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KNOTTED MILNOR FIBRES

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We show that any knot or link can arise as the real part of the Milnor fibre in a *flat* deformation of an appropriate space curve singularity. However, it seems that to achieve knotting one needs a rather high Milnor number.
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1. INTRODUCTION

We will consider reduced space curve singularities $X \subset \mathbb{C}^3$. Associated to such a germ X there are two distinct deformation theories.

- *Def*(X): one considers *flat deformations* of X, that is, one considers diagrams of the form

$$\begin{array}{ccc} X & \hookrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & S \end{array}$$

where $\mathscr{X} \to S$ is flat.

- *Unf*(X): one considers *unfoldings* of a parametrisation of X. By this we mean the following. Consider the normalisation map $n: \tilde{X} \to X$ and the composition $f = i \circ n: \tilde{X} \to \mathbb{C}^3$. Now $Unf(X) := Def(f)$, that is, we consider deformations

$$fs: \tilde{X} \times S \to \mathbb{C}^3 \times S$$

of a parametrisation whose image is X. Note that the space $\tilde{X}$ is a disjoint union of r smooth discs, where r is the number of branches.

Each of the two deformation theories has a miniversal object, unique up to (non-unique) isomorphism. The miniversal base is *smooth* in both cases. For $Def(X)$ this is because X is Cohen-Macaulay of codimension two, for $Unf(X)$ it is because no constraints are placed on deformations. The tangent space for $Def(X)$ is $T^1_X \approx Ext^1(\Omega^1_X, \vartheta_X)$ and for $Unf(X)$ it is

$$T^1_f \approx \frac{f^*(\Theta_{\mathbb{C}^3})}{df(\Theta_{\tilde{X}}) + f^{-1}(\Theta_{\mathbb{C}^3})}.$$

The fibre over a generic point of the miniversal base is a *smooth* space curve in both cases. Because in each case the versal base-space is smooth, it is not disconnected by the discriminant, and thus the generic fibres are all diffeomorphic.

Nevertheless, these deformation theories for X are *very different*. Flat deformations of X cannot in general be realised by deforming the parametrisation. Imagine a flat deformation, whose general fibre is smooth. The total space of such a smoothing is normal but not

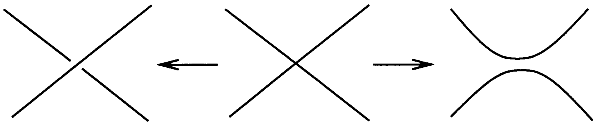


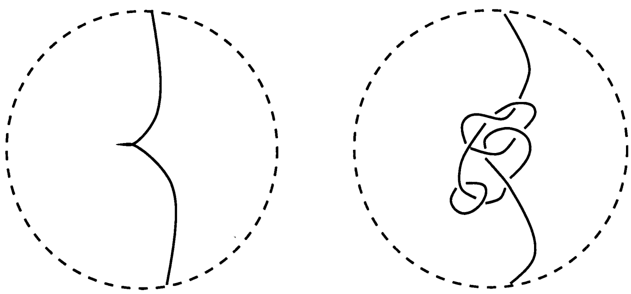
Fig. 1. Versal deformations of a nodal space curve: (i) unfolding and (ii) flat deformation.

smooth, and thus cannot be the image of any deformation of the parametrisation of the original singular curve. Similarly, an arbitrary deformation of the parameterisation of X will not induce a flat deformation of the image.†

The simplest example of this phenomenon is central to this paper: in an unfolding of the parametrisation of a node on a space curve, the two branches of the node can be separated (Fig. 1(i)), but this does not occur in a flat deformation; the fibre of any flat deformation of a curve singularity is always connected ([2], 4.2.2).‡

The generic fibre of the miniversal flat deformation of X is a piece of non-singular curve, the *Milnor fibre*. The Milnor number $\mu(X)$, the rank of the first homology group of the Milnor fibre, is an important topological invariant of the singularity.

The image of the generic member of the miniversal unfolding of $f: \tilde{X} \to X$ is topologically *uninteresting* – it is simply the image under a complex-analytic isomorphism of a disjoint union of discs. However, it becomes interesting when we consider *real* curves: those given by real equations, or, more restrictively, by real parameterisations. Now the image has a well-defined *knot-type*, since the deformation is trivial on the boundary, and thus the two ends of the curves can be joined by an arbitrary arc on the Milnor sphere.



This knot-type depends on the choice of perturbation. In fact, each component of the complement of the bifurcation set in the base of the miniversal unfolding has associated to it a knot type; moreover every knot type occurs in this way, as parametrised deformation of a suitably chosen singularity, see for example [5]. The menagerie of knot-types sprouting from a given real singularity seems to be an interesting invariant.

Example 1.1. The open trefoil knot (Fig. 2) occurs as the image of a deformation of the parametrisation $t \mapsto (t^3, t^4, t^5)$ of the (3, 4, 5) curve.

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† This is to be contrasted with the case of plane curves; the image of of an unfolding $F: \tilde{X} \times S \to (\mathbb{C}^2, 0) \times S$ of a parametrisation $f: \tilde{X} \to (\mathbb{C}^2, 0)$ of a *plane* curve singularity $X \subset (\mathbb{C}^2, 0)$ is flat over the base S of the deformation, since it is a hypersurface and so Cohen-Macaulay; but as soon as we look at curves in 3-space, this no longer holds good.

‡ The right hand drawing is misleading: the two connected components of the real smooth fibre $X_t(\mathbb{R})$ (Fig. 1 (ii)) are joined in the complex fibre: the fact that $\beta_0(X_t(\mathbb{R})) > \beta_0(X_t(\mathbb{C}))$ is compensated by the fact that $\beta_1(X_t(\mathbb{R})) < \beta_1(X_t(\mathbb{C}))$.

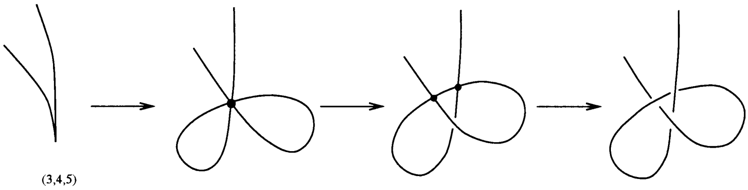

Fig. 2. The open trefoil in an unfolding of the (3, 4, 5) curve.

The first assertion follows easily from the fact that the (3, 4, 5) curve deforms, both as parametrised curve and as fibre of a flat family (see Example 4.1), to a curve with a single ordinary triple point and then in one with two nodes. The three branches of the curve passing through the triple point can be separated in a parametrised deformation, giving an open trefoil knot.

Example 1.2. In contrast to this, every Milnor fibre occurring in a flat deformation of the (3, 4, 5) curve is unknotted (Fig. 3).

In fact, we prove:

THEOREM 1.3. *No curve of multiplicity* 3 *can have a knotted real Milnor fibre.*

A proof will be given in Theorem 4.3.

Such experiments with curves of low multiplicity led us initially to the conjecture that knotting could *not* occur in the real Milnor fibres of (real) space curves. This is incorrect; in fact precisely the contrary is true:

THEOREM 1.4. *Every knot-type occurs as the real Milnor fibre of a suitably chosen isolated space-curve singularity.*

Our proof is partially constructive. In particular, we recover the old theorem of Akbulut and King [1] that every knot is algebraic, in a considerably strengthened form. The method of Akbulut and King gives no control over the complexification of the real algebraic variety whose intersection with the 3-sphere is the required knot, whereas we are placing a significant constraint on this complexification.

CONJECTURE 1.5. *Knotting does not occur in the real Milnor fbre of any space-curve singularity with Milnor number less than* 10.

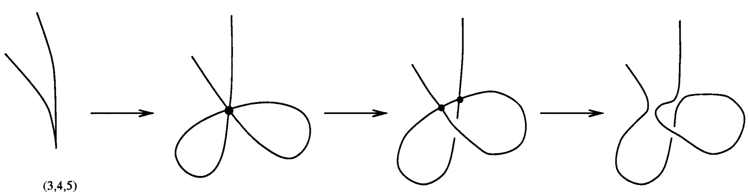

Fig. 3. Smoothing the (3, 4, 5) curve in a flat family.

2. δ-CONSTANT DEFORMATIONS

The results of this paragraph hold for arbitrary reduced curve singularities $X \subset \mathbb{C}^N$ and can be found in [2, 4] and the forthcoming [3].

Recall that the δ-invariant of a curve singularity X with normalisation $n:\tilde{X} \to X$ is defined by

$$\delta(X) = \delta(X,0) = \dim_{\mathbb{C}}(n_*(\vartheta_{\tilde{X}})/\vartheta_X).$$

The rank of the first homology group of a smoothing of X is called the Milnor number, and can be computed as

$$\mu(X) = 2\delta(X) - r + 1$$

where r denotes the number of branches of X [2].

A crucial role in the relation between the two deformation theories is played by the so-called δ-constant deformations. These form in some sense the *intersection* of the two theories.

A flat deformation $\mathscr{X}_S \to S$ with fibre X_s over $s \in S$ is called δ-constant, if the function

$$s \in S \mapsto \sum_{x \in X_s} \delta(X_s, x)$$

is constant.

An unfolding $f_S:\tilde{X} \times S \to \mathbb{C}^N \times S$ with image $f_s(\tilde{X})$ at $s \in S$ is called δ-constant, if the function

$$s \in S \mapsto \sum_{x \in f_s(\tilde{X})} \delta(f_s(\tilde{X}, x))$$

is constant.

PROPOSITION 2.1. *Let $S = (\mathbb{C}, 0)$ be smooth one-dimensional. A family $\mathscr{X} \to S$ is precisely then δ-constant if and only if it is of the form $f_S(\tilde{X} \times S) \to S$, where*

$$f_S:\tilde{X} \times S \to \mathbb{C}^N \times S$$

is a δ-constant unfolding of

$$f:\tilde{X} \to X \hookrightarrow \mathbb{C}^N.$$

Proof. (*sketch*). Let t be a parameters on S, and let $\mathscr{X} \to S$ be a δ-constant deformation of X. Let $N:\tilde{\mathscr{X}} \to \mathscr{X}$ be the normalization of the surface $\mathscr{X}$. We hve to show that the zero fibre $\tilde{\mathscr{X}}_0$ is smooth, so that $\tilde{\mathscr{X}} = \tilde{X} \times S$. But this follows from the δ-constancy of $\mathscr{X} \to S$: from the multiplication-by- t sequences and the snake lemma one concludes that the $\mathbb{C}\{t\}$-module $N_*(\vartheta_{\tilde{\mathscr{X}}})/\vartheta_{\mathscr{X}}$ is free of rank $\delta(X) = \dim_{\mathbb{C}}(n_*(\vartheta_{\tilde{\mathscr{X}}_0})/\vartheta_{\mathscr{X}})$. One concludes that $\tilde{X} = \tilde{\mathscr{X}}_0$.

Similarly, if $f_S:\tilde{X} \times S \to \mathbb{C}^N \times S$ is a δ-constant unfolding, let $\mathscr{X} := f_S(\tilde{X} \times S)$ with a *reduced structure*. We have to show that the zero-fibre X_0 of $\mathscr{X} \to S$ is isomorphic to X. The δ-constancy of the unfolding implies in a similar way the freeness of $(f_S)_*(\vartheta_{\tilde{\mathscr{X}}})/\vartheta_{\mathscr{X}}$ as $\mathbb{C}\{t\}$-module, from which it follows that $\vartheta_{X_0} \to \vartheta_{\tilde{X}}$ is injective. Hence X_0 is reduced, and hence must be equal to X. $\square$

So the δ-constant deformations of X over $S = (\mathbb{C}, 0)$ are precisely those which admit simultaneous normalisation.

2.1. Putting together flat families

Suppose that $\mathscr{X}_1 \to S$ and $\mathscr{X}_2 \to S$ are flat families of curves, with $\mathscr{X}_1, \mathscr{X}_2 \subseteq \mathbb{C}^3 \times S$. We will want on occasions to *put together* such families to construct a flat family $\mathscr{X}_1 \cup \mathscr{X}_2 \to S$. Given two curves C_1 and C_2 in 3-space, meeting at x, define an "intersection index" $I_x(C_1, C_2)$ by $I_x(C_2, C_2) = \dim_{\mathbb{C}}(\vartheta_{\mathbb{C}^3,x}/I_1 + I_2)$, where I_1 and I_2 are the (radical) ideals of C_1 and C_2. Then

Lemma 2.2. *Under these circumstances, we have*

1. $\delta(C_1 \cup C_2, x) = \delta(C_1, x) + \delta(C_2, x) + I_x(C_1, C_2)$.
2. *Given flat deformations $\mathscr{X}_1 \to \mathbb{C}$ and $\mathscr{X}_3 \to \mathbb{C}$ of the germs of C_1 and C_2 at x_0, with $\mathscr{X}_i \subseteq \mathbb{C}^3 \times \mathbb{C}$ for $i = 1, 2$. Then one has*

$$\sum_x I_x(C_{1,\,t}, C_{2,\,t}) \leqslant I_{x_0}(C_1, C_2).$$

with equality if and only if $\mathscr{X}_1 \cup \mathscr{X}_2 \to \mathbb{C}$ is a flat deformation of $C_1 \cup C_2$.

Proof. Consider the short exact sequence

$$0 \to \vartheta_{\mathscr{X}_1 \cup \mathscr{X}_2} \to \vartheta_{\mathscr{X}_1} \oplus \vartheta_{\mathscr{X}_2} \to \vartheta_{\mathscr{X}_1 \cap \mathscr{X}_2} \to 0.$$

Multiplication by the parameter t on $\mathbb{C}$ defines a morphism of this sequence to itself; let T denote the kernel of multiplication by t on $\vartheta_{\mathscr{X}_1 \cap \mathscr{X}_2}$, so that we have an exact sequence

$$0 \to T \to \vartheta_{\mathscr{X}_1 \cap \mathscr{X}_2} \xrightarrow{t\cdot} \vartheta_{\mathscr{X}_1 \cap \mathscr{X}_2} \to \vartheta_{\mathbb{C}^3, x_0}/(I_1 + I_2) \to 0,$$

from which we see that

$$\sum_x I_x(C_{1,\,t}, C_{2,\,t}) = I_{x_0}(C_1, C_2) - \dim_{\mathbb{C}} T.$$

The associated exact sequence of kernels and cokernels (i.e. coming from the snake lemma) reads

$$0 \to T \to \vartheta_{\mathscr{X}_1 \cup \mathscr{X}_2}/t\vartheta_{\mathscr{X}_1 \cup \mathscr{X}_2} \to \vartheta_{C_1, x_0} \oplus \vartheta_{C_2, x_0} \to \vartheta_{\mathbb{C}^3, x_0}/(I_1 + I_2) \to 0.$$

Here we are using the fact that $\vartheta_{\mathscr{X}_i}/t\vartheta_{\mathscr{X}_i} = \vartheta_{C_i}$. It follows that $\vartheta_{\mathscr{X}_1 \cup \mathscr{X}_2}/t\vartheta_{\mathscr{X}_1 \cup \mathscr{X}_2}$ is reduced if and only if $T = 0$. That is, the fibre of $\mathscr{X}_1 \cup \mathscr{X}_2$ over 0 is equal to $C_1 \cup C_2$ if and only if $T = 0$. □

3. HIDDEN SCAFFOLDING AND HELPING CIRCLES

One way in which one might expect to be able to construct a knot is by realising a plane projection of the given knot type as the real Milnor fibre of a plane algebraic curve, and then simply lifting apart the nodes into over- and under-crossings as required. This is unfortunately not flat, since the family then has simultaneous normalisation but does not have δ constant. However, the method of *hidden scaffolding* which we now describe provides a means of circumventing this problem.

We now introduce the basic unit of the hidden scaffolding which we use to lift apart the crossings of a plane curve in a flat family.

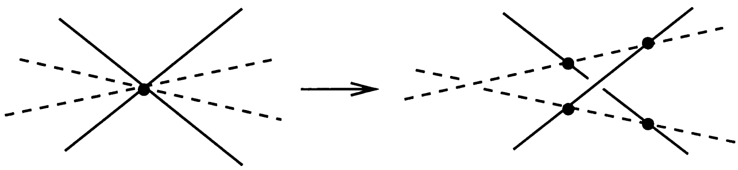

Fig. 4. Basic unit of hidden scaffolding.

Example 3.1. Let N be the node $\{(z_1,z_2)\in\mathbb{C}^2 | z_1z_2=0\}$. Consider the space ${}^2N$, the space in $\mathbb{C}^4$ defined by the real and imaginary part of the equation $z_1z_2=0$. It has components

$$
\begin{aligned}
{}^2N_{1,1} &= \{(x_1,x_2,y_1,y_2)\in\mathbb{C}^4 | x_1=y_1=0\}\\
{}^2N_{2,2} &= \{(x_1,x_2,y_1,y_2)\in\mathbb{C}^4 | x_2=y_2=0\}\\
{}^2N_{1,2} &= \{(x_1,x_2,y_1,y_2)\in\mathbb{C}^4 | x_1+iy_1=0, x_2-iy_2=0\}\\
{}^2N_{2,1} &= \{(x_1,x_2,y_1,y_2)\in\mathbb{C}^4 | x_1-iy_1=0, x_2+iy_2=0\}.
\end{aligned}
$$

The first two of these are real, the second two imaginary; note that the only real point of ${}^2N_{1,2}$ and ${}^2N_{2,1}$ is 0, and that ${}^2N_{1,1}$ and ${}^2N_{2,2}$ meet only at 0. Now take the section of ${}^2N$ by a hyperplane H_0 through 0. Provided H_0 contains none of the four 2-planes which make up ${}^2N$, we obtain a space-curve singularity consisting of two real branches ${}^2N_{1,1}\cap H_0$ and ${}^2N_{2,2}\cap H_0$ and two imaginary branches ${}^2N_{1,2}\cap H_0$ and ${}^2N_{2,1}\cap H_0$. The imaginary branches each have the unique real point 0. Let H_0 have equation h, and let $H_t=h^{-1}(t)$. The family $({}^2N,0)\xrightarrow{h}(\mathbb{C},0)$ is flat and has ${}^2N\cap H_t$ (with reduced structure) as fibre over t, since ${}^2N$ is Cohen–Macaulay. For real $t\neq 0$, ${}^2N\cap H_t$ consists of two real skew lines, and two imaginary skew lines. The family thus has simultaneous normalisation, and so must be δ-constant. A calculation shows $\delta({}^2N\cap H_0)=4$; indeed we find that in ${}^2N\cap H_t$, for $t\neq 0$, each of the imaginary lines meets each of the real lines in an imaginary point, as depicted in Fig. 1. Thus, by keeping δ constant by means of the "hidden scaffolding" $({}^2N_{1,2}\cap H_t)\cup({}^2N_{2,1}\cap H_t)$ we have succeeded in separating the two real branches of ${}^2N\cap H_0$. Indeed, as t passes through 0 the relative positions of the two real branches, with respect to a height function defining the 2-plane spanned by ${}^2N_{1,1}\cap H$ and ${}^2N_{2,2}\cap H$ have been exchanged.

Note that by 2.2 (1), the intersection index of the real part $({}^2N_{1,1}\cup{}^2N_{2,2})\cap H_0$ and the imaginary part $({}^2N_{1,2}\cup{}^2N_{2,1})\cap H_0$ of the scaffolding unit is equal to two; for $\delta({}^2N\cap H_0)=4$, while the real part and the imaginary part each consists of a pair of lines and thus has $\delta=1$. Figure 4 shows four intersection points, but of course the deformation of ${}^2N\cap H_0$ does not restrict to a flat deformation of its real and imaginary parts, so there is no contradiction. Incidentally, calculating this intersection index directly is a little tricky, since the ideal of $({}^2N_{1,1}\cup{}^2N_{2,2})\cap H_0$ is not equal to the sum of the ideal of H_0 and the ideal of ${}^2N_{1,1}\cup{}^2N_{2,2}$.

First-proof of Theorem 1.4. Now to construct a curve having real Milnor fibre with the required knot type, we proceed as follows:

- *Step* 1: Let $h: S^1\to\mathbb{R}^2$ parametrise a generic plane projection of the knot k. Then h is a stable map, and since every neighborhood of h in $C^\infty(S^1,\mathbb{R}^2)$ contains a polynomial map (i.e. the restriction of a polynomial map $\mathbb{R}^2\to\mathbb{R}^2$ to S^1), by the old Weiersraß approximation theorem, we can replace h by a polynomial map having image diffeomorphic to the image of h. This image will be an algebraic set given by an equation $f = 0$, with $f \in \mathbb{R}[x, y]$. The real part of $f = 0$ may contain isolated points, in addition to the nodal curve we want. To remove these, we add a small positive multiple of a polynomial vanishing to high order at all of the nodes on the knot projection, and taking positive values at the isolated real points of $f = 0$. Hence we see that any given generic knot projection is diffeomorphic to the real solution set of a polynomial.

- *Step* 2: Now rotate the axes so that no parallel translate of the y-axis contains more than one real double point. Choose a polynomial $p(x)$ such that all nodes lie on the graph of $y = p(x)$. The polynomial diffeomorphism

$$g : (x, y) \mapsto (x, y - p(x))$$

takes all the double points onto the x-axis, and the compromise of f with the inverse $(x, y) \mapsto (x, y + p(x))$ has zero set isotopic to the original knot projection and with all its nodes on the x-axis. Now inversion in a suitable circle transforms this algebraic set to one with all its nodes on a given circle. Hence, we see that any generic knot projection is diffeomorphic to the zero set of an $f \in \mathbb{R}[x, y]$ with the additional property that all its real double points are on the circle of radius 1 with center at the origin.

- *Step* 3: Now we assume we are in the situation of Step 2. Let $X(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 \,|\, f(x, y) = 0\}$ be the knot projection. Write $f = f_0 + f_1 + \cdots + f_d$, where f_i is homogeneous of degree i. We can assume that f_d is a product of distinct linear factors. (If the knot diagram is compact, then all these factors will be invisible over the reals.) Under the *scaling deformation*

$$f_t(x, y) = t^d f_0 + t^{d-1} f_1 + \cdots + f_d$$

the set $X_t(\mathbb{R}) := \{(x, y) \in \mathbb{R}^2 \,|\, f_t(x, y) = 0\}$ degenerates into the singularity $X_0(\mathbb{R})$ given by the product of lines $f_d = 0$. Moreover, at $t \to 0$, the nodes stay on shrinking circles $x^2 + y^2 - t^2$. In the limit $t = 0$, the circle degenerates into the imaginary line pair $x^2 + y^2 = 0$. As the angular distribution of the nodes on the circle does not change during the degeneration, the tangent lines to the circle at these nodes have pairwise distinct limiting positions.

- *Step* 4: The circle $x^2 + y^2 = t^2$ is of course the intersection of the quadratic $Q_t = \{x^2 + y^2 + z^2 = t^2\}$ with the plane $\{z = 0\}$. To the (complex) family $X_t := \{f_t(x, y) = 0\}$ add the family of lines constructed as follows: at each real node p_t of X_t, take the intersection of the quadratic Q_t with its affine tangent plane $T_{p_t} Q_t$. This intersection consists of a pair of conjugate complex lines, one in each of the two rulings of the quadric. Denote the family of added lines by L_t, and let $Y_t = X_t \cup L_t$.

We claim that the family Y_t is flat, and Y_0 has isolated singularity. Evidently X_t is flat, and L_0 is the complete intersection singularity consisting of the intersection of Q_0 with a union of planes passing through 0. Each of these planes contains the z-axis, since the affine tangent space to Q_t at any point lying in the plane $\{z = 0\}$ contains the vertical direction; it follows that the intersection of each plane with the quadratic Q_0 consists of two distinct lines, and is thus reduced. All the planes are distinct, by our assumption of the non-coincidence of the limiting directions in which the nodes approach 0. Hence L_0 has isolated singularity at 0. Of course any deformation of an ICIS is flat. In fact L_t is a deformation with simultaneous normalisation, and thus

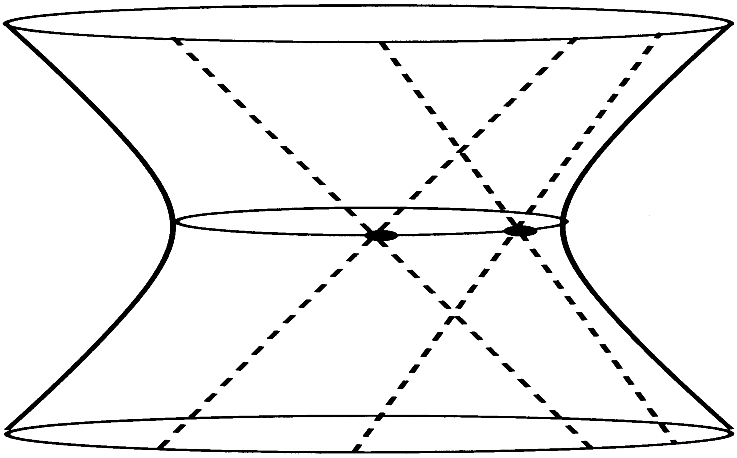

Fig. 5. Hidden scaffolding lying on a ruled surface.

must have δ constant. One calculates that $\delta(L_0) = m^2$; on Q_t, each line in one of the rulings meets each line in the other ruling, and hence there are m^2 nodes in L_t. This hidden scaffolding is represented schematically in Fig. 5, where in order to show the lines L_t we have drawn Q_t as a hyperboloid instead of the sphere described in the construction.

Thus to see that the space Y_t is a flat deformation of Y_0, it is necessary only to show that

$$\sum_{p \in L_t \cap X_t} I_p(L_t, X_t) = I_0(L_0, X_0).$$

The left-hand side of this sum is equal, by construction, to m, the number of nodes on $X_t(\mathbb{R})$, multiplied by the intersection number, equal to 2, of the real and imaginary parts of the scaffolding unit ${}^2N \cap H_0$ of Example 3.1. On the other hand,

$$I_0(L_0, X_0) = \dim \frac{\mathcal{O}_{\mathbb{C}^3, 0}}{(\prod_{i=1}^m l_i, x^2 + y^2 + z^2, z, f_0)}$$

where $l_i(x, y, z) = a_i x + b_i y$ is the equation of the i'th plane, $i = 1, \ldots, m$. Since l_i is not a factor of $x^2 + y^2$ for any $i = 1, \ldots, m$,

$$\dim \frac{\mathcal{O}_{\mathbb{C}^3, 0}}{(\prod_{i=1}^m l_i, x^2 + y^2 + z^2, z,)} = 2m$$

and thus $I_0(L_0, X_0) \leqslant 2m$. Since δ cannot go up in a deformation and we have already seen that $\sum_p I_p(L_t, X_t) = 2m$, it follows that $I_0(L_0, X_0) = 2m$ (and incidentally that $f_0 \in (\prod_{i=1}^m l_i, x^2 + y^2))$ and thus that δ is constant and the family $Y_t = L_t \cup X_t$ is flat.

- *Step* 5: The structure of Y_t at each of the nodes of the original curve X_t is isomorphic to the structure of the scaffolding unit ${}^2N \cap H_0$. It follows that the same deformations are available; in other words, at each node the two real branches can be separated into an over- or an under-crossing, and the imaginary nodes that appear (as seen in Fig. 5) can then be smoothed by an arbitrarily small deformation which will leave the real part (which is now smooth and thus stable) topologically unchanged. By openness of versality, these deformations can be realised independently of one another in a versal deformation of the curve Y_0. Hence, every knot type arising from the plane diagram

$X_t(\mathbb{R})$ by specifying over- and under-crossings, arises as real Milnor fibre of X_0. This completes the proof. □

In this construction one has $\delta(L_0) = m^2$, and $I_0(L_0, X_0) = 2m$, so $\delta(X_0 \cup L_0) = \delta(X_0) + m(m+2)$ and $\mu(Y_0) = \mu(X_0) + r - 1 + 2m(m+2) - 2m - r + 1 = \mu(X_0) + 2m\,(m+1)$.

If the knot projection is given by a polynomial of degree d, then X_0 consist of d lines, so $\mu(X_0) = (d-1)^{2\,3}$ and hence

$$\mu(Y_0) = (d-1)^2 + 2m(m+1).$$

Second proof of Theorem 1.4. We choose a plane projection of the given knot type with all its nodes on a line, and realise this is the set $X_t(\mathbb{R})$ of real zeroes of a deformation f_t of a function with isolated singularity. We suppose that all nodes lie on the line $\{x = 0\}$. Let their y-coordinate be $y_1(t), \ldots, y_m(t)$. We include the xy-plane in 3-space, so that now $X_t(\mathbb{R})$ is the set of real points of the curve defined by the ideal $(z, f_t(x, y))$. Now let

$$g_t(y,z) = \left(z - i\prod_{j=1}^{m}(y - p_j(t))\right)\left(z + i\prod_{j=1}^{m}(y - p_j(t))\right).$$

Thus, g_t is a deformation of the A_{2m-1} (Fig. 6) singularity $g_0(t) = z^2 + y^2m$. Let S_t be the curve defined by the ideal (g_t, x). Note that the real points of $g_t = 0$ are just the points $(0, p_j(t), 0)$ for $j = 1, \ldots, m$. In the neighbourhood of each point $(0, p_j(t), 0)$, the curve $Y_t := X_t \cup S_t$ is isomorphic to the scaffolding unit ${}^2N \cap H_0$. It follows once again that the two real branches of Y_t through each node can be separated in a flat deformation, and thus that we can obtain any knot or link type having $X_t(\mathbb{R})$ as a plane projection.

It remains only to show that Y_t is a flat deformation of Y_0. Since X_t and S_t are flat deformations of X_0 and S_0, respectively, flatness of Y_t is equivalent to conservation of the intersection number $I_t = \sum_{p \in X_t \cap S_t} I_p(X_t, S_t)$. Now X_t and S_t meet precisely at the points p_i,

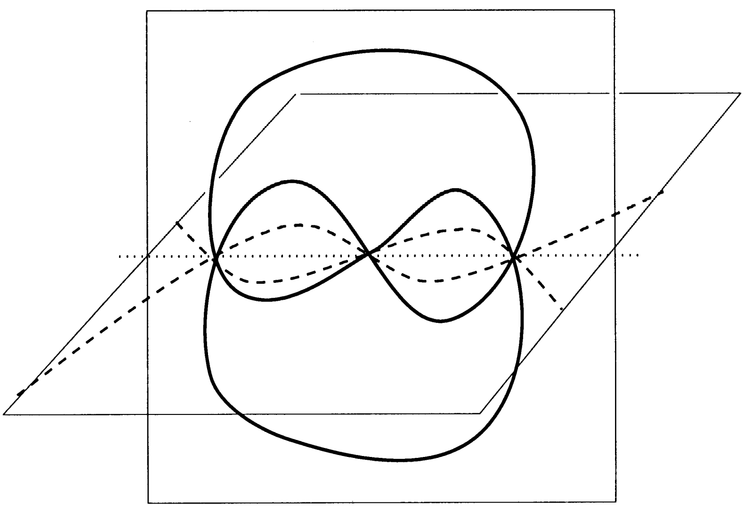

Fig. 6. Threading an imaginary A_{2m-1} through the nodes.

and $I_{p_i}(X_t, S_t) = 2$. Hence, we must show that $I_0(X_0, S_0) = 2m$. In fact

$$I_0(X_0, S_0) = \dim_{\mathbb{C}} \frac{\vartheta_{\mathbb{C}^3,0}}{(z, f_0(x,y)) + (x, z^2 + y^{2m})}$$

$$\leqslant \dim_{\mathbb{C}} \frac{\vartheta_{\mathbb{C}^3,0}}{(z, x, z^2 + y^{2m})} = 2m.$$

Once again, semi-continuity of the intersection index guarantees that this number is equal to $2m$, completing the proof that our deformation is flat. □

We remark that in this construction, we have

$$\delta(X_0 \cup S_0) = \delta(X_0) + \delta(S_0) + 2m = \delta(X_0) + 3m.$$

Since $\delta(X_0) = (\mu(X_0) + r - 1)/2$, this gives

$$\mu(X_0 \cup S_0) = \mu(X_0) + 6m - 2.$$

It will be clear by now that these constructions allow for many variations and one should choose a method that fits most easily with the knot projection.

3.1. Helping circles

Let $C_t = \{f_t(x,y) = z = 0\}$ and $D_t = \{g_t(z,y) = x = 0\}$ be smoothings of plane curve singularities, and suppose that $C_0 \cup D_0$ is an isolated singularity. The next proposition shows that if $C_t \cup D_t$ is a flat family and $C_t(\mathbb{R}) \cup D_t(\mathbb{R})$ is smooth then $C_t(\mathbb{R})$ and $D_t(\mathbb{R})$ cannot be linked. Thus, the standard link consisting of two linked circles lying in different planes cannot arise in this way.

Proposition 3.2. *If $C_t \cup D_t$ is a flat deformation of $C_0 \cup D_0$, then $C_t(\mathbb{R})$ and $D_t(\mathbb{R})$ cannot be linked.*

Proof. Flatness of $C_t \cup D_t$ implies conservation of the intersection index. Suppose that $p = (0, y_0, 0) \in C_t \cap D_t$. Then

$$I_p(C_t, D_t) = \dim_{\mathbb{C}} \frac{\vartheta_{\mathbb{C}^3,p}}{(x, z, f_t, g_t)} = \min\{\mathrm{Ord}_{y_0}(f(0,y)), \mathrm{Ord}_{y_0}(g(y,0)\}$$

$$= \min\{I_p(C_t, L), I_p(D_t, L)\}$$

where L is the y-axis. Suppose that $I_0(C_0, L) \leqslant I_0(D_0, L)$. Since the intersection multiplicity of C_t and L is conserved, we have

$$\sum_{p \in C_t \cap L} I_p(C_t, L) = I_0(C_0, L).$$

By assumption, $I_0(C_0, L) = I_0(C_0, D_0)$; both C_t and D_t are flat deformations, so by 2.2(2), flatness of $C_t \cup D_t$ implies that the intersection multiplicity of C_t and D_t is conserved. Thus

$$\sum_{p \in C_t \cap L} I_p(C_t, L) = \sum_{p \in C_t \cap D_t} I_p(C_t, D_t) = \sum_{p \in C_t \cap D_t} \min\{I_p(C_t, L), I_p(D_t, L)\}.$$

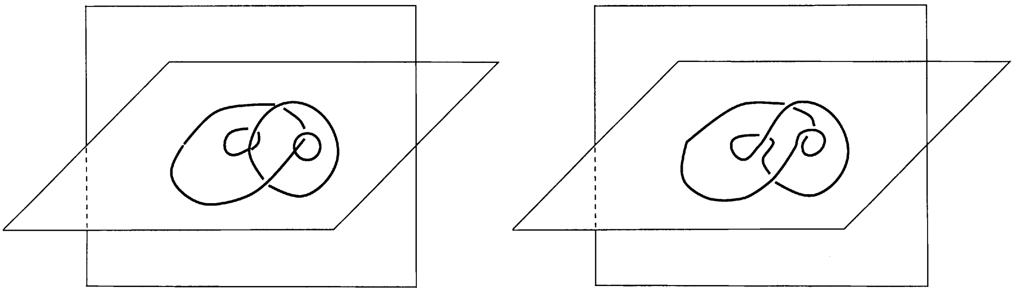

Fig. 7. (i) Singular link with helping circles and (ii) non-singular link obtained by smoothing (i).

But $C_t \cap D_t = (C_t \cap L) \cap (D_t \cap L)$, so this equation forces $D_t \cap L \supset C_t \cap L$ and $I_p(D_t, L) \geqslant I_p(C_t, L)$ at each point $p \in C_t \cap L$.

Thus every point of $C_t \cap L$ is a singular point of $C_t \cup D_t$. Since $C_t(\mathbb{R})$ must cross L in order for $C_t(\mathbb{R})$ and $D_t(\mathbb{R})$ to be linked, the conclusion follows. □

In order to produce examples of linking beginning with a curve of this type, at least one of the curves has to leave its plane.

Example 3.3. Consider the curves

$$C_t = \left\{ \left(\frac{8}{7} x^2 + \left(y + \frac{t}{6} \right)^2 - \left(\frac{t}{6} \right)^2 \right) (x^2 + y^2 - t^2) = z = 0 \right\}$$

$$D_t = \left\{ \left(\frac{8}{7} z^2 + \left(y - \frac{7}{6} t \right)^2 - \left(\frac{t}{6} \right)^2 \right) (z^2 + (y - t^2) - t^2) = x = 0 \right\}.$$

For $t > 0$, $C_t(\mathbb{R}) \cup D_t(\mathbb{R})$ is shown in Fig. 7(i). The two singular points of $C_t \cup D_t$ are ordinary nodes; smoothing them as in Fig. 7(ii) gives a pair of linked curves. A calculation shows that $\delta(C_0 \cup D_0) = 6 + 6 + 4 = 16$, and hence $\mu(C_0 \cup D_0) = 32 - 8 + 1 = 25$.

Notice that each of the nodes in Fig. 7(i) can be smoothed in two different ways, giving a total of four different but diffeomorphic links. Hence if the complement of the discriminant of $C_0(\mathbb{R}) \cup D_0(\mathbb{R})$ is simply connected then it has (at least) four connected components over which there are linked real Milnor fibres.

4. EXAMPLES

Example 4.1. The open trefoil in an unfolding of $t \mapsto (t^3, t^4, t^5)$, and in a flat deformation of the (3, 4, 5) curve together with hidden scaffolding.

Let X be the (3, 4, 5) curve, image of $f(t) = (t^3, t^4, t^5)$. We have $\delta(X) = \delta(\text{triple point}) = 2$; so a flat deformation of X to a curve with triple point can also be realised in an unfolding of f. The semi-universal deformation of X with parameters a, b, c, d, e has total space defined by the minors of the matrix

$$\begin{pmatrix} Y + a & Z + b & X^2 + c + dX + eY \\ X & Y & Z \end{pmatrix}$$

we get a triple point where all of the entries vanish simultaneously, and this forces $a = b = c = 0$. Provided $d \neq 0 \neq e$, the curve is as shown in the centre of Fig. 2.

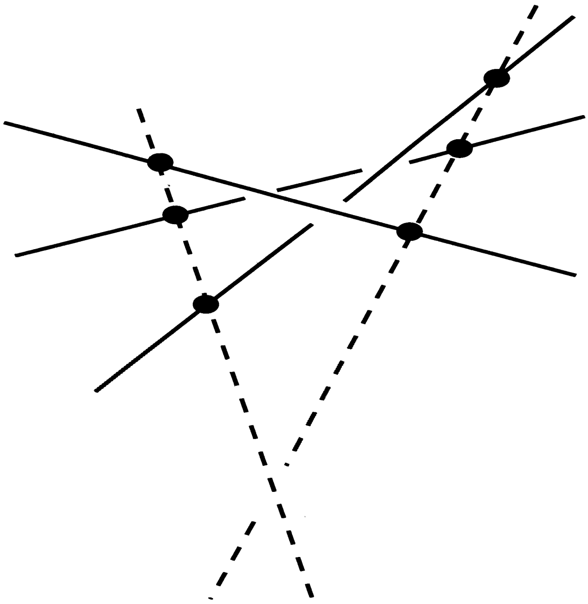

Fig. 8. Separation of the triple point using hidden scaffolding.

In an unfolding of f, this triple point can of course be separated, and the appropriate separation produces the open trefoil, as shown in Fig. 2. Such a deformation of the curve is not flat, since it has simultaneous normalisation but δ is not constant.

In order to construct a flat deformation having the same real part, we must use hidden scaffolding. First we do this for the triple point. The configuration is shown in Fig. 8; solid lines indicate real branches, dotted lines indicated purely imaginary branches, and in Fig. 8 one sees a configuration with $\delta = 6$, since each pair of real lines is joined by two imaginary lines. This configuration can be realised as a δ-constant deformation of the union of the triple point X with a plane A_1 singularity A, in such way that no three of the lines are coplanar. By 2.2 (1),

$$\delta = \delta(X) + \delta(A) + I(X, A)$$

$$= 2 + 1 + \dim \frac{\vartheta_{\mathbb{C}^3, 0}}{(x_1 x_2, x_2 x_3, x_3 x_1) + (a x_1 + b x_2 + c x_3, x_1^2 + x_2^2 x_3^2)} = 6.$$

We see one can indeed separate the real branches of the triple point as follows: to the flat deformation of Example 3.1, we add an extra real line, joining two conjugate imaginary points on the two skew imaginary lines. By appropriate choice of these points, we can ensure that as we specialise, no three of the lines become coplanar. For example, if we take the family of hyperplane sections $x_1 + y_1 + x_2 + y_2 = t$ of ${}^2N$ (see 3.1), where $X = \{z_1 z_2 = 0\}$, and our additional real line joins the point $(1, i, (it + 1 - i)/(1 + i))$ (which lies on ${}^2X_{1,2} \cap H_t$) to its complex conjugate, then the limiting direction vectors of the five lines in the configuration are $(0, 0, 1)$, $(1, -1, 0)$, $(0, 2, -1)$, $(i, -1, 1)$ and $(-i, -1, 1)$. Thus, we have arranged a deformation of real triple point in whcih the three real branches are separated.

To incorporate this into a deformation of the (3, 4, 5) curve, it is enough to check that by adding two generic lines (i.e. an A_1 singularity) to the germ of the (3, 4, 5), we obtain a singularity with $\delta = 6$. Evidently $\delta(3, 4, 5) = 2$, $\delta(A_1) = 1$, and since the two curves have

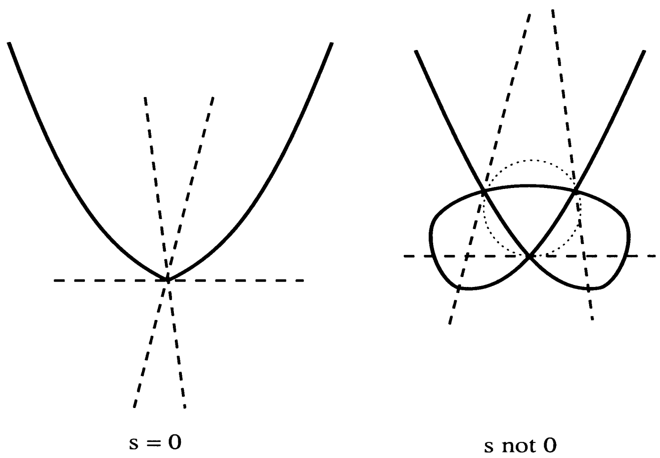

Fig. 9. Deformation of E_6.

ideals $(xy - z^2,\ yz - x^3,\ x^2y - z^2)$ and $(ax + by + cz, x^2 + y^2 + z^2)$, the intersection number I is

$$\dim \frac{\vartheta_{\mathbb{C}^3, 0}}{(xy - z^2, yz - x^3, x^2y - z^2) + (ax + by + cz, x^2 + y^2 + z^2)} = 3.$$

It follows that $\mu(X) = 2\delta - r + 1 = 10$.

Example 4.2. The open trefoil can also be obtained by the method of hidden scaffolding described in the first proof of Theorem 1.4 from a deformation of E_6, $\{x^4 - y^3 = 0\}$. To see this, consider the following 1-parameter deformation

$$(t, s) \mapsto (t^3 - s^2 t, t^4 - \tfrac{3}{2} s^2 t^2)$$

of the parameterisation $t \mapsto (t^3, t^4)$.

For generic $s \neq 0$, the image curve looks like the one in Fig. 9 below. (The double points are located at $(\frac{1}{4}\sqrt{2}s^3,\ -\frac{1}{4}s^4)$, $(-\frac{1}{4}\sqrt{2}s^3,\ -\frac{1}{4}s^4)$ and $(0\ -\frac{1}{2}s^4)$.) Any three lines at $s = 0$ can be lifted by *parallel translation* to lines going through the nodes for general s. This can

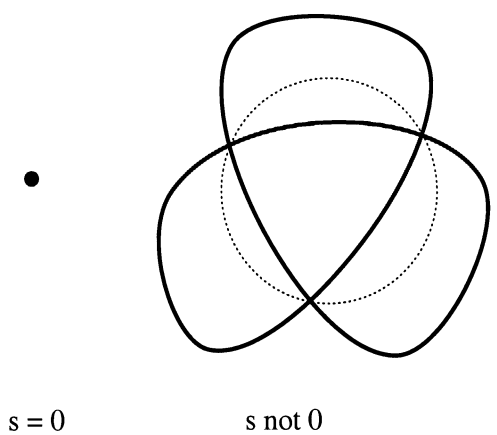

Fig. 10. The standard three nodal quartic.

even be done in such a way that the whole configuration is invariant under $x \mapsto -x$, as indicated by the dashed lines in Fig. 9. Furthermore, by symmetry, there exist a conic C_s tangent (dotted in Fig. 9) to the three lines and passing through the nodes. We insist that the conic be tangent to these three lines in order to force the tangent lines to the conic at the nodes to remain separate in the limit as s goes to 0. Now a slight variation to the first construction (in which the role of Q_s is played by a sphere intersecting the xy plane in the conic C_s) gives a space curve with $\mu = \mu(E_6) + 2{\cdot}3(3+1) = 30$, much higher than the curve in Example 4.1.

The closed trefoil can be obtained from the first construction, using the standard three nodal quartic appearing in a deformation of $x^4 + y^4$:

The Milnor number is $9 + 2{\cdot}3(3+1) = 33$, which is rather high.

We conclude by proving a *no-knotting* theorem.

THEOREM 4.3. *No curve with multiplicity* 3 *can have knotted real Milnor fibre.*

Proof. Let X_t be a Milnor fibre of X. As X has multiplicity 3, there is an open set in the Grassmannian of planes in 3-space, consisting of planes P for which $P \cap X_t$ consists of 3 points. Let P be any such plane, and consider the fat point $P_0 \cap X$, where P_0 is a parallel translate of P passing through the singular point of X. This is either a complete intersection of a line and a cubic, or is not a complete intersection. In the first case, the curve X is a plane curve, and its Milnor fibres are plane curves, and therefore not knotted.

If $X \cap P_0$ is not a complete intersection, then it is isomorphic to the fat point defined by the ideal (X^2, XY, Y^2). This has semi-universal deformation on parameters s, t, u, v with total space defined by the 2×2 minors of the matrix

$$\begin{pmatrix} X & Y+s & v \\ u & X+t & Y \end{pmatrix}.$$

In particular, all three equations are conics in X, Y, Z, and thus the three points of a smoothing of $X \cap P_0$, if distinct, can never be collinear.

We claim that this prevents $X_t(\mathbb{R})$ from being knotted (even after its two extremities are joined on the Milnor sphere). For by varying P in a parallel family, we obtain something like a braid representation of the knot type of $X_t(\mathbb{R})$ (through there are only two loose ends, to be joined by an arc on the Milnor sphere); since the three points generating the braid are never collinear, the braid cannot represent a knot. □

Given a knot or link, one can ask for the smallest Milnor number of singularity that has the given knot or link as a real Milnor fibre. We conclude the paper by giving a small list of the simplest knots and links and the lowest Milnor numbers we were able to realize for them. Probably the last entry can be improved a lot.

Knot/Link	μ	Construction
two skew lines	5	scaffolding
three skew lines	8	scaffolding
line + circle	10	help circle
circle + circle	25	help circle
open trefoil	10	scaffolding
trefoil	33	scaffolding

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