ON THE BETTI NUMBERS OF THE MILNOR FIBRE OF A CERTAIN CLASS OF HYPERSURFACE SINGULARITIES

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Introduction. For an isolated hypersurface singularity $f: (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ the following celebrated formula is valid (see [Mi], p.59):

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{x_0,\dots,x_n\}/(\partial_0 f,\dots,\partial_n f) .$$

It relates the topological invariant μ, the Milnor number to a readily computable algebraic invariant.

For a general hypersurface singularity it is improbable that there exist formulae of comparable simplicity for all Betti numbers of the Milnor fibre. However, for a more restrictive class of functions with non isolated singularities this seems to be possible. Siersma [S] studied hypersurfaces with one dimensional complete intersection singular locus along which f has (away from 0) transversally an A_1-singularity, from a topological point of view. In this paper we show that for this class of singularities the relative de Rham cohomology is torsionfree. This fact implies that for these singularities there are simple algebraic formulae for the Betti numbers of the Milnor fibre.

The proof goes as follows. In §1, we prove the coherence of the relative de Rham cohomology for so-called "concentrated singularities". In §2, we consider the spectral sequence for the Gauss-Manin system coming from the "Hodge filtration". When this spectral sequence degenerates at the E_2-level, one gets torsion freeness of the relative de Rham cohomology in the same way as Malgrange's proof of the corresponding result for isolated hypersurface singularities. In §3, finally we check by explicit calculation the degeneration of the spectral sequence for our special class of functions, using a result of Pellikaan [Pe].

§1. Coherence of Relative de Rham cohomology

In the case that $f: (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ defines an isolated singularity, Brieskorn [B], by using a projective compactification and Grauert's direct image theorem, proves that the relative hypercohomology groups $\mathbb{R}^i f_*(\Omega^\bullet_{X/S}) \simeq H^i(f_*\Omega^\bullet_{X/S})$ are coherent $\mathcal{O}_S$-modules. Here $X \xrightarrow{f} S$ is a Milnor representative of f ; i.e. $X = B_\varepsilon \cap f^{-1}(D_\eta)$ $0 < \eta << \varepsilon$ etc.

In [B-G] Buchweitz & Greuel prove a general coherence theorem for certain complexes $K^\bullet$ on an analytic space X with a flat map to a curve S , but still with the condition that the fibres have isolated singularities. They use a result of Kiehl and Verdier (see for example [D]).

In [H] Hamm proves the coherence of $\mathbb{R}^i f_*(\Omega^\bullet_{X/S})$/torsion in a quite general setting.

Here we give a coherence theorem general enough to be applied in §2 and §3. In absence of an appropriate reference, we include a proof, which is based on [B] and [B-G]. We consider map germs $(X,x) \xrightarrow{f} (S,s)$ with X an an analytic space and S a smooth curve.

<u>Definition 1.</u> A *standard representative* of the map germ $(X,x) \xrightarrow{f} (S,s)$ is a representative $X \xrightarrow{f} S$ of the form

$$X = X_{\varepsilon,\eta} := (B_\varepsilon \cap Y) \cap f^{-1}(D_\eta)$$

$$S = S_\eta \quad := D_\eta$$

with B_ε an open ε-ball in $\mathbb{C}^N$ and D_η an open η-disc in $\mathbb{C}$. For the intersection we use a fixed embedded representative $Y \subset \mathbb{C}^N$, $T \subset \mathbb{C}$ for the germ $(X,x) \xrightarrow{f} (S,s)$.

We put $\partial X := \partial B_\varepsilon \cap Y \cap f^{-1}(D_\eta)$ and $\bar{X} = X \cup \partial X$ (relative boundary and relative closure).
Note that for ε,η small enough $X_{\varepsilon,\eta}$ will be a contractible Stein space.

<u>Definition 2.</u> Let $X \xrightarrow{f} S$ be a standard representative for a germ $(X,x) \to (S,s)$ and $\mathbb{L}$ a sheaf of $\mathbb{C}$-vectorspaces on X.
$\mathbb{L}$ is called *transversally constant* (with respect to U and θ) if there exists an open neighbourhood U of $\overline{\partial X}$ in $\mathbb{C}^N$ and a C^∞-vectorfield θ on U with the following properties:

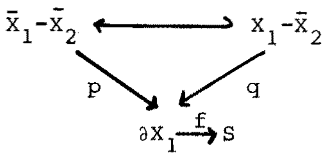
1) θ is transversal to ∂B_ε.
2) the local θ-flow in U leaves X and the fibres of f in X invariant.
3) the restriction of $\mathbb{L}$ to the local integral curves of θ is a constant sheaf.

<u>Theorem 1.</u> Let $X \xrightarrow{f} S$ be a standard representative of the germ $(X,x) \xrightarrow{f} (S,s)$.
Let $(K^\bullet, d)$ be a finite complex of sheaves on X. Assume:

1) the sheaves K^p are $\mathcal{O}_X$-coherent modules.
2) the differentials are $f^{-1}(\mathcal{O}_S)$-linear.
3) the cohomology sheaves $H^i(K^\bullet)$ are transversally constant (with respect to a single U and θ).

Then $\mathbb{R}^i f_*(K^\bullet)$ is an $\mathcal{O}_S$-coherent module.

<u>Sketch of proof</u>: Let $X = X_{\varepsilon,\eta}$. Now choose an U and θ exhibiting the $H^i(K^\bullet)$ as transversally constant sheaves. By compactness of $\overline{\partial X}$ and transversality of θ we can find $\varepsilon_2 < \varepsilon$ such that $\partial X_{\alpha,\eta} \subset U$ and $\theta \pitchfork \partial X_{\alpha,\eta}$ for all $\alpha \in [\varepsilon_2, \varepsilon]$. Choose $\varepsilon_1 \in (\varepsilon_2, \varepsilon)$. Because θ respects the f-fibres and leaves X invariant we have a commutative diagram



with $X_i = X_{\varepsilon_i,\eta}$
$\bar{X}_i = \bar{X}_{\varepsilon_i,\eta}$

Here p and q are the quotient maps induced by the local θ-flow. If $\mathbb{L}$ is a transversally constant sheaf on X (w.r.t. U and θ) then $R^i p_* \mathbb{L}\,|\bar{X}_1 - \bar{X}_2 \xrightarrow{\sim} R^i q_* \mathbb{L}|\, X_1 - \bar{X}_2$ (in fact $= 0$ for $i > 0$). By the spectral sequence for the composition of two maps we get $R^i f_* \mathbb{L}\,|\bar{X}_1 - \bar{X}_2 \xrightarrow{\sim} R^i f_* \mathbb{L}|\, X_1 - \bar{X}_2$. By Mayer-Vietoris we then get $R^i f_* \mathbb{L}\,|\bar{X}_1 \xrightarrow{\sim} R^i f_* \mathbb{L}\,|X_1$. The same argument for $X - X_1 \hookleftarrow \partial X_1$ gives $R^i f_* \mathbb{L}\,|X \xrightarrow{\sim} R^i f_* \mathbb{L}\,|\bar{X}_1 \xrightarrow{\sim} R^i f_* \mathbb{L}\,|X_1$. Apply this to $\mathbb{L} = H^i(K^\bullet)$. This gives an isomorphism of spectral sequences

$$\begin{array}{ccc} R^p f_*(\mathcal{H}^q(K^\cdot)|X) & \tilde{\to} & R^p f_*(\mathcal{H}^q(K^\cdot)|X_1) \\ \Downarrow & & \Downarrow \\ \mathbb{R}^{p+q} f_*(K^\cdot|X) & \overset{\sim}{\dashrightarrow} & \mathbb{R}^{p+q} f_*(K^\cdot|X_1) \end{array}$$

showing that shrinking of X does not change the hypercohomology. This fact implies the coherence of $\mathbb{R}^i f_*(K^\cdot)$ as $\mathcal{O}_S$-module, in exactly the same way as in ([B-G],p.250) by applying the main theorem of Kiehl & Verdier. □

Definition 3. Let $X \xrightarrow{f} S$ be a standard representative of $(X,x) \to (S,s)$. A complex of sheaves $(K^\cdot,d)$ on $X = X_{\varepsilon,\eta}$ is called *concentrated* if for all $\varepsilon' \in (0,\varepsilon]$ there exists $\eta' \in (0,\eta]$ such that the restriction of $K^\cdot$ to $X_{\varepsilon',\eta'}$ full-fills conditions 1), 2) and 3) of Theorem 1. A germ $(X,x) \to (S,s)$ is called concentrated if the relative de Rham complex $\Omega^\cdot_{X/S}$ is concentrated for some standard representative of the germ.

Examples.

1) A deformation $(X,x) \xrightarrow{f} (S,s)$ of an isolated singularity $(X_s = f^{-1}(s),x)$ is concentrated (see [B-G],p.248).
2) A hypersurface germ $f\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ with a good $\mathbb{C}^*$-action (i.e. all weights >0) is concentrated.
3) A hypersurface germ $f\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ such that for a certain representative $X \xrightarrow{f} S$ there are only a finite number of isomorphism classes of germs $(X,x) \to (S,S)$ with $x \in X$, $s = f(x)$, is concentrated.
4) The function $f = y^4 + xy^2z^2 + z^4$ does not define a concentrated germ at 0. The relative de Rham cohomology is not coherent.

We omit the proofs of these facts.

The idea is that for a concentrated complex the things really only happen in one point.

Proposition 1. Let $X \xrightarrow{f} S$ be a contractible Stein standard representative of a germ $(X,x) \to (S,s)$ and let $(K^\cdot,d)$ be a concentrated complex on X. Then:

$$\mathcal{H}^i(f_*K^\cdot)_s \tilde{\to} \mathbb{R}^i f_*(K^\cdot)_s \tilde{\to} (f_*\mathcal{H}^i(K^\cdot))_s \simeq \mathcal{H}^i(K^\cdot)_x \simeq H^i(K_x) .$$

Proof: The first isomorphism follows from the spectral sequence $\mathcal{H}^p(R^q f_* K^\cdot) \Rightarrow \mathbb{R}^{p+q} f_*(K^\cdot)$ and the fact that the K^i are coherent and

X is Stein, so $R^q f_*(K^\cdot) = 0 \quad q > 0$. For the second isomorphism we use the other spectral sequence $R^p f_*(H^q(K^\cdot)) \Rightarrow \mathbb{R}^{p+q} f_*(K^\cdot)$. By concentradness we may replace X by $\bar{X}$ and then apply ([G],II 4.11.1) to obtain $R^p f_*(H^q(K^\cdot))_s = H^p(f^{-1}(s), H^q|f^{-1}(s))$. By concentratedness again we may assume there is a contraction of $f^{-1}(s)$ to x such that the restriction of H^q to the fibres of the contraciton is constant. The proposition then follows from

<u>Lemma 1.</u> Let $\phi: X \times [0,1] \to X$ be a contraction of X to $p \in X$ by homeomorphisms (i.e.: $\phi(x,0) = x$, $\phi(x,1) = p$, $\phi(p,t) = p \quad \forall t \in [0,1]$ and $\phi(-,t): X \xrightarrow{\sim} X_t := \phi(x,t)$ homeomorphism $\forall t \in [0,1)$) . Let $\gamma_x: I \to X$; $t \to \phi(x,t)$. Let F be a sheaf on X with $F|\gamma_x([0,1))$ a constant sheaf.
Then $H^i(X,F) = 0 \quad \forall i > 0$.

<u>Proof.</u> Let $U = X-\{p\}$, $U_t = X_t-\{p\}$ and $j: U \to X$ the inclusion map. First we prove the lemma for $F = j_*G$ with G a sheaf on U . We have a spectral sequence $H^p(X,R^q j_*G) \Rightarrow H^{p+q}(U,G)$. But $H^p(X,R^q j_*G) = 0$ $p,q > 0$ because the higher direct images are concentrated at p . By constancy of G along the contraction fibres $H^{p+q}(U,G) \xrightarrow{\sim}$ $\xrightarrow{\sim} \lim_{t\to 1} H^{p+q}(U_t,G) = H^0(X,R^{p+q}j_*G)$ so we must have $H^p(X,j_*G) = 0$ for $p > 0$. Using

$$0 \to H^0_{\{p\}}(F) \to F \to \bar{F} \to 0$$

$$0 \to \bar{F} \to j_*j^*F \to H^1_{\{p\}}(F) \to 0$$

and the fact that $H^0(X,j_*j^*F) \twoheadrightarrow H^0(X,H^1_{\{p\}}(F))$ the general case follows from the special case. □

For the relative de Rham complex one has of course a link with the topology of the situation:

<u>Proposition 2.</u> Let $X \xrightarrow{f} S$ be a contractible Stein standard representative of a germ $(X,x) \to (S,s)$. Assume that $\Omega^\cdot_{X/S}$ is a concentrated complex and that $f|X-f^{-1}(s): X-f^{-1}(S) \to S-\{s\}$ is a submersion. Then there is a short exact sequence of $\mathcal{O}_S$-modules

$$0 \to (\mathbb{R}^i f_*\mathbb{C}_X) \otimes \mathcal{O}_S \to H^i(f_*\Omega^\cdot_{X/S}) \to f_*H^i(\Omega^\cdot_{X/S}) \to 0 .$$

<u>Proof.</u> Look at the spectral sequence $R^p f_*(H^q(\Omega^\cdot_{X/S})) \Rightarrow \mathbb{R}^{p+q} f_*(\Omega^\cdot_{X/S})$ and remark that $H^0(\Omega^\cdot_{X/S}) = f^{-1}\mathcal{O}_S$ and that $H^q(\Omega^\cdot_{X/S})$ is concentrated

on $f^{-1}(s)$. Use that $R^i f_* f^{-1}\mathcal{O}_S = R^i f_* \mathbb{C}_X \otimes \mathcal{O}_S$ (by an easy adaptation of [L] , p. 138) □

§2. The Gauss-Manin system

Let $X \xrightarrow{f} S$ be a standard representative of a hypersurface germ $f: (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$. The Gauss-Manin system $\mathcal{H}_X$ is a certain (complex of) $\mathcal{D}_S$-module(s), describing the behaviour of period integrals over cycles in the f-fibres (see [Ph],[S-S]).
In formula ([S-S],p.646):

$$\mathcal{H}_X = \int^{\bullet} \mathcal{O}_X = \mathbb{R}\, f_*(\Omega^{\bullet}_X[D]) \quad .$$

Here $\Omega^{\bullet}_X[D]$ is a complex of sheaves on X with differential $\underline{d}$

$$\underline{d}(\omega\cdot D^k) = d\omega\cdot D^k - df\wedge\omega\cdot D^{k+1}.$$

On this complex there is an action of t and ∂_t :

$$t\cdot(\omega\cdot D^k) = f\cdot\omega\cdot D^k - k\cdot\omega D^{k-1}$$

$$\partial_t(\omega D^k) = \omega\cdot D^{k+1} \quad .$$

One should think of the symbol $\omega\cdot D^k$ as representing the differential form

$$\operatorname{Res}_{X_t}\left(\frac{k!\,\omega}{(f-t)^{k+1}}\right)$$

on the Milnor fibre X_t . One can consider the complex $(\Omega^{\bullet}_X[D],\underline{d})$ as the associated single complex of the double complex $(K^{\bullet\bullet};d,-df\wedge)$ with $K^{pq} = \Omega^{p+q}_X$ for $q \geq 0$, $K^{pq} = 0$ for $q < 0$. This complex carries a so called "Hodge filtration", obtained by cutting off vertically.
In formula:

$$F^p\Omega^k_X[D] := \bigoplus_{k-(p+1)\geq \ell} \Omega^k_X\cdot D^{\ell} \quad .$$

This filtration gives rise to a spectral sequence.

Question. Under what conditions does this spectral sequence degenerate at E_2 ? (i.e. $d_i = 0 \quad i \geq 2$).

Is this true for concentrated singularities in the sense of §1?

<u>Remark.</u> For $f = y^4+xy^2z^2 + z^4$ it does not degenerate at E_2 .

We introduce some notation: Put $\Omega^{\bullet} = \Omega^{\bullet}_X$.

$$S^{\bullet} := \ker(df\wedge : \Omega^{\bullet} \to \Omega^{\bullet+1})$$

$$C^{\bullet} := df \wedge \Omega^{\bullet-1}$$

$$H^{\bullet} := S^{\bullet}/C^{\bullet} \quad \text{(the Koszul cohomology)}$$

$$\Omega^{\bullet}_f := \Omega^{\bullet}/C^{\bullet} \quad \text{(the relative de Rham complex)}.$$

The relations between these complexes, which carry all a differential induced and denoted by d , are summarized in the following diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & C^{\bullet} & \simeq & C^{\bullet} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & S^{\bullet} & \to & \Omega^{\bullet} & \xrightarrow{df\wedge} & C^{\bullet}[1] & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^{\bullet} & \to & \Omega^{\bullet}_f & \xrightarrow{df\wedge} & C^{\bullet}[1] & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Now the E_2-term of the spectral sequence of the Hodge filtration on $(K^{\bullet\bullet};d,-df\wedge)$ can be written as:

$$E_2^{pq} = \begin{cases} 0 & \text{if } q < 0 \\ H^p(S^{\bullet}) & \text{if } q = 0 \\ H^{p+q}(H^{\bullet}) & \text{if } q > 0 . \end{cases}$$

(Here we abbreviate $H^p(f_*S^{\bullet})$ to $H^p(S^{\bullet})$ etc.)
Thus we get a collection of maps $d_2 \colon H^p(H^{\bullet}) \to H^{p+1}(S^{\bullet})$ $p=0,\dots,n+1$.

Due to the peculiar shape of the complex $(K^{\bullet\bullet};d,-df\wedge)$ we have

<u>Lemma 2.</u> If $d_2 \colon H^p(H^{\bullet}) \to H^{p+1}(S^{\bullet})$ $p=1,\dots,n$ is the zero map, then the spectral sequence degenerates, i.e. $E_2 = E_{\infty}$.

Proof. A form $w \in \Omega^p$ represents a class in $H^p(\mathcal{H}^\cdot)$ iff $df \wedge w = 0$ and $dw = df \wedge w_1$ for a certain $w_1 \in \Omega^p$. Then $d_2[w]$ is represented by dw_1, considered as an element in $H^{p+1}(\mathcal{S}^\cdot)$. This element represents zero iff $dw_1 = d\eta$ with $df \wedge \eta = 0$ for a certain $\eta \in \Omega^p$. This means that we can change w_1 to $\tilde{w}_1 = w_1 - \eta$, which is closed. So we have: $d_2[w] = 0$ means: If $df \wedge w = 0$ and $dw = df \wedge w_1$, then we can choose w_1 closed.

Now suppose we have a form w representing a cycle for the differential d_r. This means that we can find $w_1, \ldots, w_r$ such that $df \wedge w = 0$ and $dw = df \wedge w_1$, $dw_k = df \wedge w_{k+1}$ $k=1,\ldots,r-1$ but already $dw = df \wedge w_1$ implies that we can choose w_1 closed, so we can take $w_k = 0$ $k=2,\ldots r$. Hence $d_{r+1}[w] = [dw_r] = 0$. □

Remark. $H^0(\mathcal{H}^\cdot) = H^{n+2}(\mathcal{S}^\cdot) = 0$, so the map is only interesting for $p = 1,\ldots,n$.

We will now give an alternative description of the d_2-map. Look at the long exact cohomology sequences

$$\ldots \to H^p(\mathcal{C}^\cdot) \to H^p(\mathcal{S}^\cdot) \to H^p(\mathcal{H}) \to \ldots\ldots$$

$$\ldots \to H^p(\mathcal{S}^\cdot) \to H^p(\Omega^\cdot) \to H^{p+1}(\mathcal{C}^\cdot) \to \ldots\ldots$$

coming from the diagram. If $p \geq 1$, then $H^p(\Omega^\cdot) = 0$, so we get an isomorphism $H^p(\mathcal{C}^\cdot) \overset{\sim}{\to} H^p(\mathcal{S}^\cdot)$ $(p \geq 2)$. We call this isomorphism ∂_t. If an element of $H^p(\mathcal{C}^\cdot)$ is reppresented by $df \wedge \eta$, $\eta \in \Omega^{p+1}$ then $\partial_t([df\wedge\eta]) = [d\eta]$.

We can eliminate $H^p(\mathcal{C}^\cdot)$ from the first long exact sequence using this isomorphism. So we get:

$$\begin{array}{ccccccc} \ldots \to H^p(\mathcal{H}^\cdot) & \to & H^{p+1}(\mathcal{C}^\cdot) & \overset{j}{\to} & H^{p+1}(\mathcal{S}^\cdot) & \to & \ldots\ldots \\ & \underset{\alpha}{\searrow} & \downarrow \partial_t & \nearrow \partial_t^{-1} & & & \\ & & H^{p+1}(\mathcal{S}^\cdot) & & & & \end{array} \qquad (p \geq 1)$$

Claim. $\alpha = d_2$.

Proof. The map $H^p(\mathcal{H}^\cdot) \to H^{p+1}(\mathcal{C}^\cdot)$ can be described as follows: If w represents a class in $H^p(\mathcal{H}^\cdot)$ then $df \wedge w = 0$ and there is an w_1 such that $dw = df \wedge w_1$. The image in $H^{p+1}(\mathcal{C}^\cdot)$ is then just $[dw] = [df\wedge w_1]$. Applying ∂_t to this element gives $[dw_1]$, so $\alpha([w]) = d_2([w])$. □

The map j above is induced by the inclusion $C^\bullet \subset S^\bullet$ and although the induced map $H^{p+1}(S^\bullet) \to H^{p+1}(S^\bullet)$ is not really the inverse of ∂_t, we denote it by ∂_t^{-1}. One has $\partial_t^{-1} \circ \partial_t = j$. Observe that j is $\mathcal{O}_S$-linear whereas ∂_t is a derivation over j.
Similarly we have an exact sequence and isomorphism involving $H^p(\Omega_f^\bullet)$:

$$\begin{array}{ccccccc} \cdots \to H^p(H^\bullet) & \to & H^p(\Omega_f^\bullet) & \to & H^{p+1}(C^\bullet) & \to & H^{p+1}(H^\bullet) \\ & & {\scriptstyle \partial_t^{-1}} \searrow & & \uparrow {\scriptstyle \partial_t} \wr & \nearrow & \\ & & & & H^p(\Omega_f^\bullet) & & \end{array}$$

In this diagram ∂_t is represented as follows: A class in $H^p(\Omega_f^\bullet)$ is represented by $w \in \Omega^p$ such that $dw = df \wedge \eta$. Then $\partial_t([w]) = [df \wedge \eta]$. As we have isomorphisms of the maps

$$\begin{array}{ccccc} H^p(\Omega_f^\bullet) & \overset{\approx}{\to} & H^{p+1}(C^\bullet) & \overset{\sim}{\to} & H^{p+1}(S^\bullet) \\ \downarrow \partial_t^{-1} & & \downarrow \partial_t^{-1} & & \downarrow \partial_t^{-1} \\ H^p(\Omega_f^\bullet) & \overset{\sim}{\to} & H^{p+1}(C^\bullet) & \overset{\sim}{\to} & H^{p+1}(S^\bullet) \end{array}$$

(where the horizontal maps are all called ∂_t) we get:

<u>Corollary</u>. Equivalent are

1) $d_2 : H^p(H) \to H^{p+1}(S^\bullet)$ is the zero map
2) $\partial_t^{-1} : H^p(\Omega_f^\bullet) \circlearrowleft$, $H^{p+1}(C^\bullet) \circlearrowleft$ or $H^{p+1}(S^\bullet) \circlearrowleft$ is injective
3) $H^p(\Omega_f^\bullet) \overset{j}{\to} H^{p+1}(C^\bullet)$ or $H^{p+1}(C^\bullet) \overset{j}{\to} H^{p+1}(S^\bullet)$ is injective.

Now, philosophically at least, the operator ∂_t^{-1} should be similar to multiplication by t. Injectivity of ∂_t^{-1} should learn about injectivity of t, i.e. torsion freeness of $H^p(\Omega_f^\bullet)$ as an $\mathcal{O}_S$-module. The modules $H^p(\Omega_f^\bullet)$, $H^{p+1}(C^\bullet)$ and $H^{p+1}(S^\bullet)$ are analoguous to the modules of Brieskorn [B] H, H' and H'' respectively: on $S-\{s\}$ they are locally free of rank $b_p(F)$, the p-th Betti number of the Milnor fibre $F = f^{-1}(t)$, $t \neq s$. The isomorphism on $S-\{s\}$ is given by the map $j|S-\{s\}$, so $\ker j$ and $\operatorname{cok} j$ are both modules supported on the point $\{s\}$. Further we have isomorphisms $H^p(\Omega_f^\bullet) \overset{\partial_t}{\to} H^{p+1}(S^\bullet)$ and $H^{p+1}(C^\bullet) \overset{\partial_t}{\to} H^{p+1}(S^\bullet)$. The relation $\partial_t \cdot t - t\partial_t = j$ is easily seen to hold. We repeat Malgrange's proof of the Sebastiani theorem (see [Ma], p.416): the torsion freeness of the Brieskorn module $H'' = H^{n+1}(S^\bullet)$ in the case of an isolated singularity.

Theorem 2. Assume that $H^p(\Omega_f^\cdot)$, $H^{p+1}(C^\cdot)$ and $H^{p+1}(S^\cdot)$ are coherent $\mathcal{O}_S$-modules. If $d_2: H^p(H^\cdot) \to H^{p+1}(S^\cdot)$ is the zero map, then $H^p(\Omega_f^\cdot)$, $H^{p+1}(C^\cdot)$ and $H^{p+1}(S^\cdot)$ are torsion free.

Proof. Put $E = H^{p+1}(C^\cdot)$, $F = H^{p+1}(S^\cdot)$. We have an isomorphism $E \overset{\partial_t}{\to} F$ and if $d_2 = 0$ an $\mathcal{O}_S$-linear injection $E \overset{j}{\to} F$ with $F/j(E)$ $\mathcal{O}_S$-torsion, i.e. we have an (E,F)-connection in the sense of Malgrange.
We derive a contradiction by assuming Torsion (F) $\neq 0$. So let $t\cdot\omega = 0$, $0 \neq \omega \in F$. By $E \overset{\partial_t}{\underset{\sim}{\to}} F$ we find an $\eta \in E$ such that $\partial_t\eta = \omega$. Now $t^k\eta \neq 0$ $\forall k$, because if $t^k\eta = 0$, with k smallest as possible, then $0 = \partial_t \; t^k\eta = k\cdot t^{k-1}\cdot j\cdot\eta + t^k\partial_t\eta = k\cdot t^{k-1}\cdot j\cdot\eta$. By injectivity of j it follows that $t^{k-1}\eta = 0$, so contradiction. By coherence of E as $\mathcal{O}_S$-module it follows that $\eta|S-\{s\} \neq 0$, but $\partial_t\eta|S-\{s\} = 0$. But now we use the link with the topology, by integrating η over a horizontal family of vanishing cycles $\gamma(t)$, $t \in [0,1]$. One has

$$0 = \int_{\gamma(t)} \partial_t\eta = \frac{d}{dt}\int_{\gamma(t)} \eta$$

so the period $t \to \int_{\gamma(t)}\eta$ is constant. Because η is holomorphic on the whole of X , and has closed restriction to the f-fibres, we know however that this integral has to go to zero. (Here one has to use an extension of Lemma 4.5 of [Ma] to the case of p-forms, which can be proved quite in the same way). Hence $\int_{\gamma(t)}\eta = 0$ $t \in [0,1]$. As this is true for every horizontal family of cycles we conclude that η represents the zero form. Contradiction, hence torsion (F) = 0 .
The rest of the proof is obtained by remarking that via the $\mathcal{O}_S$-linear map j $H^p(\Omega_f^\cdot)$ and $H^{p+1}(C^\cdot)$ are submodules of $H^{p+1}(S)$. □

Remark. The proof of the theorem shows that one really needs coherence modulo torsion of the module $H^{p+1}(C^\cdot)$, which follows from the results of Hamm [H]. In order to keep this paper as selfcontained as possible, we prefer to use the direct coherence theorem of §1 for the singularities we are interested in.

There is an obvious kind of converse to Theorem 2.

Proposition 3. Assume $H^p(H^\cdot)$ coherent. Then if $H^{p+1}(S^\cdot)$ is torsion free, then $d_2: H^p(H^\cdot) \to H^{p+1}(S^\cdot)$ is the zero map.

Proof. $H^p(H^\cdot)$ is an $\mathcal{O}_S$-module concentrated at s. By coherence, it is torsion. Hence the $\mathcal{O}_S$-linear map d_2 has to be zero. □

Of course, if one knows that $H^p(\Omega_f^\cdot)$ is a torsion free $\mathcal{O}_S$-module, then one gets relatively nice formulae for the Betti numbers of the Milnor fibre.

For a concentrated singularity one has the exact sequence of Proposition 2, §1:

$$0 \to R^i f_* \mathbb{C}_X \otimes \mathcal{O}_S \to H^i(f_* \Omega_f^\cdot) \to f_* H^i(\Omega_f^\cdot) \to 0 .$$

The first sheaf has stalk 0 at s and $\mathbb{C}^{b_i} \otimes \mathcal{O}_{S,t}$ at $t \neq s$ where $b_i = b_i(F)$ is the i-th Betti number of the Milnor fibre. The second sheaf is $\mathcal{O}_S$-coherent with stalk $f_* H^i(\Omega_f^\cdot) = H^i(\Omega_{f,x}^\cdot)$ at s. If we know that t acts injectively one thus finds.

$$b_i(F) = \dim_{\mathbb{C}} H^i(\Omega_{f,x}^\cdot)/t \cdot H^i(\Omega_{f,x}^\cdot) .$$

By Malgranges index theorem ([Ma],p.408) this number is also equal to $\dim_{\mathbb{C}} H^i(\Omega_{f,x}^\cdot)/\partial_t^{-1} H^i(\Omega_{f,x}^\cdot) = \dim_{\mathbb{C}} H^{i+1}(H_x^\cdot)$.

Conclusion. For a concentrated singularity where

$$\partial_t^{-1}: H^i(\Omega_{f,x}^\cdot) \hookrightarrow H^i(\Omega_{f,x}^\cdot)$$

we have: $b_i(F) = \dim_{\mathbb{C}} H^{i+1}(H_x^\cdot) \quad (i > 0)$.

§3. A special class of singularities

We now specialize our situation to the case of a hypersurface germ $f: (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ with a one dimensional singular locus. This is the simplest situation where the map d_2 of §2 can be nontrivial. (In the sequel a fixed appropriate contractible Stein representative $X \xrightarrow{f} S$ is understood).

We will give the singular locus the non reduced structure defined by the jacobi ideal $J_f = (\partial_0 f, \ldots, \partial_n f)$ and denote it by $\tilde{\Sigma}$. So we put $\mathcal{O}_{\tilde{\Sigma}} = \mathcal{O}/J_f$, where $\mathcal{O} = \mathcal{O}_X$. We also will consider the curve Σ, defined by the ideal I , which is obtained from J_f by removing the M-primary component. In other words, Σ is the largest Cohen-Macaulay curve contained in $\tilde{\Sigma}$. Thus we have an exact sequence:

$$0 \to I/J_f \to \mathcal{O}_{\tilde{\Sigma}} \to \mathcal{O}_{\Sigma} \to 0$$

where I/J_f is an M-primary $\mathcal{O}$-module. In [Pe] modules like I/J_f have been studied and they are called "jacobi-modules".

In order to study the map d_2 we first need a description of the Koszul cohomology groups H^i . One easily sees (use for instance the "Lemme d'Acyclicité, see [P-S]) that the Koszul complex on the generators $\partial_i f$, $i=0,1,\ldots n$, acting on $\mathcal{O}$, is exact except possibly in degrees 0 and 1 .
One has (where $H_i(\mathcal{O};\partial_0 f,\ldots,\partial_n f)$ denotes Koszul homology)

$$\mathcal{O}/I_f = H_0(\mathcal{O};\partial_0 f,\ldots,\partial_n f) \simeq H^{n+1} = \Omega^{n+1}/df\wedge\Omega^n$$

$$H_1(\mathcal{O};\partial_0 f,\ldots,\partial_n f) \simeq H^n = \ker(df\wedge:\Omega^n \to \Omega^{n+1})/df\wedge\Omega^{n-1}$$

$$H_i(\mathcal{O};\partial_0 f,\ldots,\partial_n f) = 0 \qquad i \geq 2 .$$

Note that H^n and H^{n+1} are $\mathcal{O}_{\tilde{\Sigma}}$-modules. The funny thing about H^n is, that although it is defined in terms of the function f , its structure as a module is only dependent on the singular locus Σ . This is always the case with the first non vanishing Koszul cohomology group. It turns out that this cohomology group as a module is always isomorphic to the dualizing module ω_Σ of the singular locus. For our purpose it is important to have an explicit isomorphism between H^n and ω_Σ . The description of this isomorphism is due to R. Pellikaan [Pe], and can be formulated as follows:

We consider the following diagram:

$$\begin{array}{ccccccccccccc}
0 & & & & & & & & & & & & \\
\uparrow & & & & & & & & & & & & \\
\mathcal{O}_\Sigma & \leftarrow & \mathcal{O} & \leftarrow & \mathcal{O}^{d_1} & \leftarrow & \mathcal{O}^{d_2} & \leftarrow\cdots & \mathcal{O}^{d_n} & \leftarrow & 0 & & \\
\uparrow & & & & \phi_1\uparrow & & \phi_2\uparrow & & \phi_n\uparrow & & & & \\
\mathcal{O}_{\tilde{\Sigma}} & \leftarrow & \mathcal{O} & \leftarrow & \Theta & \leftarrow & \Lambda^2\Theta & \leftarrow\cdots & \Lambda^n\Theta & \leftarrow & \Lambda^{n+1}\Theta & \leftarrow & 0 \\
\uparrow & & & & & & & & & & & & \\
I/J_f & & & & & & & & & & & & \\
\uparrow & & & & & & & & & & & & \\
0 & & & & & & & & & & & &
\end{array} \qquad (*)$$

In the top row we put the minimal resolution of $\mathcal{O}_\Sigma$ as an $\mathcal{O}$-module. The bottom row is a natural incarnation of the Koszul complex on the generators $\partial_i f$ $i=0,\ldots,n$; Θ is the module of tangent vectors and $\Theta \to \mathcal{O}$ is the map $\Sigma a_i\partial_i \to \Sigma a_i\partial_i f$. The vertical maps ϕ_i are induced from ϕ_1 , which expresses the fact that $J_f \subset I$. Dualizing this diagram with respect to $\mathcal{O}$ and taking homology produces a map

$$[\phi_n^T]: \operatorname{Ext}^n_{\mathcal{O}}(\mathcal{O}_\Sigma,\mathcal{O}) \to H^n .$$

<u>Theorem.</u> (R. Pellikaan [Pe],p.152)

$[\phi_n^T]$ is an isomorphism. □

So the choice of a volume form $\Omega \in \Omega^{n+1}$ will give a natural map $\omega_\Sigma \to H^n$.

We now restrict to an even more special situation: From now on we assume that Σ is a *reduced complete intersection curve.*

This is precisely the class of singularities studied by Siersma from a topological and by Pellikaan from an algebraic point of view.

Reducedness of Σ is equivalent to the condition that the function f defines a singularity which around a point $p \in \Sigma - 0$ is right equivalent to $f(x_0,\ldots,x_n) = \Sigma_{i=1}^n x_i^2$ ("generically transversal A_1 ").

If Σ is a complete intersection curve, we can write $I = (g_1,\ldots,g_n)$. From the reducedness it now follows that $f \in I^2$, so we can write $f = \frac{1}{2}\Sigma h_{ij}g_i g_j$. The function $h := \det(h_{ij})$, which is called the transversal Hessian, is non-zero on a generic point of Σ (for these facts, see [Pe]).

As Σ is a complete intersection, defined by $g_1,\ldots,g_n$, we can resolve $\mathcal{O}_\Sigma$ by the Koszul complex. This implies that in diagram (*) we can take $\phi_i = \Lambda^i\phi_1$. Using Pellikaans theorem we can write down a generator for H^n as $\mathcal{O}_\Sigma$-module as $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n$, where we put $df = \Sigma\omega_i g_i$ with $\omega_i \in \Omega^1_\Sigma$. So $H^n = \mathcal{O}_\Sigma$ $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n$.
(In concrete terms: Write $\partial_i f = \Sigma A_{ij} g_j$ with A_{ij} a $n \times (n+1)$-matrix. Then $\omega_1 \wedge \ldots \wedge \omega_n = \Sigma\, \Delta_i d\hat{x}_i$ with $\Delta_i = (-1)^i$. i-th $n{\times}n$ minor of (A_{ij}) , and $dx_i \wedge d\hat{x}_i = dx_0 \wedge \ldots \wedge dx_n$).

In order to study the map $d: H^n \to H^{n+1}$ we first project $H^{n+1} = \mathcal{O}_{\tilde{\Sigma}} \otimes \Omega^{n+1}$ to $\mathcal{O}_\Sigma \otimes \Omega^{n+1} = \Omega^{n+1}/I\cdot\Omega^{n+1}$ and study the composed map $\underline{d}: H^n \to \Omega^{n+1}/I\cdot\Omega^{n+1}$. The first step is to compute $\omega_1 \wedge \ldots \wedge \omega_n$ and $d(\omega_1 \wedge \ldots \wedge \omega_n)$ mod I .

<u>Proposition 4.</u> With the notation as above we have:

a. $\omega_1 \wedge \ldots \wedge \omega_n = h\cdot dg_1 \wedge \ldots \wedge dg_n \mod I\cdot\Omega^n$

b. $d(\omega_1 \wedge \ldots \wedge \omega_n) = \tfrac{1}{2}dh \wedge dg_1 \wedge \ldots \wedge dg_n \mod I\cdot\Omega^{n+1}$.

<u>Proof.</u> Write $f = \tfrac{1}{2}\Sigma h_{ij} g_i \cdot g_j$. Then we have

$$df = \sum_{i,j} (h_{ij}dg_j + \tfrac{1}{2}dh_{ij}g_j)g_i$$

so we can take

$$\omega_i = \sum_j (h_{ij}dg_j + \tfrac{1}{2}dh_{ij}\cdot g_j) \ .$$

Hence

$$\omega_i = \sum_j h_{ij}dg_j \mod I\cdot\Omega^1$$

$$d\omega_i = \sum dh_{ij}\wedge dg_j - \tfrac{1}{2}dh_{ij}\wedge dg_j = \tfrac{1}{2} \sum_j dh_{ij}\wedge dg_j \ .$$

So $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n = \det(h_{ij})dg_1 \wedge \ldots \wedge dg_n \mod I\cdot\Omega^n$ and

$$d(\omega_1 \wedge \ldots \wedge \omega_n) = \sum_i (-1)^i\ \omega_1 \wedge \ldots \wedge d\omega_i \wedge \ldots \wedge \omega_n =$$

$$= \sum_i (-1)^i\ (\Sigma\ h_{1j}dg_j) \wedge \ldots \wedge (\Sigma\ \tfrac{1}{2}dh_{ij}\wedge dg_j) \wedge \ldots$$

$$\ldots \wedge (\Sigma\ h_{nj}dg_j) \mod I\cdot\Omega^{n+1} = \tfrac{1}{2}dh \wedge dg_1 \wedge \ldots \wedge dg_n \ .$$ □

Using this proposition, we can compute $\underline{d}$:

$$\underline{d}(P\omega_1 \wedge \ldots \wedge \omega_n) = dP \wedge \omega_1 \wedge \ldots \wedge \omega_n + Pd(\omega_1 \wedge \ldots \wedge \omega_n)$$

$$= hdP \wedge dg_1 \wedge \ldots \wedge dg_n + \tfrac{1}{2}P\cdot dh \wedge dg_1 \wedge \ldots \wedge dg_n \ .$$

Introducing the vectorfield θ , dual to $dg_1 \wedge \ldots \wedge dg_n$ (i.e.: $i_\theta(dx_0\wedge\ldots\wedge dx_n) = dg_1 \wedge \ldots \wedge dg_n$ where i_θ is the contraction operator) we can interpret $\underline{d}$ as a map $D: \mathcal{O}_\Sigma \to \mathcal{O}_\Sigma$; $P \to D(P) =$ $= h\cdot\theta(P) + \tfrac{1}{2}\theta(h)\cdot P$ making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{H}^n & \xrightarrow{\ \underline{d}\ } & \Omega^{n+1}/I\Omega^{n+1} \\ \uparrow & & \uparrow \\ \mathcal{O}_\Sigma & \xrightarrow{\ D\ } & \mathcal{O}_\Sigma \end{array} \ .$$

Here $\mathcal{O}_\Sigma \xrightarrow{\sim} H^n$ is given by $P \to P\cdot\omega_1 \wedge \ldots \wedge \omega_n$ and $\mathcal{O}_\Sigma \xrightarrow{\sim} \Omega^{n+1}/I\Omega^{n+1}$ by $P \to P\cdot dx_0 \wedge \ldots \wedge dx_n$.
The vectorfield θ is tangent to Σ and non-zero on $\Sigma - \{0\}$.

Now we can prove:

Theorem 3. Let $f\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ define a singularity which has a one dimensional singular locus, which is a reduced complete intersection. Write $f = \frac{1}{2}\Sigma\, h_{ij} g_i g_j$ with $I = (g_1,\ldots,g_n)$ the ideal of Σ and put $h = \det(h_{ij})$. Then

If h is not a unit then $H^n \xrightarrow{d} H^{n+1}$ is injective

If h is a unit then $H^n \to H^{n+1}$ has a one dimensional kernel, which can be represented by a closed form.

Proof. Let $P\cdot\omega_1 \wedge \ldots \wedge \omega_n \in H^n$ be an element in the kernel of the operator d . Then also $\underline{d}(P\omega_1 \wedge \ldots \wedge \omega_n) = 0$ i.e.: $D(P) = 0$. In the ring $\mathcal{O}_\Sigma[h^{\frac{1}{2}}]$ we can write the operator D as follows:

$$D(P) = h\theta(P) + \tfrac{1}{2}\,\theta(h)P = h^{\frac{1}{2}}\cdot\theta(h^{\frac{1}{2}}\cdot P) \ .$$

Because h is a function that is non-zero on $\Sigma-\{0\}$ we conclude $\theta(h^{\frac{1}{2}}\cdot P) = 0$. Because θ is a vectorfield that is tangent to Σ and non-vanishing on $\Sigma-\{0\}$ it follows that $h^{\frac{1}{2}}\cdot P = C \bmod I\cdot\mathcal{O}_\Sigma[h^{\frac{1}{2}}]$, where C is a constant. If this constant is non-zero, then one must have that h is a unit in $\mathcal{O}_{\Sigma,0}$. If this constant is zero it follows that $P \in I$, i.e. $P\,\omega_1 \wedge \ldots \wedge \omega_n$ represents zero hence $d\colon H^n \to H^{n+1}$ is injective.
If h is a unit, then we can "diagonalize" the matrix h_{ij} by a change of generators for the ideal I from the g_i to $\tilde{g}_i$, achieving the form $f = \frac{1}{2}\,\Sigma\tilde{g}_i^2$ for our function f . (see [S],p.23). But then $df = \Sigma\, d\tilde{g}_i\cdot\tilde{g}_i$, hence the generator of H^n is represented by $d\tilde{g}_1 \wedge \ldots \wedge d\tilde{g}_n$ which is a closed form. It is easy to see that every element in the kernel is a scalar multiple of $d\tilde{g}_1 \wedge \ldots \wedge dg_n$. □

Corollary. Under the hypothesis of theorem 3 and with notations of §2 we have:
1) $H^n(\Omega_f^\bullet)$, $H^{n+1}(C^\bullet)$ and $H^{n+1}(S^\bullet)$ are free $\mathcal{O}_S$-modules of rank $b_n(F)$.
2) $H^{n+1}(\Omega_f^\bullet)$, $H^n(C^\bullet)$ and $H^n(S^\bullet)$ are free $\mathcal{O}_S$-modules of rank $b_{n-1}(F)$.

3) $b_n(F) = \dim_{\mathbb{C}} H^{n+1}(H^\cdot) = \dim_{\mathbb{C}}(\Omega^{n+1}/df \wedge \Omega^n + dH^n)$.

4) $b_{n-1}(F) = 1$ if h is a unit
$\phantom{b_{n-1}(F)} = 0$ if h is not a unit.

The corollary follows by remarking that the complexes $\Omega_f^\cdot$, $C^\cdot$ and $S^\cdot$ are concentrated for these singularities, and the fact that the d_2-map is the zeromap, as follows from the fact that the kernel of $d: H^n \to H^{n+1}$ can be represented by a closed form.

It is interesting to note that $\Omega^{n+1}/df \wedge \Omega^n + dH^n$, which is a vector space of dimension $b_n(F)$, does not have a structure of an $\mathcal{O}_X$-module, as in the case of an isolated singularity.
The proof of Theorem 3 shows a bit more: if h is not a unit then $dH^n \cap I\Omega^{n+1} = 0$. This fact gives an exact sequence

$$0 \to I/J_f \otimes \Omega^{n+1} \to \Omega^{n+1}/df \wedge \Omega^n + dH^n \to \Omega^{n+1}/I\Omega^{n+1} + dH^n \to 0$$

leading to the formula

$$b_n(F) = \dim_{\mathbb{C}}(I/J_f) + \dim_{\mathbb{C}}(\mathcal{O}_\Sigma/D(\mathcal{O}_\Sigma)) .$$

The first part, $\dim(I/J_f)$, is called the jacobi number of f .
Pellikaan has proved a conjecture of Siersma, stating that this number j_f is equal to $\#A_1$-points $+\#D_\infty$-points in a generic approximation of f , making the singular locus into a smooth curve.
The second part, $\dim(\mathcal{O}_\Sigma/D(\mathcal{O}_\Sigma))$ has to be equal to $\mu(\Sigma) + \#D_\infty - 1$, by comparison with Siersma's formula ([S],p.4). We will give an algebraic proof of this fact.

First note the formula of Buchweitz and Greuel for the Milnor number of a curve: $\mu(\Sigma) = \dim(\omega_\Sigma/d\mathcal{O}_\Sigma)$ (see [B-G],p.244). Secondly, the number of D_∞ points in a deformation can be computed as $\dim(\mathcal{O}_\Sigma/h\cdot\mathcal{O}_\Sigma)$ (see [Pe],p.83).
Now assume that $h^{\frac{1}{2}} \in \mathcal{O}_\Sigma$. Then it is easy to see that we can consider $D: \mathcal{O}_\Sigma \to \mathcal{O}_\Sigma$ as the composition of the following four maps

$$\mathcal{O}_\Sigma \xrightarrow{h^{\frac{1}{2}}} \mathcal{O}_\Sigma \xrightarrow{d} \omega_\Sigma \xrightarrow{h^{\frac{1}{2}}} \omega_\Sigma \approx \mathcal{O}_\Sigma$$

where the first and the third maps are multiplications and the last one is the identification of ω_Σ with $\mathcal{O}_\Sigma$ by the generator $[dx_0 \wedge \ldots \wedge dx_n/dg_1 \wedge \ldots \wedge dg_n]$. By additivity of the index we find:

$$\mathrm{Index}(D) = \mathrm{Index}(h^{\frac{1}{2}}) + \mathrm{Index}(d) + \mathrm{Index}(h^{\frac{1}{2}}) .$$

$$\dim(\mathcal{O}_\Sigma/D(\mathcal{O}_\Sigma)) = \dim(\mathcal{O}_\Sigma/h\cdot\mathcal{O}_\Sigma) + \mu(\Sigma)-1 .$$

The proof in the case that $h^{\frac{1}{2}} \notin \mathcal{O}_\Sigma$ is similar.

In the case of a line singularity, i.e. Σ is a smooth curve, one can choose coordinates $(x,y_1,\dots,y_n)$ such that $I = (y_1,\dots,y_n)$ and $h = x^\alpha$. As in this case $\theta = \partial_x$ we get a particularly nice form for the operator: $D = x^{\alpha-1}\cdot(x\partial_x + \frac{\alpha}{2})$.

Concluding remarks and questions.

1) There should be some clear "geometry" in the map $d: H^n \to H^{n+1}$. The expression $D = x^{\alpha-1}(x\partial_x + \frac{\alpha}{2})$ for line singularities suggests that it describes the monodromy of the transversal vanishing cycle by a connection on Σ. However, in general Σ is singular and can have several irreducible components and it is not clear in what sense d is a connection.
2) It is a shame that this theory does not cover the case of $f = x\cdot y\cdot z$; the singular locus is not a complete intersection. Here $b_1(F) = 2$. Is it always true that $b_{n-1}(F) \le$ Gorenstein type (Σ) when Σ is a reduced curve? Numerous examples confirm this guess.
3) There are many other examples of function for which one can verify the degeneration of the spectral sequence. For example for the singularities studied by T. de Jong in [dJ] one can check this often.
4) The vectorbundle $H^{n+1}(f_*S^\cdot)$ sitting in the Gauss-Manin system does not seem to play the same rôle as in the isolated singularities case in the sense of characteristic exponents. We will study this in in a later paper.

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