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Real Line Arrangements and Surfaces with Many Real Nodes

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Summary. A long standing question is if the maximum number $\mu(d)$ of nodes on a surface of degree d in $\mathbb{P}^3(\mathbb{C})$ can be achieved by a surface defined over the reals which has only real singularities. The currently best known asymptotic lower bound, $\mu(d) \gtrapprox \frac{5}{12}d^3$, is provided by Chmutov's construction from 1992 which gives surfaces whose nodes have non-real coordinates.

Using explicit constructions of certain real line arrangements we show that Chmutov's construction can be adapted to give only real singularities. All currently best known constructions which exceed Chmutov's lower bound (i.e., for $d = 3, 4, \ldots, 8, 10, 12$) can also be realized with only real singularities. Thus, our result shows that, up to now, all known lower bounds for $\mu(d)$ can be attained with only real singularities.

We conclude with an application of the theory of real line arrangements which shows that our arrangements are aymptotically the best possible ones for the purpose of constructing surfaces with many nodes. This proves a special case of a conjecture of Chmutov.

3.1 Introduction

A node (or A_1 singularity) in $\mathbb{C}^3$ is a singular point which can be written in the form $x^2 + y^2 + z^2 = 0$ in some local coordinates. We denote by $\mu(d)$ the maximum possible number of nodes on a surface in $\mathbb{P}^3(\mathbb{C})$. The question of determining $\mu(d)$ has a long and rich history. Currently, $\mu(d)$ is only known for $d = 1, 2, \ldots, 6$ (see [1, 12] for sextics and [15] for a recent improvement for septics).

In this paper, we consider the relationship between $\mu(d)$ and the maximum possible number of real nodes on a surface in $\mathbb{P}^3(\mathbb{R})$ which we denote by $\mu^{\mathbb{R}}(d)$. Obviously, $\mu^{\mathbb{R}}(d) \leq \mu(d)$, but do we even have $\mu^{\mathbb{R}}(d) = \mu(d)$? In other words: Can the maximum number of nodes be achieved with real surfaces with real singularities?

The previous question arises naturally because all results in low degree $d \leq 12$ suggest that it could be true (see [1, 8, 9, 15, 19] and table 3.1). But the best known asymptotic lower bound, $\mu(d) \gtrapprox \frac{5}{12}d^3$, follows from Chmutov's construction [5] which yields only singularities with non-real coordinates. In this paper, we show that his construction can be adapted to give surfaces with only real singularities (see table 3.1). In the real case we can distinguish between two types of nodes, *conical nodes* ($x^2+y^2-z^2=0$) and *solitary points* ($x^2+y^2+z^2=0$): Our construction produces only conical nodes.

Notice that in general there are no better real upper bounds for $\mu^{\mathbb{R}}(d)$ known than the well-known complex ones of Miyaoka [17] and Varchenko [20]. But in some cases, for solitary points there exist better bounds via the relation to the zero$^{\text{th}}$ Betti number. E.g., it has been shown by Nikulin that a K3 surface cannot have more than 10 solitary points (although it can have 16 conical nodes). For quartic surfaces in $\mathbb{P}^3$ this result is probably due to R.W.H.T. Hudson (see [7] for an overview on related results).

We show an upper bound of $\approx \frac{5}{6}d^2$ for the maximum number of real critical points on two levels of real simple line arrangements consisting of d lines; here, *simple* means that no three lines meet in a common point. In [6], Chmutov conjectured this to be the maximum number for all complex plane curves of degree d. He also noticed [5] that such a bound directly implies an upper bound for the number of real nodes of certain surfaces. Our upper bound shows that our examples are asymptotically the best possible real line arrangements for this purpose.

d	1	2	3	4	5	6	7	8	9	10	11	12	13	d
$\mu(d), \mu^{\mathbb{R}}(d) \leq$	0	1	4	16	31	65	104	174	246	360	480	645	832	$\frac{4}{9}d(d-1)^2$
$\mu(d), \mu^{\mathbb{R}}(d) \geq$	0	1	4	16	31	65	99	168	**216**	345	**425**	600	**732**	$\approx \mathbf{\frac{5}{12}d^3}$

Table 3.1. The currently known bounds for the maximum number $\mu(d)$ (resp. $\mu^{\mathbb{R}}(d)$) of nodes on a surface of degree d in $\mathbb{P}^3(\mathbb{C})$ (resp. $\mathbb{P}^3(\mathbb{R})$) are equal. The bold numbers indicate in which cases our result improves the previously known lower bound for $\mu^{\mathbb{R}}(d)$.

3.2 Variants of Chmutov's Surfaces with Many Real Nodes

Let $T_d(z) \in \mathbb{R}[z]$ be the Tchebychev polynomial of degree d with critical values -1 and $+1$ (see fig. 3.2). This can either be defined recursively by $T_0(z) := 1,\ T_1(z) := z,\ T_d(z) := 2{\cdot}z{\cdot}T_{d-1}(z) - T_{d-2}(z)$ for $d \geq 2$, or implicitly by $T_d(\cos(z)) = \cos(dz)$. Chmutov [5] uses them together with the so-called *folding polynomials* $F_d^{A_2}(x,y) \in \mathbb{R}[x,y]$ associated to the root-system A_2 to construct surfaces

$$\mathrm{Chm}_d^{A_2}(x,y,z) := F_d^{A_2}(x,y) + \frac{1}{2}(T_d(z)+1)$$

with many nodes. These folding polynomials are defined as follows:

$$F_d^{A_2}(x,y) := 2 + \det \begin{pmatrix} x & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 2y & x & \ddots & \ddots & & & \vdots \\ 3 & y & \ddots & \ddots & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & y & x \end{pmatrix} + \det \begin{pmatrix} y & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 2x & y & \ddots & \ddots & & & \vdots \\ 3 & x & \ddots & \ddots & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & x & y \end{pmatrix}. \tag{3.1}$$

The $F_d^{A_2}(x,y)$ have critical points with only three different critical values: $0, -1$, and 8. Thus, the surface $\mathrm{Chm}_d^{A_2}(x,y,z)$ is singular exactly at those points at which the critical values of $F_d^{A_2}(x,y)$ and $\frac{1}{2}(T_d(z)+1)$ sum up to zero (i.e., either both are 0 or the first is -1 and the second is $+1$).

Notice that the plane curve defined by $F_d^{A_2}(x,y)$ consists in fact of d lines. But these are not real lines and the critical points of this folding polynomial also have non-real coordinates. It is natural to ask whether there is a real line arrangement which leads to the same number of critical points. The term *folding polynomials* was introduced in [21] (here we use a slightly different definition). In his article, Withers also described many of their properties, but it was Chmutov [5] who noticed that $F_d^{A_2}(x,y)$ has only few different critical values. In [3], the first author computed the critical points of the other folding polynomials. Among these, there are the following examples which are the real line arrangements we have been looking for (see [3, p. 87–89]):

We define the real folding polynomial $F_{\mathbb{R},d}^{A_2}(x,y) \in \mathbb{R}[x,y]$ associated to the root system A_2 as (see also fig. 3.2)

$$F_{\mathbb{R},d}^{A_2}(x,y) := F_d^{A_2}(x+iy,\ x-iy), \tag{3.2}$$

where i is the imaginary number. It is easy to see that the $F_{\mathbb{R},d}^{A_2}(x,y)$ have indeed real coefficients. The numbers of critical points are the same as those of $F_d^{A_2}(x,y)$; but now they have real coordinates as the following lemma shows:

Lemma 1. *The real folding polynomial $F_{\mathbb{R},d}^{A_2}(x,y)$ associated to the root system A_2 has $\binom{d}{2}$ real critical points with critical value 0 and*

$$\frac{1}{3}d^2 - d \quad \text{if } d \equiv 0 \mod 3, \qquad \frac{1}{3}d^2 - d + \frac{2}{3} \quad \text{otherwise} \tag{3.3}$$

real critical points with critical value -1. The other critical points also have real coordinates and have critical value 8.

Proof. We proceed similar to the case discussed in [5], see [3, p. 87–95] for details. To calculate the critical points of the real folding polynomial $F_{\mathbb{R},d}^{A_2}$, we use the map $h^1 : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$(u,v) \mapsto \begin{pmatrix} \cos(2\pi(u+v)) + \cos(2\pi u) + \cos(2\pi v) \\ \sin(2\pi(u+v)) - \sin(2\pi u) - \sin(2\pi v) \end{pmatrix}.$$

This is in fact just the real and imaginary part of the first component of the generalized cosine h considered by Withers [21] and Chmutov [5]. It is easy to see that h^1 is a coordinate change if $u-v>0$, $u+2v>0$, and $2u+v<1$. It transforms the polynomial $F^{A_2}_{\mathbb{R},d}$ into the function $G^{A_2}_d : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$G^{A_2}_d(u,v) := F^{A_2}_{\mathbb{R},d}(h^1(u,v)) = 2\cos(2\pi du)+2\cos(2\pi dv)+2\cos(2\pi d(u+v))+2.$$

The calculation of the critical points of $G^{A_2}_d$ is exactly the same as the one performed in [5]. As the function $G^{A_2}_d$ has $(d-1)^2$ distinct real critical points in the region defined by $u-v>0$, $u+2v>0$, and $2u+v<1$, the images of these points under the map h^1 are all the critical points of the real folding polynomial $F^{A_2}_{\mathbb{R},d}$ of degree d. In contrast to [5], we get real critical points because h^1 is a map from $\mathbb{R}^2$ into itself.

None of the other root systems yield more critical points on two levels. But as mentioned in [16], the real folding polynomials associated to the root system B_2 give hypersurfaces in $\mathbb{P}^n$, $n \geq 5$, which improve the previously known lower bounds for the maximum number of nodes in higher dimensions slightly (see [16]; [3] gives a detailed discussion of all these folding polynomials and their critical points).

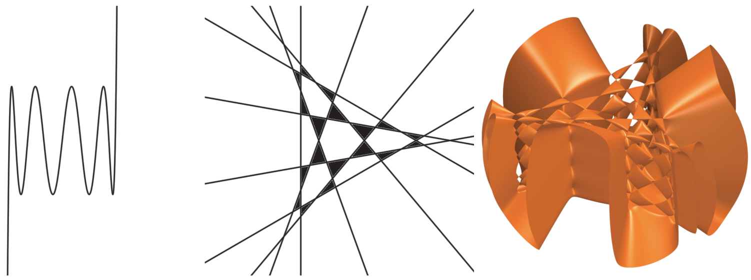

Fig. 3.1. For degree $d = 9$ we show the Tchebychev polynomial $T_9(z)$, the real folding polynomial $F^{A_2}_{\mathbb{R},9}(x,y)$ associated to the root system A_2, and the surface $\mathrm{Chm}^{A_2}_{\mathbb{R},9}(x,y,z)$. The bounded regions in which $F^{A_2}_{\mathbb{R},9}(x,y)$ takes negative values are marked in black.

The lemma immediately gives the following variant of Chmutov's nodal surfaces:

Theorem 2. *Let $d \in \mathbb{N}$. The real projective surface of degree d defined by*

$$\mathrm{Chm}^{A_2}_{\mathbb{R},d}(x,y,z) \;:=\; F^{A_2}_{\mathbb{R},d}(x,y) + \frac{1}{2}(T_d(z)+1) \;\in\; \mathbb{R}[x,y,z] \tag{3.4}$$

has the following number of real nodes:

$$\begin{array}{ll} \frac{1}{12}\left(5d^3-13d^2+12d\right) & \text{if } d \equiv 0 \mod 6, \\ \frac{1}{12}\left(5d^3-13d^2+16d-8\right) & \text{if } d \equiv 2,4 \mod 6, \\ \frac{1}{12}\left(5d^3-14d^2+13d-4\right) & \text{if } d \equiv 1,5 \mod 6, \\ \frac{1}{12}\left(5d^3-14d^2+9d\right) & \text{if } d \equiv 3 \mod 6. \end{array} \tag{3.5}$$

These numbers are the same as the numbers of complex nodes of Chmutov's surfaces $\mathrm{Chm}_d^{A_2}(x,y,z)$. To our knowledge, the result gives new lower bounds for the maximum number $\mu^{\mathbb{R}}(d)$ of real singularities on a surface of degree d in $\mathbb{P}^3(\mathbb{R})$ for $d = 9, 11$ and $d \geq 13$, see table 3.1. Notice that all best known lower bounds for $\mu^{\mathbb{R}}(d)$ are attained by surfaces with only conical nodes which is not astonishing in view of the upper bounds for solitary points mentioned in the introduction.

3.3 On Two-Colorings of Real Simple Line Arrangements

The real folding polynomials $F_{\mathbb{R},d}^{A_2}(x,y)$ used in the previous section are in fact *real simple (straight) line arrangements* in $\mathbb{R}^2$, i.e., lines no three of which meet in a point. Such arrangements can be 2-colored in a natural way (see fig. 3.2): We label in black those regions (*cells*) of $\mathbb{R}^2 \setminus \{F_{\mathbb{R},d}^{A_2}(x,y) = 0\}$ in which $F_{\mathbb{R},d}^{A_2}(x,y)$ takes negative values, the others in white. The bounded black regions in fig. 3.2 contain exactly one critical point with critical value -1 each.

Harborth has shown in [11] that the maximum number $M_b(d)$ of black cells in such real simple line arrangements of d lines satisfies:

$$M_b(d) \leq \begin{cases} \frac{1}{3}d^2 + \frac{1}{3}d, \ d \text{ odd}, \\ \frac{1}{3}d^2 + \frac{1}{6}d, \ d \text{ even}. \end{cases} \tag{3.6}$$

d of these cells are unbounded. This is a purely combinatorial result which is strongly related to the problem of determining the maximum number of triangles in such arrangements which has a long and rich history (see [10]). Notice that this bound is better than the one obtained by Kharlamov using Hodge theory [13]. It is known that the bound (3.6) is exact for infinitely many values of d. The real folding polynomials $F_{\mathbb{R},d}^{A_2}(x,y)$ almost achieve this bound. Moreover, our arrangements have the very special property that all critical points with a negative (resp. positive) critical value have the same critical value -1 (resp. $+8$).

To translate the upper bound on the number of black cells into an upper bound on critical points we use the following lemma:

Lemma 3 (see Lemme 10, 11 in [18]). *Let f be a real simple line arrangement consisting of $d \geq 3$ lines. Then f has exactly $\binom{d-1}{2}$ bounded open cells each of which contains exactly one critical point. Moreover, all the critical points of f are non-degenerate. No unbounded open cell contains a critical point.*

It is easy to prove the lemma, e.g. by counting the number of bounded cells and by observing that each such cell contains at least one critical point. Comparing this with the number $(d-1)^2 - \binom{d}{2} = \binom{d-1}{2}$ of all critical points with non-zero critical values gives the result. Now we can show that our real line arrangements are asymptotically the best possible ones for constructing surfaces with many singularities:

Theorem 4. *The maximum number of critical points with the same non-zero real critical value $0 \neq v \in \mathbb{R}$ of a real simple line arrangement is bounded by $M_b(d) - d$, where d is the number of lines. In particular, the maximum number of critical points on two levels of such an arrangement does not exceed $\binom{d}{2} + M_b(d) - d \approx \frac{5}{6}d^2$.*

Proof. By the preceding lemma, the number of critical points with non-zero critical value equals the number of bounded cells of the real simple line arrangement. The upper bound (3.6) for the maximum number $M_b(d)$ of black cells of a real simple line arrangement now gives the result, because the line arrangement has exactly $\binom{d}{2}$ critical points with critical value 0.

Chmutov showed a much more general result ([4], see [6] for the case of non-degenerate critical points): For a plane curve of degree d the maximum number of critical points on two levels does not exceed $\approx \frac{7}{8}d^2$. In [6], he conjectured $\approx \frac{5}{6}d^2$ to be the actual maximum which is attained by the complex line arrangements $F_d^{A_2}(x, y)$ he used for his construction (and also by the real line arrangement $F_{\mathbb{R},d}^{A_2}(x, y)$). Thus, our theorem 4 is the verification of Chmutov's conjecture in the particular case of real simple line arrangements. As Chmutov remarked in [5], such an upper bound immediately implies an upper bound on the maximum number of nodes on a surface in separated variables:

Corollary 5. *A surface of the form $p(x, y) + q(z) = 0$ cannot have more than $\approx \frac{1}{2}d^2 \cdot \frac{1}{2}d + \frac{1}{3}d^2 \cdot \frac{1}{2}d = \frac{5}{12}d^3$ nodes if $p(x, y)$ is a real simple line arrangement. This number is attained by the surfaces $\mathrm{Chm}_{\mathbb{R},d}^{A_2}(x, y, z)$ defined in theorem 2.*

3.4 Concluding Remarks

Comparing our bound from corollary 5 to the upper bound $\approx \frac{5}{12}d^3$ on the zero$^{\text{th}}$ Betti number (see e.g., [2] or [7]) one is tempted to ask if it is possible to deform our singular surfaces to get examples with many real connected components. But our surfaces $\mathrm{Chm}_{\mathbb{R},d}^{A_2}(x, y, z)$ only contain A_1^- singularities which locally look like a cone ($x^2 + y^2 - z^2 = 0$). When removing the singularities from the zero-set of the surface every connected component contains at least three of the singularities. Thus, the zero$^{\text{th}}$ Betti number of a small deformation of our surfaces are not larger than $\approx \frac{5}{3 \cdot 12}d^3$ which is far below the number $\approx \frac{13}{36}d^3$ resulting from Bihan's construction [2].

Conversely, we may ask if it is always possible to move the lines of a simple real line arrangement in such a way that all critical points which have a critical value of the same sign can be chosen to have the same critical value. If this were true then it would be possible to improve our lower bound for the maximum number $\mu^{\mathbb{R}}(d)$ of real nodes on a real surface of degree d slightly because it is known that the upper bounds for the maximum number $M_b(d)$ of black cells are in fact exact for infinitely many d. E.g., in the already cited article [11], Harborth gave an explicit arrangement of 13 straight lines which has $\frac{1}{3} \cdot 13^2 + \frac{1}{3} \cdot 13 - 13 = 47$ bounded black regions. When regarding this arrangement as a polynomial of degree $d = 13$ it has exactly one critical point with a negative critical value within each of the black regions. If all these negative critical values can be chosen to be the same then such a polynomial will lead to a surface with $\binom{13}{2} \cdot \lceil \frac{13-1}{2} \rceil + 47 \cdot \lfloor \frac{13-1}{2} \rfloor = 750 > 732$ nodes. Similarly, such a surface of degree 9 would have $228 > 216$ nodes. In the case of degree 7 the construction would only yield 96 nodes which is less than the number 99 found in [15].

Notice that it is not clear that line arrangements are the best plane curves for our purpose, and we may ask: Is it possible to exceed the number of critical points on two levels of the line arrangements $F^{A_2}_{\mathbb{R},d}(x, y)$ using irreducible curves of higher degrees? Either in the real or in the complex case? This is not true for the real folding polynomials. E.g., those associated to the root system B_2 consist of many ellipses and yield surfaces with fewer singularities (see [3]).

References

1. W. Barth. Two Projective Surfaces with Many Nodes, Admitting the Symmetry of the Icosahedron. *J. Algebraic Geom.*, 5(1):173–186, 1996.
2. F. Bihan. Asymptotic behaviour of Betti numbers of real algebraic surfaces. *Comment. Math. Helv.*, 78:227–244, 2003.
3. S. Breske. Konstruktion von Flächen mit vielen reellen Singularitäten mit Hilfe von Faltungspolynomen. Diploma Thesis. University of Mainz, 2005. Available from [14].
4. S.V. Chmutov. Spectrum and equivariant deformations of critical points. *Uspekhi mat. nauk*, 39(4):113–114, 1984. In Russian.
5. S.V. Chmutov. Examples of Projective Surfaces with Many Singularities. *J. Algebraic Geom.*, 1(2):191–196, 1992.
6. S.V. Chmutov. Extremal distributions of critical points and critical values. In D. T. Lê, K. Saito, and B. Teissier, editors, *Singularity Theory*, pages 192–205, 1995.
7. A. Degtyarev and V. Kharlamov. Topological properties of real algebraic varieties: Du côté de chez Rokhlin. *Russ. Math. Surv.*, 55(4):735–814, 2000.
8. S. Endraß. Flächen mit vielen Doppelpunkten. *DMV-Mitteilungen*, 4(4):17–20, 1995.
9. S. Endraß. A Projective Surface of Degree Eight with 168 Nodes. *J. Algebraic Geom.*, 6(2):325–334, 1997.
10. J.E. Goodman and J. O'Rourke, editors. *Handbook of Computational Geometry*, chapter 5: Pseudoline Arrangements. Chapman & Hall/CRC, 2nd edition, 2004.
11. H. Harborth. Two-Colorings of Simple Arrangements. In *Finite and Infinite Sets*, number 37 in Colloquia Mathematica Societatis János Bolyai, pages 371–378. North-Holland, 1981.
12. D.B. Jaffe and D. Ruberman. A Sextic Surface cannot have 66 Nodes. *J. Algebraic Geom.*, 6(1):151–168, 1997.
13. V. Kharlamov. Overview of topological properties of real algebraic surfaces. math.AG/0502127, 2005.
14. O. Labs. Algebraic Surface Homepage. Information, Images and Tools on Algebraic Surfaces. *www.AlgebraicSurface.net*, 2003.
15. O. Labs. A Septic with 99 Real Nodes. Preprint, math.AG/0409348, to appear in: *Rend. Sem. Mat. Univ. Pad.*, 2004.
16. O. Labs. Dessins D'Enfants and Hypersurfaces in $\mathbb{P}^3$ with many A_j-Singularities. Preprint, math.AG/0505022, 2005.
17. Y. Miyaoka. The Maximal Number of Quotient Singularities on Surfaces with Given Numerical Invariants. *Math. Ann.*, 268:159–171, 1984.
18. A. Ortiz-Rodriguez. Quelques aspects sur la géométrie des surfaces algébriques réelles. *Bull. Sci. Math.*, 127:149–177, 2003.

19. A. Sarti. Pencils of symmetric surfaces in $\mathbb{P}^3$. *J. Algebra*, 246(1):429–452, 2001.
20. A.N. Varchenko. On the Semicontinuity of the Spectrum and an Upper Bound for the Number of Singular Points of a Projective Hypersurface. *J. Soviet Math.*, 270:735–739, 1983.
21. W.D. Withers. Folding Polynomials and Their Dynamics. *Amer. Math. Monthly*, 95:399–413, 1988.

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