

## ONE-FORMS ON SINGULAR CURVES AND THE TOPOLOGY OF REAL CURVE SINGULARITIES

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### INTRODUCTION

LET  $f: \mathbb{R}^n, 0 \rightarrow \mathbb{R}^{n-1}, 0$  be a real analytic map germ, with  $f^{-1}(0)$  a reduced curve germ. In a recent paper [1], Aoki, Fukuda and Nishimura produced a remarkable algebraic method for computing the number branches of this curve. Their method is, briefly, to associate to  $f$  a map-germ  $F: \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0$  whose topological degree is equal to the number of branches of  $f^{-1}(0)$ , and then to use the Eisenbud–Levine theorem [7] to calculate the degree of  $F$  as the signature of a quadratic form on the local algebra of  $F$ . (We describe it in more detail in §2.)

The aim of this paper is to generalize the method of Aoki *et al.* to apply to the case where the curve is not a complete intersection. In the case of a complete intersection, local duality comes into play in the use of the Eisenbud–Levine theorem on  $F$ . However, since there is no such map  $F$  in the general case, we were led to use local duality and residues on the curve. As usual, the more general setting clarifies the special one.

Given any meromorphic form  $\alpha$  on a curve  $\mathcal{C}$ , we use the module of *Rosenlicht differentials*  $\omega_{\mathcal{C}}$  of the curve to define two “ramification modules” which measure in some sense the zeros and poles of the form respectively. These modules are finite dimensional vector spaces, and we prove in §1 that the difference in dimension is preserved under deformation of both the form and the curve. In the case that  $\alpha = dg$  for some holomorphic function  $g$ , this enables us to find the number of critical points of a small generic deformation of  $g$ . Further, we give a simple proof of the fact that the jump in Milnor number (as defined in [3] and [6]) in a flat family of curve singularities is equal to the vanishing Euler characteristic.

In the case that  $\mathcal{C}$  and  $\alpha$  are *real*, the 1-form defines an orientation on each connected component of  $\mathcal{C} - \{p\}$  (= half-branch), where  $p$  is the base point of  $\mathcal{C}$ . Some of these half-branches will be oriented outwards and some inwards. Moreover, the two ramification modules come with real valued non-degenerate quadratic forms. We show in §2 that the sum of signatures of these two forms is equal to the difference between the numbers of branches oriented outwards and those oriented inwards. This is related to the classical method of Hermite for calculating the number of real roots of a polynomial as the signature of a quadratic form (see [12]). We remark that it seems surprising that the two important features of these ramification modules are the *difference* of the dimensions, but the *sum* of the signatures.

### §1. THE RAMIFICATION MODULES OF A ONE-FORM

Usually,  $\mathcal{C}$  will be a germ of a reduced analytic curve with base point  $p$  and local ring  $\mathcal{O}_{\mathcal{C}}$ , in this section defined over  $\mathbb{C}$  but in §2 defined over  $\mathbb{R}$ . The normalization of the curve  $\mathcal{C}$  will

be denoted by  $n: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ . We will be interested in three basic local invariants of curve singularities. First,  $r$  is the number of irreducible components of  $\mathcal{C}$ , or what is the same, the number of points in  $n^{-1}(p) \subset \tilde{\mathcal{C}}$ . Second,  $\delta = \dim_{\mathbb{C}}(\mathcal{O}_{\tilde{\mathcal{C}}}/\mathcal{O}_{\mathcal{C}})$ , which can be interpreted heuristically as the number of double points concentrated at  $p$ . Third is the Milnor number  $\mu$ , which is equal to  $2\delta - r + 1$ . We extend  $\delta$  and  $\mu$  additively to multigerms and to global curves (with finitely many singular points), which means that the relation becomes  $\mu = 2\delta - \Sigma(r - 1)$ , where the summation runs over the singular points of  $\mathcal{C}$ .

Let  $\Omega_{\mathcal{C}}$  be the  $\mathcal{O}_{\mathcal{C}}$ -module of Kähler one-forms on  $\mathcal{C}$  and let  $\Omega_{\mathcal{C}}(*) = \mathcal{O}_{\mathcal{C}}(*) \otimes \Omega_{\mathcal{C}}$  be the  $\mathcal{O}_{\mathcal{C}}(*)$ -module of meromorphic one-forms on  $\mathcal{C}$  (or, what is the same, on  $\tilde{\mathcal{C}}$ ), where  $\mathcal{O}_{\mathcal{C}}(*)$  is the total fraction ring of  $\mathcal{O}_{\mathcal{C}}$ . There is a (weakly) non-degenerate bilinear form:

$$\begin{aligned} \text{Res}: \Omega_{\mathcal{C}}(*) \times \mathcal{O}_{\mathcal{C}}(*) &\rightarrow \mathbb{C} \\ \omega, h &\mapsto \text{Res}_{\{p\}}(h, \omega) \end{aligned} \tag{1.1}$$

which we call the *residue pairing*. The residue can be defined as

$$\text{Res}_{\{p\}}(\alpha) = (2\pi i)^{-1} \int_{\partial \mathcal{C}} \alpha$$

where  $\partial \mathcal{C}$  is the boundary of an appropriately small representative of  $\mathcal{C}$ . If  $\mathcal{C}$  is a multigerms then the total residue is the sum of the residues at the base points. Indeed, another way to define the residue is to pull back the form to the normalization and then add the residues over the points lying above  $p$  (see [11]).

The  $\mathcal{O}_{\mathcal{C}}$ -module of *Rosenlicht differentials*  $\omega_{\mathcal{C}}$  is defined by

$$\omega_{\mathcal{C}} := \mathcal{O}_{\mathcal{C}}^{\perp} = \{ \omega \in \Omega_{\mathcal{C}}(*) \mid \text{Res}(\omega, \mathcal{O}_{\mathcal{C}}) = 0 \}.$$

This module is the *dualizing module* of the curve. If  $\mathcal{C}$  is mapped finitely to  $\mathbb{C}^{n+1}$ ,  $n \geq 0$ , then this module is naturally isomorphic to  $\mathcal{E}xt_{\mathbb{C}}^n(\mathcal{O}_{\mathcal{C}}, \Omega^{n+1})$ , where  $\Omega^{n+1}$  is the  $\mathcal{O} = \mathcal{O}_{\mathbb{C}^{n+1}}$ -module of  $(n + 1)$ -forms. The curve  $\mathcal{C}$  is said to be *Gorenstein* if  $\omega_{\mathcal{C}}$  is generated over  $\mathcal{O}_{\mathcal{C}}$  by one element. Such a generating element is called a *Gorenstein generator*.

The pairing (1.1) descends to a non-degenerate pairing between  $\omega_{\mathcal{C}}/\omega_{\tilde{\mathcal{C}}}$  and  $\mathcal{O}_{\tilde{\mathcal{C}}}/\mathcal{O}_{\mathcal{C}}$ , so the former also has dimension  $\delta$ . Furthermore, by reducing to a finite dimensional situation (essentially working modulo the conductor ideal) one can show that  $\mathcal{O}_{\mathcal{C}} = \omega_{\mathcal{C}}^{\perp}$ .

*Definition (1.2).* (i) Let  $\alpha \in \Omega_{\mathcal{C}}(*)$ . We say that  $\alpha$  is a *finite form* (or  $\alpha$  is *finite*) if its restriction to each branch is not identically zero. Such a form induces an isomorphism  $\Omega_{\mathcal{C}}(*) \xrightarrow{\cong} \mathcal{O}_{\mathcal{C}}(*)$ ;  $\omega \mapsto \omega/\alpha$ . Composing this with the residue pairing (1.1) gives a symmetric bilinear form:

$$\begin{aligned} \Psi &= \Psi_{\alpha}: \Omega_{\mathcal{C}}(*) \times \Omega_{\mathcal{C}}(*) \rightarrow \mathbb{C} \\ \omega_1, \omega_2 &\mapsto \text{Res} \left( \frac{\omega_1}{\alpha}, \omega_2 \right). \end{aligned} \tag{1.3}$$

(ii) Let  $\alpha$  be a finite form. We define the *ramification modules*:

$$\begin{aligned} R^+ &= R^+(\alpha) := \omega_{\mathcal{C}}/\omega_{\mathcal{C}} \cap \mathcal{O}_{\mathcal{C}} \cdot \alpha \\ R^- &= R^-(\alpha) := \mathcal{O}_{\mathcal{C}} \cdot \alpha/\omega_{\mathcal{C}} \cap \mathcal{O}_{\mathcal{C}} \cdot \alpha \end{aligned}$$

and the integer  $\rho(\alpha) := \dim_{\mathbb{C}} R^+(\alpha) - \dim_{\mathbb{C}} R^-(\alpha)$ .

(iii) For a finite form  $\alpha$ ,  $\Psi$  descends to two bilinear forms:

$$\psi_{\alpha}^{\pm}: R^{\pm}(\alpha) \times R^{\pm}(\alpha) \rightarrow \mathbb{C}.$$

Remarks (1.4). A finite form  $\alpha$  has on each branch an expansion as

$$\alpha = \sum_{k \geq N} \alpha_k t^k dt$$

for some  $N \in \mathbb{Z}$  (the order of  $\alpha$  on that branch), where  $t$  is a local parameter and  $\alpha_N \neq 0$ . It is easy to see that for a finite form the ramification modules are finite dimensional vector spaces over  $\mathbb{C}$ . It is worth noting that the number  $\rho(\alpha)$  can also be computed as:

$$\rho(\alpha) = \dim_{\mathbb{C}}(\omega_{\mathcal{C}}/\mathcal{L}) - \dim_{\mathbb{C}}(\mathcal{C}_{\mathcal{C}} \cdot \alpha/\mathcal{L}),$$

where  $\mathcal{L}$  is any subspace  $\omega_{\mathcal{C}} \cap \mathcal{C}_{\mathcal{C}} \cdot \alpha$  of finite codimension. Finally, since  $\mathcal{C}_{\mathcal{C}} = \omega_{\mathcal{C}}^{-1}$  it follows that the quadratic forms  $\psi_{\alpha}^{\pm}$  are well-defined and non-degenerate.

Special cases (1.5). (i) Let  $\mathcal{C}$  be smooth, so  $\mathcal{C}_{\mathcal{C}} = \mathbb{C}\{t\}$  and  $\omega_{\mathcal{C}} = \mathbb{C}\{t\} \cdot dt$ , and let  $\alpha$  be a meromorphic 1-form of order  $N$ . Then  $\mathcal{C}_{\mathcal{C}} \cdot \alpha = t^N \mathbb{C}\{t\} dt$ , so if  $N \geq 0$  then  $\dim R^+(\alpha) = N$ ,  $\dim R^-(\alpha) = 0$ , while if  $N \leq 0$  then  $\dim R^+ = 0$  and  $\dim R^- = -N$ . In particular  $\rho(\alpha) = N$ .

(ii) It is clear from the definitions that  $R^+(\alpha) = R^-(\alpha) = 0$  if and only if  $\omega_{\mathcal{C}} = \mathcal{C}_{\mathcal{C}} \cdot \alpha$ , that is  $\mathcal{C}$  is Gorenstein with generator  $\alpha$ .

(iii) Suppose  $\alpha = dg$ , with  $g \in \mathcal{C}_{\mathcal{C}}$ . Then  $\alpha$  is finite precisely when  $g$  defines a finite map  $g: \mathcal{C} \rightarrow \mathcal{D}$ , where  $\mathcal{D}$  is a germ of the complex line, and

$$R^+(dg) = \omega_{\mathcal{C}}/\mathcal{C}_{\mathcal{C}} \cdot dg,$$

$$R^-(dg) = 0.$$

In this situation  $\omega_{\mathcal{C}}$  is equal to  $\mathcal{L}_{\mathcal{C}/\mathcal{D}} \cdot dg$ , where  $\mathcal{L}_{\mathcal{C}/\mathcal{D}} = \{f \in \mathcal{C}_{\mathcal{C}}(*) \mid \text{trace}(f \cdot \mathcal{C}_{\mathcal{C}}) \subset \mathcal{C}_{\mathcal{D}}\}$  is the classical complementary module, so  $R^+(dg) \approx \mathcal{L}_{\mathcal{C}/\mathcal{D}}/\mathcal{C}_{\mathcal{C}}$ . In [8] Herzog and Waldi relate the dimension of this space to the cotangent complex of  $\mathcal{C} \rightarrow \mathcal{D}$ .

(iv) Now suppose that  $\mathcal{C}$  is Gorenstein with generator  $\omega$  and  $g \in \mathcal{C}_{\mathcal{C}}$ . In this case  $g$  has a Jacobian defined as  $Jac(g) = dg/\omega$ . Then

$$R^+(dg) \approx \mathcal{C}_{\mathcal{C}}/Jac(g),$$

so the choice of  $\omega$  gives  $R^+(dg)$  an algebra structure. In particular, suppose  $f: \mathbb{C}^n, 0 \rightarrow \mathbb{C}^{n-1}, 0$  defines an isolated complete intersection singularity. Then  $\omega_{\mathcal{C}}$  is generated by any  $\omega$  satisfying  $\omega \wedge f^* \omega_{n-1} = \omega_n$  (where  $\omega_r$  is a holomorphic volume form on  $\mathbb{C}^r$ ), so

$$Jac(g) = (dg \wedge f^* \omega_{n-1})/\omega_n = \det \begin{bmatrix} \partial f_k / \partial x_i \\ \dots \dots \dots \\ \partial g / \partial x_i \end{bmatrix}.$$

Thus,

$$R^+(dg) \approx \mathbb{C}\{x_1, \dots, x_n\}/(f_1, \dots, f_{n-1}, Jac(g)),$$

which is precisely (the complexification of) the local algebra of  $F$  considered in [1].

LEMMA (1.6). Let  $n: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$  be the normalization map and  $\alpha$  any finite 1-form on  $\mathcal{C}$ . Then

$$\rho(n^* \alpha) = \rho(\alpha) - 2\delta$$

where  $\delta$  is the  $\delta$ -invariant of  $\mathcal{C}$ .

*Proof.* Let  $A, B, C, D$  be finite dimensional vector spaces then in the ring of formal vector spaces:

$$A - D = (B - C) + (A - B) + (C - D).$$

Apply this to  $A = \omega_{\mathcal{C}}/M, B = \omega_{\tilde{\mathcal{C}}}/M, C = \mathcal{O}_{\tilde{\mathcal{C}}}.n^*\alpha/M, D = \mathcal{O}_{\mathcal{C}}.\alpha/M$ , where  $M$  is  $\omega_{\tilde{\mathcal{C}}} \cap \mathcal{O}_{\mathcal{C}}.\alpha$ . Note that  $\dim(A - B) = \dim(C - D) = \delta$ , so taking dimensions gives the result.  $\square$

We now consider the behaviour of  $\rho$  under deformations of the curve  $\mathcal{C}$  and the form  $\alpha$ . So we have a cartesian diagram of germs:

$$\begin{array}{ccc} (\mathcal{C}, p) & \hookrightarrow & (\mathfrak{X}, p) \\ \downarrow & & \downarrow \Pi \\ \{0\} & \hookrightarrow & (S, 0) \end{array}$$

Here  $S$  is a smooth curve germ,  $\Pi$  is a flat map, and  $\mathfrak{X}$  is the total space of the deformation of  $\mathcal{C}$ , i.e.  $\mathfrak{X}$  is a surface germ. We choose good representatives of all these germs (for good representatives see [9], 2.B or [6]), and we will be sloppy with the distinction between germs and global section of sheaves. The fibres of  $\Pi$  are curves, which we denote by  $\mathcal{C}_s := \Pi^{-1}(s)$ . Consider further an analytic family of 1-forms  $\alpha_s$  on the fibres  $\mathcal{C}_s$ , i.e. an element  $A \in \Omega_{\mathfrak{X}/S}(* )$  such that  $A|_{\mathcal{C}_s} = \alpha_s$ . Because the form  $\alpha = \alpha_0$  is finite, we may assume after a possible shrinking of  $S$  that  $A$  has no vertical zero or pole components, i.e. all  $\alpha_s$  are finite forms everywhere on  $\mathcal{C}_s$ . Define the following function on  $S$ :

$$\begin{aligned} \rho : S &\rightarrow \mathbb{Z} \\ s &\mapsto \rho(\alpha_s) = \sum_{q \in \mathcal{C}_s} \rho(\alpha_s, q) \end{aligned}$$

Here  $\rho(\alpha_s, q)$  is the  $\rho$  invariant of (1.4) (iii) of the form  $\alpha_s$  on the curve germ  $(\mathcal{C}_s, q)$ .

**THEOREM (1.7).** The function  $\rho : S \rightarrow \mathbb{Z}$  is constant.

*Proof.* First we choose a function  $H \in \mathcal{O}_{\mathfrak{X}}$  such that  $A.H \in \omega_{\mathfrak{X}/S}$  and  $\Pi_*(\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}.H)$  is a free  $\mathcal{O}_S$ -module of finite rank. This is possible as  $A$  restricts to finite forms on the fibres and the deformation is flat. Here  $\omega_{\mathfrak{X}/S}$  is the so-called *relative dualizing module*, which can be considered as a subsheaf of  $\Omega_{\mathfrak{X}/S}(* )$ . For a flat family  $\mathfrak{X} \rightarrow S$ , the sheaf  $\omega_{\mathfrak{X}/S}$  is  $S$ -flat and restricts to  $\omega_{\mathcal{C}_s}$  on the fibre  $\mathcal{C}_s$  (see for example [6]). Since now both  $\omega_{\mathfrak{X}/S}$  and  $\mathcal{O}_{\mathfrak{X}}$  are flat over  $S$  and specialize to  $\omega_{\mathcal{C}}$  and  $\mathcal{O}_{\mathcal{C}}$  on the special fibre  $\mathcal{C}$  we see that  $\Pi_*(\omega_{\mathfrak{X}/S}/\mathcal{O}_{\mathfrak{X}}.H.A)$  is also a free  $\mathcal{O}_S$ -module of finite rank. Hence, because  $\mathcal{O}_{\mathfrak{X}} \approx \mathcal{O}_{\mathfrak{X}}.A$  etc., one has for all  $s \in S$ :

$$\begin{aligned} \text{rank } \Pi_*(\mathcal{O}_{\mathfrak{X}}.A/\mathcal{O}_{\mathfrak{X}}.H.A) &= \dim_{\mathbb{C}}(\mathcal{O}_{\mathcal{C}_s}.\alpha_s/\mathcal{O}_{\mathcal{C}_s}.H_s.\alpha_s) \\ \text{rank } \Pi_*(\omega_{\mathfrak{X}/S}/\mathcal{O}_{\mathfrak{X}}.H.A) &= \dim_{\mathbb{C}}(\omega_{\mathcal{C}_s}/\mathcal{O}_{\mathcal{C}_s}.H_s.\alpha_s) \end{aligned}$$

where  $H_s = H|_{\mathcal{C}_s}$ . But  $\mathcal{O}_{\mathcal{C}_s}.\alpha_s.H_s \subset \omega_{\mathcal{C}_s} \cap \mathcal{O}_{\mathcal{C}_s}.\alpha_s$ , so using Remark (1.4) we see that the value of  $\rho(s)$  is independent of  $s$ .  $\square$

*Remark.* It is clear that the dimensions of  $R^+$  and  $R^-$  are not themselves constant under deformation: even on a smooth curve poles and zeros can annihilate.

As might be guessed from the invariance of  $\rho$  under deformation, this number has a clear topological meaning, which we now explain. Let  $\mathcal{C}$  be a curve—either a small representative of a (multi-)germ or a global curve—with boundary  $\partial\mathcal{C}$ . ( $\partial\mathcal{C}$  must be disjoint from the singularities of  $\mathcal{C}$ .) Let  $\alpha$  be a finite meromorphic 1-form on  $\mathcal{C}$ . At a smooth point  $p$  of  $\mathcal{C}$ , the

real part  $Re(\alpha)$  has a singularity if and only if  $\alpha$  has a zero or a pole. A simple calculation shows that if  $\alpha$  has order  $N$  at  $p$ , then the index  $i(Re(\alpha); p)$  of  $Re(\alpha)$  at  $p$  (the usual winding number of the associated section of the circle bundle) is  $-N$ . That is, at a smooth point of  $\mathcal{C}$  (cf. 1.5(i))

$$i(Re(\alpha); p) = -\rho(\alpha; p). \tag{1.4}$$

In [2], Arnol'd introduced a local index  $i_+$  associated to a boundary singularity of a 1-form. (A form has a boundary singularity if its restriction to the boundary has a singularity, but the form itself does not.) He showed that if  $M$  is any compact manifold with boundary, and  $\beta$  is any 1-form on  $M$  with only boundary singularities at the boundary, then

$$\chi(M) = I(\beta) + I_+(\beta), \tag{1.5}$$

Here  $I$  and  $I_+$  are the sums of the indices  $i$  and  $i_+$  respectively and  $\chi$  the Euler characteristic.

To apply this to our case, we consider the normalization  $\tilde{\mathcal{C}}$  of  $\mathcal{C}$ , which is smooth. Then from (1.4) and (1.5) we obtain:

$$\chi(\tilde{\mathcal{C}}) = -\rho(n^*\alpha) + I_+(Re(n^*\alpha)). \tag{1.6}$$

Now by Lemma (1.6),  $\rho(n^*\alpha) = \rho(\alpha) - 2\delta$ , and for topological reasons,  $\chi(\tilde{\mathcal{C}}) = \chi(\mathcal{C}) + \sum(r - 1)$ . Moreover, away from the singular points of  $\mathcal{C}$  the normalization is an isomorphism. In particular this is the case in a neighbourhood of  $\partial\mathcal{C}$ . So (1.6) becomes:

$$\chi(\mathcal{C}) + \sum(r - 1) = -\rho(\alpha) + 2\delta + I_+(Re(\alpha)),$$

or, by definition of  $\mu$ :

$$\chi(\mathcal{C}) - \mu(\mathcal{C}) = -\rho(\alpha) + I_+(Re(\alpha)). \tag{1.7}$$

In particular:

PROPOSITION (1.8). *Let  $\mathcal{C}$  be a compact curve without boundary, and  $\alpha$  any finite 1-form on  $\mathcal{C}$ . Then*

$$\rho(\alpha) = \mu(\mathcal{C}) - \chi(\mathcal{C}),$$

where  $\mu(\mathcal{C})$  is the sum of the local Milnor numbers at the singular points of  $\mathcal{C}$  and  $\chi(\mathcal{C})$  is the Euler characteristic. ⊠

Now let  $g: \mathcal{C} \rightarrow \mathcal{D}$  be a finite map of degree  $d$  on all irreducible components of  $\mathcal{D}$ , and suppose that  $g$  is unramified over  $\partial\mathcal{C} = g^{-1}\partial\mathcal{D}$ . Then for any finite 1-form  $\alpha$  on  $\mathcal{D}$ , non-singular in a neighbourhood of  $\partial\mathcal{D}$ ,  $I_+(Re(g^*\alpha)) = d \cdot I_+(Re(\alpha))$ . Thus (1.7) gives the following singular analogue of the Zeuthen–Hurwitz formula of Riemann:

PROPOSITION (1.9). *With  $g: \mathcal{C} \rightarrow \mathcal{D}$  as above,*

$$\rho(g^*\alpha) - d \cdot \rho(\alpha) = \mu(\mathcal{C}) - \chi(\mathcal{C}) - d \cdot (\mu(\mathcal{D}) - \chi(\mathcal{D})).$$

*In particular, if  $\mathcal{C}$  is a germ,  $\mathcal{D}$  is smooth and  $\alpha = dt$ , then one has:*

$$\rho(dg) = \mu(\mathcal{C}) + d - 1. \tag{1.9} \quad \text{⊠}$$

*Remark.* The invariant  $\mu(\mathcal{C}) + d - 1$  has also been considered in [6], Lemma 6.2.8 and in [5], “On Zariski’s criterion . . .”, prop. 2.2, which say that it is constant under deformations of  $\mathcal{C}$ . (In [6] for  $\mathcal{C}$  a complete intersection and in [5] for general  $\mathcal{C}$ .) Here the constancy follows from Theorem (1.7) and the above proposition. The number is equal to the multiplicity of the discriminant of  $g$ , as defined in [5].

This differential-topological point of view gives another proof of a theorem of Milnor [10] for plane curves, Bassein [3] for smoothable curves and Buchweitz and Greuel [6] in the general case. (In fact, it is close to Milnor’s original proof.)

**THEOREM (1.10).** Let  $\Pi: \mathfrak{X} \rightarrow S$  be (a good representative of) a flat deformation of the curve germ  $\mathcal{C} = \mathcal{C}_0$ . Then for all  $s \in S$

$$\mu(\mathcal{C}_s) - \mu(\mathcal{C}_0) = \chi(\mathcal{C}_s) - \chi(\mathcal{C}_0),$$

where  $\chi$  is the topological Euler characteristic and  $\mu$  is the sum of the local Milnor numbers over the curve.

*Remark.* In fact, it is not hard to show that under a flat deformation a reduced curve germ remains connected (see e.g. [6]), so one can replace  $\chi$  by  $-\dim H^1$  in the formula above.

*Proof.* After possibly shrinking  $\mathfrak{X}$  and  $S$  we can assume that:

- (a)  $\Pi|_{\partial\mathfrak{X}}$  is a smooth fibration ( $\partial\mathfrak{X} = \bigcup \partial\mathcal{C}_s$ );
- (b) we have a holomorphic 1-form  $A$  on  $\mathfrak{X}$  whose restrictions  $\alpha_s$  are finite;
- (c) for all  $s$  the zeros of  $\alpha_s$  do not meet  $\partial\mathcal{C}_s$ .

The theorem then follows from (1.7) since  $I_+(Re(\alpha_s))$  is constant (Arnol’d’s boundary index  $i_+$  is constant under homotopy provided no singular points of the 1-form cross the boundary), and  $\rho(\alpha_s)$  is constant by Theorem (1.7). □

We turn to another consequence of the deformation Theorem (1.7) concerning the multiplicity of the critical point of a function on a singular curve germ. In [4], Bruce and Roberts define for certain singular spaces and functions  $g$  on them a “stratified Milnor number”  $\mu(g)$  in terms of the Jacobian ideal generated by vector fields tangent to  $X$  acting on  $g$ . They show that if the so-called logarithmic characteristic variety  $LC^-(X)$  is Cohen–Macaulay, then this Milnor number is continuous under deformation of  $g$  ([4], prop. 5.4). However, even in the simplest examples,  $\mu(g)$  is not constant under deformations of  $X$ :

*Example.*  $X = \{(x, y) \in \mathbb{C}^2 | x, y = 0\}$ ;  $g = x + y$ . Then

$$\mu(g) = \dim \mathbb{C}\{x, y\} / (xy, x\partial_x(x + y), y\partial_y(x + y)) = 1,$$

whereas the number of critical points of  $g$  on  $xy = \varepsilon$  is clearly 2. (And of course,  $\rho(dg) = 2$ .)

**COROLLARY (1.11)** (of Theorem (1.7) and Proposition (1.9)). *Let  $g \in m_{\mathcal{C}}$  define a finite mapping of degree  $d$ . Then for generic  $L \in m_{\mathcal{C}} - m_{\mathcal{C}}^2$  and sufficiently small  $\lambda \neq 0$  the function  $g + \lambda.L$  has  $d - m(\mathcal{C})$  critical points away from 0, where  $m(\mathcal{C})$  is the multiplicity of  $\mathcal{C}$ .*

*Proof.* For a finite mapping  $h: \mathcal{C} \rightarrow \mathbb{C}$ , we have by Proposition (1.9):  $\rho(dh) = \mu(\mathcal{C}) + \text{degree}(h) - 1$ . Thus for generic  $h \in m_{\mathcal{C}} - m_{\mathcal{C}}^2$ ,  $\rho(dh) = \mu(\mathcal{C}) + m(\mathcal{C}) - 1$ . Thus for generic  $L$  this equation is satisfied by both  $L$  and  $\lambda.L + g$  for sufficiently small  $\lambda$ . By Theorem (1.7),  $\rho(dg + \lambda.dL)$  summed over critical points is independent of  $\lambda$ . A further genericity condition on  $L$  ensures that all the critical points away from 0 have multiplicity one. Thus the number of these critical points is:  $\rho(dg) - \rho(dg + \lambda.dL, 0) = (\mu(\mathcal{C}) + d - 1) - (\mu(\mathcal{C}) + m(\mathcal{C}) - 1)$ . □

*Remark.* The number  $\text{deg}(g) - m(\mathcal{C})$  is called the number of vertical tangents of  $g$ , see also [6].

§2. REAL CURVE SINGULARITIES

In this section we consider real 1-forms on real analytic curve germs. We find that the signatures of the quadratic forms defined in Definition (1.2) are related to the orientation induced on the curve. Throughout this section  $\mathcal{C}$  denotes a germ of a real analytic curve. The local ring  $\mathcal{O}_{\mathcal{C}}$  now is an  $\mathbb{R}$ -algebra of the form:

$$\mathcal{O}_{\mathcal{C}} \approx \mathbb{R}\{x_1, \dots, x_n\}/I$$

for some  $n$  and ideal  $I$ . The complexification  $\mathcal{C}_{\mathbb{C}}$  has local ring  $\mathcal{O}_{\mathcal{C}} \otimes_{\mathbb{R}} \mathbb{C}$ . The complexification of an  $\mathbb{R}$ -algebra or module has a natural complex conjugation on it, and one can identify the original with the subspace of fixed points of this conjugation. Whenever we say that  $\mathcal{C}$  is reduced, irreducible or Gorenstein, we mean that  $\mathcal{C}_{\mathbb{C}}$  is. Note that all the  $\mathcal{O}_{\mathcal{C}}$ -modules defined in §1 are already defined over  $\mathbb{R}$ . Let  $\alpha \in \Omega_{\mathcal{C}}(*)$  be finite. Then on each half-branch of  $\mathcal{C}$  (that is, connected component of  $\mathcal{C} - \{p\}$ )  $\alpha$  defines an orientation. Given  $\alpha$ , we say that a half-branch is *inbound* or *outbound* accordingly as the orientation is towards or away from the base point  $p$ . We have from Definition (1.2) two real artinian  $\mathcal{O}_{\mathcal{C}}$ -modules  $R^{\pm}(\alpha)$  with non-degenerate quadratic forms  $\psi_{\alpha}^{\pm}$ .

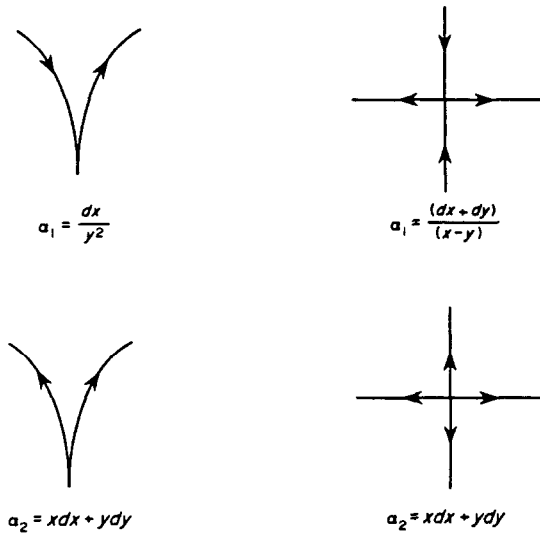


Fig. 1. Examples of orientations induced by 1-forms on singular curves. The curves are  $y^3 - x^2 = 0$  and  $xy = 0$ . In each case  $\alpha_1$  is a Gorenstein generator.

**THEOREM (2.1).** *Let  $\mathcal{C}$  be a reduced real analytic curve germ and  $\alpha$  be a finite meromorphic 1-form on  $\mathcal{C}$ . Then:*

$$\# \{\text{outbound half-branches}\} - \# \{\text{inbound half-branches}\} = 2 \text{Sig}(\psi_{\alpha}^{+}) + 2 \text{Sig}(\psi_{\alpha}^{-}),$$

where *Sig* denotes the signature of a quadratic form, that is the difference between the number of positive and the number of negative eigenvalues.

**COROLLARY (2.2).** *For  $\alpha = \sum x_i dx_i$  we have  $R^{-}(\alpha) = 0$  and every half-branch is outbound, so the number of half-branches of  $\mathcal{C}$  is equal to  $2 \text{Sig}(\psi_{\alpha}^{+})$ .  $\square$*

Before proving this theorem we show that in the case of an isolated complete intersection curve it reduces to the theorem of Aoki, Fukuda and Nishimura [1]. Let

$$f = (f_1, \dots, f_{n-1}): \mathbb{R}^n; 0 \rightarrow \mathbb{R}^{n-1}, 0$$

define an isolated complete intersection germ  $\mathcal{C} = f^{-1}(0)$  and let  $g \in \mathfrak{m}_{\mathcal{C}}$  define a finite mapping. From (1.4)(iv),  $R^+(dg) = \mathcal{O}_{\mathcal{C}}/\text{Jac}(g)$ , which is the local algebra of the finite map  $F = (f_1, \dots, f_{n-1}, \text{Jac}(g))$  and  $R^-(dg) = 0$ . The Jacobian of  $F$  is then  $\text{Jac}(\text{Jac}(g))$ .

**THEOREM** (Aoki, Fukuda, Nishimura). *Let  $f$  be as above and put  $g = \sum x_i^2$ . If  $\varphi: \mathcal{O}_{\mathcal{C}}/\text{Jac}(g) \rightarrow \mathbb{R}$  is any linear functional with  $\varphi(\text{Jac}(\text{Jac}(g))) > 0$  and if  $B_{\varphi}$  is the symmetric bilinear form defined by  $B_{\varphi}(a, b) = \varphi(a \cdot b)$  then*

$$2 \text{Sig } B_{\varphi} = \# \{ \text{half-branches of } \mathcal{C} \} \quad \square$$

We show that our quadratic form  $\psi_{\alpha}^+$ ,  $\alpha = dg$  is of the form  $B_{\varphi}$  for a suitable  $\varphi$ . Define  $\varphi: \mathcal{O}_{\mathcal{C}}/\text{Jac}(g) \rightarrow \mathbb{R}$  by

$$\varphi(h) = \text{Res} \left( \frac{h \cdot \omega}{\text{Jac}(g)} \right)$$

where  $\omega$  is a Gorenstein generator as in (1.4) (iv). Then

$$\varphi(\text{Jac}(\text{Jac}(g))) = \text{Res} \left( \frac{d \text{Jac}(g)}{\text{Jac}(g)} \right) > 0.$$

*Remark* (2.3). In fact, the method of Aoki *et al.* generalizes to the following: Let  $f \in \mathfrak{m}_{\mathcal{C}}$  define a finite mapping, then  $2 \text{Sig } B_{\varphi}$  is the difference between the number of half-branches with  $g > 0$  and the number of half-branches on which  $g < 0$ . Note further that the signature of the quadratic form as  $B_{\varphi}$  can be computed as the dimension of  $\mathcal{O}_{\mathcal{C}}/\text{Jac}(g)$  minus twice the dimension of a maximal square-zero ideal, comparing Theorems 1.1 and 1.2 of Eisenbud and Levine [7].

*Proof of Theorem* (2.1). Let  $n: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$  be the normalization of  $\mathcal{C}$  and let  $\beta = n^* \alpha$  be the pull back of the 1-form  $\alpha$  to the normalization.

**LEMMA** (2.4).  $\text{Sig } \psi_{\alpha}^+ + \text{Sig } \psi_{\alpha}^- = \text{Sig } \psi_{\beta}^+ + \text{Sig } \psi_{\beta}^-$ .

*Proof.* Consider the subspace  $\mathcal{V} = \omega_{\mathcal{C}} + \mathcal{O}_{\tilde{\mathcal{C}}}\beta$  of  $\Omega_{\tilde{\mathcal{C}}}(\ast) = \Omega_{\mathcal{C}}(\ast)$ . On  $\mathcal{V}$  there is the quadratic form  $\Psi = \Psi_{\alpha} = \Psi_{\beta}$  defined in (1.3), from which the  $\psi$ 's are induced. With respect to  $\Psi$  one has  $\omega_{\mathcal{C}} = (\mathcal{O}_{\mathcal{C}}\alpha)^{\perp}$  and  $\omega_{\tilde{\mathcal{C}}} = (\mathcal{O}_{\tilde{\mathcal{C}}}\beta)^{\perp}$ . So the statement follows from the following lemma:

**LEMMA** (2.5). *Let  $\Psi$  be a quadratic form on a vector space  $\mathcal{V}$  whose null space has finite codimension in  $\mathcal{V}$ . Then for any subspace  $\mathcal{L}$  of  $\mathcal{V}$  one has:*

$$\text{Sig } \Psi|_{\mathcal{L}} + \text{Sig } \Psi|_{\mathcal{L}^{\perp}} = \text{Sig } \Psi \quad \square \square$$

The above two lemmas show that our sum of signatures does not change under normalization of  $\mathcal{C}$ . We have reduced the proof of Theorem (2.1) to the special case where  $\mathcal{C}$  is smooth: since the normalization map is an isomorphism away from the singular point, the numbers of outbound and of inbound half-branches are the same on  $\mathcal{C}$  and  $\tilde{\mathcal{C}}$ . So we can assume:

$$\mathcal{O}_{\mathcal{C}} = \bigoplus_{i=1}^a \mathbb{R}\{t_i\} \bigoplus_{j=1}^b \mathbb{R}\{s_j, u_j\}/(u_j^2 + 1).$$

Here  $a$  is the number of real branches and  $b$  is the number of complex conjugate pairs, so



$r = a + 2b$  is the total number of branches of  $\mathcal{C}$ . Similarly, since  $\mathcal{C}$  is smooth:

$$\omega_{\mathcal{C}} = \bigoplus_{i=1}^a \mathbb{R}\{t_i\}.dt_i \oplus \bigoplus_{j=1}^b \mathbb{R}\{s_j, u_j\}.ds_j/(u_j^2 + 1).$$

The modules  $R^{\pm}$  split into corresponding direct sums, the summands being pairwise orthogonal with respect to  $\psi^{\pm}$ , so we reduce to the case where  $\mathcal{C}_{\mathcal{C}}$  has only one summand.

LEMMA (2.6). *Let  $\mathcal{C}_{\mathcal{C}} = \mathbb{R}\{t\}$ ,  $\omega_{\mathcal{C}} = \mathbb{R}\{t\}.dt$ . Consider a finite 1-form  $\alpha = \sum_{j \geq N} a_j t^j .dt$  of order  $N$  (so  $a_N \neq 0$ ). Then*

$$\text{Sig } \psi_{\alpha} = \begin{cases} 0 & \text{if } N \text{ is even} \\ \text{sign}(a_N) & \text{if } N \text{ is odd} \end{cases} .$$

Here  $\psi = \psi^+$  if  $N \geq 0$  and  $\psi = \psi^-$  if  $N \leq 0$ .

*Proof.* Suppose  $N \geq 0$ . Then  $R_x^- = 0$  and  $\dim R_x^+ = N$ . We can choose as a basis for  $R^+ : \{t^{-N}\alpha, t^{-N+1}\alpha, \dots, t^{-1}\alpha\}$ , with respect to which the matrix of  $\psi^+$  has a very simple form with the number  $a_N$  on and zeros below the anti-diagonal. From this one reads off the signature immediately. For  $N \leq 0$  the proof is of course similar. ⊠

LEMMA (2.7). *Let  $\mathcal{C}_{\mathcal{C}} = \mathbb{R}\{s, u\}/(u^2 + 1)$ ,  $\omega_{\mathcal{C}} = \mathcal{C}_{\mathcal{C}}.ds$ . Consider a finite 1-form  $\alpha$  on  $\mathcal{C}$ . Then  $\text{Sig } \psi_{\alpha}^{\pm} = 0$ .*

*Proof.* On the  $\mathcal{C}_{\mathcal{C}}$ -modules  $R^{\pm}$  we now have a transformation  $U: \omega \mapsto u.\omega$ , whose square is  $-1$ . Thus  $\psi^{\pm}(u\omega_1, u\omega_2) = \psi^{\pm}(u^2.\omega_1, \omega_2) = -\psi^{\pm}(\omega_1, \omega_2)$ , so  $\psi^{\pm}$  is under the automorphism  $U$  equivalent to  $-\psi^{\pm}$ . Consequently  $\text{Sig } \psi_{\alpha}^{\pm} = 0$ . ⊠

We have now proved Theorem (2.1), since by Lemma (2.7) the complex conjugate branches of the complexification of  $\mathcal{C}$  give no contribution to the signature, while by Lemma (2.6) the real half-branches give a contribution in agreement with the orientation: if the order  $N$  of a 1-form  $\alpha$  on a branch is even, then this branch has one inbound and one outbound half-branch, while if  $N$  is odd both half-branches are outbound or inbound accordingly as the sign of  $a_N$  is positive or negative. ⊠

COROLLARY (2.8). *Let  $\mathcal{C}$  be a real Gorenstein curve and let  $\omega$  be a Gorenstein generator. Then the number of inbound half-branches with respect to  $\omega$  is the same as the number of outbound half-branches.*

*Proof.* In this case  $R^+ = R^- = 0$  by (1.4) (ii). Hence the result follows from Theorem (2.1). ⊠

Remarks (2.9). Corollary (2.8) should be seen as an expression of a “geometric symmetry” for Gorenstein curves. In the case of an isolated complete intersection case (2.8) can be proved more directly as follows: Let  $f: \mathbb{R}^n, 0 \rightarrow \mathbb{R}^{n-1}, 0$  define  $\mathcal{C}$  and consider a small sphere  $S$  around  $0$  in  $\mathbb{R}^n$ , transverse to  $\mathcal{C}$ . The map  $f|_S$  has  $0$  as a regular value and for  $x \in f|_S^{-1}(0)$  the jacobian of  $f|_S$  at  $x$  is positive or negative accordingly as  $x$  lies on an outbound or inbound half-branch. (This can be seen from the formula for  $\omega$  given in (1.4)(iv).) Since the degree of any map  $S \rightarrow \mathbb{R}^{n-1}$  is zero, the result follows. More generally, for a smoothable Gorenstein curve singularity (2.8) is clear.

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