upper semicontinuous and hence $\dim(\bar{K} \cap V_0)$ and $\dim(\bar{K} \cap V_{>0})$ are upper semicontinuous too.

(6.3) Corollary. *The invariants* $\tau - q$ *and* $\tau - q'$ *are upper semicontinuous and the invariants* α *and* $\alpha + \beta$ *are lower semicontinuous on the* μ*-constant stratum of a 2-dimensional isolated hypersurface singularity* (see also [13], (1.13.2)).

Proof. It follows from the preceding formulas that

$$\mu - \tau + q = p_g - g + \alpha + \beta = \dim Q^f/\bar{K} \cap V_0;$$

$$\mu - \tau + q' = p_g + \alpha = \dim Q^f/\bar{K} \cap V_{>0}.$$

Remark. The invariants above admit obvious generalizations to arbitrary dimension in such a way that [13] Theorem (1.9) and Proposition (6.2) above remain valid. For the corresponding formula for $\mu - \tau$, valid for complete intersections, see [5].

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