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Komplexe Algebraische Geometrie

Organised by
Fabrizio Catanese, Bayreuth
Yujiro Kawamata, Tokyo
Gang Tian, Princeton
Eckart Viehweg, Essen

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Abstracts

On the Quantisation of Completely Integrable Hamiltonian Systems

Duco van Straten
(joint work with Mauricio Garay)

Classical mechanics is described by a hamiltonian function that induces a flow in a phase space. The mathematical model is that of a symplectic manifold M, where the symplectic form ω defines an identification ϕ between the cotangent bundle Ω_M and the tangent bundle Θ_M; a function H on M defines a flow by integrating the hamiltonian vector field $\phi(dH)$, [1].

We consider the case $M = \mathbb{C}^{2n}$ with canonical coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ such that $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. The dynamics is described by the Hamilton equations

$$\dot{p_i} = -\partial H/\partial q_i, \;\; \dot{q_i} = \partial H/\partial p_i$$

where the hamiltonian H is a function of the $2n$ coordinates (p, q). The time derivative of an arbitrary function is then given by $\dot{F} = \{H, F\}$, where

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}$$

is the *Poisson-bracket* of F and G. F is called a *conserved quantity* if $\dot{F} = 0$, or, what is the same F *Poisson commutes* with H, $\{F, H\} = 0$.

In general we call $I_1, I_2, \ldots, I_n \in R := \mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n]$ which are functionally independent and with $\{I_i, I_j\} = 0$ for all i, j a *(polynomial classical) integrable system*. Although they are rare and hard to construct, several examples are known, like the tops of Euler, Lagrange, Kovalevskaya; special cases of the Henon-Heiles system, the Calogero-Moser systems, to mention a few. In many cases the fibres of the map $I := (I_1, \ldots, I_n) : \mathbb{C}^{2n} \longrightarrow \mathbb{C}^n$ are affine pieces of abelian varieties, see [6] for an overview. In algebraic geometry one encounteres the integrable Hitchin system, the systems of Beauville-Mukai, which the global situation of Lagrangian fibrations on hyperkähler manifolds.

In their 1925 paper [2], Born and Jordan realised that quantum mechanics is a *non-commutative deformation of classical mechanics*: the ring $R = \mathbb{C}[p, q]$ is replaced by the non-commutative Heisenberg algebra $Q := \mathbb{C} < \hbar, p, q >$ with the relation

$$pq - qp = \hbar, \hbar := \frac{h}{2\pi i}, \quad (h = 6.10^{-34} Js)$$

$\hbar$ should be considered as a central element, and classical mechanics is recovered by putting $\hbar = 0$. Indeed, one can consider R as a quotient of Q: $Q/\hbar Q = R$. It was observed by Dirac, that the Poisson-bracket is recovered from the commutator via

$$\{f,g\} := \frac{1}{\hbar}[F,G] \mod \hbar Q$$

Question: *Given a integrable system $I_1,\ldots,I_n \in R$, do there exist $J_1,\ldots,J_n \in Q$ such that $[J_i,J_j]=0$ and $J_i = I_i \mod \hbar$?*

If we can find such commuting $J_1,\ldots,J_n$, we will say the system is *quantum completely integrable*. We have no general answer to this question, but for many integrable systems explicit quantisations are known. The quantisation of the Hitchin system plays a central role in the *geometric Langlands program* [3].

It is natural to work order by order in $\hbar$ and put $Q_k := Q/\hbar^k Q$ and replace Q by the completion $\hat{Q} = \lim_{\leftarrow k} Q_k$. We consider the polynomial ring $A = \mathbb{C}[I_1,\ldots,I_n] \stackrel{\iota_1}{\hookrightarrow} Q_1 = R$ which we try to lift ι_1 order by order to $A \stackrel{\iota_2}{\hookrightarrow} Q_2, \ldots, A \stackrel{\iota_k}{\hookrightarrow} Q_k$. The Poisson-commutativity of the I_i is equivalent to the liftability of ι_1 to ι_2.

Let $\Theta_A := Der(A,A) = \bigoplus_{i=1} A\frac{\partial}{\partial I_i}$ and put $C^p := R \otimes_A \wedge^p \Theta_A$. We have n commuting derivations $f \mapsto \{I_i, f\}$ of R, which combine to define a differential

$$\delta : C^p \longrightarrow C^{p+1}, \quad fw \mapsto \sum_{i=1}^{n} \{f, I_i\} \frac{\partial}{\partial I_i} \wedge w$$

Proposition [5]**:** Consider $\iota_k : A \longrightarrow Q_k$ and a lifting to $\iota_{k+1} : A \longrightarrow Q_{k+1}$. Then there exists a well-defined obstruction element

$$\Xi = \Xi(\iota_k) \in H^2(C^\bullet, \delta).$$

with the following property: ι_k can be lifted to $\iota_{k+2} : A \longrightarrow Q_{k+2}$ by changing the lift ι_{k+1} if and only if $\Xi(\iota_k) = 0$.

We put $X = Spec(R) = \mathbb{C}^{2n}$, $S = Spec(A) = \mathbb{C}^n$ and let $I : X \longrightarrow S$ the corresponding map. There is a discriminant set $\Sigma \subset S$, such that the pull-back $I' : X' \longrightarrow S' := S \setminus \Sigma$ is smooth and for $s \in S'$ the fibre X_s is a smooth Lagrangian subvariety of X. The complex $(C^\bullet, \delta)$ can be sheafied to a sheaf complex $\mathcal{C}^\bullet$ on X.

Proposition [5]**:** There is a natural map of complexes

$$\rho : (\Omega^\bullet_{X/S}, d) \longrightarrow (\mathcal{C}^\bullet, \delta)$$

which is an isomorphism on X'.

As a consequence, the obstruction class Ξ induces for $s \in S'$ an element

$$\Xi_s \in H^2(\Omega_{X_s}) = H^2(X_s, \mathbb{C})$$

If one makes reasonable assumptions on the structure of the singularities, one can show coherence of the cohomology, using the classical Kiehl-Verdier approach:

Theorem [4]**:** If $I : X \longrightarrow S$ is *pyramidal*, then $H^i(\mathcal{C}^\bullet, \delta)$ are $\mathcal{O}_S$-coherent.

Corollary: If $H^2(\mathcal{C}^\bullet, \delta)$ is torsion free, then the obstruction Ξ is zero if and only if $\Xi_s = 0$ for generic $s \in S'$.

In fact, the modules H^i are in fact free modules in all examples we calculated.

The classical Darboux-Givental'-Weinstein theorem says that in the C^∞ context, a neighbourhood of a Lagrange submanifold L is symplectomorphic to a neighbourhood in the cotangent bundle T^*L. The same is true in our situation for $L = X_s \subset X$, because L is a Stein space. As a consequence of the rigidity of of the Poisson structure, it seems one can construct a formal quantisation on a formal generic fibre. This *Quantum Darboux theorem* would imply the vanisishing of Ξ_s for s generic. One would obtain the following corollary: If $I : X \longrightarrow S$ is pyramidal and $H^2(C^\bullet, \delta)$ is torsion free, then there I lifts to a formal quantum integrable system: we find $J_i \in \hat{Q}$, $[J_1, J_j] = 0$ and $J_i = I_i \mod \hbar$.

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Reporter: Sönke Rollenske