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Projective resolutions associated to projections

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Abstract. In this paper we will describe projective resolutions of d dimensional Cohen–Macaulay spaces X by means of a projection of X to a hypersurface in $d+1$-dimensional space. We will show that for a certain class of projections, the resulting resolution is minimal.

1. Introduction

Let X be a d-dimensional germ of an analytic space and let $\phi : X \to \mathbb{C}^{d+1}$ be a finite map. Via ϕ we can consider $\mathcal{O}_X$ as an $\mathcal{O} := \mathcal{O}_{\mathbb{C}^{d+1}}$-module. If X is *Cohen–Macaulay*, then $\mathcal{O}_X$ has a free resolution as $\mathcal{O}$-module of the form:

$$0 \to G \xrightarrow{\Phi} F \to \mathcal{O}_X \to 0 \tag{1}$$

where $F = \oplus_{k=0}^{r} \mathcal{O} \cdot f_k$ and $G = \oplus_{k=0}^{r} \mathcal{O} \cdot g_k$ are free $\mathcal{O}$-modules of rank $r+1$. The determinant f of the matrix (Φ_{ij}) can be used as a defining equation for the image Y of X in $\mathbb{C}^{d+1}$, see [7]. Now $\mathcal{O}_X$ is not only a $\mathcal{O}$-module, but even a *$\mathcal{O}$-algebra*, due to the fact that $\mathcal{O}_X$ is a *ring*. Let f_k be mapped to u_k in $\mathcal{O}_X$. We may suppose that $u_0 = 1$. We get a surjection:

$$\mathcal{O}[f_1, \dots, f_r] \to \mathcal{O}_X \to 0 \tag{2}$$

of $\mathcal{O}$-algebras, or equivalently, an embedding $X \hookrightarrow \mathbb{C}^{d+1} \times \mathbb{C}^r$. The equations of X in this embedding come into two types:

$$\sum_{i=0}^{r} \Phi_{ij} f_i = 0 \tag{3}$$

$$f_i f_j - \sum_{k=0}^{r} M_{ijk} f_k = 0 \tag{4}$$

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The equations (4) are the “module-equations” between the u_i that follow from 1. The equations 4 are the “multiplication-equations”. They express the product $u_i u_j$ in the module basis. The M_{ijk} are certain elements of $\mathcal{O}$ and could be called the structure constants, cf. [2], [6]. Another way of looking at the equations (3) and (4) is to say that the left hand side of these equations generate the kernel of the surjection of (2) as an $S := \mathcal{O}[f_1, \ldots, f_r]$-module.

In the first part of this article we will extend this to a description of a projective resolution of $\mathcal{O}_X$ as an S-module. It turns out that this resolution has the form:

$$0 \to \mathcal{L}_{r+1} \to \mathcal{L}_r \to \ldots \to \mathcal{L}_1 \to S \to \mathcal{O}_X \to 0 \tag{5}$$

where $\mathcal{L}_k$ is a free S-module of rank $k \cdot \binom{r+2}{k+1}$. Note that these are the well-known ranks occuring in the minimal resolutions of varieties of minimal multiplicity, [8], [3]. Our complex involves Φ, certain maps L and M describing the algebra structure of $\mathcal{O}_X$ on the complex (1) and a certain homotopy H expressing the associativity of the multiplication in $\mathcal{O}_X$. The construction follows the steps taken in [3], where a similar complex was constructed associated to a map $X \to \mathbb{C}^d$, representing (in the case that X is Cohen–Macaulay) $\mathcal{O}_X$ as a free $\mathcal{O}_{\mathbb{C}^d}$-module.

In the second part of the article we treat the special case that the map $\phi : X \to \mathbb{C}^{d+1}$ is generically $1-1$. In that case the image space Y will contain a subscheme Σ, defined by the *conductor ideal* $I = \mathrm{Hom}_Y(\mathcal{O}_X, \mathcal{O}_Y) \subset \mathcal{O}_Y$. This subscheme Σ will be Cohen–Macaulay of codimension 2 in $\mathbb{C}^{d+1}$ and is contained in the singular locus of Y. Conversely, when $\Sigma \subset Y$ is given, we can reconstruct X. This is reviewed in the third section.

If moreover the conductor ideal $I \subset \mathcal{O}$ is *radical* then a hypersurface defined by a $g \in I$ is singular along Σ if and only if g is in the second symbolic power $I^{(2)}$. This $I^{(2)}$ contains the ordinary second power I^2. So in this situation the defining function f is in $I^{(2)}$. In [4] “generic” mappings $\phi : X \to \mathbb{C}^3$ were studied, where X is a normal surface germ. It was shown there that the module $M(X, \phi) := I^{(2)}/(I^2 + (f))$ is *independent* of the chosen ϕ, as it can be identified with the dual of $\mathrm{Ext}^1_X(\omega_X, \mathcal{O}_X)$. (The ideal I and the equation $f = 0$ of the image depend very much on ϕ, however.) In particular, one sees from this fact that if X is a Gorenstein singularity, then $M(X, \phi) = 0$. In other words, $I^{(2)}/I^2$ is a cyclic module with generator f. This was also proved in [6]. Now it is well known that the minimal resolution of a Gorenstein germ can be taken to be a symmetric complex. This implies that the complex (5) is in such cases never minimal (unless $r = 0$, i.e. $X = Y$). The other extreme somehow is represented by those X for which the invariant $M(X, \phi)$ is as big as possible for a given Σ. In other words, if $f \in I^2$. In the fourth section we turn our attention to this case. It turns out that in this case one can express the maps L, M and H *explicitly* in terms of the matrix Φ_{ij}. As a consequence, we get that in this case the resolution (5) is *minimal*.

It is not so clear what the geometric meaning of “$f \in I^2$” is. In any case, it represents a property of X and ϕ, and *not* of X alone. The complex considered in [3] was shown to be minimal in the case that the singularity has *minimal multiplicity* with respect to its embedding dimension. Strange enough, the condition $f \in I^2$ seems to be totally unrelated to this condition. In fact, if $f \in I^2$ then in almost all cases the space X will *not* be of minimal multiplicity. The most optimistic guess on minimality is that the complex (5) is always minimal, *unless* f is a generator of $I^{(2)}/I^2$, but we have been unable to prove anything more in this direction.

2. A projective resolution

We consider a commutative ring R with 1, and E a finitely generated projective R-module. We put $S := \sum_k S_k(E)$, where S_k is the k-th symmetric power of E. The "diagonal" map Δ is the map:

$$\Delta : \wedge^k(E) \to \wedge^k(E) \otimes E$$

defined on generators by:

$$\Delta(e_1 \wedge \ldots \wedge e_k) = \sum_i (-1)^{i-1} e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge e_k \otimes e_i .$$

Here and in the sequel the tensor products are over the ring R. We define for any S-module M a map:

$$d_M : \wedge^k(E) \otimes M \to \wedge^{k-1}(E) \otimes M$$

by $d_M := (1 \otimes m)(\Delta \otimes 1)$, where $m : S \otimes M \to M$ is the multiplication map. By abuse of notation, the map $M \otimes \wedge^k(E) \to M \otimes \wedge^{k-1}(E)$ defined by $s d_M s$, where s is the swap that interchanges the tensor factors, is also denoted by d_M. Note that $d_M d_M = 0$.

Proposition 1. *Let M be an S-module which is finitely generated as an R-module. Put $K_k := S \otimes \wedge^k(E) \otimes M$ and $d := d_S \otimes 1 - 1 \otimes d_M : K_k \to K_{k-1}$. Then $d^2 = 0$ and*

$$\mathbf{K}(M) : 0 \to K_r \to K_{r-1} \to \ldots K_1 \to K_0 = S \otimes M \to 0$$

is a resolution of M as S-module.

Proof. For a proof see [3], Theorem 1.1 (In this theorem it is assumed that M is projective, but this is not needed in the proof of the above statement.) □

In case that M is a projective R-module, the above complex $\mathbf{K}(M)$ is an S-projective resolution of M. Special such S-modules arise as R-algebras of the form $R \oplus E$ as considered in [3]. We will consider the case of R-algebras A given by an exact sequence of projective R-modules:

Diagram 2.

$$0 \to G \xrightarrow{\Phi} R \oplus E \to A \to 0$$

where $rk(E) = r$ and $rk(G) = r + 1$. We abbreviate $R \oplus E$ to F.

Because A is (in general) no longer a projective R-module, the resolution in Proposition 1 with $M = A$ does not give us a projective resolution of A as an S-module. We will replace A by "$G \xrightarrow{\Phi} F$", but the differential needs special care. In order to define this differential we introduce some maps expressing the commutativity and associativity of A. Consider the following commutative diagram:

Diagram 3.

$$\begin{array}{ccccccccccc}
0 & \to & \wedge^2(G) & \to & F\otimes G & \to & S_2(F) & \to & S_2(A) & \to & 0\\
 & & \downarrow & & \downarrow m_2 & & \downarrow m_1 & & \downarrow m & & \\
 & & 0 & \to & G & \to & F & \to & A & \to & 0
\end{array}$$

The first row is a projective resolution of the second symmetric power $S_2(A)$ of A, m is the multiplication map of the algebra structure of A, which is lifted to maps m_1 and m_2 of complexes. Because $F = R \oplus E$, we have decompositions:

$$S_2(F) = F \oplus S_2(E),$$

$$F \otimes G = G \oplus E \otimes G.$$

Therefore we can decompose m_1 and m_2 as follows:

$$m_1 = Id_F \oplus M \text{ where } M : S_2(E) \to F,$$

$$m_2 = Id_G \oplus L \text{ where } L : E \otimes G \to G.$$

By composition we get a map $E \otimes E \to S_2(E) \to F$ that we also denote by M.

In order to express the associativity of the multiplication on A, we consider the following commutative diagram:

Diagram 4.

$$\begin{array}{ccccccccc}
0 & \to & \wedge^2(E)\otimes G & \stackrel{1\otimes\Phi}{\longrightarrow} & \wedge^2(E)\otimes F & \to & \wedge^2(E)\otimes A & \to & 0\\
 & & [L,L]\downarrow & & [M,M]\downarrow & & 0\downarrow & & \\
0 & \to & G & \stackrel{\Phi}{\longrightarrow} & F & \to & A & \to & 0
\end{array}$$

The map $[M, M]$ is defined as $M(1 \otimes M)(\Delta \otimes 1)$, so

$$[M, M](e_1 \wedge e_2 \otimes f) := M(e_1 \otimes M(e_2 \otimes f)) - M(e_2 \otimes M(e_1 \otimes f)).$$

The map $[L, L]$ is defined similarly.

The commutativity of the left hand square follows from the commutativity of Diagram 3, whereas the commutativity of the right hand square expresses the associativity and commutativity of the algebra A. It follows that there is a *homotopy* $H : \wedge^2(E) \otimes F \to G$ with $\Phi H = [M, M]$ and $H(1 \otimes \Phi) = [L, L]$.

Proposition 5. *Let* $\mathcal{A}_k = S \otimes \wedge^k(E) \otimes F \oplus S \otimes \wedge^{k-1}(E) \otimes G$ *and*

$$\partial := \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} : \mathcal{A}_k \to \mathcal{A}_{k-1}$$

with

1. $d_1 = d_S \otimes 1 - (1 \otimes M)(\Delta \otimes 1)$;
2. $d_2 = 1 \otimes \Phi$;
3. $d_3 = (1 \otimes H)(\Delta \otimes 1)(\Delta \otimes 1)$;
4. $d_4 = -d_S \otimes 1 + (1 \otimes L)(\Delta \otimes 1)$

Then one has that $\partial\partial = 0$, i.e. $\mathbf{A} := (A., \partial)$ is a complex.

Proof. This is for the a straightforward calculation, and is an expression of the various commutations of maps. We indicate what is involved.

1. For $d_1^2 + d_2 d_3 = 0$, use $\phi H = [M, M]$.
2. For $d_3 d_2 + d_4^2 = 0$, use $H(1 \otimes \Phi) = [L, L]$.
3. For $d_1 d_2 + d_2 d_4 = 0$, use $M(1 \otimes \Phi) = \Phi L$.
4. The most difficult one is to show that $d_3 d_1 + d_4 d_3 = 0$. For this it turns out that one has to use the commutativity of the following diagram:

$$\begin{tikzcd}
\wedge^3(E) \otimes F \arrow[r, "\Delta\otimes 1"] \arrow[d, "\Delta \otimes 1"'] & \wedge^2(E) \otimes E \otimes F \arrow[r, "s\otimes 1"] & E \otimes \wedge^2(E) \otimes F \arrow[r, "1\otimes H"] & E \otimes G \arrow[d, "L"] \\
\wedge^2(E) \otimes E \otimes F \arrow[r, "1\otimes m_1"] & \wedge^2(E) \otimes F \arrow[rr, "H"] & & G
\end{tikzcd}$$

This commutativity can be checked by composing with the injective map Φ. After doing this, the commutativity comes down to the relations $\Phi H = [M, M]$ and $\Phi L = M(1 \otimes \Phi)$, together with the equality of maps $\wedge^3(E) \otimes F \to F$:

$$[M, M](1 \otimes m_1)(\Delta \otimes 1) = m_1([M, M] \otimes 1)(s \otimes 1)(\Delta \otimes 1)$$

which is checked by direct computation. □

Lemma 6. *Let $B = \oplus B_k$ be a $\mathbb{Z}$-graded Abelian group with a map δ of degree -1. (Not necessarily $\delta\delta = 0$!) Consider the "mapping cone" $\mathbf{C} := (C., d)$ where $C_k := B_k \oplus B_{k-1}$ and $d = \begin{pmatrix} \delta & Id \\ -\delta\delta & -\delta \end{pmatrix}$. Then $d^2 = 0$, and $\mathbf{C}$ is an exact complex.*

Proof. To show that $d^2 = 0$ is a simple computation. To show that $\mathbf{C}$ is exact, we establish the homotopy between the zero map and the identity map of $\mathbf{C}$ by $\begin{pmatrix} 0 & 0 \\ Id & 0 \end{pmatrix} : C_k \to C_{k+1}$. □

Proposition 7. *The complex:*

$$\mathbf{A} : 0 \to \mathcal{A}_{r+1} \xrightarrow{\partial} \mathcal{A}_r \xrightarrow{\partial} \ldots \xrightarrow{\partial} \mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0 = S \otimes F \to 0$$

is an S-projective resolution of A.

Proof. We apply Lemma 6 with $B_k = S \otimes \wedge^k(E) \otimes G$ and $\delta = d_S \otimes 1 - (1 \otimes L)(\Delta \otimes 1)$ and get an exact mapping cone complex $\mathbf{C}$. We have an *injective* map of complexes $\mathbf{C} \hookrightarrow \mathbf{A}$, given in degree k by:

$$(1 \otimes \Phi) \oplus Id : S \otimes \wedge^k(E) \otimes G \oplus S \otimes \wedge^{k-1}(E) \otimes G \to S \otimes \wedge^k(E) \otimes F \oplus S \otimes \wedge^{k-1}(E) \otimes G.$$

The cokernel of this map can be identified with the complex $\mathbf{K}(A)$ of Proposition 1. So we have a short exact sequence of complexes:

$$0 \to \mathbf{C} \to \mathbf{A} \to \mathbf{K}(A) \to 0$$

Because $\mathbf{C}$ is exact by Lemma 6 and $\mathbf{K}(A)$ is a resolution of A by Proposition 1, it follows from the long exact homology sequence that $\mathbf{A}$ is an S-projective resolution of A. □

3. A smaller resolution

Although the complex **A** has the right length, it is usually not minimal. In [3] it is described how to obtain from $\mathbf{K}(A)$ a smaller complex. We will use their ideas to prune our complex **A** in a similar way. We will therefore be brief.

Definition 8. *(see also [1], [3]) Let $\pi : F = R \oplus E \to E$ be the Cartesian projection and define maps:*

$$in := (\wedge^k \pi \otimes 1)\Delta : \wedge^{k+1}(F) \to \wedge^k(E) \otimes F$$

as the compostion of the diagonal map and the induced projection.

The commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \to & \wedge^k(E) & \to & \wedge^{k+1}(F) & \to & \wedge^{k+1}(E) & \to 0 \\ & =\downarrow & & \text{in} \downarrow & & \Delta \downarrow & \\ 0 \to & \wedge^k(E) & \to & \wedge^k(E) \otimes F & \to & \wedge^k(E) \otimes E & \to 0 \end{array}$$

shows that $\mathrm{Coker(in)} \cong \mathrm{Coker}(\Delta)$. We denote this common cokernel by $L^k := L_2^k := \mathrm{Coker}(\Delta : \wedge^{k+1}(E) \to \wedge^k(E) \otimes E)$. The module L^k is projective and has rank $k \cdot \binom{r+1}{k+1}$.

Consider the inclusion $F = R \oplus E \hookrightarrow S$ and the induced map $S \otimes F \to S$. The Koszul complex $\mathbf{P} := (P., \delta)$ on this map with terms $P_k := S \otimes \wedge^k(F)$ and the usual differential, is exact.

Proposition 9. *Let j be the map:*

$$j := (1 \otimes in) \oplus 0 : S \otimes \wedge^{k+1}(F) \to S \otimes \wedge^k(E) \otimes F \oplus S \otimes \wedge^{k-1}(E) \otimes G$$

Then the diagram

$$\begin{array}{ccc} P_{k+1} & \xrightarrow{\delta} & P_k \\ j \downarrow & & j \downarrow \\ \mathcal{A}_k & \xrightarrow{\partial} & \mathcal{A}_{k-1} \end{array}$$

is anti-commutative. We therefore have an induced differential $\partial : \mathcal{L}_k \to \mathcal{L}_{k-1}$, $k \geq 2$, where $\mathcal{L}_k := \mathrm{Coker}(j) = S \otimes L^k \oplus S \otimes \wedge^{k-1}(E) \otimes G$. Note that the rank of $\mathcal{L}$ is equal to $k \cdot \binom{r+2}{k+1}$.

Proof. The anti-commutativity of the diagram

$$\begin{array}{ccc} S \otimes \wedge^{k+1}(F) & \xrightarrow{\delta} & S \otimes \wedge^k(F) \\ 1 \otimes in \downarrow & & 1 \otimes in \downarrow \\ S \otimes \wedge^k(E) \otimes F & \xrightarrow{d_1} & S \otimes \wedge^{k-1}(E) \otimes F \end{array}$$

can be proved as in [3], Lemma 3.1. So to prove the statement of the proposition, we have to show that the composition:

$$S \otimes \wedge^{k+1}(F) \xrightarrow{1 \otimes in} S \otimes \wedge^k(E) \otimes F \xrightarrow{d_3} S \otimes \wedge^{k-2}(E) \otimes G$$

is the zero map. A direct computation (use $\Phi H = [M, M]$) shows that the composition of this map with the injective map $1 \otimes \Phi$ maps the element $s \otimes e_1 \wedge e_2 \wedge \ldots \wedge e_{k+1}$ to $\sum_{i<j<k} (-1)^{i+j+k} s \otimes (e_i \wedge e_j \wedge e_k) \otimes \gamma$ where

$$\begin{aligned}\gamma = &- M(e_i \otimes M(e_j \otimes e_k)) + M(e_i \otimes M(e_k \otimes e_j)) \\ &- M(e_j \otimes M(e_k \otimes e_i)) + M(e_j \otimes M(e_i \otimes e_k)) \\ &- M(e_k \otimes M(e_i \otimes e_j)) + M(e_k \otimes M(e_j \otimes e_i)).\end{aligned}$$

This is zero due to the symmetry of the map M. $\square$

Theorem 10. *The complex*

$$\mathbf{L} = (\mathcal{L}., \partial):\ 0 \to \mathcal{L}_{r+1} \xrightarrow{\partial} \mathcal{L}_r \xrightarrow{\partial} \ldots \xrightarrow{\partial} \mathcal{L}_2 \xrightarrow{\partial} \mathcal{L}_1 \xrightarrow{\partial} \mathcal{L}_0 := S \to 0$$

with $\partial : \mathcal{L}_k \to \mathcal{L}_{k-1}, k \geq 2$ *as in Proposition 9 and*

$$\partial : \mathcal{L}_1 = S \otimes S_2(E) \oplus S \otimes G \to \mathcal{L}_0 = S$$

given by: $\partial(s \otimes e_1 \otimes e_2 \oplus t \otimes g) = s(e_1 e_2 - M(e_1 \otimes e_2)) + t\Phi(g)$ *is an* S*-projective resolution of* A*. Furthermore, if the ring* R *is local with maximal ideal* $\mathbf{m}$*, then the resolution is minimal (after localization at* $(\mathbf{m}, E) \subset S$*) if* $\Phi(G) \subset \mathbf{m}F,\ L(E \otimes G) \subset \mathbf{m}G,\ M(E \otimes E) \subset \mathbf{m}F$ *and* $H(\wedge^2(E) \otimes E) \subset \mathbf{m}G$.

Proof. As in [3], Theorem 3.2. $\square$

4. Projections

In [4] the following situation was studied:

$$\begin{array}{ccc} & X & \\ & \downarrow \rho & \\ \Sigma \stackrel{i}{\to} & Y & \subset Z \end{array}$$

Here $Y = \mathrm{Spec}(B)$ is a hypersurface in a smooth ambient space $Z = \mathrm{Spec}(R)$. If $\rho : X \to Y$ is a generically $1-1$ map from a Cohen–Macaulay space $X = \mathrm{Spec}(A)$ to Y, then the conductor $I = \mathrm{Hom}_B(A, B)$ defines a subspace $\Sigma = \mathrm{Spec}(C)$; $C = B/I$ of Y. From the inclusion $i : \Sigma \to Y$ one can reconstruct A as a B-module via $A = \mathrm{Hom}_B(I, B)$. The *ring structure* on A is translated into the fact that the ideal I satisfies the *ring condition* (R.C.):

Lemma 11 (ring condition).

$$\mathrm{Hom}_B(I, I) \cong \mathrm{Hom}_B(I, B)$$

Conversely, any ideal $I \subset B$ that satisfies this ring condition gives rise to an algebra structur on the module $\mathrm{Hom}_B(I, B)$, which as an R-module has a projective resolution as in Diagram 2. The ring condition can also be interpreted by saying that the hypersurface Y has to be *singular* along Σ. For example, if for local equation $f = 0$ for Y we have that $f \in I_R^2$, (where I_R is the ideal of Σ in R), then $\Sigma \subset Y$ satisfies (R.C.). This particular cased will be studied in more detail in Section five.

Below we will describe how, in the case of a "generic" projection $X \to Y \subset Z$, the algebra structure on A is determined by the map Φ. So we start with diagram 2. The ideal of the image is constructed as follows: the map Φ induces a map $\wedge^{r+1}(\Phi) : \wedge^{r+1}(G) \to \wedge^{r+1}(F)$ and by transposition an injective map $i : \mathcal{L} \hookrightarrow R$, where $\mathcal{L} := \wedge^{r+1}(G) \otimes \wedge^{r+1}(F^*)$ is an invertible module. We define $B := R/\mathcal{L}$. Now A is a B-module. This is Cramer's rule, and an intrinsic way of saying this is by looking at the map $\wedge^r(\Phi) : \wedge^r(G) \to \wedge^r(F)$, which by transposition and the natural isomorphism $\wedge^r(F) \cong \wedge^{r+1}(F^*) \otimes F$ gives a map $\Psi : F \otimes \mathcal{L} \to G$. One has $\Phi\Psi = Id \otimes i$, so Ψ is a homotopy expressing the fact that multiplication with elements of $\mathcal{L}$ is zero on A. From now on we will make the following assumption:

Assumption 12. *The canonical map:*

$$\mathrm{Hom}_B(A, B) \xrightarrow{\text{can}} B$$
$$a \mapsto a(1)$$

is injective.

Therefore $\mathrm{Hom}_B(A, B)$ is via "can" an ideal I in B (and in A) and is called the conductor of the ring map $B \to A$. The map "can" sits in the following diagram with exact rows and columns:

Diagram 13.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & E^* \otimes \mathcal{L} & \xrightarrow{Id} & E^* \otimes \mathcal{L} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow \phi \otimes 1 & & \downarrow & & \\
0 & \to & F^* \otimes \mathcal{L} & \xrightarrow{\Phi^* \otimes 1} & G^* \otimes \mathcal{L} & \longrightarrow & \mathrm{Hom}_B(A, B) & \to & 0 \\
 & & \downarrow p \otimes 1 & & \downarrow \Delta & & \downarrow \text{can} & & \\
0 & \to & \mathcal{L} & \longrightarrow & R & \longrightarrow & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & C & \longrightarrow & C & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The second row is a presentation of $\mathrm{Hom}_B(A, B)$ and can be obtained from diagram 2 essentially by dualization. The third row is the definition of B. The map $p : F^* \to R$ is induced by the inclusion $R \hookrightarrow F$, and the map $\Delta : G^* \otimes \mathcal{L} \to R$ is induced by the composition $R \hookrightarrow F \xrightarrow{\Psi} G \otimes \mathcal{L}^*$ by transposition. The columns of the diagram are obtained by the snake lemma.

The decomposition $F = R \oplus E$ decomposes the map $\Phi : G \to F$ into two maps:

$$\alpha : G \longrightarrow R,$$

$$\phi : G \longrightarrow E.$$

The diagonal map $\alpha^* : \mathcal{L} \to G^* \otimes \mathcal{L}$ is induced by α by transposition and tensoring with $\mathcal{L}$.

The module A can be obtained back as $A \cong \mathrm{Hom}_B(\mathrm{Hom}_B(A, B), B)$ and under this isomorphims the element 1 corresponds to the map *can*. The ring condition (R.C.) $\mathrm{Hom}_B(I, I) = \mathrm{Hom}_B(I, B)$ therefore means that every element $a \in A$, corresponding to an element

$$\hat{a} : \mathrm{Hom}_B(A, B) \to B;\ \phi \mapsto \phi(a)$$

in $\mathrm{Hom}_B(\mathrm{Hom}_B(A, B), B)$ and represented by (a_F, a_G) can be lifted to (b_F, b_G), representing $\hat{b} \in \mathrm{Hom}_B(\mathrm{Hom}_B(A, B), \mathrm{Hom}_B(A, B))$, making the following diagram commutative:

$$\begin{array}{ccccccccc}
 & 0 \to & F^* \otimes \mathcal{L} & \longrightarrow & G^* \otimes \mathcal{L} & \longrightarrow & \mathrm{Hom}_B(A, B) & \longrightarrow & 0 \\
 & b_F \swarrow & | & b_G \swarrow & | & \hat{b} \swarrow & | & & \\
0 \to F^* \otimes \mathcal{L} & \longrightarrow & G^* \otimes \mathcal{L} & \longrightarrow & \mathrm{Hom}_B(A, B) & \longrightarrow 0 & & & \\
 & p \otimes 1 \searrow & \downarrow a_F & \Delta \searrow & \downarrow a_G & \mathrm{can} \searrow & \downarrow \hat{a} & & \\
 & 0 \to & \mathcal{L} & \longrightarrow & R & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

By transposition b_F and b_G induce maps $M(a) : F \to F$ and $L(a) : G \to G$, representing the multiplication by a on A. The maps a_F and a_G are determined by a as follows:

Proposition 14. 1. *The transposition a_F^* is a lift of $a \in A$.*
2. *The transposition $a_G^* \in G \otimes \mathcal{L}^*$ is equal to $\tilde{\Psi} : F \to G \otimes \mathcal{L}^*$ is the map induced by Ψ.*

The proof is left to the reader.

In short, the maps $L(a)$ and $M(a)$ describing the multiplication by $a \in A$ are determined by the following steps:

1. Lift a to $a_F^* \in F$ and get $a_F : F^* \otimes \mathcal{L} \to \mathcal{L}$.
2. Compute a_G^* as $\tilde{\Psi}(a_F^*) \in G \otimes \mathcal{L}^*$ and get $a_G : G^* \otimes \mathcal{L} \to R$.
3. Lift the map a_G over the map $\Delta : G^* \otimes \mathcal{L} \to R$ to get a map $b_G : G^* \otimes \mathcal{L} \to G^* \otimes \mathcal{L}$ and by transposition $L(a) : G \to G$. This is the essential step, and the condition to be able to do this is of course (R.C.).
4. Lift the composition $b_G(\Phi^* \otimes 1)$ over $\Phi^* \otimes 1$ to get $b_F : F^* \otimes \mathcal{L} \to F^* \otimes \mathcal{L}$ and by transposition $M(a) : F \to F$. As the map “can” is injective, this is possible for any choice of b_G in step 3.

5. A particular case

A particular case in which the ring condition, (see Lemma 11) is satisfied arises as follows. Suppose that we are given the R-resolution of Σ of the form:

$$0 \to E^* \xrightarrow{\phi^*} G^* \xrightarrow{\Delta} R \longrightarrow C \to 0 \tag{6}$$

where we assume (for reasons of simplicity) that E and G are *free* R-modules. We choose bases $\{f_k\}$ and $\{g_k\}$ $(k = 0, \dots, r)$ for $F = R \oplus E$ resp. G and assume that $f_0 = 1 \in R$. The map $\phi : G \to E$ has as matrix (ϕ_{ij}), i.e. $\phi(g_j) = \phi_{ij} f_i$. Here and in the sequel we use the Einstein summation convention: indices occuring twice are summed over.

The module $\mathcal{L}$ is trivial and the component $\Delta_i := \Delta(g_i^*)$ can be obtained as the i-th minor of (ϕ_{ij}). Let I_R be the ideal in R generated by the Δ_i. The particular case we want to discuss in some more detail is the case $f \in I_R^2$. We will moreover assume that f is a non-zerodivisor in R. Such an f can always be written as:

$$f = h_{ij}\Delta_i\Delta_j$$

where h_{ij} is a symmetric matrix of elements of R. (If one does not want to assume that E and G are free, then the matrix h_{ij} should be considered as an element h of $S_2(G^*) \otimes \mathcal{L}$.) We now take $\alpha_i = h_{ij}\Delta_j$ and let $\Phi : G \longrightarrow F$ be the map defined by the following matrix:

$$(\Phi_{ij}) = \begin{pmatrix} \alpha_0 & \dots & \alpha_r \\ \phi_{10} & \dots & \phi_{1r} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \phi_{r0} & \dots & \phi_{rr} \end{pmatrix}$$

So $f = \det(\Phi)$, which is a generator for the ideal of a space Y. We will determine the maps L, M and H of Diagram 4 expressing the ring structure of $A = \mathrm{Coker}(\Phi)$. To do this we need some elementary relations between minors of matrices.

Definition 15. *Let $\Phi = (\Phi_{ij})_{0 \le i,j \le r-1}$ be a square matrix of size $r+1$. Let $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_p)$ be strictly increasing sequences of numbers smaller than r. We define $\Psi_{I,J} = (-1)^k \det(\Phi^{I,J})$, where $k = i_1 + \dots + i_p + j_1 + \dots + j_p$ and $\Phi^{I,J}$ is obtained from the matrix Φ by deleting columns $i_1, \dots, i_p$ and rows $j_1, \dots, j_p$. The $\Psi_{I,J}$ for non-strictly increasing sequences of numbers are defined by making $\Psi_{I,J}$ anti-symmetric in both I and J.*

Lemma 16. *One has the following identities:*

1L. $\Phi_{ij}\Psi_{jk} = det(\Phi)\delta_{ik}$
1R. $\Psi_{jk}\Phi_{ij} = det(\Phi)\delta_{ik},$
2L. $\Phi_{ij}\Psi_{jkmn} = \Psi_{kn}\delta_{im} - \Psi_{km}\delta_{in},$
2R. $\Psi_{nmkj}\Phi_{ji} = \Psi_{nk}\delta_{mi} - \Psi_{mk}\delta_{ni},$
3L. $\Phi_{ij}\Psi_{jkmnpq} = \Psi_{kmpq}\delta_{in} + \Psi_{kmqn}\delta_{ip} + \Psi_{kmnp}\delta_{iq},$
3R. $\Psi_{qpnmkj}\Phi_{ji} = \Psi_{qpmk}\delta_{ni} + \Psi_{nqmk}\delta_{pi} + \Psi_{pnmk}\delta_{qi},$

Proof. 1L is Cramer's rule. 2L is obtained by expanding the determinant of Φ by deleting column k and rows m and n and concatenating with the i-th row of Φ with respect to its j-th column. 3L is obtained similarly. The "R-identities" are obtained by "reflection". $\square$

Because of the special shape of the matrix Φ we find it useful to use the following notation.

Definition 17. *We put:*

$$\Delta_i = \Psi_{i0};\ \Delta_{ijk} := \Psi_{ij0k};\ \Delta_{ijkmn} := \Psi_{ijk0mn}.$$

Note that the Δ_i are in fact the components of the map $\Delta : G^* \to R$. The ideal generated by these Δ_i is exactly I, and the ring condition (R.C.) is exactly that $\Psi_{ij} \in I$ for all i and j. The identities we will use all follow from Lemma 16 by putting some index equal to zero are are summarized in:

Lemma 18. 1. $\alpha_k \Delta_{kij} = \Psi_{ij},\ \ j \geq 1,$
2. $\phi_{ij}\Delta_{jkm} = -\Delta_k \delta_{im},$
3. $\Delta_{ijk}\phi_{km} = \Delta_i \delta_{jm} - \Delta_j \delta_{im},$
4. $\Delta_{qpnkj}\phi_{ji} = \Delta_{qpk}\delta_{ni} + \Delta_{nkq}\delta_{pi} + \Delta_{pnk}\delta_{qi}.$

Theorem 19. *Matrices L^p and M^p, representing multiplication by f_p, i.e. making a commutative diagram:*

$$\begin{array}{ccccccc} 0 \to & G & \xrightarrow{\Phi} & F & \longrightarrow & A & \longrightarrow 0 \\ & \downarrow L^p & & \downarrow M^p & & \downarrow \cdot f_p & \\ 0 \to & G & \xrightarrow{\Phi} & F & \longrightarrow & A & \longrightarrow 0 \end{array}$$

are given by

$$L^p_{ij} = h_{jk}\Delta_{kip}$$

$$M^p_{ij} = \begin{cases} 1 & \text{for } j = 0 \text{ and } i = p \\ \frac{1}{2}Tr(L^p L^j) & \text{for } i = 0 \text{ and } j > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Substituting $\alpha_k = h_{km}\Delta_m$ in the first identity of Lemma 18 we obtain:

$$\Psi_{ip} = h_{km}\Delta_m \Delta_{kip}$$

As explained in the fourth section, the map $L^{p*} : G^* \to G^*$ is a lift of Ψ_p over Δ, where $\Psi_p(g_i^*) = \Psi_{ip}$, i.e. we have a commutative diagram:

$$\begin{array}{ccc} & & G^* \\ & L^{p*} \nearrow & \downarrow \Delta \\ G^* & \xrightarrow{\Psi_p} & R. \end{array}$$

So we can take $L^{p*}_{mi} = h_{km}\Delta_{kip}$. This gives the first statement. To prove the statement about M^p we have to show the commutativity of the diagram in the statement of the theorem. Because of the special shape of the matrix M^p this is equivalent to the statements:

A. $(\phi L^p)(g_j) = \alpha_j f_p$,
B. The following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & E \\ L^p \downarrow & & \mu \downarrow \\ G & \xrightarrow{\alpha} & R \end{array}$$

where $\mu(f_i) = \frac{1}{2}Tr(L^p L^i)$.

Indeed $(\phi L^p)_{ij} = \phi_{ik}L^p_{kj} = \phi_{ik}h_{jm}\Delta_{mkp}$. By the second identity in Lemma 18 this is equal to $h_{jm}\Delta_m\delta_{ip}$. This gives A. To prove B, we calculate $2\mu\phi(g_j)$. This by definition is equal to $\phi_{ij}h_{ab}\Delta_{bci}h_{cd}\Delta_{dap}$. We use the third identity in Lemma 18 to rewrite this as $h_{jd}\Delta_{dap}h_{ab}\Delta_b - \Delta_c h_{cd}\Delta_{dap}h_{aj}$. Because of the symmetry of h_{ij} and the anti-symmetry of Δ_{dap} in d and a we see that the above expression is equal to $2hjd\Delta_{dap}h_{ab}\Delta_b$ which by definition is $2\alpha_a L^p_{aj}$ which proves B. □

Theorem 20. *The homotopy $H : \wedge^2(E) \otimes F \to G$ has as matrix $\frac{1}{2}(\mathcal{E}_{pqij} - \mathcal{E}_{qpij})$ where $\mathcal{E}_{pqij} = h_{ab}\Delta_{bcp}h_{cd}\Delta_{daiqj}$.*

Proof. We have to prove $H(1 \otimes \Phi) = [L, L]$ and $\Phi H = [M, M]$. It suffices to prove the first equality because from this it follows that $\Phi H(1 \otimes \Phi) = \Phi[L, L] = [M, M](1 \otimes \Phi)$. We compose with Ψ and conclude that $\Phi H(1 \otimes f \cdot Id) = [M, M](1 \otimes f \cdot Id)$. As we assume f to be a non-zerodivisor the second equality follows. We compute:

$$\mathcal{E}_{pqim}\Phi_{mk} = h_{ab}\Delta_{bcp}h_{cd}\Delta_{daiqm}\Phi_{mk}.$$

By the fourth identity in Lemma 18 this is equal to:

$$\begin{aligned} &h_{ab}\Delta_{bcp}h_{cd}(\Delta_{daq}\delta_{ik} + \Delta_{idq}\delta_{ak} + \Delta_{aiq}\delta_{dk}) \\ &\qquad = h_{ab}\Delta_{bcp}h_{cd}\Delta_{daq}\delta_{ik} + 2h_{kb}\Delta_{bcp}h_{cd}\Delta_{idq} \end{aligned}$$

by relabeling the indices in the last term and using the anti-symmetry of the Δ's. Note that the first term in the last expression is symmetric in p and q, and therefore vanishes if one computes $H(1 \otimes \Phi)$. On the other hand we compute $(L^p L^q)_{ik} = L^p_{ic}L^q_{ck} = h_{cd}\Delta_{dip}h_{kb}\Delta_{bcq}$. After relabeling the indices one sees that $H_{pqim}\Phi_{mk} = [L^p, L^q]_{ik}$.

Remark 21. The maps L, M and H can be described intrinsically in terms of Φ and $h \in S_2(G^*) \otimes \mathcal{L}$. However, to prove the commutativities expressed by Theorems 19 and 20 this basis free approach seems to be of no help. Rather, the notation with *diagrammatic tensors* [5] is appropriate for this type of calculations.

Corollary 22. *If $(R, \mathbf{m})$ is a local ring, the entries of ϕ_{ij} are in $\mathbf{m}$ and $f \in I_R^2$ as above, then the complex $\mathbf{L}.$ of Theorem 10 is a minimal resolution of $A = \mathrm{Cok}(\Phi)$ as S-module (after localizing at $(\mathbf{m}, E)$).*

Proof. This follows from Theorem 10 and the explicit formulas for L, M and H given in Theorems 19 and 20. □

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