Monodromy calculations of fourth order equations of Calabi-Yau type

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Abstract

This paper contains a preliminary study of the monodromy of certain fourth order differential equations, that were called of Calabi-Yau type in [3]. Some of these equations can be interpreted as the Picard-Fuchs equations of a Calabi-Yau manifold with one complex modulus, which links up the observed integrality to the conjectured integrality of the Gopakumar-Vafa invariants. A natural question is if in the other cases such a geometrical interpretation is also possible. Our investigations of the monodromies are intended as a first step in answering this question. We use a numerical approach combined with some ideas from homological mirror symmetry to determine the monodromy for some further one-parameter models. Furthermore, we present a conjectural identification of the Picard-Fuchs equation for 5 new examples from Borceas list and conjecture the existence of some new Calabi-Yau three folds. The paper does not contain any theorems or proofs but is, we think, nevertheless of interest.

1 Introduction

A differential operator of order n on $\mathbb{P}^1$ has the form

$$L := a_n(z)\frac{d^n}{dz^n} + a_{n-1}(z)\frac{d^{n-1}}{dz^{n-1}} + \cdots + a_0(z), \tag{1}$$

where the $a_i(z)$ are polynomials. The set $\Sigma \subset \mathbb{P}^1$ of singular points is given by the zeros of $a_n(z)$ and possibly $z = \infty$. The solutions to the equation $Ly = 0$ can be considered as a $\mathbb{C}$-local system $\mathbb{L}$ of rank n on $S := \mathbb{P}^1 \setminus \Sigma$. After the choice of a base point $s \in \mathbb{P}^1 \setminus \Sigma$, the information of $\mathbb{L}$ is given by the monodromy representation

$$\pi_1(S, s) \longrightarrow \mathrm{Aut}(\mathbb{L}_s) = \mathrm{Gl}_n(\mathbb{C})$$

A power series $y_0(x) \in \mathbb{Z}[[x]]$ that satisfies a homogeneous linear differential equation as above is a G-function and a folklore conjecture that goes back to Bombieri and Dwork states that all such power series and differential operators have a *geometrical origin* (see [28]). This means that the operator should occur as a *factor* of a Picard-Fuchs operator describing the variation of a cohomology

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of a family $\rho : \mathcal{Y} \longrightarrow \mathbb{P}^1$, with singular fibres over Σ and defined over a number field. The local system $\mathbb{L}$ should then be a summand of a local system $\mathbb{L}_{\mathbb{C}} := R^d\rho_*(\mathbb{C}_{\mathcal{Y}})_{|S}$, where d is the complex dimension of the fibres of ρ. It follows among other things that the equation has regular singularities with all exponents rational. It can be shown that the set of power series of geometric origin in this sense is closed under the ordinary product of power series and under the coefficientwise Hadamard product of series. On the level of local systems, the Cauchy product corresponds to the tensor product, whereas the Hadamard product correspond to the convolution of local systems. We refer to the books [5] and [22] for details.

The fourth order equations in [3] were collected with a stricter notion of geometrical origin in mind: by requiring that the operator admits an invariant symplectic form and gives rise to integral instanton numbers, it starts making sense asking for the existence of a one-parameter family $\mathcal{Y} \longrightarrow \mathbb{P}^1$ of *Calabi-Yau three folds*, whose associated Picard-Fuchs operator for $\mathrm{H}^3(Y_s)$ is the given one. The instanton numbers then should have the interpretation of counting curves on a mirror manifold X with Picard number one. The first 14 equations in the list are in fact the much studied hypergeometric cases (see [14], [32], [25], [9], [34], [15]). Mirror pairs of Calabi-Yau threefolds obtained from Batyrev's polar duality of reflexive polytopes [7] yield a plethora of examples but usually with high Picard number (see [24]). By taking restrictions to carefully chosen one-dimensional sub-loci these examples sometimes give rise to equations of Calabi-Yau type, but the instanton numbers computed in this way represent *sums over different homology classes* and there will not exist a Calabi-Yau three fold X with Picard number one with the given instanton numbers. Case 15 is an example of this phenomenon: it is the equation belonging to the diagonal restriction of Calabi-Yau family in $\mathbb{P}^3 \times \mathbb{P}^3$ (see [9]). The list contains many more of such examples. The question is how can one see this from the differential equation alone.

In order to find the cases that are potentially of strict geometric origin, we remark that a geometrical local system $\mathbb{L}_{\mathbb{C}}$ carries a integral lattice $\mathbb{L}_{\mathbb{Z}} = R^d\rho_*(\mathbb{Z}_{\mathcal{Y}})|_S$ and that Poincaré-duality provides it with a *unimodular* pairing $\langle \cdot, \cdot \rangle$, which in our case is alternating. Hence the monodromy representation is in the symplectic group $\mathrm{Sp}(4, \mathbb{Z})$. For differential equations of hypergeometric type, the monodromy representation is explicitly known, essentially because the associated local system is rigid (Levelt's theorem, [12],[22]). This leads to the 14 hypergeometric cases mentioned above. For equations with three singular points which are not of hypergeometric type or for equations with more than three singular points, the monodromy representation is in general not determined by local data alone and we have the problem of accesory parameters. We do not know of any general method to determine the monodromy representation in such cases. We use a brute force numerical approach combined with ideas from homological mirror symmetry to conjecturally determine the monodromy for some further one-parameter models.

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2 Sketch of Homological Mirror Symmetry

According to Kontsevich [26], the phenomenon of mirror symmetry between Calabi-Yau spaces X and Y should be formulated in terms of equivalence of categories. To a Calabi-Yau space X one can associate two triangulated categories, namely the derived category of coherent sheaves $D^b(X)$ and a derived Fukaya-category $D\mathcal{F}(X)$ of lagrangian cycles (graded, with local systems on them) in X (see [17]). The first category depends only on the holomorphic moduli, the second only on the symplectic (or Kähler) moduli. Mirror symmetry between Calabi-Yau spaces X and Y is then expressed as equivalences of categories.

$$\mathrm{Mir} : D^b(X) \xrightarrow{\approx} D\mathcal{F}(Y), \quad D^b(Y) \xrightarrow{\approx} D\mathcal{F}(X)$$

These equivalences induce isomorphisms between the corresponding K-groups. Via the Chern character they descend to cohomology:

$$\mathrm{mir} : \mathrm{H}^{\mathrm{ev}}(X, \mathbb{Q}) \xrightarrow{\approx} \mathrm{H}^d(Y, \mathbb{Q}), \quad \mathrm{H}^{\mathrm{ev}}(Y, \mathbb{Q}) \xrightarrow{\approx} \mathrm{H}^d(X, \mathbb{Q}),$$

where $d = \dim_C X = \dim_C Y$. This also induces an isomorphism between the Kähler moduli $\mathrm{H}^{1,1}(X)$ of X and the complex moduli of $\mathrm{H}^{d-1,1}(Y)$ of Y. In the Strominger-Yau-Zaslow picture of mirror symmetry (see [36], [20]) X and Y are represented as (real) singular torus fibration over a common base B. The fibres are dual tori and mirror symmetry should correspond to fibrewise T-duality. From this one can get some intuitive understanding of the mirror transformation on objects. In particular, the structure sheaf $\mathcal{O}_p$ of a point $p \in X$ gets mapped to a SYZ-fibre $\mathbf{T}$ (with a local system on it) in Y and the structure sheaf $\mathcal{O}_X$ should map to the image $\mathbf{S}$ of a section $\sigma : B \longrightarrow Y$ of the fibration.

For any pair $(\mathcal{E}, \mathcal{F})$ of objects of $D^b(X)$ the Euler bilinear form is defined by

$$\langle \mathcal{E}, \mathcal{F} \rangle := \chi(\mathcal{E}, \mathcal{F}) = \sum_i (-1)^i \dim \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i]).$$

which by Serre duality and triviality of the canonical bundle is $(-1)^d$ symmetric. It descends via the Chern-character to a bilinear form $\langle \cdot, \cdot \rangle$ on the cohomology $\mathrm{H}^{\mathrm{ev}}(X, \mathbb{Q})$ of X, which by Riemann-Roch is given by

$$\langle \alpha, \beta \rangle = \int_X \tilde{\alpha} \cup \beta \cup \mathrm{td}(X),$$

where $\tilde{\alpha} = (-1)^k \alpha$ for $\alpha \in \mathrm{H}^{2k}(X, \mathbb{Q})$.

Under the mirror transformation the form $\langle \cdot, \cdot \rangle$ should correspond to the intersection form $\langle \cdot, \cdot \rangle$ of the corresponding lagrangians. One instance of this can easily be checked

$$\langle \mathcal{O}_p, \mathcal{O}_X \rangle = 1 = \langle \mathbf{T}, \mathbf{S} \rangle.$$

3 Monodromy in one-parameter models

From now on we assume that X and Y are strict Calabi-Yau three-folds and furthermore that they satisfy $h^{2,1}(Y) = 1 = h^{1,1}(X)$. This is the case of so called one-parameter models: Y varies in a one-dimensional moduli space and X has one Kähler modulus, i.e., $\mathrm{Pic}(X) = \mathbb{Z}$. In such a case one has $\dim \mathrm{H}^3(Y) = 4 = \dim \mathrm{H}^{\mathrm{ev}}(X)$.

To be specific, we assume that we have a proper map $\rho : \mathcal{Y} \longrightarrow \mathbb{P}^1$, smooth outside singular fibres that sit over points from $\Sigma \subset \mathbb{P}^1$ and furthermore that Y is the fibre over a base-point $s \in \mathbb{P}^1 \backslash \Sigma =: S$. As the geometrical monodromy along a path $\gamma \in \pi_1(S, s)$ can be realised as a symplectic map $M(\gamma) : Y_s \longrightarrow Y_s$, $M(\gamma)$ induces an autoequivalence of its symplectic invariant $D\mathcal{F}(Y)$, thus setting up a homomorphism

$$\pi_1(S, s) \longrightarrow \mathrm{Auteq}(D\mathcal{F}(Y))$$

which is a refined version of the ordinary monodromy representation of $\pi_1(S, s)$ on $\mathrm{H}^{\mathrm{odd}}(Y)$. The group $\pi_1(S, s)$ is generated by paths that encircle one of the singular fibres of the family. The induced transformation is determined by the specific properties of the singular fibre. If the fibre aquires the simplest type of singularity, namely an A_1-singularity ('conifold'), there is a vanishing lagrangian 3-sphere. The geometrical monodromy is then a Dehn-twist along this sphere and its effect on homology is given by the classical Picard-Lefschetz transformation [29], [6], [31]:

$$\alpha \mapsto S_\delta(\alpha) := \alpha - \langle \delta, \alpha \rangle \delta$$

where δ is the homology class of the vanishing cycle. In the situation of mirror symmetry there also will be a point of degeneration with maximal unipotent monodromy. The fibre will typically have normal crossing singularities and there will be a 'vanishing n-torus', invariant under the monodromy.

Using the mirror equivalence Mir we get a representation

$$\pi_1(S, s) \longrightarrow \mathrm{Auteq}(D^b(X))$$

and one may ask what sort of autoequivalences correspond to specific types of degenerations of Y.

In [35] Seidel and Thomas described a type of autoequivalence in $D^b(X)$ to mirror a symplectic Dehn-twist. It is the Seidel-Thomas twist $T_\mathcal{E}$ by a so called spherical object $\mathcal{E}$ of $D^b(X)$, which has the property that $\dim(\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})) = \dim \mathrm{H}^*(\mathbf{S})$ and is given by the triangle

$$\longrightarrow (\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow T_\mathcal{E}(\mathcal{F}) \xrightarrow{+1}$$

The structure sheaf $\mathcal{O}_X$ is the basic spherical object in $D^b(X)$, but also each line bundle $L \in \mathrm{Pic}(X)$ is spherical. Another particularly simple type of autoequivalence is the operation $\otimes L$ of tensoring with a line bundle L. Note that $\mathcal{O}_p \otimes L = \mathcal{O}_p$. This fits on the mirror side to the monodromy tranformation around a point of maximal unipotent monodromy, with invariant vanishing torus $\mathbf{T}$.

Let us write out these transformations on the level of cohomology. Let $L = \mathcal{O}(H)$ be the ample generator of $\mathrm{Pic}(X)$. The powers $1, H, H^2, H^3$ form a basis for $\mathrm{H}^{\mathrm{ev}}(X, \mathbb{Q})$. With respect to this basis, the matrix T of tensoring with L is given by

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 & 1 \end{pmatrix}, \qquad (2)$$

as easily follows from $\mathrm{ch}(L\otimes\mathcal{E})=\mathrm{ch}(L)\cup\mathrm{ch}(\mathcal{E})=e^H\cup\mathrm{ch}(\mathcal{E})$.

The twist $T_{\mathcal{O}_X}$ on the level of cohomology is given by $\gamma\mapsto\gamma-\int_X\gamma\cup td(X)\cdot 1$ and hence its matrix is given by

$$S=\begin{pmatrix}1 & -c & 0 & -d\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{3}$$

where

$$d:=H^3,\qquad c:=c_2\cdot H/12.$$

The matrix Q representing the bilinear form $\langle\cdot,\cdot\rangle$ in this basis is given by

$$Q=\begin{pmatrix}0 & c & 0 & d\\ -c & 0 & -d & 0\\ 0 & d & 0 & 0\\ -d & 0 & 0 & 0\end{pmatrix}$$

Now Kontsevich [27] observed the miracle that for the quintic and its mirror the matrices T and S indeed correspond to monodromy matrices of the Picard-Fuchs operator

$$\theta^4-5^5z\big(\theta+\tfrac{1}{5}\big)\big(\theta+\tfrac{2}{5}\big)\big(\theta+\tfrac{3}{5}\big)\big(\theta+\tfrac{4}{5}\big).$$

It has 0, $1/5^5$ and ∞ as singular points. In an appropriate base, the monodromy around 0 is given by T and around $1/5^5$ by S.

We see that apparently the following happens: there is a point of maximal unipotent monodromy, corresponding to $\otimes\mathcal{O}(H)$ in $\mathrm{Auteq}(D^b(X))$ and there is a conifold point, corresponding to the twist along $\mathcal{O}_X$.

Similar things occur in all the 14 hypergeometric cases. As there are only three singular points in these cases, these two monodromies generate the monodromy group. We refer to [21] for a generalisation to Calabi-Yaus in more general toric manifolds.

Calabi-Yau spaces with Picard number one seem to be rather scarse. Apart from the 14 hypergeometric cases there is there is a list (not claiming completeness in any sense) by Borcea [13] containing 11 further cases. The examples are ramified covers and complete intersections in Fano-varieties with Picard-number one. We know of a few other cases. Basic invariants for such X are the *degree* $d:=H^3$, the *second Chern class* $c_2\cdot H$ and the Euler number $c_3=\chi_{top}$, of which the first two can be read off from the matrix S.

It is sometimes more convenient to work with a different representation based on the one used by C. Doran and J. Morgan (see [15]). That basis can be obtained from the one above using the coordinate transformation given by the matrix

$$W=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & -1\\ 0 & \frac{1}{d} & -\frac{1}{2} & \frac{1}{3}-\frac{c}{d}\\ \frac{1}{d} & 0 & -\frac{c}{d} & \frac{c}{d}\end{pmatrix}$$

where c and d are as above. This yields the following representation:

$$T_{\mathrm{DM}} = W^{-1}TW = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & d & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_{\mathrm{DM}} = W^{-1}SW = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$Q_{\mathrm{DM}} = W^{t}QW = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -k \\ -1 & -1 & k & 0 \end{pmatrix}$$

Here $d = H^3$ and $k = \frac{c_2 \cdot H}{12} + \frac{H^3}{6}$. This last number has a simple interpretation as the dimension $\dim(H^0(X, \mathcal{O}(H))$ of the linear system $|H|$.

4 Computation of the monodromy

Our starting point for the computation of the monodromy is the following working hypothesis

Hypothesis 1 *Any differential equation of Calabi-Yau type which is strictly geometrical and for which the instanton numbers have an interpretation as the numbers of curves on a mirror manifold, the monodromy should satisfy the following conditions:*

(H1) There is a point of maximal unipotent monodromy, corresponding to $\otimes\mathcal{O}(H)$ *in* $\mathrm{Auteq}(D^b(X))$.

(H2) There is a conifold point, corresponding to the twist along $\mathcal{O}_X$.

By construction all the equations in the list from [2] have a point of maximal unipotent monodromy at $z = 0$. The non-obvious part is to find a conifold point. We observed that in the cases where we know the conifold point the *spectrum*, i.e., the set of zeros of the indicial equation at that point, was $\{0, 1, 1, 2\}$. This is also suggested by Hodge theory. Therefore as a practical selection criterium, we computed the indicial equations at the singular points of all equations and found the equations with at least one singular point with spectrum $\{0, 1, 1, 2\}$. As of the time of writing of this article there were 178 such equations in our database. In many cases there are several such points, but there are also some notable exceptions, where no such singular point exists. An example is equation 32, which is related to $\zeta(4)$ (see [3]). For the moment, we are unable to find integral or even just rational lattices for these cases.

For all the 178 equations that do have at least one singular point with spectrum $\{0, 1, 1, 2\}$ we computed high precision numerical approximations for a set of generators of the monodromy group. These computations were done in `Maple`. The first step was to determine the critical points $z_1, \ldots, z_\ell$ and to choose a reference point p. Next for each of the critical points z_i except the point $z = \infty$ we choose a piecewise linear loop starting and ending at the reference point p and enclosing only one critical point, namely z_i (see Figure 1).

Using the `Maple`-function `dsolve` we can numerically integrate the differential equation along these paths. It turns out to be a bit tricky to obtain the precision needed for the next steps. We used the following options: method=gear,

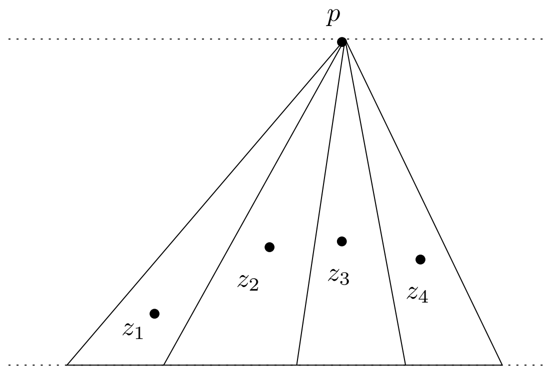

Figure 1: Piecewise linear loops around the critical points z_i

relerr=10^{-15}, abserr=10^{-15} and also increased `Digits` to 100. This yielded the monodromy matrices with respect to an arbitrary basis and produces fully filled 4×4-matrices with seemingly random complex entries.

At this point there is a simple consistency check that we can do. If there exists an integral lattice, the characteristic polynomial of each of the monodromies should be a polynomial with integral coefficients. As a further check, the roots of the indicial equations at the corresponding singular points, should be logarithms of the roots of the characteristic polynomial. For the MUM-point and the points with spectrum $\{0, 1, 1, 2\}$ the characteristic polynomial should be $(1 - \lambda)^4$. This provides an indication of the precision we have achieved.

The next step is to try and find a simultaneous base change that makes all matrices integral, i.e., to find a monodromy invariant lattice Λ. The crucial observation is the following. The monodromy S around an A_1-singularity has the property that $\mathrm{rk}(S - \mathrm{Id}) = 1$. The one-dimensional image of $S - \mathrm{Id}$ is the span of the vanishing cycle. Now choose one of the singular points with spectrum $\{0, 1, 1, 2\}$ and call the monodromy around the loop enclosing this singular point S. As we are working with numerical approximations we cannot expect $S - \mathrm{Id}$ to have rank 1, but we can hope that the columns of the matrix $S - \mathrm{Id}$ are nearly proportional. In that case we can pick an arbitrary vector and apply $S - \mathrm{Id}$ to it. In this way we find a vector v_0 that should be a good approximation to a lattice vector.

Further lattice vectors $v_1, \ldots, v_k$ can be obtained by applying words in the numerically computed monodromy matrices to v_0. By picking n independent vectors among the ones found in this way, we should find a basis for $\Lambda \otimes \mathbb{Q}$. When we transform the monodromy matrices to this basis, the resulting matrices should have rational entries. Of course this will not be exact, but we can try to find rational matrices close to the matrices that we do find. For this we used continued fractions. It may happen that we get very large denominators or that the rational approximation is not very accurate. In that case we can try another set of n independent vectors among the v_i. If that is not successful, we can try another point with spectrum $\{0, 1, 1, 2\}$, if there is any. As a consistency check, we can compute the characteristic polynomials of these rational matrices and check that they have integral coefficients. As noticed above, at the MUM-point and the conifold point the characteristic polynomial should be $(1-\lambda)^4$, which we can also check. If any of these checks fails, we have to try again with a different basis or a different singular point with spectrum $\{0,1,1,2\}$. However, it can and does happen that we try all potential conifold points and several choices of a basis in each case, but do not find a rational basis. We did find a rational basis in 143 of the 178 investigated cases.

The rational basis found in this way is still rather arbitrary. However, a major advantage is that at this point we expect to be working with the *exact* monodromy matrices. This allows us to do linear algebra without worrying about the extra complications of working with non exact numerical approximations. Provided that the monodromy matrices around the MUM-point and the conifold point have the right Jordan structure, we can find a new basis such that with respect to this basis they have the standard form (2) and (3). In a geometrical situation we expect the transformed matrices to be integral. This happens in 64 cases. When we have the monodromies around the MUM-point and the conifold point in the standard form, we can read off the invariants H^3 and $c_2 \cdot H$ and try to match the invariants with those of known Calabi-Yau spaces.

Despite our efforts to identify equivalent Calabi-Yau equations our list probably still includes some Calabi-Yau equations that correspond to the same geometrical situation. Transformations in the parameter z are a way of constructing seemingly different equations that actually describe the same geometrical situation. In a geometrical language this corresponds to pullback under a map $f : \mathbb{P}^1 \to \mathbb{P}^1$. If the map f is not injective, this may increase the number of singular points. As long as the map f is unramified around the MUM-point and the conifold point it does not change the monodromies and we ought to find the same H^3 and $c_2 \cdot H$. So as practical way of trying to group together the equations that correspond to the same geometry, we sort the 64 integral equations we found according to H^3 and $c_2 \cdot H$. If we find several equations with the same H^3 and $c_2 \cdot H$, it turns out that the genus zero instanton numbers also coincide. This is a strong indication that these equations are equivalent.

5 Conifold-period and Euler characteristic

If Ω is a family of holomorphic three forms on Y_s (that is, a section of $\mathcal{L} := \rho_*(\omega_{\mathcal{Y}}/S)$) and Γ is a horizontal family of cycles, then the *periods*

$$\int_\Gamma \Omega$$

are the solutions of the associated Picard-Fuchs equation. In our situation we identified two cycles, namely the torus $\mathbf{T}$ near the MUM-point, and the vanishing sphere $\mathbf{S}$ near the conifold point z_c. Correspondingly we have the fundamental period $\int_{\mathbf{T}} \Omega$ which is the unique holomorphic solution near the MUM-point. Equally important is the period $\int_{\mathbf{S}} \Omega$ which we call the *conifold-period* and which was called $z_2(t)$ in the paper [14]. As the local monodromy around the conifold point is supposed to be a symplectic reflection in S, this special period can be determined directly from the differential equation as follows. At such a conifold point there exists a basis of solutions to the Calabi-Yau-equation around this point that consists of three power series solutions and one solution of the form

$$y(z) = f(z)\log(z - z_c) + g(z).$$

Going around z_c once y is replaced by $y(z) + 2\pi i f(z)$. The power series $f(z)$ represents a special solution around $z = z_c$ that is determined up to a multiplicative scalar and which we call the *conifold-period*. This function can be continued analytically around an arbitrary path which avoids the singularities of the differential equation. It is a remarkable fact that in all (but one, namely nr. 224) of the examples we know, the point z_c is the singular point that is *closest to the origin*. So there is a preferred path from z_c to 0 by going along a straight line and we consider the analytic continuation along this path. In [14] the expansion of the conifold period around 0 is derived for the quintic. It has the form

$$z_2(t) = \frac{H^3}{6}t^3 + \frac{c_2 \cdot H}{24}t + \frac{c_3}{(2\pi i)^3}\zeta(3) + O(q). \tag{4}$$

The term $O(q)$ stands for any terms containing $q = e^{2\pi i t}$ and $t = \frac{1}{2\pi i}\frac{y_1(z)}{y_0(z)} = \frac{1}{2\pi i}\log z + \frac{1}{2\pi i}\frac{f_1(z)}{f_0(z)}$ instead of (6). Remarkable here is the 'constant term' $\frac{c_3}{(2\pi i)^3}\zeta(3)$. This term is related to the four-loop correction to the free energy F_0 introduced in [14].[1]

One can conjecture this expansion to hold in all cases, which leads to the following algorithm to determine c_3. One can easily compute an expansion of $y(z)$ to an arbitrary number of terms, e.g., using the `Maple`-function `formal_sol` from the `DEtools`-package, as we did. This allows us to find $f(z)$ as the coefficient of $\log(z - z_c)$. Around the MUM-point $z = 0$ we can compute expansions of the elements $y_i(z)$ of the Frobenius basis (see Appendix B). We suppose that the domains of convergence of the solutions around $z = 0$ and those around $z = z_c$ overlap. That enables us to pick some point z_* where both expansions converge. Computing numerically $f^{(k)}(z_*)$ $(k = 0, \ldots, 3)$ and $y_i^{(k)}(z_*)$ $(k, i = 0, \ldots, 3)$, we can consider the equations

$$f^{(k)}(z_*) = \sum_{i=0}^{3} c_i y_i^{(k)}(z_*).$$

These equations can be solved for the c_i and determine the analytic continuation z_2 around 0 of $f(z)$ as a linear combination of the $y_i(z)$

$$z_2 = \sum_{i=0}^{3} c_i y_i(z).$$

From this we can readily read of the expansion of z_2 in t. At this point we can already check that the coefficient of t^2 vanishes. As the conifold-period $f(z)$ was only determined up to a constant, of course $z_2(t)$ ist determined up to a

[1] To be precise, one has an expansion (see [23]):

$$F_0 = H^3\frac{t^3}{3!} + (c_2 \cdot H)t + \frac{\chi}{2}\zeta(3) + \sum_{d=1}^{\infty} n_d^0 Li_3(q^d)$$

where $\mathrm{Li}_3(x) := \sum_{k=1}^{\infty} k^{-3}x^k$ is the classical trilogarithm.

constant. If we suppose that H^3 is known, then one can multiply the expansion for $z_2(t)$ by a constant such that the coefficient of t^3 is $\frac{H^3}{6}$. We can then read off $c_2 \cdot H$ and c_3. In praxis we find H^3 as discussed above from the monodromy generators. This also yields $c_2 \cdot H$, so we have one more consistency check. It is remarkable that in all cases we indeed find an integral value of c_3!

6 Comments on the table of Calabi-Yau-equations

In Table 1 the heading Sings denotes the number of singular point of the (first mentioned) differential equation. An additional $*$ indicates, that an apparent singularity is present, around which there is no monodromy. The notation $X(\dots)$ denotes a complete intersection of the indicated degrees in the indicated manifold. Apart from the familiar 13 hypergeometric cases and the cases from complete intersection in Grassmanians that were studied in in [8], one finds a few notable further cases. First there is the elusive 14th hypergeometric case, observed in [4] and [15]. Any complete intersection $X(2,12)$ inside $\mathbb{P}(1,1,1,1,4,6)$ has a singular point of type $A_1/(\mathbb{Z}/2)$, which does not admit a crepant resolution. The case $X \xrightarrow{2:1} B_5$ is the Calabi-Yau double cover of the Fano-threefold B_5, which is nothing but the three-dimensional section of Grass$(2,5)$, which is no. 14 in the list of Borcea. We found a fit with the equation 51 from [2]. A mirror for this Calabi-Yau is not known, but we conjecture the Picard-Fuchs equation to be the indicated one. We find similar fits for

$X(1,1,1,1,1,1,2) \subset X_{10}$: Here $X_{10} \subset \mathbb{P}^{15}$ is the celebrated 10-dimensional spinor variety of isotropic 4-planes in the 8-dimensional quadric.

$X(1,1,2) \subset$ LGrass$(3,6)$: LGrass$(3,6) = \mathrm{Sp}(3,\mathbb{C})/P(\alpha_3) \subset$ Grass$(3,6)$ is the Lagrangian Grassmanian.

$X(1,2) \subset X_5$: Here $X_5 = G_2/P(\alpha_{\mathrm{long}}) \subset$ Grass$(5,7)$ is the space of 5-dimensional subspaces isotropic for a 4-form on a 7-dimensional space.

These are complete intersections inside homogeneous spaces. In principle one can calculate the Picard-Fuchs equation for the instanton numbers for these cases and verify our conjecture. The first method consist in computing the quantum cohomology of these homogeneous examples (for example by fixed point localisation) and then use the quantum Lefschetz hyperplane principle. A second method consists of finding a toric degeneration and then using polar duality. Such toric degenerations have been constructed for all spherical varieties in [1]. Both methods were used in [8] for the case of complete intersections in Grassmannians.

In his thesis [39], F. Tonoli considers Calabi-Yau varieties in $\mathbb{P}^6$ of degree 12 up to 17. The first one is the complete intersection $X(2,2,3)$, the second one the 5×5-Pfaffian, for which we found a fit with the data from equation 99. The 7×7-Pfaffian was considered in [33]. The remaining three case are new Calabi-Yau threefolds for which we have not yet found corresponding Picard-Fuchs equations.

The column for the Euler characteristic c_3 was determined using the expansion of the conifold-period around the MUM-point. It is a miracle that we found integral values in all cases (except 224). This checked with the known Euler

Table 1: **Calabi-Yau equations with integral monodromy**

H^3	$c_2 \cdot H$	c_3	$\lvert H\rvert$	Sings	Database	Description	Reference
1	10	48?	1	4*	225		
1	22	-120	2	3	13	$X(6,6) \subset \mathbb{P}^5(1,1,2,2,3,3)$	[25]
1	34	-288	3	3	2	$X(10) \subset \mathbb{P}^4(1,1,1,2,5)$	[32]
1	46	-484	4	3	9	$X(2,12) \subset \mathbb{P}^5(1,1,1,1,4,6)$(?)	[4],[15]
2	20	-44	2	4*	271		
2	32	-156	3	3	12	$X(3,4) \subset \mathbb{P}^5(1,1,1,1,1,2)$	[25]
2	44	-296	4	3	7	$X(8) \subset \mathbb{P}^5(1,1,1,1,4)$	[32]
3	42	-204	4	3	8, 125	$X(6) \subset \mathbb{P}^4(1,1,1,1,2)$	[32]
4	40	-144	4	3	10	$X(4,4) \subset \mathbb{P}^5(1,1,1,1,2,2)$	[25]
4	52	-256	5	3	14, 85, 86	$X(2,6) \subset \mathbb{P}^5(1,1,1,1,1,3)$	[25]
5	38	-100*	4	4*	302		
5	50	-200	5	3	1, 79, 87, 128	$X(5) \subset \mathbb{P}^4$	[14]
5	62	-310	6	4	63		
6	36	-72	4	4*	33		
6	48	-156	5	3	11, 95	$X(4,6) \subset \mathbb{P}^5(1,1,1,2,2,3)$	[25]
7	46	-120*	5	4*	109		
8	32	-8	4	4	291		
8	56	-176	6	3	6, 75, 76, 96	$X(2,4) \subset \mathbb{P}^6$	[30]
9	30	12?	4	4	73		
9	54	-144	6	3	4	$X(3,3) \subset \mathbb{P}^5$	[30]
10	40	-50	5	5*	118		
10	40	-32	5	4*	292		
10	52	-116*	6	4*	263		
10	64	-200	7	4	51	**Conj:** $X \xrightarrow{2:1} B_5$	[13, nr. 14]
12	36	-32	5	5*	117		
12	48	-60	6	5*	267		
12	60	-144	7	3	5, 90, 91, 93	$X(2,2,3) \subset \mathbb{P}^6$	[30]
13	58	-120	7	4*	99	**Conj:** 5×5-Pfaffian $\subset \mathbb{P}^6$	[39]
14	56	-98	7	5*	222	7×7-Pfaffian $\subset \mathbb{P}^6$	[33]
14	56	-100	7	5*	289		
15	54	-78	7	?	?	$\mathrm{To}_{15} \subset \mathbb{P}^6$	[39]
15	66	-150	8	4	24	$X(1,1,3) \subset \mathrm{Grass}(2,5)$	[8]
16	52	-60	7	?	?	$\mathrm{To}_{16} \subset \mathbb{P}^6$	[39]
16	64	-128	8	3	3, 72, 224	$X(2,2,2,2) \subset \mathbb{P}^7$	[30]
17	50	-44	7	?	?	$\mathrm{To}_{17} \subset \mathbb{P}^6$	[39]
18	60	-88	8	4	266		
20	68	-120	9	4	25	$X(1,2,2) \subset \mathrm{Grass}(2,5)$	[8]
21	66	-102	9	5*	254		
21	66	-100	9	5*	270		
24	72	-116	10	4	29	**Conj:** $X(1,1,1,1,1,1,2) \subset X_{10}$	[13, nr. 6]
25	70	-100*	10	5*	101		
28	76	-116	11	4	26	$X(1,1,1,1,2) \subset \mathrm{Grass}(2,6)$	
29	74	-100*	11	5*	256		
32	80	-116	12	4	42	**Conj:** $X(1,1,2) \subset \mathrm{LGrass}(3,6)$	[13, nr. 8]
33	78	-102*	12	5*	259		
34	76	-88	12	4*	255		
36	72	-72	12	5*	100		
36	84	-120	13	4	184	**Conj:** $X(1,2) \subset X_5$	[13, nr. 9]
42	84	-98	14	6	27	$X(1,1,1,1,1,1,1) \subset \mathrm{Grass}(2,7)$	[8]
42	84	-96	14	4	28	$X(1,1,1,1,1,1) \subset \mathrm{Grass}(3,6)$	[8]
44	92	-128	15	?	?	$X \xrightarrow{2:1} A_{22}$ or A'_{22}	[13, nr. 10]
47	86	-90*	15	6**	257		
56	92	-92	17	?	?	$X(1,1,1,1) \subset F_1(Q_5)$	[13, nr. 24]
57	90	-84	17	5*	247	Tjøtta's example	[37]

number in those cases where a geometrical interpretation was known. However, there are two notable cases where we get a *positive* value for c_3, which excludes an interpretation as a Calabi-Yau space with Picard number one. Furthermore, the conjectural integrality of elliptic intanton numbers implies some congruence property on c_3. In most cases this was satisfied, giving a strong indication that a Calabi-Yau threefold with the indicated invariants should exist. In some cases however, we found non-integral in this way n^1_d. This is indicated with a $*$ after the value for c_3.

In the database column we indicate the number of the equation in the electronic database of Calabi-Yau equations that can be found at the web address

http://enriques.mathematik.uni-mainz.de/enckevort/db

Up to 180 these numbers coincide with the ones used in [2]. For higher numbers one should check the source field in the electronic database. If it contains Almkvist[n] the corresponding number in [2] is n.

7 Some Examples

Let us now discuss a few typical examples from Table 1 in more detail. For full information on the other cases, we refer to the database mentioned above.

Example 1

The first equation we want to study is equation 28 from [2], which is given by the following operator

$$L = \theta^4 - z(65\,\theta^4 + 130\,\theta^3 + 105\,\theta^2 + 40\,\theta + 6) + 4z^2(4\,\theta + 3)(\theta + 1)^2(4\,\theta + 5),$$

where $\theta = z\frac{d}{dz}$. This differential operator has four singular points, namely 0, 1/64, 1, and ∞. The Riemann scheme is

$$P\left\{\begin{array}{cccc} 0 & 1/64 & 1 & \infty \\ \hline 0 & 0 & 0 & 3/4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 5/4 \end{array}\right\}$$

Here the columns are the spectra, i.e., the set of solutions to the indicial equation at the singular point indicated above the line. The points 1/64 and 1 have spectrum $\{0, 1, 1, 2\}$, so they are potential conifold points. Using the algorithm discussed above we computed the monodromies around the critical points and found an integral lattice. With respect to this lattice the monodromy matrices are as follows

$$T = T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 42 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = T_{\frac{1}{64}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -14 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 37 & 12 & -252 & 156 \\ -126 & -41 & 882 & -546 \\ -12 & -4 & 85 & -52 \\ -18 & -6 & 126 & -77 \end{pmatrix},$$

$$T_\infty = (T_1 T_{\frac{1}{64}} T_0)^{-1} = \begin{pmatrix} 77 & 29 & -588 & 348 \\ -112 & -41 & 840 & -504 \\ -6 & -2 & 43 & -27 \\ -17 & -6 & 126 & -77 \end{pmatrix}.$$

From this, one can read off the invariants

$$H^3 = 42, \quad c_2 \cdot H = 84.$$

In this case we know that the equation is the Picard-Fuchs equation of the complete intersection $X(1,1,1,1,1,1)$ in Grass$(3,6)$ and we can easily check that these numbers coincide with the ones computed from the geometry. Of course, the value c_3 computed from the expansion of the conifold-period gives the right value -96.

One can easily check that $T_{\frac{1}{64}}$ and T_1 are of the Picard-Lefschetz form $S_{\lambda,v}$, with the vector v given by

$$v_{\frac{1}{64}} = \begin{pmatrix} 0 \\ 14 \\ 1 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 6 \\ -21 \\ -2 \\ -3 \end{pmatrix}.$$

For $T_{\frac{1}{64}}$ we have $\lambda = 1$, but for T_1 we have $\lambda = 2$. So the critical point $z = 1$ is *not* an ordinary conifold point. This $\lambda = 2$ is exactly what is needed to get integral genus one instanton numbers with the recipe from Appendix B. The first few elliptic instanton numbers that we find in this way are

$$n_1^1 = n_2^1 = n_3^1 = n_4^1 = 0, \quad n_5^1 = 84, \quad n_6^1 = 74382, \quad n_7^1 = 8161452.$$

So it appears that there is a $\mathbb{R}P^3$ vanishing at the point 1. The derived category of coherent sheaves in a Grassmannian is reasonably well understood (see [22]) and so one can hope to study in detail what happens in $D^b(X)$. This will be persued at another place, [40].

Example 2

Our second example has been discussed in [33, 38]. It is interesting because there are two points with maximal unipotent monodromy both of which have a geometrical interpretation. Because our convention is to have the point of maximal unipotent monodromy that we are considering at $z = 0$ this example occurs twice in our list: once as 27 and once as 222.

In the former case the differential operator is given by

$$\begin{aligned}L = {} & 3^2\,\theta^4 - 3z(173\,\theta^4 + 340\,\theta^3 + 272\,\theta^2 + 102\,\theta + 15) \\ & - 2z^2(1129\,\theta^4 + 5032\,\theta^3 + 7597\,\theta^2 + 4773\,\theta + 1083) \\ & + 2z^3(843\,\theta^4 + 2628\,\theta^3 + 2353\,\theta^2 + 675\,\theta + 6) \\ & - z^4(295\,\theta^4 + 608\,\theta^3 + 478\,\theta^2 + 174\,\theta + 26) + z^5(\theta+1)^4\end{aligned}$$

The Riemann scheme of equation 27 is

$$P\left\{\begin{array}{cccccc} \zeta_1 & 0 & \zeta_2 & 3 & \zeta_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\},$$

where $\zeta_1 < \zeta_2 < \zeta_3$ are the (real) roots of $z^3 - 289z^2 - 57z + 1$. The monodromies can be determined with our usual recipe

$$T_{\zeta_1} = \begin{pmatrix} 15 & 7 & -98 & 49 \\ -28 & -13 & 196 & -98 \\ -2 & -1 & 15 & -7 \\ -4 & -2 & 28 & -13 \end{pmatrix}, \quad T = T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 42 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S = T_{\zeta_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -14 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad T_3 = \mathrm{Id}, \quad T_{\zeta_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -84 & 1 & 392 & -392 \\ -9 & 0 & 43 & -42 \\ -9 & 0 & 42 & -41 \end{pmatrix},$$

$$T_\infty = (T_{\zeta_3} T_3 T_{\zeta_2} T_0 T_{\zeta_1})^{-1} = \begin{pmatrix} 85 & 6 & -448 & 399 \\ -266 & -13 & 1330 & -1232 \\ -26 & -1 & 127 & -120 \\ -42 & -2 & 210 & -195 \end{pmatrix}.$$

Here the monodromy operators T_{ζ_1}, T_{ζ_2}, and T_{ζ_3} can be written in the Picard-Lefschetz form $S_{1,v}$ with the vector v given by

$$v_{\zeta_1} = \begin{pmatrix} 7 \\ -14 \\ -2 \\ -2 \end{pmatrix}, \quad v_{\zeta_2} = \begin{pmatrix} 0 \\ 14 \\ 1 \\ 1 \end{pmatrix}, \quad v_{\zeta_3} = \begin{pmatrix} 0 \\ 28 \\ 3 \\ 3 \end{pmatrix}.$$

The operator for equation 222 can be obtained by replacing $y(z)$ by $w^{-1}y(w^{-1})$ where $w = 1/z$. In this way one finds the operator

$$\begin{aligned}L = {} & \theta^4 - z(295\,\theta^4 + 572\,\theta^3 + 424\,\theta^2 + 138\,\theta + 17) \\ & + 2z^2(843\,\theta^4 + 744\,\theta^3 - 473\,\theta^2 - 481\,\theta - 101) \\ & - 2z^3(1129\,\theta^4 - 516\,\theta^3 - 725\,\theta^2 - 159\,\theta + 4) \\ & - 3z^4(173\,\theta^4 + 352\,\theta^3 + 290\,\theta^2 + 114\,\theta + 18) + 3^2 z^5(\theta+1)^4.\end{aligned}$$

The Riemann scheme of 222 also follows from that of 27

$$P\left\{\begin{array}{cccccc} 1/\zeta_1 & 0 & 1/\zeta_3 & 1/3 & 1/\zeta_2 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\}.$$

For the monodromies the relation is not so obvious. Doing the standard computation we find

$$T_{\zeta_1^{-1}} = \begin{pmatrix} 29 & 14 & -98 & 49 \\ -56 & -27 & 196 & -98 \\ -8 & -4 & 29 & -14 \\ -16 & -8 & 56 & -27 \end{pmatrix}, \quad T = T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 14 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S = T_{\zeta_3^{-1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad T_{1/3} = \mathrm{Id}, \quad T_{\zeta_2^{-1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -105 & 1 & 294 & -294 \\ -25 & 0 & 71 & -70 \\ -25 & 0 & 70 & -69 \end{pmatrix},$$

$$T_\infty = (T_{\zeta_2^{-1}} T_{1/3} T_{\zeta_3^{-1}} T_0 T_{\zeta_1^{-1}})^{-1} = \begin{pmatrix} 155 & 13 & -476 & 427 \\ -420 & -27 & 1260 & -1162 \\ -76 & -4 & 225 & -211 \\ -126 & -8 & 378 & -349 \end{pmatrix}.$$

Again the monodromies $T_{\zeta_1^{-1}}$, $T_{\zeta_3^{-1}}$, and $T_{\zeta_2^{-1}}$ can be written in the Picard-Lefschetz form $S_{1,v}$ with the vector v given by

$$v_{\zeta_1^{-1}} = \begin{pmatrix} 7 \\ -14 \\ -2 \\ -4 \end{pmatrix}, \quad v_{\zeta_3^{-1}} = \begin{pmatrix} 0 \\ 7 \\ 1 \\ 1 \end{pmatrix}, \quad v_{\zeta_2^{-1}} = \begin{pmatrix} 0 \\ 21 \\ 5 \\ 5 \end{pmatrix}.$$

Despite the fact that we are really dealing with the same equation in a different formulation, the monodromies look rather different. Of course the monodromy groups generated by these matrices are isomorphic, but it is not so easy to see.

Example 3

The next example is equation 29 from [2]. The operator is

$$L = \theta^4 - 2z(2\,\theta + 1)^2(17\,\theta^2 + 17\,\theta + 5) + 2^2 z^2 (2\,\theta + 1)(\theta + 1)^2(2\,\theta + 3).$$

In this case the Riemann scheme is

$$P\left\{\begin{array}{cccc} 0 & \zeta_1 & \zeta_2 & \infty \\ \hline 0 & 0 & 0 & 1/2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 3/2 \end{array}\right\},$$

where $\zeta_1 < \zeta_2$ are the (real) roots of $1 - 136z + 16z^2$. The monodromy matrices are

$$T = T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 24 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = T_{\zeta_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -10 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{\zeta_2} = \begin{pmatrix} 51 & 20 & -240 & 140 \\ -130 & -51 & 624 & -364 \\ -15 & -6 & 73 & -42 \\ -25 & -10 & 120 & -69 \end{pmatrix},$$

$$T_\infty = (T_{\zeta_2} T_{\zeta_1} T_0)^{-1} = \begin{pmatrix} 71 & 31 & -360 & 200 \\ -120 & -51 & 600 & -340 \\ -10 & -4 & 49 & -29 \\ -24 & -10 & 120 & -69 \end{pmatrix}.$$

One can check that the T_{ζ_i} can be written as $S_{1,v}$ with v given by

$$v_{\zeta_1} = \begin{pmatrix} 0 \\ 10 \\ 1 \\ 1 \end{pmatrix}, \quad v_{\zeta_2} = \begin{pmatrix} 10 \\ -26 \\ -3 \\ -5 \end{pmatrix}.$$

From the expressions for T and S we find $H^3 = 24$, $c_2 \cdot H = 72$. We also have enough information to compute the elliptic instanton numbers as a function of c_3. By equating $n_1^1 = 0$ we find $c_3 = -116$ and all the n_i^1 we computed are integral. Thes same value for c_3 is obtained from the expansion of the conifold-period. It turns out that we are *lucky* and that there is exactly one 1-parameter Calabi-Yau known with these invariants, namely $X(1,1,1,1,1,1,2) \subset X_{10}$ (see [13]). So we conjecture that equation 29 is the Picard-Fuchs equation corresponding to this Calabi-Yau. In the same way we conjecturally identified the Picard-Fuchs equations of four more 1-parameter Calabi-Yau spaces. We labelled these equations in Table 1 by writing **Conj:** in front of the conjectured Calabi-Yau.

Example 4

As our final example we will use equation 270 (218 in the numbering from [2]) which is given by the diffential operator

$$\begin{aligned} L = {} & 7^2\,\theta^4 - 42z(192\,\theta^4 + 396\,\theta^3 + 303\,\theta^2 + 105\,\theta + 14) \\ & + 2^2 \cdot 3z^2(1188\,\theta^4 + 11736\,\theta^3 + 20431\,\theta^2 + 12152\,\theta + 2436) \\ & + 2^2 \cdot 3^3 z^3(532\,\theta^4 + 504\,\theta^3 - 3455\,\theta^2 - 3829\,\theta - 1036) \\ & - 6^4 z^4(2\,\theta + 1)(36\,\theta^3 + 306\,\theta^2 + 421\,\theta + 156) \\ & - 2^6 \cdot 3^4 z^5(2\,\theta + 1)(3\,\theta + 2)(3\,\theta + 4)(2\,\theta + 3). \end{aligned}$$

The Riemann scheme is

$$P\left\{\begin{array}{cccccc} -7/12 & 0 & \zeta_1 & \zeta_2 & \zeta_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 0 & 1 & 1 & 1 & 2/3 \\ 3 & 0 & 1 & 1 & 1 & 4/3 \\ 4 & 0 & 2 & 2 & 2 & 3/2 \end{array}\right\},$$

where ζ_1 is the real root of $1296z^3 - 864z^2 + 168z - 1$ and ζ_2, ζ_3 are its complex roots with $\operatorname{im}\zeta_2 < 0$ and $\zeta_3 = \bar{\zeta}_2$. The monodromies can be computed and turn out to be integral

$$T_{-\frac{7}{12}} = \mathrm{Id}, \quad T = T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 21 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_{\zeta_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -9 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{\zeta_2} = \begin{pmatrix} 16 & 5 & -60 & 40 \\ -45 & -14 & 180 & -120 \\ -6 & -2 & 25 & -16 \\ -9 & -3 & 36 & -23 \end{pmatrix}, \quad T_{\zeta_3} = \begin{pmatrix} 11 & 5 & -45 & 25 \\ -18 & -8 & 81 & -45 \\ -2 & -1 & 10 & -5 \\ -4 & -2 & 18 & -9 \end{pmatrix},$$

$$T_\infty = (T_{\zeta_3}T_{\zeta_2}T_{\zeta_1}T_0T_{-\frac{7}{12}})^{-1} = \begin{pmatrix} 8 & 5 & -45 & 21 \\ -12 & -5 & 60 & -36 \\ -1 & 0 & 4 & -4 \\ -3 & -1 & 15 & -10 \end{pmatrix}.$$

So we find $H^3 = 21$ and $c_2 \cdot H = 66$. The T_{ζ_i} can be written in Picard-Lefschetz form with $\lambda = 1$ and v given by

$$v_{\zeta_1} = \begin{pmatrix} 0 \\ 9 \\ 1 \\ 1 \end{pmatrix}, \quad v_{\zeta_2} = \begin{pmatrix} 5 \\ -15 \\ -2 \\ -3 \end{pmatrix}, \quad v_{\zeta_3} = \begin{pmatrix} 5 \\ -9 \\ -1 \\ -2 \end{pmatrix}.$$

We can also compute the elliptic instanton numbers. Setting $n_1^1 = 0$ we find $c_3 = -100$ and with this value of c_3 all computed n_d^1 turn out to be integers. The same value of c_3 was obtained from the expansion of the conifold-period. So we have a Calabi-Yau equation that as far as we can check looks like the Picard-Fuchs equation of a Calabi-Yau manifold. However, we do not know any 1-parameter Calabi-Yau with the geometric invariants that we computed. In Table 1 there are some more equations which look geometrical in every respect, but for which we have not found any Calabi-Yau yet.

8 Open problems

The work described in this paper is no more than a start and there are many open problems left. We have found quite a few Calabi-Yau equations that look in every respect like the Picard-Fuchs equation of a Calabi-Yau manifold, but for which we do not know if a Calabi-Yau manifold exists. We know the degree, the second Chern class, the Euler characteristic and the instanton numbers.

To determine an integral lattice we need to single out two singular points, where we bring the monodromies into the the standard forms T_{DM} and S_{DM}.

When there are no singular points with spectrum $\{0,1,1,2\}$ we do not have a good candidate for S_{DM} and cannot even start our procedure for determining an integral lattice. It would be interesting to see what can be done in such cases. We also did not look for other integral lattices as in [15].

The conjectural appearance of the constant term $c_3\zeta(3)/(2\pi i)^3$ in the expansion of the conifold period (and the free energy) is very intriguing. Is this a mathematical theorem?

The key obstacle to computing the elliptic instanton numbers is finding the holomorphic function $f(z)$ in (11). Our ansatz in combination with our recipe for determining the exponents works a many cases, but it is no more than an educated guess. A better understanding of the genus one computation in terms of the BCOV-torsion as in [16] will probably be helpful.

Many of the equations from the list in [2] come from Hadamard products. The singular points of a Hadamard product are given by products of the singular points of the factors. Maybe it is also possible to determine the monodromies of the Hadamard product in terms of the monodromies of the factors.

A Orbifolds of A_1

In many examples one encounters monodromy tranformations that are not described by the usual Picard-Lefschetz formula, but rather are *powers* of such operations. We offer a possible explanation of this phenomonen, which is only visible on the integral level.

Consider a lattice Λ with bilinear form $\langle\cdot,\cdot\rangle$. For $\beta\in\Lambda$ and $\lambda\in\mathbb{Z}$ consider the the transformation

$$S_{\lambda,\beta}:\Lambda\longrightarrow\Lambda,\quad S_{\lambda,\beta}(\alpha)=\alpha-\lambda\langle\beta,\alpha\rangle\beta. \tag{5}$$

The transformation $S_{\lambda,\beta}$ preserves $\langle\cdot,\cdot\rangle$ in the symmetric case only when $\lambda=2/Q(\beta,\beta)$ (or $\lambda=0$). In that case $S_{\lambda,\beta}$ has order two and is a reflection. When $\langle\cdot,\cdot\rangle$ is antisymmetric, there is no restriction on λ and $S_{\lambda,\beta}\circ S_{\lambda',\beta}=S_{\lambda+\lambda',\beta}$. So in that case $S_{\lambda,\beta}$ does not have finite order.

Such transformations occur as monodromy transformations where not a sphere, but rather a quotient S^3/G by a finite group G is vanishing, as we will explain now. Consider the function defining the three-dimensional A_1-singularity:

$$f:\mathbb{C}^4\longrightarrow\mathbb{C},\quad f(x,y,z,t)=x^2+y^2+z^2+t^2$$

The fibre F_s of f over $s\in\mathbb{C}\backslash 0$ is called the Milnor fibre and can be identified with the cotangent bundel to the sphere $\{(x,y,z,t)\in\mathbb{R}^4\mid x^2+y^2+z^2+t^2=s\}$, which is vanishing when $s\to 0$. We choose an orientation and let δ be the homology class of this sphere. There is also a covanishing cycle ϵ in the dual group $\mathrm{H}_3^{\mathrm{cl}}(F_s,\mathbb{Z})$ (homology with closed support). One has

$$\mathrm{H}_3(F_s,\mathbb{Z})=\mathbb{Z}\delta,\quad \mathrm{H}_3^{\mathrm{cl}}(F_s,\mathbb{Z})=\mathbb{Z}\epsilon,\quad \langle\delta,\epsilon\rangle=1$$

Let $G\subset\mathrm{SU}(2)=S^3$ be a finite subgroup. G then acts linearly on $\mathbb{R}^4$ and by complexification on $\mathbb{C}^4$, leaving invariant the function f defining the A_1-singularity. Consider the quotient map $\pi:\mathbb{C}^4\longrightarrow X:=\mathbb{C}^4/G$. The space X will be singular, but f descends to a function $g:X\longrightarrow\mathbb{C}$, such that $f=\pi\circ g$. So the fibre $G_s:=g^{-1}(s)$ is the quotient of F_s by G. In the fibre G_s there is a cycle S^3/G vanishing when $s \to 0$, with homology class $d \in \mathrm{H}_3(G_s, \mathbb{Z})$. As above there also exists a covanishing cycle $e \in \mathrm{H}_3^{\mathrm{cl}}(G_s, \mathbb{Z})$ such that

$$\mathrm{H}_3(G_s, \mathbb{Z}) = \mathbb{Z}d, \quad \mathrm{H}_3^{\mathrm{cl}}(G_s, \mathbb{Z}) = \mathbb{Z}e, \quad \langle d, e \rangle = 1.$$

The map π induces maps π^* and π_* between the homology groups of F_s and G_s and one easily sees that

$$\pi_*(\delta) = |G|d, \quad \pi_*(\epsilon) = e, \quad \pi^*(d) = \delta, \quad \pi^*(e) = |G|\epsilon$$

The Picard-Lefschetz formula tells us that under the monodromy of f the cycle δ remains fixed, whereas the cycle ϵ gets mapped to $\epsilon - \delta$. From the fact that the monodromy commutes with the group action we obtain, by taking π_*, that d remains fixed, whereas e gets mapped to $e - |G|d$. From this one deduces in the usual way that the occurence of a singularity of type A_1/G will lead the monodromy transformation described by the modified Picard-Lefschetz formula (see [6])

$$\gamma \mapsto \gamma - |G|\langle d, \gamma \rangle d$$

The cycle d should should give rise to a spherical object in the derived category of the mirror, but only the $|G|$th power of the Seidel-Thomas twist would arise from a monodromy transformation.

B Computation of instanton numbers

According to [18, 19] (see also [23]) we have the following expansion for the partition function F of the topological string

$$F = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g = \sum_{g=0}^{\infty} \sum_{d} \sum_{m=0}^{\infty} n_d^g \frac{1}{m} \left(2 \sin \frac{m\lambda}{2} \right)^{2g-2} q^{dm}.$$

The partion function F can be defined physically or mathematically using Gromov-Witten invariants. The above formula can then be considered to define the Gopakumar-Vafa invariants n_d^g. In contrast to e.g., the Gromov-Witten invariants, the Gopakumar-Vafa invariants are conjectured to be always integral.

We will restrict to the genus zero and genus one invariants. Furthermore, we will only consider the 1-parameter case. In that case we have the following formulas (see [23])

$$\partial_t^3 F_0 = n_0^0 + \sum_{\ell=1}^{\infty} \frac{n_\ell^0 \ell^3 q^\ell}{1 - q^\ell}$$

with $n_0^0 = H^3$ and

$$\partial_t F_1 = \frac{c_2 \cdot H}{24} + \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} \left(\tfrac{1}{12} n_d^0 + n_d^1 \right) d q^{kd}.$$

Here we define the coordinate t by $q = e^{-t}$. So if we can compute the left hand sides of these equations the invariants n_d^0 and n_d^1 can easily be determined.

To do so, we first introduce a special basis of solutions for the equation (1) around a point of maximal unipotent monodromy, i.e., a singular point where

$\lambda = 0$ is the only solution to the indicial equation. In physical terms such a point (also called MUM-point) corresponds to a large radius limit point.

Suppose $z = 0$ is a MUM-point. Then we can use the Frobenius method. The idea is to consider a solution with values in the ring $\mathbb{C}[\rho]/(\rho^n)$. We make the following ansatz for such a solution

$$\tilde{y}(z) = \sum_{n=0}^{\infty} A(n,\rho) z^{n+\rho} = y_0(z) + y_1(z)\rho + \cdots + y_{n-1}(z)\rho^{n-1},$$

where we define

$$z^\rho = e^{\log z \cdot \rho} = 1 + \log z \cdot \rho + \frac{\log^2 z}{\rho} \cdot \rho^2 + \cdots + \frac{\log^{n-1} z}{(n-1)!} \cdot \rho^{n-1}.$$

Using $\theta z^{n+\rho} = (n+\rho) z^{n+\rho}$, where $\theta = z\frac{d}{dz}$, we can translate the equation $L\tilde{y} = 0$ into a recursion relation for the $A(n,\rho)$. As initial condition for the recursion we use $A(0,\rho) = 1$. The y_i we find in this way are called the *Frobenius basis.*

Define power series f_i by the following expression

$$\sum_{n=0}^{\infty} A(n,\rho) z^n = f_0(z) + f_1(z)\rho + \cdots + f_{n-1}(z)\rho^{n-1}.$$

Because $z^\rho \sum_{i=0}^{n-1} f_i(z)\rho^i = \sum_{i=0}^{n-1} y_i(z)\rho^i$, we find

$$y_i(z) = \sum_{j=0}^{i} \frac{\log^i z}{i!} f_{j-i}(z).$$

Using the Frobenius base we can define a new coordinate

$$t = y_1(z)/y_0(z) = \log z + \frac{f_1(z)}{f_0(z)}. \tag{6}$$

There are basically two ways to compute $\partial_t^3 F_0$. The starting point of the first one is the *Yukawa coupling* in the z coordinate

$$K_{zzz} = \exp\left(-\tfrac{1}{2}\int a_3(z)\mathrm{d}z\right),$$

where $a_3(z)$ is one of the coefficients from (1). The claim is that $\partial_t^3 F$ is the following transformation of this function to the t-coordinate defined in (6)

$$\partial_t^3 F(t) = \frac{K_{zzz}(z(t))}{y_0^2(z(t))\left(\frac{dt}{dz}\right)^3} \tag{7}$$

Now recall that a Calabi-Yau equation has to satisfy a list of conditions (see [3, 2]). One of these can be written as

$$a_1 = \frac{1}{2}a_2 a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3 a_3' - \frac{1}{2}a_3'' \tag{8}$$

According to Proposition 1 from [3] this condition is equivalent to the two conditions

$$\frac{d^2}{dt^2}\frac{y_2}{y_0} = \frac{\exp\left(-\frac{1}{2}\int a_3(z)\mathrm{d}z\right)}{y_0^2\left(\frac{dt}{dz}\right)^3}, \tag{9}$$

$$\frac{d^2}{dt^2}\frac{y_3}{y_0} = t\frac{d^2}{dt^2}\frac{y_2}{y_0}. \tag{10}$$

The second condition is equivalent to the existence of a function G and a constant c such that

$$\Pi(t) := \frac{1}{y_0}\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ t \\ \partial_t G - c \\ t\partial_t G - 2G \end{pmatrix}.$$

The vector $\Pi(t)$ is called the *normalized period vector*. Using the second condition (7) translates to

$$\partial_t^3 F_0 = \partial_t^3 G(t) = \partial_t^2 \frac{y_2}{y_0}.$$

This yields a different way of computing $\partial_t^3 F_0$ and therefore also the instanton numbers n_d^0.

To compute $\partial_t F_1$ we use the recipe from [10, 11] based on an analysis of the so called holomorphic anomaly. We will use the following formula from [10] (using our notation and adapted slightly for the case we are studying):

$$\partial_t F_1 = \partial_t \log\left(\frac{z^{1+\frac{c_2\cdot H}{12}} f(z)}{y_0^{4-\frac{c_3}{12}}\frac{\partial t}{\partial z}}\right). \tag{11}$$

In this formula one needs the geometrical data $c_2 \cdot H$ and c_3 which can usually be determined from the monodromy calculation and/or conifold period. However, the main problem with this formula is the function f which is a holomorphic function of z that still has to be determined. We will use an ansatz for f to reduce this problem to the determination of a finite number of parameters. To describe this ansatz note that because of the special form of a Calabi-Yau equation we can write

$$a_4(z) = z^4\Delta(z) = z^4\prod_i(\Delta_i(z))^{k_i},$$

for some polynomial $\Delta(z)$, which we call the *discriminant*. The $\Delta_i(z)$ are the irreducible factors (over $\mathbb{R}$) of $\Delta(z)$. Our ansatz is then the following

$$f(z) = \prod_i(\Delta_i(z))^{s_i},$$

where the exponents $s_i \in \mathbb{Q}$ still have to be determined. The (apparent) singular points of the operator are the zeros of the discriminant $\Delta(z)$ (and 0 and ∞). So each of the factors $\Delta_i(z)$ corresponds via its zeros to one or more (apparent) singular points. To determine the exponents we look at the monodromies around the corresponding singular points. When the singular point is a conifold, i.e., the monodromy is of the form $S_{1,v}$, then the exponent is generally assumed to be $-\frac{1}{6}$. We generalize this to $-\frac{\lambda}{6}$ for monodromies of the form $S_{\lambda,v}$ for arbitrary λ. When the monodromy is the identity, we put the exponent to zero. These rules already allow us to deal with many equations. However, monodromies of other types for which we do not know of a sensible guess do occur.

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