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THE INTERMEDIATE JACOBIANS OF THE THETA DIVISORS OF FOUR-DIMENSIONAL PRINCIPALLY POLARIZED ABELIAN VARIETIES

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Abstract

Let A be a principally polarized abelian variety of dimension four and let $\Theta \subset A$ be a symmetric theta-divisor, which we assume to be smooth. Using the Hodge structure on $H^3(\Theta)$ we associate to A two abelian subvarieties $J(\mathbf{K}) \subset J(\mathbf{H})$ of the intermediate jacobian $J(\Theta)$ of Θ of dimensions five and nine respectively. We show that $J(\mathbf{H})$ is generated by the image under the Abel-Jacobi map of the family $\mathscr{F}$ of Prym-embedded curves in Θ and that there is a commutative diagram

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & J(\Theta) \\ \downarrow & & \downarrow \\ \mathscr{P}^{-1}(A) & \longrightarrow & J(\mathbf{Q}) \end{array}$$

where $J(\mathbf{Q})$ is the dual abelian variety of $J(\mathbf{K})$, $\mathscr{P} : \mathscr{R}_5 \longrightarrow \mathscr{A}_4$ is the *Prym map*, the two vertical arrows are onto and the image of $\mathscr{P}^{-1}(A)$ generates $J(\mathbf{Q})$.

Introduction

Let A be a principally polarized abelian variety (*ppav*) of dimension four, let Θ be a symmetric theta divisor for A, and assume that Θ is smooth. The cohomology group

$$H^3(\Theta, \mathbb{Z})$$

contains a natural rank 10 sublattice

$$\mathbf{K} := \operatorname{Ker}(H^3(\Theta, \mathbb{Z}) \xrightarrow{\cup\theta} H^5(A, \mathbb{Z})).$$

So we obtain a five-dimensional complex subtorus $J(\mathbf{K})$ of the intermediate jacobian $J(\Theta)$ of Θ. It could be called the "primitive intermediate jacobian of Θ". We denote the dual torus of $J(\mathbf{K})$ by $J(\mathbf{Q})$, this is a quotient of $J(\Theta)$. We also define a complex torus $J(\mathbf{H}) \subset J(\Theta)$ which

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sits in an exact sequence

$$0 \longrightarrow J(\mathbf{K}) \longrightarrow J(\mathbf{H}) \longrightarrow A \longrightarrow 0.$$

These complex tori (all except $J(\Theta)$) are in fact abelian varieties. In general, the intermediate jacobian $J(V)$ of a threefold V with $h^{3,0}(V) \neq 0$ does not contain any nonzero abelian subvarieties. Grothendieck's version of the Hodge conjecture states that if there is a nonzero abelian variety in $J(V)$, then it should be generated by the image under the Abel-Jacobi map of some family of curves in V. We use the Prym map to show that the conjecture is true for $J(\mathbf{H}) \subset J(\Theta)$ and hence for $J(\mathbf{K}) \subset J(\Theta)$:

To A one can associate a smooth cubic threefold $T \subset \mathbf{P}^4 = \mathbf{P}(\Gamma_{00})$ ([15]). The intermediate jacobian $J(T)$ of T is an abelian variety of dimension five, isomorphic to the Albanese variety of the variety F parametrizing the family of lines in T ([8]). The fiber $\mathscr{P}^{-1}(A)$ of the Prym map at A maps onto F with generically finite fibers of cardinality 2 ([15]). There is an involution $\lambda : (\widetilde{X}, X) \mapsto (\widetilde{X}_\lambda, X_\lambda)$ acting in the fibers of the map $\mathscr{P}^{-1}(A) \longrightarrow F$ such that (see [15]):

"*The curve* $\widetilde{X}_\lambda$ *parametrizes exactly the Prym-embeddings of* $\widetilde{X}$ *into* $\Theta \subset A$".

The variety $\mathscr{F}$ parametrizing the family of Prym-embedded curves in Θ therefore maps onto $\mathscr{P}^{-1}(A)$ with fiber $\widetilde{X}_\lambda$ at $(\widetilde{X}, X)$. We show that the image of $\mathscr{F}$ in $J(\Theta)$ by the Abel-Jacobi mapping generates $J(\mathbf{H})$ and that we have a commutative diagram

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & J(\Theta) \\ \downarrow & & \downarrow \\ \mathscr{P}^{-1}(A) & \longrightarrow & J(\mathbf{Q}) \end{array}$$

such that the image of $\mathscr{P}^{-1}(A)$ generates $J(Q)$. When A is generic, it is well-known that $\mathscr{P}^{-1}(A)$ is smooth. We show that in this case $\mathscr{F}$ is also smooth. The above diagram induces therefore the commutative diagram:

$$\begin{array}{ccc} \mathrm{Alb}(\mathscr{F}) & \longrightarrow & \widetilde{J(\Theta)} \\ \downarrow & & \downarrow \\ \mathrm{Alb}(\mathscr{P}^{-1}(A)) & \longrightarrow & J(Q) \end{array}$$

We show that the bottom horizontal map is actually an isomorphism. When A is generic, $\mathscr{P}^{-1}(A)$ is actually an étale double cover of F (see [10] and [15]) and $\mathrm{Alb}(\mathscr{P}^{-1}(A))$ is also a double cover of $\mathrm{Alb}(F) \cong J(T)$. So we deduce that $J(\mathbf{Q})$ is a double cover of $J(T)$.

The structure of the paper is as follows. In §1 we review the basics about Lefschetz theory, intermediate jacobians and polarizations that we need. In §2 we show the existence of the above diagrams and the fact that the image of $\mathscr{P}^{-1}(A)$ generates $J(\mathbf{Q})$. This is done in the usual way by a computation at the level of tangent spaces. In §3 we show that $\mathscr{F}$ is smooth and its Albanese variety has dimension 9 when A is generic. In §4 we show that the map $\mathrm{Alb}(\mathscr{P}^{-1}(A)) \longrightarrow J(\mathbf{Q})$ is an isomorphism and that the image of $\mathrm{Alb}(\mathscr{F})$ is the torus $J(\mathbf{H})$ for A generic. This is done by degeneration to the case where A is the jacobian of a smooth curve of genus 4. We then deduce from this that for *any* A with smooth theta divisor the image of $\mathscr{F}$ by the Abel-Jacobi map generates $J(\mathbf{H})$.

Conventions. Unless otherwise stated, all homology and cohomology groups are with integer coefficients. All varieties we consider are over the field $\mathbb{C}$ of complex numbers.

1. The primitive intermediate jacobian

1.1. Lefschetz theory. Let A be an abelian variety of dimension $n+1$ over $\mathbb{C}$ and let $\Theta \subset A$ be a smooth and ample hypersurface. There is a strong relation between the cohomologies of A and Θ. For instance, one has:

Proposition 1.1 (Weak Lefschetz theorem). *Let $\Theta \subset A$ be smooth and ample. Then*

$j_* : H_k(\Theta) \longrightarrow H_k(A)$	$j^! : H^k(\Theta) \longrightarrow H^{k+2}(A)$
$j^* : H^k(A) \longrightarrow H^k(\Theta)$	$j_! : H_{k+2}(A) \longrightarrow H_k(\Theta)$
are isomorphisms for $k < n$	*are isomorphisms for $k > n$*

Also j_* and $j_!$ are surjective for $k = n$, j^* and $j^!$ are injective for $k = n$.
Furthermore, one has:

$$j^! = P_\Theta \cdot j^* \cdot P_A^{-1}, \quad j_! = P_A^{-1} \cdot j_* \cdot P_\Theta, \quad \cup[\Theta] = j_! \cdot j^*.$$

Here $P_\Theta : H^k(\Theta) \longrightarrow H_{2n-k}(\Theta)$ and $P_A : H^k(A) \longrightarrow H_{2n+2-k}(A)$ are the Poincaré duality maps, and $\cup[\Theta]$ is the cup product with the fundamental class of Θ.

Also, if the (co)homology of A has no torsion, then the (co)homology of Θ has no torsion.

Proof. In the case where Θ is very ample see, for instance, [16] or [21]. Since the proofs only depend on the fact that the complement of Θ is an affine variety, they work for an ample smooth divisor in an abelian variety (see [19]). The statement about the torsion follows from the universal coefficient theorem and Poincaré duality: $\operatorname{tors}(H_{k-1}) = \operatorname{tors}(H^k) = \operatorname{tors}(H_{2n-k})$ ($H = H(\Theta)$). As the lower homology of Θ is that of A, the result follows. ◊

So there is only one "new" group $H^n(\Theta)$, with two maps

$$H^n(A) \hookrightarrow H^n(\Theta) \longrightarrow H^{n+2}(A)$$

whose composition is equal to $\cup[\Theta]$. The hard Lefschetz theorem says that this map is an isomorphism over $\mathbb{Q}$, so it is injective over $\mathbb{Z}$ (with finite cokernel), because there is no torsion. We make the following

Definition 1.2.

$$\begin{aligned} \mathbf{K} &:= \operatorname{Ker}(j_! : H^n(\Theta) \longrightarrow H^{n+2}(A)), \\ \mathbf{Q} &:= \operatorname{Coker}(j^* : H^n(A) \hookrightarrow H^n(\Theta)). \end{aligned}$$

Proposition 1.3. (i) *With respect to the intersection pairing*

$$Q : H^n(\Theta) \otimes H^n(\Theta) \longrightarrow \mathbb{Z}$$

the inclusion $\mathbf{K} \hookrightarrow H^n(\Theta)$ *is dual to the surjection* $H^n(\Theta) \longrightarrow \mathbf{Q}$.

(ii) *There is an exact sequence of the form*

$$0 \longrightarrow \mathbf{K} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{T} \longrightarrow 0$$

where

$$\mathbf{T} := \operatorname{Coker}([\Theta]\cup : H^n(A) \hookrightarrow H^{n+2}(A))$$

is a torsion group, which inherits a nondegenerate pairing

$$\mathbf{T} \otimes \mathbf{T} \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. From the definition of $\mathbf{K}$ and $\mathbf{Q}$ it follows that (i) is equivalent to the formula $j^* = P_\Theta^{-1} \cdot j^! \cdot P_A$. The first statement in (ii) follows from a straightforward diagram chase. By taking $\operatorname{Hom}(-, \mathbb{Z})$ of the sequence in (ii) we arrive at the statement about the pairing on $\mathbf{T}$. ◊

1.2. Intermediate jacobians. Let $H = (H_\mathbb{Z}, F^\cdot)$ be a $\mathbb{Z}$-Hodge structure of weight n. This means that we are given a lattice $H_\mathbb{Z}$ and a Hodge filtration

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^0 = H_\mathbb{C}$$

on the complexification $H_\mathbb{C} = H_\mathbb{Z} \otimes \mathbb{C}$. If $n := 2m + 1$ is an **odd** number, what we will suppose from now on, then the *intermediate jacobian* of H

is defined to be (see [13], page 9)

$$J(H) := F^{m+1}\backslash H_{\mathbb{C}}/H_{\mathbb{Z}}.$$

A *polarization* of H is a nondegenerate alternating bilinear form

$$Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \longrightarrow \mathbb{Z}.$$

Such a polarization will **not** in general induce a polarization on the intermediate jacobian: the Hodge form is not necessarily positive definite. In any case, Q defines a map q from $J(H)$ to the **dual torus** $\widehat{J(H)}$. As Q is nondegenerate, this is an isogeny.

We can apply this to the Hodge structure $H^n(\Theta)$ considered in the previous section, and construct the intermediate Jacobian $J(\Theta) := J(H^n(\Theta))$. Now take a look at the sequence

$$0 \longrightarrow \mathbf{K} \longrightarrow H^n(\Theta) \xrightarrow{j_!} H^{n+2}(A) \longrightarrow 0.$$

As $j_!$ is a morphism of Hodge structures (of type (1,1)), one can give $\mathbf{K}$ the structure of a $\mathbb{Z}$-Hodge structure, polarized by the restriction of Q to $\mathbf{K}$. In particular, $(\mathbf{K}_{\mathbb{C}})^{p,q} := \mathrm{Ker}(H^{p,q}(\Theta) \longrightarrow H^{p+1,q+1}(A))$. So we can apply the above construction to $\mathbf{K}$ and define

$$J(\mathbf{K}) := F^{m+1}(\mathbf{K})\backslash \mathbf{K}_{\mathbb{C}}/\mathbf{K}.$$

One defines $J(\mathbf{Q})$ in an analogous way. Then we have

Proposition 1.4. *The theta group of* $J(\mathbf{K})$ *is isomorphic to* $\mathbf{T}$, *that is, we have an exact sequence*

$$0 \longrightarrow \mathbf{T} \longrightarrow J(\mathbf{K}) \xrightarrow{q} \widehat{J(\mathbf{K})} \longrightarrow 0.$$

Furthermore, there is a canonical isomorphism $J(\mathbf{Q}) \cong \widehat{J(\mathbf{K})}$.

Proof. From the snake lemma it follows that for any morphism of a torus to its dual torus, the theta group is equal to $\mathrm{Coker}(Q : \mathbf{K} \longrightarrow \mathbf{K}^*)$. As we have $\mathbf{K}^* = \mathbf{Q}$ one gets $\mathbf{K}^*/\mathbf{K} \cong \mathrm{T}$. $\Diamond$

The rank of the lattice $\mathbf{K}$ can be computed from the knowledge of $h^*(A) := \dim H^*(A)$ and the Euler characteristic of Θ. In fact it follows easily from Proposition 1.1 that

$$\mathrm{rank}(\mathbf{K}) = h^{n+2} - h^{n+1} + (-1)^n(\chi(\Theta) - \chi(A)).$$

1.3. The case of the abelian fourfold. From now on we assume that A is a ppav of dimension four, and Θ its theta divisor. For an open dense subset of $\mathscr{A}_4$ this will be a *smooth* threefold. Its Euler characteristic is equal to

$$\text{coefficient of } \Theta^4 \text{ in } \Theta/(1+\Theta) = -4! = -24.$$

As $h^5(A) = 56$ and $h^4(A) = 70$ we find

$$\mathrm{rank}(\mathbf{K}) = 10.$$

From the exact sequence

$$0 \longrightarrow \Omega^4_A \longrightarrow \Omega^4_A(\Theta) \xrightarrow{\mathrm{Res}} \Omega^3_\Theta \longrightarrow 0$$

it follows that $h^0(\Omega^3_\Theta) = 4$ $(= h^0(\Omega^3_A))$, from which it follows that $\dim(\mathbf{K}_\mathbb{C})^{0,3} = 0$. This implies that

$$\dim(\mathbf{K}^{1,2}_\mathbb{C}) = \dim(\mathbf{K}^{2,1}_\mathbb{C}) = 5.$$

So the Hodge-form is definite, and the torus $J(\mathbf{K})$ is an abelian variety. To determine the type of its polarization, we need the following:

Proposition 1.5. *The cup product homomorphism* $H^1(A, \mathbb{Z}) \xrightarrow{\eta\cup} H^5(A, \mathbb{Z})$ *with* $\eta := (1/2)\cdot[\Theta]^2 \in H^4(A, \mathbb{Z})$ *induces a natural isomorphism*

$$(\mathbb{Z}/2)^8 \cong H^1(A, \mathbb{Z}/2) = \mathrm{Coker}(H^3(A, \mathbb{Z}) \xrightarrow{\cup\Theta} H^5(A, \mathbb{Z})).$$

Proof. One takes a standard symplectic basis A_i, B_i, $i = 1, 2, 3, 4$ for $H^1(A, \mathbb{Z})$. The group $H^p(A, \mathbb{Z})$ can be identified with the p-th exterior power of $H^1(A, \mathbb{Z})$ and thereby one obtains an induced basis of $H^*(A, \mathbb{Z})$. The element $[\Theta] \in H^2(A, \mathbb{Z})$ is represented by $\sum_{i=1}^4 A_iB_i$, the element η by $\sum_{i<j} A_iB_iA_jB_j$. The statement follows now from a straightforward computation. ◊

Corollary 1.6. *The polarization on the torus* $J(\mathbf{K})$ *is of type* (1,2,2,2,2). *Hence the torus* $J(\mathbf{Q})$ *can be given a natural polarization of type* (2,1,1,1,1).

Proof. This follows immediately from Propositions 1.3, 1.4 and 1.5. One can give $J(\mathbf{Q})$ a polarization using the dual isogeny, [19]. ◊

1.4. A nine-dimensional torus. Apart from the five-dimensional torus $J(\mathbf{K})$ (and it dual $J(\mathbf{Q})$) there is also a nine-dimensional torus $J(\mathbf{H})$ that will play a role in the sequel.

Definition 1.7.

$$\mathbf{H} := \mathrm{Ker}(H^3(\Theta) \longrightarrow P^3A^*),$$
$$\mathbf{H}^* := \mathrm{Coker}(P^3A \longrightarrow H^3(\Theta)).$$

Here $P^3A := \ker(H^3(A) \xrightarrow{\cup\eta} H^7(A))$ is the third primitive cohomology lattice, and $P^3A^* \cong \mathrm{Coker}(H^1(A) \xrightarrow{\cup\eta} H^5(A))$ its dual. So $\mathbf{H}$ and $\mathbf{H}^*$ are dual rank nine lattices. It is clear from the construction that these lattices underlie natural $\mathbb{Z}$-Hodge structures, so we can consider the intermediate jacobians $J(\mathbf{H})$ and $J(\mathbf{H}^*)$. Again, the polarization type can be determined by a diagram chase, using the principal polarization on $H^3(\Theta)$. Instead of spelling out the details, we summarize the basic facts in the following proposition.

Proposition 1.8. A. *There is a pair of exact sequences:*

$$0 \longrightarrow \mathbf{K} \longrightarrow \mathbf{H} \longrightarrow H^1(A) \longrightarrow 0$$

$$0 \longrightarrow H^1(A) \longrightarrow \mathbf{H} \longrightarrow \mathbf{Q} \longrightarrow 0$$

and a similar pair obtained by dualizing.

B. *There is an exact sequence:*

$$0 \longrightarrow \mathbf{H} \longrightarrow \mathbf{H}^* \longrightarrow H^1(A, \mathbb{Z}/3) \longrightarrow 0.$$

So the torus $J(\mathbf{H})$ *has a natural polarization of type* $(3, 3, 3, 3, 1, 1, 1, 1, 1)$.

Proof. This follows from straightforward diagram chases and the fact that the principal polarization $H^3(\Theta)$ induces a polarization with theta group $H^1(A, \mathbb{Z}/3)$ on the primitive cohomology. ◊

The upshot of all this is that the intermediate jacobian $J(\Theta)$ contains a *sub*torus $J(\mathbf{H})$ of dimension nine which sits in an exact sequence:

$$0 \longrightarrow A \longrightarrow J(\mathbf{H}) \longrightarrow J(\mathbf{Q}) \longrightarrow 0.$$

The torus $J(\mathbf{H})$ also contains $J(\mathbf{K})$ and sits in an exact sequence

$$0 \longrightarrow J(\mathbf{K}) \longrightarrow J(\mathbf{H}) \longrightarrow A \longrightarrow 0.$$

2. The infinitesimal Abel-Jacobi mapping

Assume we have a family $\mathcal{F}$ of curves in a smooth threefold Θ. That is, we have a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{p} & \Theta \\ q\downarrow & & \\ \mathcal{F} & & \end{array}$$

Here $\mathcal{C}$ is the "universal curve" over $\mathcal{F}$, and the restriction of p to the fibers of q is an embedding (or finite map) of a curve into Θ. Choosing a base point $t_0 \in \mathscr{F}$, one obtains an Abel-Jacobi map

$$\begin{array}{cccc} \mathrm{AJ} : & \mathscr{F} & \longrightarrow & J(\Theta) \\ & t & \mapsto & \int_{\mathscr{C}_{t_0}}^{\mathscr{C}_t} \end{array}$$

where $\mathscr{C}_t$ is the fiber of $\mathscr{C} \longrightarrow \mathscr{F}$ at $t \in \mathscr{F}$ and $\int_{\mathscr{C}_{t_0}}^{\mathscr{C}_t}$ is the linear form on $H^{3,0}(\Theta) \oplus H^{2,1}(\Theta)$ which associates to ω its integral on a three-cycle with boundary $\mathscr{C}_t - \mathscr{C}_{t_0}$.

The image of AJ generates an abelian subvariety of $J(\Theta)$ and Grothendieck's version of the Hodge conjecture (see e.g. [5], page 292) states that all abelian subvarieties of $J(\Theta)$ are generated by images of Abel-Jacobi mappings for appropriate families of curves.

We have seen in §1 that, for the theta-divisor Θ of a ppav A of dimension four, there are two nice abelian subvarieties $J(\mathbf{K})$ and $J(\mathbf{H})$ in $J(\Theta)$. The question now is: can one find families of curves whose images by the Abel-Jacobi map generate them?

Since every ppav A of dimension four contains Prym-embedded curves, it is natural to take a look at these.

2.1. Prym-embedded curves in A and Θ. Let $\mathscr{P} : \mathscr{R}_5 \longrightarrow \mathscr{A}_4$ be the Prym map, i.e., $\mathscr{P}$ associates to each admissible double cover $(\pi : \widetilde{X} \longrightarrow X)$ of a stable curve X of genus five its Prym variety

$$\begin{aligned} P(\widetilde{X}, X) &:= \mathrm{Im}(1 - \sigma^* : J(\widetilde{X}) \longrightarrow J(\widetilde{X})) \\ &= \mathrm{Ker}^0(\nu : J(\widetilde{X}) \longrightarrow J(X)) \end{aligned}$$

where σ is the involution interchanging the two sheets of π, $\nu : \mathrm{Pic}(\widetilde{X}) \longrightarrow \mathrm{Pic}(X)$ is the norm map and by $\mathrm{Ker}^0(\nu)$ we mean the component of the identity in the kernel of ν. For general background on the Prym construction we refer to [2] and [18]. The Prym map $\mathscr{P}$ is surjective [2], and because $\dim(\mathscr{A}_4) = 10$ and $\dim(\mathscr{R}_5) = 12$, the fiber $\mathscr{P}^{-1}(A)$ for A generic in $\mathscr{A}_4$ is a smooth surface. When Θ is smooth, the fiber is always a surface and the generic elements of any component of the fiber are double covers of smooth curves (see [15]).

There is a useful parametrization of the Prym of a covering. Consider the following subvarieties of $\mathrm{Pic}^8(\widetilde{X})$

$$A^+ := \{D \in \mathrm{Pic}^8(\widetilde{X}) | \nu(D) \cong \omega_X,\ h^0(D) \text{ even}\},$$

$$A^- := \{D \in \mathrm{Pic}^8(\widetilde{X}) | \nu(D) \cong \omega_X,\ h^0(D) \text{ odd}\}.$$

Both are principal homogeneous spaces over A. The divisor Θ is a translate of

$$\Theta^+ = \{L \in A^+ | h^0(L) > 0\}$$

For each $D \in A^-$ one gets an embedding

$$\phi_D : \widetilde{X} \longrightarrow A^+ \subset J(\widetilde{X})\,;\, x \mapsto D(L_x - \sigma(L_x))$$

where L_x is an effective Cartier divisor of degree 1 on $\widetilde{X}$ with support x. The image of such a morphism is called a Prym-embedding of $\widetilde{X}$ or a Prym-embedded curve.

For a generic ppav, R. Donagi [10] has discovered a strange involution

$$\lambda : \mathscr{P}^{-1}(A) \longrightarrow \mathscr{P}^{-1}(A).$$

The first named author (see [15]) has extended this involution to "most" of $\mathscr{R}_5$ and has also given the following nice geometric interpretation of the involution in terms of Prym-curves and the theta-divisor.

Theorem 2.1 [15, Theorem 1]. *Let $\Theta \subset A$ be a symmetric θ-divisor. Suppose that A is neither decomposable nor the jacobian of a hyperelliptic curve. Then the Prym-embeddings of $\widetilde{X}$ inside Θ are exactly parametrized by the curve $\widetilde{X}_\lambda$.*

Definition 2.2. From now on we let $\mathscr{F} :=$ scheme parametrizing the family of Prym-embedded curves inside Θ.

So we see that the above theorem tells us that the fiber of the natural projection

$$\mathscr{F} \longrightarrow \mathscr{P}^{-1}(A)$$

over the point $(\widetilde{X}, X) \in \mathscr{P}^{-1}(A)$ is precisely the curve $\widetilde{X}_\lambda$. In particular, the dimension of $\mathscr{F}$ is three. We will study the Abel-Jacobi map

$$AJ : \mathscr{F} \longrightarrow J(\Theta)$$

of this family. In general, $\mathscr{F}$ might be singular, but for A generic $\mathscr{F}$ will be smooth, see §3.

2.2. The image of a fiber. Let $\widetilde{X} \subset \mathscr{F}$ be a fiber of $\mathscr{F} \longrightarrow \mathscr{P}^{-1}(A)$ (so $\widetilde{X}$ maps to $(\widetilde{X}_\lambda, X_\lambda) \in \mathscr{P}^{-1}(A)$). The Abel-Jacobi map induces a morphism $J\widetilde{X} \longrightarrow J(\Theta)$. Composing this with $\pi^* : JX \longrightarrow J\widetilde{X}$, we get a morphism $JX \longrightarrow J(\Theta)$. For generic choices of A and $(\widetilde{X}_\lambda, X_\lambda) \in \mathscr{P}^{-1}(A)$, JX is a simple abelian variety hence this morphism is either 0 or an isogeny onto its image. The latter is not possible because the family of abelian subvarieties of $J(\Theta)$ is discrete. So the image of JX in $J(\Theta)$ is 0 for generic and hence for all choices of A and $(\widetilde{X}, X)$. So there is an induced morphism $A = J\widetilde{X}/JX \longrightarrow J(\Theta)$. Let us compute the image of $H_1(A, \mathbb{Z}) \subset H_1(\widetilde{X}, \mathbb{Z}) \subset H_1(\mathcal{F}, \mathbb{Z})$ in $H_3(A, \mathbb{Z}) = H^5(A, \mathbb{Z}) = H^3(\Theta, \mathbb{Z})/\mathbf{K}$:

Choose a symplectic basis $\{a_1, \dots, a_4, b_1, \dots, b_4\}$ of $H_1(A, \mathbb{Z})$, so that the homology class of Θ is

$$\theta^* = \sum_{i,j,k \text{ distinct}} a_i \times b_i \times a_j \times b_j \times a_k \times b_k$$

where " $\times$" is Pontrjagin product. Then the homology class of any Prym-embedding of $\widetilde{X}$ in A is

$$(\theta^*)^3/3 = 2. \sum_{1 \le i \le 4} a_i \times b_i.$$

Let us compute, for instance, the image of a_1 in $H_3(A, \mathbb{Z})$. Since we are just translating the curve $\widetilde{X}$ along the loop a_1, the image of a_1 is just

$$2 \cdot \sum_{1 \le i \le 4} a_1 \times a_i \times b_i.$$

So, in particular, the image of A in $J(\Theta)$ is nonzero.

2.3. The tangent space to J(Q). It follows from [21] pages 444-445 that there is an exact sequence

$$0 \longrightarrow \Omega^2_\Theta \longrightarrow \Omega^3_A(\Theta)|_\Theta \xrightarrow{d} \Omega^4_A(2\Theta)|_\Theta \longrightarrow 0.$$

Here the first map is induced by cup product with $d\theta/\theta$, where $\theta = 0$ is a local equation of $\Theta \subset A$. Part of the long exact cohomology sequence associated to the above sequence is:

$$0 \longrightarrow H^0(\Omega^2_\Theta) \longrightarrow H^0(\Omega^3_A(\Theta)|_\Theta) \longrightarrow H^0(\Omega^4_A(2\Theta)|_\Theta) \longrightarrow H^1(\Omega^2_\Theta) \longrightarrow \dots.$$

The cotangent space $T_0^* J(\mathbf{Q})$ to $J(\mathbf{Q})$ at 0 can be identified with the "primitive part" of $H^1(\Omega^2_\Theta)$, that is, the image of $H^0(\Omega^4_A(2\Theta)|_\Theta)$. Dualizing the above sequence, we see that the tangent space $T_0 J(Q)$ to $J(Q)$ at 0 is the image of $H^1(\Omega^2_\Theta)^*$ in $H^0(\Omega^4_A(2\Theta)|_\Theta)^*$. As $H^0(\Omega^2_\Theta) \cong \bigwedge^2 H^0(\mathcal{O}_\Theta(\Theta))$ and $H^0(\Omega^3_A(\Theta)|_\Theta) \cong H^0(\mathcal{O}_\Theta(\Theta))^{\otimes 2}$, the cokernel of $H^0(\Omega^2_\Theta) \hookrightarrow H^0(\Omega^3_A(\Theta)|_\Theta)$ can be identified with $S^2 H^0(\mathcal{O}_\Theta(\Theta))$. Thus we get the following presentation of $T_0^* J(\mathbf{Q})$:

$$0 \longrightarrow S^2 H^0(\mathcal{O}_\Theta(\Theta)) \xrightarrow{m} H^0(\mathcal{O}_\Theta(2\Theta)) \cong H^0(\Omega^4_A(2\Theta)|_\Theta) \longrightarrow T_0^* J(\mathbf{Q}) \longrightarrow 0$$

where the map m is induced by multiplication of sections. We see that indeed $\dim T_0^* J(\mathbf{Q}) = h^0(\mathscr{O}_\Theta(2\Theta)) - \dim S^2 H^0(\mathscr{O}_\Theta(\Theta)) = 15 - 10 = 5$, as it should be.

2.4. The normal bundle to $\widetilde{X}$. We choose an element $(\widetilde{X}, X)$ of $\mathscr{P}^{-1}(A)$, a generic Prym-embedding of $\widetilde{X}$ in Θ and identify it with $\widetilde{X}$. We suppose that X is *smooth*. In order to study the infinitesimal Abel-Jacobi map, we need good control over the normal bundle of $\widetilde{X}$ in Θ. We put

$$N := N_{\widetilde{X}/\Theta}.$$

Lemma 2.3. *The dimension of $H^0(N)$ is* 3.

Proof. First notice that $h^0(N)$ is at least 3 because $\mathscr{F}$ is three-dimensional.

For $a \in A$ let Θ_a be the translate of Θ by a, i.e., $\Theta_a = t_{-a}^* \Theta$ where $t_a : A \longrightarrow A$ is translation by a. Choose $a \in A$ such that $\widetilde{X} \subset \Theta_a$ and a is generic for this property. Then $\widetilde{X} \subset (\Theta \cap \Theta_a)$ which is smooth by the proof of Corollary 2.16 in [15] and we have the exact sequence:

$$0 \longrightarrow N_{\widetilde{X}/\Theta\cap\Theta_a} \longrightarrow N \longrightarrow N_{\Theta\cap\Theta_a/\Theta}|_{\widetilde{X}} \longrightarrow 0$$

From the exact sequence

$$0 \longrightarrow T_{\widetilde{X}} \longrightarrow T_{\Theta\cap\Theta_a}|_{\widetilde{X}} \longrightarrow N_{\widetilde{X}/\Theta\cap\Theta_a} \longrightarrow 0$$

we deduce $N_{\widetilde{X}/\Theta\cap\Theta_a} \cong \omega_{\widetilde{X}}(-\Theta - \Theta_a)$ which is easily seen to have degree 0. It is nontrivial because a is generic in a curve ([15], Theorem 1). Hence $h^0(N_{\widetilde{X}/\Theta\cap\Theta_a}) = 0$ and $h^0(N) \leq h^0(N_{\Theta\cap\Theta_a/\Theta}|_{\widetilde{X}})$. Then we see that $N_{\Theta\cap\Theta_a/\Theta}|_{\widetilde{X}} \cong \mathscr{O}_{\widetilde{X}}(\Theta_a)$ and it is shown in [15] (just before Proposition 2.11) that, since $\widetilde{X} \subset \Theta_a$, one has $h^0(\mathscr{O}_{\widetilde{X}}(\Theta_a)) = h^1(\mathscr{O}_{\widetilde{X}}(\Theta_a)) = 3$. $\diamond$

2.5. The Clemens-Welters diagram. The infinitesimal Abel-Jacobi map, i.e., the differential of the Abel-Jacobi map

$$AJ : \mathscr{F} \longrightarrow J(\Theta)$$

at the point $\widetilde{X} \in \mathscr{F}$ is the map

$$dAJ : H^0(\widetilde{X}, N) \longrightarrow H^1(\Omega_\Theta^2)^*.$$

We have seen that the image of the map $H^1(\Omega_\Theta^2)^* \longrightarrow H^0(\Omega_A^4(2\Theta)|_\Theta)^* \subset H^0(\Omega_A^4(2\Theta))^*$ is the tangent space $T_0 J(\mathbf{Q})$ to $J(\mathbf{Q})$. We will now study the composed map

$$\gamma : H^0(N) \longrightarrow H^0(\Omega_A^4(2\Theta))^*.$$

We use the following commutative diagram, due to G. Welters (see [6], [24]):

$$\begin{array}{ccc} H^0(N)\otimes H^0(\Omega_A^4(2\Theta)) & \xrightarrow{\text{Residue}} & H^0(N)\otimes H^0(N_{\Theta/A}\otimes\Omega_\Theta^3) \\ \downarrow & & \downarrow \\ H^0(N)\otimes H^1(\widehat{\Omega}_\Theta^2) & & H^0(N)\otimes H^1(N\otimes\Omega_\Theta^3) \\ & \searrow \quad H^1(\omega_{\widetilde{X}})\cong\mathbb{C} \quad \swarrow & \end{array}$$

Here $\widehat{\Omega}^p$ is the subsheaf of Ω^p generated by the closed forms. The vertical map in the left-hand column is induced by the composition of the maps $H^0(\Omega_A^4(2\Theta)) \longrightarrow H^1(\widehat{\Omega}_A^3(\Theta)) \longrightarrow H^1(\widehat{\Omega}_\Theta^2)$, where the second map is given by taking residues and the first map is the first connecting homomorphism in the cohomology sequence of the exact sequence:

$$0 \longrightarrow \widehat{\Omega}_A^3(\Theta) \longrightarrow \Omega_A^3(\Theta) \xrightarrow{d} \Omega_A^4(2\Theta) \longrightarrow 0.$$

From the inclusion $\widehat{\Omega}_\Theta^2 \longrightarrow \Omega_\Theta^2$ we get a map $H^1(\widehat{\Omega}_\Theta^2) \longrightarrow H^1(\Omega_\Theta^2)$. The slanted map on the left is induced by the infinitesimal Abel-Jacobi mapping composed with the dual of the map $H^1(\widehat{\Omega}_\Theta^2) \longrightarrow H^1(\Omega_\Theta^2)$. On the right-hand side, the mapping $H^0(N_{\Theta/A}\otimes\Omega_\Theta^3) \longrightarrow H^1(N\otimes\Omega_\Theta^3)$ is the composition of the restriction map

$$H^0(N_{\Theta/A}\otimes\Omega_\Theta^3) \longrightarrow H^0(N_{\Theta/A}|_{\widetilde{X}}\otimes\Omega_\Theta^3)$$

with the map induced by the first connecting homomorphism of the cohomology sequence of the normal bundle sequence

$$0 \longrightarrow N \longrightarrow N_{\widetilde{X}/A} \longrightarrow N_{\Theta/A}|_{\widetilde{X}} \longrightarrow 0$$

after tensoring with Ω_Θ^3.

It is easy to show (see [6]) that the map $H^0(N)\otimes H^1(N\otimes\Omega_\Theta^3) \longrightarrow H^1(\omega_{\widetilde{X}})\cong\mathbb{C}$ is a perfect pairing. In particular, $H^0(N)$ and $H^1(N\otimes\Omega_\Theta^3)$ have the same dimension, i.e., 3.

We need the following

Lemma 2.4. *The map*

$$H^0(N_{\Theta/A}\otimes\Omega_\Theta^3) \longrightarrow H^0(N_{\Theta/A}|_{\widetilde{X}}\otimes\Omega_\Theta^3)$$

is surjective.

Proof. As the normal bundle $N_{\Theta/A}$ is isomorphic to $\mathscr{O}_\Theta(\Theta)$ and $\Omega^3_\Theta \cong \mathscr{O}_\Theta \otimes \Omega^4_A(\Theta) \cong \mathscr{O}_\Theta(\Theta)$, the map can be identified with the restriction map

$$H^0(\mathscr{O}_\Theta(2\Theta)) \longrightarrow H^0(\mathscr{O}_{\widetilde{X}}(2\Theta)).$$

Since the map $H^0(\mathscr{O}_A(2\Theta)) \longrightarrow H^0(\mathscr{O}_{\widetilde{X}}(2\Theta))$ is the composition of the two restriction maps $H^0(\mathscr{O}_A(2\Theta)) \longrightarrow H^0(\mathscr{O}_\Theta(2\Theta))$ and $H^0(\mathscr{O}_\Theta(2\Theta)) \longrightarrow H^0(\mathscr{O}_{\widetilde{X}}(2\Theta))$, it is enough to show that the map

$$H^0(\mathscr{O}_A(2\Theta)) \longrightarrow H^0(\mathscr{O}_{\widetilde{X}}(2\Theta))$$

is onto.

Fix an element D of A^+ (see 2.1). Then there is an element E of A^- such that the embedding of $\widetilde{X}$ in Θ is given by: $p \mapsto E \otimes D^{-1}(p - \sigma p)$. Let $\Theta \subset A$ be the inverse image of Θ^+ by the morphism

$$\begin{array}{ccc} A & \longrightarrow & A^+ \\ x & \mapsto & x \otimes D \end{array}$$

where we identify $x \in A$ with the corresponding invertible sheaf on $\widetilde{X}$. Then, for all $x \in A$, $\mathscr{O}_{\widetilde{X}}(\Theta_x) \cong E \otimes x$. The map

$$H^0(\mathscr{O}_A(2\Theta)) \longrightarrow H^0(\mathscr{O}_{\widetilde{X}}(2\Theta))$$

is onto by the following argument which was told us by A. Beauville:

The argument consists in finding elements in the image of $H^0(\mathscr{O}_A(2\Theta))$ which form a basis of $H^0(\mathscr{O}_{\widetilde{X}}(2\Theta))$.

Let A' be the connected component of $\mathrm{Ker}(\nu : J\widetilde{X} \longrightarrow JX)$ which does not contain 0. Consider the morphism

$$\begin{array}{ccc} A' & \longrightarrow & A^+ \\ \alpha & \mapsto & \alpha \otimes E \end{array}$$

Choose $\alpha \in A'$ such that $\alpha \otimes E \in \Theta^+$ and $\alpha^{-1} \otimes E \notin \Theta^+$: this is possible because E is generic in $\widetilde{X}_\lambda = \{E : h^0(E) \geq 3\} \subset A^-$ (see [15]), so that $E \ncong \sigma^* E$ ($\alpha^{-1} \otimes E \in \Theta^+ \iff \alpha \otimes \sigma^* E \in \Theta^+$). For such generic α, the linear system $|\alpha \otimes E|$ has no base points. Hence, since it is positive-dimensional, it contains a divisor $E_\alpha = \sum_{1 \leq i \leq 8} x_i$ such that the points x_i are all distinct. Put $E_{\alpha,j} = E_\alpha - x_j + \sigma x_j = \sigma x_j + \sum_{i \neq j} x_i$ for j between 1 and 8. Since we supposed α generic in a three-dimensional family, we have $h^0(E_{\alpha,j}) = 1$ for all j because $\widetilde{X}_\lambda$ above is one-dimensional. Since

$2E(-E_{\alpha,j}) = E \otimes \alpha^{-1}(x_j - \sigma x_j)$, we also have $h^0(2E(-E_{\alpha,j})) = 1$. Let $E'_{\alpha,j}$ be the unique effective divisor of $|2E(-E_{\alpha,j})|$. Then the divisor $E_{\alpha,j} + E'_{\alpha,j}$ is cut on $\widetilde{X}$ by $\Theta_a + \Theta_{-a}$ where $a = \alpha(\sigma x_j - x_j)$. Hence, if s_j is a section of $H^0(\mathcal{O}_{\widetilde{X}}(2\Theta))$ with divisor $E_{\alpha,j} + E'_{\alpha,j}$, then s_j is in the image of $H^0(\mathcal{O}_A(2\Theta))$. It is now easily seen that the s_j's are linearly independant hence generate $H^0(\mathcal{O}_{\widetilde{X}}(2\Theta))$. ◊

Proposition 2.5. *The rank of the map*

$$H^0(N_{\Theta/A}|_{\widetilde{X}} \otimes \Omega^3_\Theta) \longrightarrow H^1(N \otimes \Omega^3_\Theta)$$

is at least 2.

Proof. It is enough to prove that $h^1(N_{\widetilde{X}/A} \otimes \Omega^3_\Theta)$ is at most 1. Choose two elements a and b of A such that $\widetilde{X} \subset \Theta_a \cap \Theta_b$. Since $\widetilde{X} \subset \Theta \cap \Theta_a \cap \Theta_b$, one has an exact sequence

$$0 \longrightarrow N_{\widetilde{X}/A} \longrightarrow \mathcal{O}_{\widetilde{X}}(\Theta) \oplus \mathcal{O}_{\widetilde{X}}(\Theta_a) \oplus \mathcal{O}_{\widetilde{X}}(\Theta_b) \longrightarrow \mathbf{sk} \longrightarrow 0$$

where $\mathbf{sk}$ is a skyscraper sheaf such that $h^0(\mathbf{sk}) = 8$ (because the degree of $N_{\widetilde{X}/A}$ is 16 while the degree of $\mathcal{O}_{\widetilde{X}}(\Theta) \oplus \mathcal{O}_{\widetilde{X}}(\Theta_a) \oplus \mathcal{O}_{\widetilde{X}}(\Theta_b)$ is 24). Write $\Theta \cap \Theta_a \cap \Theta_b = \widetilde{X} \cup S$. Then $\mathbf{sk}$ is supported on $S \cap \widetilde{X}$ and the degree of $S \cap \widetilde{X}$ is also 8. By [15] Proposition 2.14 there is exactly one more Prym-embedding of $\widetilde{X}$ in $\Theta_a \cap \Theta_b$, call it $\widetilde{X}'$. It follows easily from [15] section 1.4 that one can choose a and b in such a way that there is an element c of A with $\Theta_a \cap \Theta_b \cap \Theta_c = S \cup \widetilde{X}'$. Then the divisor $S \cap \widetilde{X}$ on $\widetilde{X}$ is cut on $\widetilde{X}$ by Θ_c. Tensor the exact sequence above with $\Omega^3_\Theta \cong \mathcal{O}_\Theta(\Theta)$ to obtain the exact sequence of cohomology groups

$$0 \longrightarrow H^0(N_{\widetilde{X}/A} \otimes \Omega^3_\Theta) \longrightarrow H^0(\widetilde{X}, 2\Theta) \oplus H^0(\widetilde{X}, \Theta + \Theta_a) \oplus H^0(\widetilde{X}, \Theta + \Theta_b)$$

$$\longrightarrow H^0(\mathbf{sk}) \longrightarrow H^1(N_{\widetilde{X}/A} \otimes \Omega^3_\Theta) \longrightarrow 0.$$

So, in order to show that $h^1(N_{\widetilde{X}/A} \otimes \Omega^3_\Theta) \leq 1$, it is enough to show that the rank of the map $H^0(\widetilde{X}, 2\Theta) \longrightarrow H^0(\mathbf{sk})$ is 7. Both of these spaces have dimension 8. So we have to show that the kernel of this map is one-dimensional. Let s be an element of $H^0(\widetilde{X}, 2\Theta)$ such that its image in $H^0(\mathbf{sk})$ is 0. Let $Z(s)$ be the divisor of zeros of s in $\widetilde{X}$. Then, $Z(s) - \Theta_c \cdot \widetilde{X} \equiv \Theta_{-c} \cdot \widetilde{X}$. Since Θ_c and Θ_{-c} do not contain $\widetilde{X}$, we have (see [15] before Proposition 2.11) that $h^0(\Theta_c \cdot \widetilde{X}) = h^0(\Theta_{-c} \cdot \widetilde{X}) = 1$. Also,

$Z(s) - \Theta_c \cdot \widetilde{X}$ is effective by hypothesis. Hence $Z(s) = \Theta_c \cdot \widetilde{X} + \Theta_{-c} \cdot \widetilde{X}$ and s is unique up to multiplication by a scalar. ◊

Corollary 2.6. *The rank of the map γ is at least 2 and at most 3.*

Proof. This follows from the commutativity of the Clemens-Welters diagram, Lemma 2.4 and Proposition 2.5. ◊

The space $H^0(N)$ sits in the canonical exact sequence:

$$0 \longrightarrow H^0(T_\Theta|_{\widetilde{X}}) \longrightarrow H^0(N) \longrightarrow H^1(T_{\widetilde{X}})$$

obtained from the exact sequence

$$0 \longrightarrow T_{\widetilde{X}} \longrightarrow T_\Theta|_{\widetilde{X}} \longrightarrow N \longrightarrow 0.$$

The space $H^0(T_\Theta|_{\widetilde{X}})$ parametrizes exactly the infinitesimal deformations of $\widetilde{X}$ that are trivial in moduli. Since $\widetilde{X}_\lambda$ parametrizes deformations of $\widetilde{X}$ in Θ that are trivial in moduli, $H^0(T_\Theta|_{\widetilde{X}})$ contains the tangent space to $\widetilde{X}_\lambda$ at the point E defined in the proof of 2.4. Hence $h^0(T_\Theta|_{\widetilde{X}})$ is at least 1. The image of $H^0(N)$ in $H^1(T_{\widetilde{X}})$ can be canonically identified with the tangent space to $\mathscr{P}^{-1}(A)$ at $(\widetilde{X}_\lambda, X_\lambda)$. Since the dimension of this tangent space is at least 2, and $h^0(N) = 3$, we see that $h^0(T_\Theta|_{\widetilde{X}}) = 1$ and the tangent space to $\mathscr{P}^{-1}(A)$ at $(\widetilde{X}_\lambda, X_\lambda)$ has dimension 2. So we can canonically identify $H^0(T_\Theta|_{\widetilde{X}})$ with $T_E \widetilde{X}_\lambda$. As $\widetilde{X}_\lambda$ generates A inside $J(\Theta)$, the infinitesimal Abel-Jacobi mapping is nonzero on $H^0(T_\Theta|_{\widetilde{X}})$. The following implies that $H^0(T_\Theta|_{\widetilde{X}})$ is contained in the kernel of γ and hence that the rank of γ is exactly 2.

Corollary 2.7. *The composition $A \longrightarrow J(\Theta) \longrightarrow J(\mathbf{Q})$ is zero. Hence the map $\mathscr{F} \longrightarrow J(Q)$ factors through $\mathscr{P}^{-1}(A)$. In other words, we have a commutative diagram*

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & J(\Theta) \\ \downarrow & & \downarrow \\ \mathscr{P}^{-1}(A) & \longrightarrow & J(\mathbf{Q}) \end{array}$$

Proof. Suppose that the map $A \longrightarrow J(Q)$ is nonzero. Let B be its cokernel. Then the map $\mathscr{F} \longrightarrow B$ factors through $\mathscr{P}^{-1}(A)$. When A is generic, $\mathscr{P}^{-1}(A)$ is smooth and we obtain an induced morphism of abelian varieties: $\mathrm{Alb}\,\mathscr{P}^{-1}(A) \longrightarrow B$. Since A is generic, $\mathrm{Alb}\,\mathscr{P}^{-1}(A)$ is isogenous to the simple abelian variety $J(T)$ (see [10], [15] and [8]). Hence $\mathrm{Alb}\,\mathscr{P}^{-1}(A)$ is simple. The kernel of the morphism $\mathrm{Alb}\,\mathscr{P}^{-1}(A) \longrightarrow B$ is an abelian subvariety of $\mathrm{Alb}\,\mathscr{P}^{-1}(A)$ and, since the dimension of B is less than 5, it is non zero. So the kernel of $\mathrm{Alb}\,\mathscr{P}^{-1}(A) \longrightarrow B$ is all of

$\mathrm{Alb}\mathcal{P}^{-1}(A)$ and the morphism $\mathrm{Alb}\mathcal{P}^{-1}(A) \longrightarrow B$ is zero. Now consider the canonical exact sequence

$$0 \longrightarrow H^0(T_\Theta|_{\tilde{X}}) \longrightarrow H^0(N) \longrightarrow T_{(\tilde{X},X)}\mathcal{P}^{-1}(A) \longrightarrow 0$$

deduced from the map $\mathcal{F} \longrightarrow \mathcal{P}^{-1}(A)$. We deduce from the above that the map of tangent spaces $T_{(\tilde{X},X)}\mathcal{P}^{-1}(A) \longrightarrow T_0 B$ is 0. Hence the rank of $\gamma : H^0(N) \longrightarrow T_0 J(Q)$ is 1: this contradicts Corollary 2.6. ◊

Since we saw above that the image of $H^0(T_\Theta|_{\tilde{X}})$ by the infinitesimal Abel-Jacobi map is nonzero, it easily follows that

Corollary 2.8. *The infinitesimal Abel-Jacobi mapping has rank* 3. *The image of* $\mathcal{F}$ *generates an abelian variety of dimension* 9 *which surjects onto* $J(\mathbf{Q})$. *In particular, since the image of* A *in* $J(\mathbf{Q})$ *is zero* (*Corollary* 2.7), *the image of* $\mathcal{P}^{-1}(A)$ *generates* $J(\mathbf{Q})$.

Let JH' be the abelian subvariety of $J(\Theta)$ generated by the image of $\mathcal{F}$. Then it follows from the above that either the intersection of JH' with $J(\mathbf{H})$ is the image of A or $J(\mathbf{H}) = JH'$. We will see in Corollary 4.8 that the latter is true.

3. The space of Prym-embedded curves in Θ

We now take a closer look at the space $\mathcal{F}$ of Prym–embedded curves in Θ and state some properties of the global Abel-Jacobi map

$$AJ : \mathcal{F} \longrightarrow J(\Theta).$$

Proposition 3.1. *If* A *is generic in* $\mathcal{A}_4$, *then* $\mathcal{F}$ *is smooth.*

Proof. Since we are supposing that A is generic and since the locus of curves with automorphisms has codimension at least 3 in the moduli space $\overline{\mathcal{M}}_5$ of stable curves of genus 5, the curve X and also $(\tilde{X}, X)$ have no automorphisms for any element $(\tilde{X}, X)$ of $\mathcal{P}^{-1}(A)$. Also, A has no nontrivial automorphisms. Let $\tilde{\mathcal{C}}_5 \longrightarrow \mathcal{C}_5$ be the universal double cover over the locus $\mathcal{R}_5^0$ in $\mathcal{R}_5$ of Prym-curves $(\tilde{X}, X)$ such that X has no automorphisms. We can therefore identify the tangent spaces to $\overline{\mathcal{M}}_5$, $\mathcal{A}_4$, $\mathcal{R}_5$, $\tilde{\mathcal{C}}_5$ and $\mathcal{P}^{-1}(A)$ at X, A, $(\tilde{X}, X)$, $(\tilde{X}, X, p \in \tilde{X})$ and $(\tilde{X}, X)$ respectively with the corresponding spaces of first order infinitesimal deformations. The variety $\mathcal{F}$ is clearly smooth outside the nodes of its fibers over $\mathcal{P}^{-1}(A)$ (since $\mathcal{P}^{-1}(A)$ is smooth). Let p be a node of

$\widetilde{X} \subset \mathscr{F}$. Then there is an exact sequence

$$T^*_{(\widetilde{X}, X)}\mathscr{R}_5 \longrightarrow T^*_{(\widetilde{X}, X, p)}\widetilde{\mathscr{C}}_5 \longrightarrow T^*_p\widetilde{X} \longrightarrow 0.$$

In particular, since $\mathscr{R}_5$ is smooth at $(\widetilde{X}, X)$ (see, e.g., [11]) and the image of $T^*_{(\widetilde{X}, X)}\mathscr{R}_5$ has corank 1 in $T^*_{(\widetilde{X}, X, p)}\widetilde{\mathscr{C}}_5$, $\widetilde{\mathscr{C}}_5$ is smooth at $(\widetilde{X}, X, p)$ (see also [20] page 305).

The elements of $\mathscr{P}^{-1}(A)$ are irreducible (i.e., $\widetilde{X}$ is irreducible) and have at worst two nodes. There is a one-dimensional family of elements with one node and a finite number of elements with two nodes. Let Δ_0 and Δ_{00} be the loci in $\mathscr{R}_5^0$ which parametrize respectively irreducible elements with exactly one node and irreducible elements with exactly two nodes. Then we have the commutative diagram

$$\begin{array}{ccc} T^*_A\mathscr{A}_4 & \hookrightarrow & T^*_{(\widetilde{X}, X)}\mathscr{R}_5 \\ & \searrow \quad \swarrow & \\ & T^*_{(\widetilde{X}, X)}\Delta_0 & \end{array}$$

when X has one node, and the commutative diagram

$$\begin{array}{ccc} T^*_A\mathscr{A}_4 & \hookrightarrow & T^*_{(\widetilde{X}, X)}\mathscr{R}_5 \\ & \searrow \quad \swarrow & \\ & T^*_{(\widetilde{X}, X)}\Delta_{00} & \end{array}$$

when X has two nodes. (The horizontal maps are injective because, since A is generic, the Prym map has maximal rank everywhere on $\mathscr{P}^{-1}(A)$.)

It follows easily from [15] section 2, that the two right-hand slanted arrows above are surjective and the left-hand slanted arrows are injective. Let $\mathscr{D}_0$ and $\mathscr{D}_{00}$ be the loci of the nodes of the fibers of $\widetilde{\mathscr{C}}_5|_{\Delta_0} \longrightarrow \Delta_0$ and $\widetilde{\mathscr{C}}_5|_{\Delta_{00}} \longrightarrow \Delta_{00}$ respectively. Then $\mathscr{D}_0 \longrightarrow \Delta_0$ is an isomorphism and $\mathscr{D}_{00} \longrightarrow \Delta_{00}$ is an étale double cover. So we can add to the above diagrams two surjections (these are the rightmost slanted maps below):

$$\begin{array}{ccccc} T^*_A\mathscr{A}_4 & \hookrightarrow & T^*_{(\widetilde{X}, X)}\mathscr{R}_5 & \longrightarrow & T^*_{(\widetilde{X}, X, p)}\widetilde{\mathscr{C}}_5 \\ & \searrow & \downarrow & \swarrow & \\ & & T^*_{(\widetilde{X}, X)}\Delta_0 & & \end{array}$$

and

$$\begin{array}{ccccc} T^*_A\mathscr{A}_4 & \hookrightarrow & T^*_{(\widetilde{X}, X)}\mathscr{R}_5 & \longrightarrow & T^*_{(\widetilde{X}, X, p)}\widetilde{\mathscr{C}}_5 \\ & \searrow & \downarrow & \swarrow & \\ & & T^*_{(\widetilde{X}, X)}\Delta_{00} & & \end{array}$$

Since the leftmost maps are injective, it easily follows from the commutativity of the above diagrams that $T^*_A\mathscr{A}_4$ injects into $T^*_{(\tilde{X},X,p)}\widetilde{\mathscr{C}}_5$. So the map $\widetilde{\mathscr{C}}_5 \longrightarrow \mathscr{A}_4$ has maximal rank at $p \in \tilde{X} \subset \widetilde{\mathscr{C}}_5$. Since $\widetilde{\mathscr{C}}_5$ is smooth at $(\tilde{X}, X, p)$ and $\mathscr{A}_4$ is smooth at A, $\mathscr{F}$ is smooth at p. $\diamondsuit$

Consider again the diagram of the universal curve over $\mathscr{F}$

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{p} & \Theta \\ q\downarrow & & \\ \mathscr{F} & & \end{array}$$

There is an induced map

$$\alpha := q_! p^* : H^3(\Theta, Z) \longrightarrow H^1(\mathscr{F}, \mathbb{Z})$$

which is the transpose of the map

$$\beta := p_* q^! : H_1(\mathscr{F}, \mathbb{Z}) \longrightarrow H_3(\Theta, \mathbb{Z})$$

obtained by taking the full inverse image of a one-cycle in $\mathscr{F}$ and mapping it into Θ. If $\mathscr{F}$ is smooth, then $H^1(\mathscr{F}, \mathbb{Z})$ carries the structure of a pure Hodge structure of weight 1, and as α is a morphism of Hodge structures of type $(-1, -1)$, we obtain an induced map of tori

$$\mathrm{AJ} : \mathrm{Alb}(\mathscr{F}) \longrightarrow J(\Theta).$$

The results of §2 now give factorizations:

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & J(\Theta) \\ \downarrow & & \downarrow \\ \mathscr{P}^{-1}(A) & \longrightarrow & J(\mathbf{Q}) \end{array} \qquad \begin{array}{ccc} \mathrm{Alb}\mathscr{F} & \longrightarrow & J(\Theta) \\ \downarrow & & \downarrow \\ \mathrm{Alb}\mathscr{P}^{-1}(A) & \longrightarrow & J(\mathbf{Q}) \end{array}$$

Since $\mathrm{Alb}\mathscr{P}^{-1}(A)$ and $J(Q)$ have the same dimension, the right bottom map (defined only when $\mathscr{P}^{-1}(A)$ is smooth) $\mathrm{Alb}\mathscr{P}^{-1}(A) \longrightarrow J(\mathbf{Q})$ is an isogeny. In particular, for A generic, $J(\mathbf{Q})$ and $J(\mathbf{K})$ are both simple, because $\mathrm{Alb}\mathscr{P}^{-1}(A)$ is isogenous to the simple abelian variety $J(T)$ ([10], [15] and [8]). We will see in the next section that the map $\mathrm{Alb}\mathscr{P}^{-1}(A) \longrightarrow J(\mathbf{Q})$ is an isomorphism. We now determine the dimension of $\mathrm{Alb}(\mathscr{F})$.

Proposition 3.2. *The Albanese variety* $\mathrm{Alb}(\mathscr{F})$ *of* $\mathscr{F}$ *has dimension nine.*

Proof. We have to compute $h^1(\mathscr{F}, \mathbb{Q})$. To do this, we use the Leray spectral sequence of the map $\rho : \mathscr{F} \longrightarrow \mathscr{P}^{-1}(A)$. One obtains an exact sequence:

$$0 \longrightarrow H^1(\mathscr{P}^{-1}(A)) \longrightarrow H^1(\mathscr{F}) \xrightarrow{t} H^0(R^1\rho_*(\mathbb{Z}_{\mathscr{F}})) \\ \longrightarrow H^2(\mathscr{P}^{-1}(A)) \longrightarrow \dots.$$

Let $U \subset \mathscr{P}^{-1}(A)$ be the subset of points over which the fiber of ρ is smooth, $i : U \hookrightarrow \mathscr{P}^{-1}(A)$ the inclusion and Δ the complement of U in $\mathscr{P}^{-1}(A)$. As the general fibre of ρ is smooth, Δ is a curve. Furthermore, put $H = R^1\rho_*(\mathbb{Z}_{\mathscr{F}})$ and $S := \mathscr{P}^{-1}(A)$.

Claim: The map $H \longrightarrow i_*i^*H$ is injective.

This follows from the fact that we are dealing with an H^1, and from the fact that $\mathscr{F}$ is smooth. Take a point $p \in \Delta$, and consider the factorization $i = k.j$, where $k : S - \{p\} \hookrightarrow S$ and $j : U \hookrightarrow S - \{p\}$. Let B be a small ball around p. The sections of the kernel of $H \longrightarrow k_*k^*H$ over B is just the kernel of the restriction map $H^1(\rho^{-1}(B)) \longrightarrow H^1(\rho^{-1}(B - \{p\}))$. This map sits in an exact sequence:

$$\dots \longrightarrow H^1(\rho^{-1}(B), \rho^{-1}(B - \{p\})) \longrightarrow H^1(\rho^{-1}(B)) \\ \longrightarrow H^1(\rho^{-1}(B - \{p\})) \longrightarrow \dots.$$

However,

$$H^1(\rho^{-1}(B), \rho^{-1}(B - \{p\})) \cong H^1(\rho^{-1}(B), \partial\rho^{-1}(B)) \cong H_5(\rho^{-1}(B)) \\ \cong H_5(\rho^{-1}(p)) = 0$$

by Lefschetz duality and contraction to the fibre which is (real) two-dimensional. It follows that the map $H \longrightarrow k_*k^*H$ is injective. (In fact, it is an isomorphism.) Without loss of generality, we may suppose that the map ρ is, in a neigbourhood of any point $q \in \Delta - \{p\}$, topologically isomorphic to the product of (Δ, q) with the preimage $\rho^{-1}(D)$ of a small disc D transverse to Δ at q. Then we can apply the same reasoning to the complex surface $\rho^{-1}(D)$ mapping to D and the point q to conclude that the map $k^*H \longrightarrow j_*j^*k^*H$ is injective. From these two facts the injectivity of $H \longrightarrow i_*i^*H$ follows. ◊

From this we see that we have an inclusion

$$H^0(R^1\rho_*(\mathbb{Z}_{\mathscr{F}})) \subset H^0(R^1\rho_*(\mathbb{Z}_{\mathscr{F}}|U)).$$

The sheaf $R^1\rho_*(\mathbb{Z}_{\mathscr{F}})|U$ is a local system, and its global sections are the invariants under the monodromy. The invariant cycle theorem [9] states that the composition of t with restriction to U is surjective after tensoring with $\mathbb{Q}$. In other words, we have an exact sequence:

$$0 \longrightarrow H^1(\mathscr{P}^{-1}(A), \mathbb{Q}) \longrightarrow H^1(\mathscr{F}, \mathbb{Q}) \xrightarrow{\iota} H^0(R^1\rho_*(\mathbb{Q}_{\mathscr{F}})|U) \longrightarrow 0.$$

Claim: $H^0(R^1\rho_*(\mathbb{Q}_{\mathscr{F}})|U) = H^1(A, \mathbb{Q})$.

Proof. Clearly, $H^1(A, \mathbb{Q}) \subset H^0(R^1\rho_*(\mathbb{Q}_{\mathscr{F}})|U)$, because $H^1(A, \mathbb{Q})$ is *invariant*. On the other hand, $H^0(R^1\rho_*(\mathbb{Q}_{\mathscr{F}})|U)$ is a Hodge structure, and the restriction maps $r_s : H^0(R^1\rho_*(\mathbb{Q}_{\mathscr{F}})|U) \longrightarrow H^1(\rho^{-1}(s), \mathbb{Q})$ are morphisms of Hodge structures, [9]. Put $\rho^{-1}(s) = \widetilde{X}_s$; we have an exact sequence

$$0 \longrightarrow H^1(A, \mathbb{Q}) \longrightarrow H^1(\widetilde{X}_s, \mathbb{Q}) \longrightarrow H^1(X_s, \mathbb{Q}) \longrightarrow 0.$$

We obtain an induced map of Hodge structures $H^0(R^1\rho_*(\mathbb{Q}_{\mathscr{F}})|U) \longrightarrow H^1(X_s, \mathbb{Q})$ by composition. However, this map must be zero, as the first Hodge structure is constant, whereas the second varies with s and is generically simple. The claim follows, and hence the proposition. ◊

Remark. It seems very probable that the above result in fact is valid over the integers, that is, we would have exact sequences

$$0 \longrightarrow H^1(\mathscr{P}^{-1}(A)) \longrightarrow H^1(\mathscr{F}) \longrightarrow H^1(A) \longrightarrow 0,$$

$$0 \longrightarrow A \longrightarrow \mathrm{Alb}(\mathscr{F}) \longrightarrow \mathrm{Alb}\,\mathscr{P}^{-1}(A) \longrightarrow 0.$$

It seems however that this stronger statement does not follow from the above arguments.

4. Degeneration to the jacobian case

In §2 and §3 we have seen that, when $\mathscr{F}$ and $\mathscr{P}^{-1}(A)$ are smooth, have a commutative diagram

$$\begin{array}{ccc} \mathrm{Alb}(\mathscr{F}) & \longrightarrow & J(\Theta) \\ \downarrow & & \downarrow \\ \mathrm{Alb}(\mathscr{P}^{-1}(A)) & \longrightarrow & J(\mathbf{Q}) \end{array}$$

where $\mathrm{Alb}(\mathscr{P}^{-1}(A)) \longrightarrow J(\mathbf{Q})$ is an *isogeny*.

For generic A, the endomorphism ring of $\mathrm{Alb}(\mathscr{P}^{-1}(A))$ is isomorphic to $\mathbb{Z}$. So, since both $\mathrm{Alb}(\mathscr{P}^{-1}(A))$ and $J(\mathbf{Q})$ have the *same* polarization type, ($\mathrm{Alb}(\mathscr{P}^{-1}(A))$ is an étale double cover of the ppav $J(T)$ by [10] and [15]), one can conclude that the isogeny is multiplication by some natural number n. In this section we will show:

Theorem 4.1. *$n = 1$, that is, the isogeny* $\mathrm{Alb}(\mathscr{P}^{-1}(A)) \longrightarrow J(\mathbf{Q})$ *is an isomorphism for A generic.*

To obtain this result, we degenerate A to the jacobian of a curve C of genus four. We will do several computations in cohomology rings of symmetric products of C. For this we use some results of Macdonald [17], which for the convenience of the reader we have gathered in an appendix.

4.1. The theta-divisor in the jacobian case. By a g_d^r on a curve we mean a linear system of degree d and (projective) dimension r.

Let C be a smooth curve of genus four with two distinct g_3^1's. The canonical model κC of C is the complete intersection of a quadric and a cubic in $\mathbf{P}^3$. Since C has two distinct g_3^1's, the quadric is smooth and its rulings cut the divisors of the two g_3^1's on C. Let $C^{(3)}$ be the third symmetric product of C. The image of the map

$$C^{(3)} \longrightarrow \mathrm{Pic}^3(C); \quad (p, q, r) \mapsto \mathscr{O}_C(p+q+r)$$

is the variety W_3 of effective divisor classes. Let P_1 and P_2 be the two smooth rational curves in $C^{(3)}$ which parametrize the divisors of the two g_3^1's. The curves P_1 and P_2 are contracted to the singular points of W_3 [1]. The tangent cones to W_3 at these points can be identified with the quadric containing κC [1]. So these two points are nodes on W_3. Let $A := J(C)$ be the Jacobian of C. Choosing a theta-characteristic κ on C, we obtain two isomorphisms

$$\mathrm{Pic}^3(C) \xrightarrow{\cong} A; \quad \mathscr{O}_C(D) \mapsto \mathscr{O}_C(D-\kappa),$$

$$W_3 \xrightarrow{\cong} \Theta.$$

From now on we will identify these spaces. Note that the map

$$C^{(3)} \longrightarrow \Theta$$

is a canonical *small resolution* of the singularities of Θ. This will enable us to relate the (co)-homologies of $C^{(3)}$ and Θ.

4.2. Homology of the theta-divisor. Since $C^{(3)}$ is a small resolution of Θ, we have an exact sequence (see, for instance, [4] pages 119-120):

$$0 \longrightarrow H_3(C^{(3)}) \longrightarrow H_3(\Theta) \longrightarrow \mathbf{M} \longrightarrow H_2(C^{(3)}) \longrightarrow H_2(\Theta) \longrightarrow 0$$

where $\mathbf{M} := \mathbb{Z}\cdot P_1 \oplus \mathbb{Z}\cdot P_2$ is the free abelian group generated by P_1 and P_2. The map

$$\mathbf{M} \longrightarrow H_2(C^{(3)})$$

sends P_i to its homology class in $C^{(3)}$. Now, by a result of Macdonald [17], (see also the appendix), the curves P_1 and P_2 are homologous in $C^{(3)}$, and hence we obtain an exact sequence:

$$0 \longrightarrow H_3(C^{(3)}) \longrightarrow H_3(\Theta) \longrightarrow \mathbb{Z}.\Lambda \longrightarrow 0$$

where $\Lambda = P_1 - P_2 \in \mathbf{M}$. Any three-chain Γ in $C^{(3)}$ with boundary $P_1 - P_2$ maps to Λ. The dual cohomology sequence of the above sequence is:

$$0 \longrightarrow (\mathbb{Z}.\Lambda)^* \longrightarrow H^3(\Theta) \longrightarrow H^3(C^{(3)}) \longrightarrow 0.$$

4.3. The fiber of the Prym map at JC. Suppose C is automorphism-free. The fiber $\mathscr{P}^{-1}(JC)$ has two irreducible components both isomorphic to $C^{(2)}$.

FIRST COMPONENT: Let $p+q$ be an element of $C^{(2)}$. Define $\widetilde{V} := (V_1 \sqcup V_2)/\sim$ where $V_1 \cong V_2 \cong C$ and "$\sim$" identifies p on V_i with q on V_{3-i}. Let V be the quotient of $\widetilde{V}$ by the involution interchanging V_1 and V_2. Then V is isomorphic to C with the points p and q identified. The cover $(\widetilde{V}, V)$ is called a Wirtinger double cover and is admissible in the sense of [2]. One has (see, for instance, [2])

$$\mathscr{P}(\widetilde{V}, V) = J(C) = A.$$

SECOND COMPONENT: Again choose $p+q \in C^{(2)}$. The linear system $|K_C - p - q|$ is a g_4^1 on C. The trigonal construction of Recillas [22] realizes the Jacobian of a curve with a g_4^1 as the Prym of a double cover of a trigonal curve. The construction is as follows. Let $\widetilde{W} \subset C^{(2)}$ be the curve

$$\{p+q : h^0(g_4^1 - p - q) > 0\}.$$

Define the involution $\imath$ on $\widetilde{W}$ in the following way: if

$$K_C \equiv p+q+s+t+u+v,$$

then $\imath(s+t) = u+v$. We denote $\widetilde{W}/\imath = W$. Then

$$\mathscr{P}(\widetilde{W}, W) = J(C) = A.$$

As there are three distinct ways to divide four points into two sets of two points, the curve W comes naturally with a g_3^1.

The two distinct g_3^1's on C give rise to two embeddings j_1 and j_2 of the curve C in $C^{(2)}$:

$$j_i : C \longrightarrow C^{(2)}; \quad s \mapsto u+v \text{ whenever } s+u+v \in P_i; \; i = 1, 2.$$

Let us denote the images of C by these maps by C_1 and C_2. Notice that these two curves do not intersect each other. (In fact the class $[C_i]$ is equal to $\sum_{i=1}^4 A_i B_i - 2\eta$ (see the appendix); one computes the self intersection to be zero.) The surface $\mathscr{P}^{-1}(A)$ is obtained by glueing two copies of $C^{(2)}$ together, identifying C_1 on one copy with C_2 on the other (see [11] and [2]).

4.4. Homology of the fiber of $\mathscr{P}$ at JC. From the above description it is easy to compute the homology of $\mathscr{P}^{-1}(A)$ using the Mayer-Vietoris exact sequence. It is as follows

$$\ldots H_1(C) \oplus H_1(C) \longrightarrow H_1(C^{(2)}) \oplus H_1(C^{(2)}) \longrightarrow H_1(\mathscr{P}^{-1}(A)) \longrightarrow$$

$$H_0(C) \oplus H_0(C) \longrightarrow H_0(C^{(2)}) \oplus H_0(C^{(2)}) \longrightarrow H_0(\mathscr{P}^{-1}(A)) \longrightarrow 0.$$

Furthermore, we have the following

Lemma 4.2. *The map*

$$(j_i)_* : H_1(C) \longrightarrow H_1(C^{(2)})$$

is an isomorphism for $i = 1, 2$.

Proof. This is the same as saying that $\cup[C_i] : H^1(C) \longrightarrow H^3(C^{(2)})$ is an isomorphism over $\mathbb{Z}$. However, $[C_i] = \sum_{i=1}^4 A_i B_i - 2\eta$ and thus $A_1 \cdot [C_i] = A_1 \cdot \eta$, etc, which is a basis for $H^3(C^{(2)})$ (see the appendix). $\Diamond$

Hence, the image of $H_1(C) \oplus H_1(C)$ is the diagonal of $H_1(C^{(2)}) \oplus H_1(C^{(2)})$. So we obtain the exact sequence

$$0 \longrightarrow H_1(C^{(2)}) \longrightarrow H_1(\mathscr{P}^{-1}(A)) \longrightarrow \mathbb{Z}.\lambda \longrightarrow 0$$

where λ is any cycle that generates the image of $H_1(\mathscr{P}^{-1}(A))$ in $H_0(C) \oplus H_0(C)$ (this is easily seen to be isomorphic to $\mathbb{Z}$).

4.5. Prym-embedded curves in Θ and the universal curve. Let $(\widetilde{V}, V)$ and $(\widetilde{W}, W)$ be as in section 4.3. Then, by [15], $\widetilde{V}$ parametrizes the Prym-embeddings of $\widetilde{W}$ into Θ and conversely. The Prym-embeddings are as follows:

Let a be a point of C. Associate to it *two* maps $\widetilde{W} \longrightarrow C^{(3)}$:

$$\text{one}_a : s + t \mapsto s + t + a,$$

$$\text{two}_a : s + t \mapsto |K_C - \iota(s + t) - a|.$$

As $\text{one}_p = \text{two}_q$ and $\text{one}_q = \text{two}_p$, we see that we obtain embeddings of $\widetilde{W}$ in $C^{(3)}$ parametrized by $\widetilde{V}$. Also, given a point $s + t \in \widetilde{W}$, we obtain a map $\widetilde{V} \longrightarrow C^{(3)}$ as follows:

$$V_1 \ni a \mapsto s + t + a,$$

$$V_2 \ni a \mapsto |K_C - \iota(s + t) - a|.$$

Composing these embeddings with the map $C^{(3)} \longrightarrow \Theta$ gives us Prym-embedded curves in Θ, and it can be shown that *all* Prym-embedded curves are obtained in this way, [15]. So we see that the family $\mathscr{F} \longrightarrow \mathscr{P}^{-1}(A)$ of Prym-embedded curves in Θ has *three* components. Two of these lie over the component of $\mathscr{P}^{-1}(A)$ which parametrizes smooth Prym-curves:

$$\mathscr{F}_{11} \cong C \times C^{(2)}, \quad \mathscr{F}_{12} \cong C \times C^{(2)}.$$

The point $(p, p + q) \in \mathscr{F}_{11}$ is identified with $(q, p + q) \in \mathscr{F}_{12}$. We put $\mathscr{F}_1 = \mathscr{F}_{11} \cup \mathscr{F}_{12}$.

Over the component parametrizing the singular Prym–curves, there is only one component

$$\mathscr{F}_2 = P := \{(p + q, r + s) \in C^{(2)} \times C^{(2)} | h^0(K_C - p - q - s - t) > 0\}$$

Note that $\mathscr{F}_2$ also is glued to $\mathscr{F}_{11}$ and $\mathscr{F}_{12}$.

From this description of $\mathscr{F}$ one can also see that the universal curve $\mathscr{C}$ over $\mathscr{F}$ has four components, each of which is isomorphic to $C \times P$:

$$\mathscr{C} = \mathscr{C}_{11} \cup \mathscr{C}_{12} \cup \mathscr{C}_{21} \cup \mathscr{C}_{22}.$$

Here $\mathscr{C}_{ij}$ maps to $\mathscr{F}_i$. (For our purposes it is not necessary to write down the explicit glueings between these components.)

For example, the restriction of the universal curve over $\mathscr{F}_{11}$ and its mapping to $C^{(3)}$ and $\mathscr{F}_{11}$ are explicitly given by

$$\begin{array}{ccccc} \mathscr{C}_{11} & = & C \times P & \longrightarrow & C^{(3)} \\ & & (t, p + q, r + s) & \mapsto & t + r + s \end{array}$$

$$\begin{array}{ccccccc} \mathscr{C}_{11} & = & C \times P & \longrightarrow & \mathscr{F}_{11} & = & C \times C^{(2)} \\ & & (t, p + q, r + s) & \mapsto & (t, p + q) & & \end{array}$$

We will need this later for our explicit calculations.

4.6. The degeneration argument. Now consider a generic one-parameter family of abelian varieties over a small disc T.

$$\mathscr{A} \xrightarrow{\pi} T.$$

We denote the fibers $\pi^{-1}(t)$ by A_t, and assume that $A_0 = A = JC$. We can associate to this family the families

$$\Theta_T \longrightarrow T$$
$$\mathscr{F}_T \longrightarrow T$$
$$\mathscr{P}^{-1}(\mathscr{A}) \longrightarrow T.$$

We may assume that the fibers Θ_t, $\mathscr{F}_t$ and $\mathscr{P}^{-1}(\mathscr{A})_t = \mathscr{P}^{-1}(A_t)$ are smooth for $t \in T$, $t \neq 0$. For $t = 0$ the fibers $\Theta = \Theta_0$ and $\mathscr{P}^{-1}(JC)$ are singular and were described above. Under these circumstances, there is a *vanishing homology* sequence relating the homology of $\Theta = \Theta_0$ and Θ_t, $t \neq 0$ (see, for instance, [4]). It is as follows:

$$0 \longrightarrow H_4(\Theta_t) \longrightarrow H_4(\Theta) \longrightarrow \mathbf{N} \longrightarrow H_3(\Theta_t) \longrightarrow H_3(\Theta) \longrightarrow 0.$$

Here $\mathbf{N} := \mathbb{Z} \oplus \mathbb{Z}$ is the free abelian group with basis the two nodes of Θ. The map

$$\mathbf{N} \longrightarrow H_3(\Theta_t)$$

maps a node to the corresponding vanishing cycle in the nearby fiber Θ_t. Since P_1 and P_2 are homologous, the two vanishing cycles are homologous in Θ_t. So we have an exact sequence

$$0 \longrightarrow \mathbb{Z}.V \longrightarrow H_3(\Theta_t) \longrightarrow H_3(\Theta) \longrightarrow 0,$$

where V is any of the two vanishing cycles.

Recall that we also have a sequence:

$$0 \longrightarrow H_3(C^{(3)}) \longrightarrow H_3(\Theta) \longrightarrow \mathbb{Z}.\Lambda \longrightarrow 0.$$

Similarly, there is a vanishing homology sequence for $\mathscr{P}^{-1}(\mathscr{A})$:

$$0 \longrightarrow \mathbb{Z}.\nu \longrightarrow H_1(\mathscr{P}^{-1}(A_t)) \longrightarrow H_1(\mathscr{P}^{-1}(A)) \longrightarrow 0,$$

which complements the sequence

$$0 \longrightarrow H_1(C^{(2)}) \longrightarrow H_1(\mathscr{P}^{-1}(A)) \longrightarrow \mathbb{Z}.\lambda \longrightarrow 0.$$

The Hodge structures $H_3(\Theta_t)$ and $H_1(\mathscr{P}^{-1}(A_t))$ depend on the choice of $t \neq 0$, but it follows from the work of Clemens ([3]) and Steenbrink ([23])

that one can define a *limes mixed Hodge structure* on the homology of the general fiber in such a way that the above sequence becomes a sequence of mixed Hodge structures. The relation between $C^{(3)}$ and Θ_t can then be expressed as follows (with a slightly nonstandard convention for the weights):

$$\begin{aligned} Gr_4^W(H_3(\Theta_t)) &= \mathbb{Z}\cdot V, \\ Gr_3^W(H_3(\Theta_t)) &= H_3(C^{(3)}, \mathbb{Z}), \\ Gr_2^W(H_3(\Theta_t)) &= \mathbb{Z}\cdot\Lambda. \end{aligned}$$

Similarly

$$\begin{aligned} Gr_2^W(H_1(\mathscr{P}^{-1}(A_t))) &= \mathbb{Z}\cdot\nu, \\ Gr_1^W(H_1(\mathscr{P}^{-1}(A_t))) &= H_1(C^{(2)}, \mathbb{Z}), \\ Gr_0^W(H_1(\mathscr{P}^{-1}(A_t))) &= \mathbb{Z}\cdot\lambda. \end{aligned}$$

Now, for each $t \neq 0$, we have an Abel-Jacobi mapping

$$\mathrm{AJ}_t : \mathrm{Alb}(\mathscr{F}_t) \longrightarrow J(\Theta_t)$$

which is induced by a morphism of $\mathbb{Z}$-Hodge structures of weight $(-1, -1)$:

$$\beta_t : H_1(\mathscr{F}_t) \longrightarrow H_3(\Theta_t).$$

This then induces a morphism of limes mixed Hodge structures, and we obtain a commutative diagram:

$$\begin{array}{ccc} Gr_1^W(H_1(\mathscr{F})) & \longrightarrow & Gr_3^W(H_3(\Theta_t)) \\ \downarrow & & \downarrow \\ Gr_1^W(H_1(\mathscr{P}^{-1}(A_t))) & \longrightarrow & Gr_3^W(\mathbf{Q}) \end{array}$$

So, since the morphism from $\mathrm{Alb}\,\mathscr{P}^{-1}(A)$ to $J(\mathbf{Q})$ is multiplication by n, in order to prove that the Abel-Jacobi mapping induces an isomorphism between $\mathrm{Alb}(\mathscr{P}^{-1}(A))$ and $J(\mathbf{Q})$, it suffices to show that it induces an isomorphism on the pure weight three part, which can be related directly to symmetric products of the curve C.

4.7. Cohomology of Θ. From the above exact sequences and the results of Macdonald, we can get a complete description of $H^*(\Theta_t, \mathbb{Z})$ in terms of the curve C. For the Betti-numbers one gets:

k	0	1	2	3	4	5	6
rank $H^k(\Theta_t)$	1	8	28	66	28	8	1

In fact, one has:

$$H^2(\Theta_t) \cong \{x \in H^2(C^{(3)}) | x \cdot c = 0\},$$

$$H^4(\Theta_t) \cong H^2(C^{(3)})/\mathbb{Z}\cdot c,$$

where $c = \sum_{i=1}^4 A_i B_i \eta - 3\eta^2$ is the class of P_1 and P_2. For the description of $H^3(\Theta_t)$ we have to choose a lift of of V^* to $H^3(\Theta_t)$. By Poincaré duality, this can be done in such a way that $\Lambda^* \cdot V^* = 1$ One can then write:

$$H^3(\Theta_t) \cong H^3(C^{(3)}) \oplus \mathbb{Z}\cdot\Lambda^* \oplus \mathbb{Z}\cdot V^*.$$

The ring structure of $H^*(\Theta_t)$ is now obvious. Using this, it becomes a matter of straightforward linear algebra to write down bases for all the lattices involved. We just state the result.

Proposition 4.3. *With the notation of the appendix, one has the following bases:*

For $\mathbf{K}$:

$$A_i(2\eta - \theta),\ B_i(2\eta - \theta),\ \Lambda^*,\ V^*.$$

For $\mathbf{H}$:

$$A_i\cdot\eta,\ B_i\cdot\eta,\ A_i\cdot\theta,\ B_i\cdot\theta,\ \Lambda^*,\ V^*.$$

The dual lattices are generated by the following elements:

For $\mathbf{Q}$:

$$A_i\cdot\eta,\ B_i\cdot\eta,\ \Lambda^*,\ V^*.$$

For $\mathbf{H}^*$:

$$A_i(A_j\cdot B_j),\ B_i(A_j\cdot B_j),\ A_i\cdot\eta,\ B_i\cdot\eta,\ \Lambda^*,\ V^*/ \qquad A_i(A_j\cdot B_j) \equiv A_i(A_k\cdot B_k),\ etc.$$

Using these descriptions, one also can check the results of §1 for a *generic* A.

4.8. An Abel-Jacobi mapping. We now want to compute explicitly the Abel-Jacobi mapping

$$\beta : H_1(\mathscr{P}^{-1}(A)) \longrightarrow \mathbf{Q}$$

in the Jacobian case. Geometrically, this map arises as follows: Consider the diagram

$$\begin{array}{ccc} \mathscr{C} & \longrightarrow & C^{(3)} \\ \downarrow & & \\ \mathscr{F} & & \end{array}$$

By taking the full inverse image of a cycle $\gamma \in H_1(\mathscr{F})$ we obtain a three cycle in $\mathscr{C}$, which is mapped to a three cycle $\in H_3(C^{(3)})$. We can restrict this family to the component $\mathscr{F}_{11}$ of $\mathscr{F}$. For each choice of a point t_0

on C, the map $\mathcal{F}_{11} \longrightarrow C_1^{(2)}$; $(t, r+s) \mapsto r+s$ has a *section*:

$$C_1^{(2)} \longrightarrow \mathcal{F}_{11}; p+q \mapsto (t_0, p+q).$$

In this way we obtain maps

$$H_1(C^{(2)}) \longrightarrow H_1(\mathcal{F}) \longrightarrow H_1(\mathcal{P}^{-1}(A)),$$

whose composition can be checked to be injective. When we restrict the family to this section, we obtain a diagram:

$$\begin{array}{ccc} P & \longrightarrow & C^{(3)} \\ \downarrow & & \\ C^{(2)} & & \end{array}$$

where the map $P \longrightarrow C^{(3)}$ is the composition of the maps

$$P \longrightarrow C^{(2)}; (p+q, r+s) \mapsto r+s$$

and

$$C^{(2)} \longrightarrow C^{(3)}; r+s \mapsto t_0 + r + s.$$

(Recall that $P := \{(p+q, r+s) \in C^{(2)} \times C^{(2)} | h^0(K_C - p - q - s - t) > 0\}$.)

Now consider the diagram

$$\begin{array}{ccc} P & \xrightarrow{p_2} & C^{(2)} \\ {\scriptstyle p_1}\downarrow & & \\ C^{(2)} & & \end{array}$$

where the maps p_i are the restrictions of the natural projections q_i : $C^{(2)} \times C^{(2)} \longrightarrow C^{(2)}$, $i = 1, 2$.

Proposition 4.4. *The mapping*

$$p_{2!}p_1^* : H^3(C^{(2)}) \longrightarrow H^1(C^{(2)})$$

is the inverse of the mapping

$$\eta \cup : H^1(C^{(2)}) \longrightarrow H^3(C^{(2)}).$$

In particular, it is an isomorphism.

Proof. First note that our cycle P is the pull-back of the cycle $D \subset C^{(4)}$ consisting of coplanar four-tuples of points on the canonical curve. So, by the formula of Macdonald, its class in $H^2(C^{(4)})$ is

$$\sum_{i=1}^{4} A_i B_i - \eta.$$

Using the Künneth-isomorphism

$$H^*(C^{(2)} \times C^{(2)}) \longrightarrow H^*(C^{(2)}) \otimes H^*(C^{(2)})$$

we can write the class of P as:

$$\sum_{i=1}^{4} (A_i \otimes 1 + 1 \otimes A_i) \cdot (B_i \otimes 1 + 1 \otimes B_i) - (\eta \otimes 1 + 1 \otimes \eta).$$

Take any element of $H^3(C^{(2)})$, say $A_1 \cdot \eta$. Clearly one has :

$$p_{2!} p_1^*(A_1 \cdot \eta) = q_{2!}([P] \cdot q_1^*(A_1 \cdot \eta)).$$

Now, $q_1^*(A_1 \cdot \eta) = A_1 \cdot \eta \otimes 1$, so by a computation in the ring $H^*(C^{(2)})$ we find that

$$[P] \cdot q_1^*(A_1 \cdot \eta) = \eta^2 \otimes A_1 + \sum_{i=1}^{4} A_1 \cdot \eta \otimes A_i B_i - A_1 \cdot \eta \otimes \eta.$$

To compute the image of this by $q_{2!}$, one has to use the projection formula:

$$(q_{2!}\alpha) \cdot \beta = \alpha \cdot (q_2^* \beta)$$

(for all $\alpha \in H^5(C^{(2)} \times C^{(2)})$ and all $\beta \in H^3(C^{(2)})$). Now take:

$$\alpha = [P] \cdot q_1^*(A_1 \cdot \eta)$$

and let β run over a basis of $H^3(C^{(2)})$, that is, $A_i \cdot \eta$ and $B_i \cdot \eta$, $i = 1, 2, 3, 4$. We obtain:

$$(q_{2!}([P] \cdot A_1 \cdot \eta \otimes 1)) \cdot (1 \otimes A_i \cdot \eta) = 0$$

for $i = 1, 2, 3, 4$, but

$$(q_{2!}([P] \cdot A_1 \cdot \eta \otimes 1)) \cdot (1 \otimes B_i \cdot \eta) = 0$$

only for $i = 2, 3, 4$ and equal to 1 for $i = 1$. As $A_i \cdot \eta$, $B_i \cdot \eta \in H^3(C^{(2)})$ is exactly the dual basis to B_i, $A_i \in H^1(C^{(2)})$ we conclude that

$$p_{2!}(p_1^*(A_1 \cdot \eta)) = A_1.$$

Hence, $p_{2!} p_1$ is indeed the inverse of $\eta \cup$. $\Diamond$

Notice that the proposition also says that the mapping

$$p_{2*}p_1^! : H_1(C^{(2)}) \longrightarrow H_3(C^{(2)})$$

is an isomorphism (over $\mathbb{Z}$).

Corollary 4.5. *The Abel-Jacobi mapping*

$$H_1(C^{(2)}) \cong H^3(C^{(2)}) \longrightarrow H_3(C^{(3)}) \cong H^3(C^{(3)})$$

is obtained by sending an element to the element with the same name, but now considered in the other ring (by the natural inclusion $H^*(C^{(2)}) \longrightarrow H^*(C^{(3)})$*).*

Proof. The class of $C^{(2)} + t_0 \subset C^{(3)}$ is just η, so the result follows from the proposition. ◊

Corollary 4.6. *The induced map*

$$Gr_1^W(H_1(P^{-1}(A)) \longrightarrow Gr_3^W(\mathbf{Q})$$

is an isomorphism.

Corollary 4.7. *If* A *is generic, then the morphism induced by the Abel-Jacobi mapping*

$$\mathrm{Alb}(P^{-1}(A)) \longrightarrow J(\mathbf{Q})$$

is an isomorphism of polarized abelian varieties.

Corollary 4.8. *For* any *abelian variety with smooth theta divisor, the image of* $\mathscr{F}$ *by the Abel-Jacobi mapping generates* $J(\mathbf{H})$. *If* $\mathscr{F}$ *is smooth, then there is an isogeny* $\mathrm{Alb}(\mathscr{F}) \longrightarrow J(\mathbf{H})$.

Proof. By our observation at the end of section 2 it suffices to observe that it follows from Corollary 4.5 that, for A generic, the intersection of JH' and $J(\mathbf{H})$ is not A. Hence $J(\mathbf{H}) = JH'$ for generic and hence for all A with smooth theta divisor. ◊

Remark 4.9. A calculation similar to 4.4 gives $A_1\theta$ as the image of

$$A_1 \otimes \eta^2 \in H^1(C) \otimes H^4(C^2) \subset H^5(\mathscr{F}_{11})$$

which describes the cohomology class of a cycle *in* a fiber. This is in accordance with what we found in 2.2.

Remark 4.10. For a generic abelian variety, the result of section 2 could have been deduced from the results of this section. However, it cannot be deduced from the results of this section for *any* abelian variety with smooth theta divisor.

Some open problems. We would like to mention a few open problems.

Is it true that the Abel-Jacobi mapping induces an isomorphism $\mathrm{Alb}(\mathscr{F}) \longrightarrow J(\mathbf{H})$?

Our construction gives a nice family of curves that generate the torus $J(\mathbf{H})$. By taking restrictions we obtain families that generate $J(\mathbf{K})$. But is there a *nice* family of curves around that generates $J(\mathbf{K})$?

Is there a simple construction of the point of order two in $J(\mathbf{Q})$ that corresponds to the strange involution λ on the fibres of the Prym map?

Is there a direct description of the Theta-divisor of $J(\mathbf{K})$ or $J(\mathbf{Q})$?

Generalizations. One might ask what happens with the picture described in this paper if we consider abelian varieties of dimension different from four. As explained in §1, if A is an abelian $(n+1)$-fold with smooth Θ-divisor, then the Hodge structure $H^n(\Theta)$ has an interesting sub-Hodge structure $\mathbf{K} = \ker(H^n(\Theta) \longrightarrow H^{n+2}(A))$. If n is odd, we can form its intermediate jacobian $J(\mathbf{K})$. However, for $n > 3$, the complex torus $J(K)$ will no longer be an abelian variety because there will be consecutive nonzero $H^{p,q}$'s and thus the Hodge form will no longer be positive definite. In any case, the level (that is, $\max(|p-q|, H^{p,q} \neq 0)$) of the Hodge structure $\mathbf{K}$ will be $n-2$, and the generalized Hodge conjecture would imply that this sub-Hodge structure comes from some nontrivial family of subvarieties of Θ via an Abel-Jacobi mapping, the cohomology of the parameter space of the family being basically $\mathbf{K}$. We are unable to produce such a nontrivial family of subvarieties inside Θ in higher dimensions and this seems to be an interesting problem for further research. If n is even, say $n = 2p$, then one can try to see whether the primitive cohomology classes of Hodge type (p, p) come from subvarieties of Θ. As for abelian varieties of dimension 1, 2 and 3 $\mathbf{K}$ is trivial or uninteresting, the case of abelian fourfolds is really the first nontrivial case.

5. Appendix: Cohomology of a symmetric product

We recall a result of Macdonald ([17]) which describes the cohomology ring $H^*(C^{(n)})$ in terms of the cohomology of the curve C. Let $\alpha_1, \dots, \alpha_4; \beta_1, \dots, \beta_4$ be a symplectic basis for $H^1(C)$, that is, the products

$$\alpha_i \cdot \beta_i = \lambda = -\beta_i \cdot \alpha_i$$

are the only nonzero intersection products (here λ is a generator of $H^2(C)$). Consider now the following elements in the n-th symmetric product of $H^1(C)$:

$$A_i := \alpha_i \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \alpha_i \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \alpha_i,$$

$$B_i := \beta_i \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \beta_i \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \beta_i ,$$

$$\eta := \lambda \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \lambda \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \lambda .$$

These elements can be considered as living in $H^*(C^{(n)})$, and in fact these generate the cohomology as a (graded) ring:

$$H^*(C^{(n)}) = \mathbb{Z}[A_1, \dots, A_4, B_1, \dots, B_4, \eta]/I_n$$

where $\deg(A_i) = \deg(B_i) = 1$; $\deg(\eta) = 2$, and the ideal I_n is generated by the elements of the form

$$A_I \cdot B_J (AB - \eta)_K \cdot \eta^q$$

where $I = \{i_1, i_2, \dots\}$, $J = \{j_1, j_2, \dots\}$, and $K = \{k_1, k_2, \dots\}$ are disjoint index sets with $|I| + |J| + 2|K| + q = n + 1$, $A_I = A_{i_1} A_{i_2} \dots$, $B_J = B_{j_1} B_{j_2} \dots$, $(AB - \eta)_K = (A_{k_1} B_{k_1} - \eta)(A_{k_2} B_{k_2} - \eta) \dots$.

From this the ranks and an additive basis for the cohomology are easily obtained. For instance, for a curve of genus four one finds:

Group	Rank	Basis
$H^0(C^{(2)})$	1	1
$H^1(C^{(2)})$	8	A_i, B_i
$H^2(C^{(2)})$	29	$A_i A_j, A_i B_j, B_i B_j, \eta$
$H^3(C^{(2)})$	8	$A_i \eta, B_i \eta$
$H^4(C^{(2)})$	1	η^2

with relations $A_i A_j B_j = A_i \eta$, $A_i B_i \eta = \eta^2$, etc. For the third symmetric product one has:

Group	Rank	Basis
$H^0(C^{(3)})$	1	1
$H^1(C^{(3)})$	8	A_i, B_i
$H^2(C^{(3)})$	29	$A_i A_j, A_i B_j, B_i B_j, \eta$
$H^3(C^{(3)})$	64	$C_i C_j C_k, A_i \eta, B_i \eta$
$H^4(C^{(3)})$	29	$A_i A_j \eta, A_i B_j \eta, B_i B_j \eta, \eta^2$
$H^5(C^{(3)})$	8	$A_i \eta^2, B_i \eta^2$
$H^6(C^{(3)})$	1	η^3

where $C_i = A_i$ or B_i. Relations: $A_1 B_1 A_2 B_2 = A_1 B_1 \eta + A_2 B_2 \eta - \eta^2$, $A_1 A_2 B_2 \eta = A_1 \eta^2$, $A_1 B_1 A_2 B_2 A_3 B_3 = \eta^3$, etc.

Furthermore, the class of the subvariety of $C^{(n)}$ parametrizing the divisors $D \in C^{(n)}$ such that $h^0(g_N^r - D) > 0$ for some fixed g_N^r on C is equal to the coefficient of t^{n-r} in the expression

$$(1+\eta.t)^{N-r-g}\prod_{i=1}^{g}(1+A_iB_i.t)$$

where g is the genus of the curve.

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