# WEAK WHITNEY REGULARITY IMPLIES EQUIMULTIPLICITY FOR FAMILIES OF COMPLEX HYPERSURFACES

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## 1. INTRODUCTION

A weakened form of Whitney's condition $(b)$ was introduced by K. Bekka and the first author [B1, BT1, BT2]. It was initially motivated by work of M. Ferrarotti on metric properties of Whitney stratified sets [Fe1, Fe2], and by the observation that the logarithmic or slow spiral has finite length. These weakly Whitney stratified sets retain many properties of Whitney stratified sets, including for example the property that any submanifold transverse to a stratum $Y$ is transverse to all strata in some neighbourhood of $Y$: this is an easy consequence of Whitney $(a)$-regularity. Also Thom's first isotopy lemma still applies. This requires a delicate adaptation of Mather's proof for Whitney stratifications [Ma], which was carried out by K. Bekka in his 1988 thesis [B1] and published in [B2]. It follows in particular that weakly Whitney stratifications are locally topologically trivial and are triangulable. They also have many of the same metric properties as Whitney stratified sets, as shown in [BT3]. Bekka and Trotman, Orro and Trotman [OT1, OT2], Parusinski [Pa], Pflaum [Pf] and Schürmann [S] have obtained further properties of weakly Whitney stratified sets.

It was shown in [BT2] that there exist real algebraic varieties with weakly Whitney regular stratifications which are not Whitney regular. No examples are known among complex analytic varieties, so that the question arises as to whether the two notions of Whitney regularity and weak Whitney regularity coincide in the complex case. As a test, it is natural to check the Briançon-Speder examples of families of complex surface singularities in $\mathbf{C}^3$ which have constant Milnor number but which are not Whitney regular in $\mathbf{C}^4$ [BS1]. Calculations by K. Bekka and the first author show that none of the Briançon-Speder examples (of which there are infinitely many) are weakly Whitney regular [BT4].

As further evidence that weak Whitney regularity and Whitney regularity might be equivalent for complex analytic stratifications, or at least for complex analytic hypersurfaces, we show here that equimultiplicity of a family of complex analytic hypersurfaces follows from weak Whitney regularity of the family over the parameter space. That equimultiplicity follows from Whitney regularity was proved for general complex analytic spaces by Hironaka in 1969 [Hi]. In 1976 [BS2] Briançon and Speder gave a different proof, valid for families of complex hypersurfaces with isolated singularities, and Navarro Aznar generalised their proof to the general complex case in 1980 [N].

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## 2. Weak Whitney regularity.

**Definition 2.1.**
Let $X, Y$ be two submanifolds of a riemannian manifold $M$ with $X \cap Y = \emptyset$ and take $y \in \overline{X} \cap Y$.
**Condition** $(a)$: The triple $(X, Y, y)$ is said to satisfy Whitney's condition $(a)$ if for each sequence of points $x_i$ of $X$ converging to $y \in Y$ such that $T_{x_i}X$ converges to $\tau$ (in the corresponding grassmannian in $TM$), then $T_yY \subset \tau$.

**Condition** $(b)$: The triple $(X, Y, y)$ is said to satisfy Whitney's condition $(b)$ if there exists a local diffeomorphism $h : \mathbb{R}^n \to M$ onto a neighbourhood $U$ of $y$ in $M$ such that for each sequence of points $(x_i, y_i)$ of $h^{-1}(X) \times h^{-1}(Y)$ converging to $(h^{-1}(y), h^{-1}(y))$, such that the sequence $T_{x_i}h^{-1}(X)$ converges to $\tau$ in the corresponding grassmannian and the sequence $\overline{x_i y_i}$ converges to $\ell$ in $\mathbb{P}^{n-1}(\mathbb{R})$, then $\ell \subset \tau$.

One says that the pair $(X, Y)$ satisfies condition $(a)$ (resp.$(b)$) if $(X, Y, y)$ satisfies $(a)$ (resp. $(b)$) at each $y \in \overline{X} \cap Y$.

**Definition 2.2**.
**Condition** $(b^\pi)$**:** If $\pi$ is a local $C^1$ retraction associated to a $C^1$ tubular neighbourhood of $Y$ near $y$, a condition $(b^\pi)$ is obtained from the definition of $(b)$ by replacing the sequence $y_i$ by the sequence $\pi(x_i)$ (cf. [Wh1, Th2]).

It is well-known and straightforward to show that condition $(b)$ implies condition $(a)$ (see [Ma], or [Wa]). In fact $(b)$ is equivalent to the combination of $(a)$ and $(b^\pi)$ [NT], which we shall denote by $(a + b^\pi)$. More generally whenever two equisingularity conditions $(E_1)$ and $(E_2)$ are satisfied we shall say that $(E_1 + E_2)$ is satisfied.

We now recall the regularity condition called $(\delta)$ introduced by K. Bekka and the first author. It is a weakening of condition $(b)$.

**Definition 2.3.** Given a euclidean vector space $V$, and two vectors $v_1, v_2 \in V^* = V - \{0\}$, define the sine of the acute angle $\theta(v_1, v_2)$ between them by :

$$\sin\theta(v_1, v_2) = \frac{\|v_1 \wedge v_2\|}{\|v_1\|.\|v_2\|}$$

where $v_1 \wedge v_2$ is the usual vector product and $\|.\|$ is the norm on $V$ induced by the euclidean structure.

Given two vector subspaces $S$ and $T$ of $V$ define the sine of the angle between them by :

$$\sin\theta(S, T) = \sup\{\sin\theta(s, T) : s \in S^*\}$$

where

$$\sin\theta(s, T) = \inf\{\sin\theta(s, t) : t \in T^*\}.$$

If $\pi_T : V \longrightarrow T^\perp$ is the orthogonal projection onto the orthogonal complement of $T$, then $\sin\theta(s, T) = \frac{||\pi_T(s)||}{||s||}$. The definition of sine for lines is the same as for vectors : take unit vectors on the lines.

One verifies easily that : $\sin\theta(v_1, v_3) \leq \sin\theta(v_1, v_2) + \sin\theta(v_2, v_3)$ for all $v_1, v_2, v_3 \in V^*$, and $\sin\theta(S_1 + S_2, T) \leq \sin\theta(S_1, T) + \sin\theta(S_2, T)$, for subspaces $S_1, S_2, T$ of $V$ such that $S_1$ is orthogonal to $S_2$.

**Definition 2.4.**
**Condition** $(\delta)$**:** We say that the triple $(X, Y, y)$ satisfies *condition* $(\delta)$ if there exist a local diffeomorphism $h : \mathbb{R}^n \longrightarrow M$ to a neighbourhood $U$ of $y$ in $M$, and a real number $\delta_y$, $0 \leq \delta_y < 1$, such that for every sequence $(x_i, y_i)$ of $h^{-1}(X) \times h^{-1}(Y)$ which converges to $(h^{-1}(y), h^{-1}(y))$ such that the sequence $\overline{x_i y_i}$ converges to $\ell$ in $\mathbb{P}^{n-1}(\mathbb{R})$ and the sequence $T_{x_i} h^{-1}(X)$ converges to $\tau$, then $\sin\theta(l, \tau) \leq \delta_y$.

Clearly condition $(b)$ implies $(\delta)$ : just take $\delta_y = 0$.

**Definition 2.5.** A *weakly Whitney regular stratification* of a subspace $A$ of a $C^1$ manifold $M$ is a locally finite partition of $A$ into connected $C^1$ submanifolds, called the *strata*, such that :

1 ) - Frontier Condition : if $X$ and $Y$ are distinct strata such that $\overline{X} \cap Y \neq \emptyset$, then $Y \subset \overline{X}$. $X$ and $Y$ are then said to be *adjacent*.

2 ) - Each pair of adjacent strata satisfies condition $(a)$.

3 ) - Each pair of adjacent strata satisfies condition $(\delta)$.

In fact the frontier condition turns out to follow from conditions $(a)$ and $(\delta)$, in exactly the same way as it follows from condition $(b)$, because in both cases one is able to apply Thom's first isotopy lemma to prove local topological triviality along strata.

**Remark 2.6.** If $\pi$ is a local $C^1$ retraction associated to a $C^1$ tubular neighbourhood of $Y$ near $y$, a condition $(\delta^\pi)$ is obtained from the definition 2.4 of $(\delta)$ by replacing the sequence $y_i$ by the sequence $\pi(x_i)$.

**Lemma 2.7.** $(a + \delta) \iff (a + \delta^\pi)$.
**Proof.** Clearly $(\delta) \implies (\delta^\pi)$, so it suffices to show that $(a + \delta^\pi) \implies (\delta)$. In the definition of $(\delta)$ decompose the limiting vector $l$ as the sum of a vector $l_1$ tangent to $Y$ at $y$, and a vector $l_2$ tangent to $\pi^{-1}(y)$ at $y$. Then $\sin\theta(l, \tau) = \sin\theta(l_1 + l_2, \tau) \leq \sin\theta(l_1, \tau) + \sin\theta(l_2, \tau)$. By condition $(a)$, $\sin\theta(l_1, \tau) = 0$, hence $\sin\theta(l, \tau) \leq \sin\theta(l_2, \tau)$ (as we observed above in Definition 2.3) which is less than or equal to $\delta_y$ by hypothesis, implying $(\delta)$. $\square$

It is obvious from the definition of $(\delta)$ that $(b)$ implies $(a + \delta)$, so justifying the terminology "weakly Whitney" for $(a + \delta)$. In [BT2] there are real algebraic examples illustrating that $(a + \delta)$ does not imply $(b)$, and that $(\delta)$ does not imply $(a)$. There is currently no example known of a weakly Whitney regular complex analytic stratification which is not Whitney regular.

## 3. THE BAD SET OF LIMITS FOR $(b^\pi)$-REGULARITY

Let $Y = 0^{n+1} \times \mathbb{C}^m \subset \mathbb{C}^{m+n+1}$ and let $V$ be a complex analytic subset of dimension $d$ in $\mathbb{C}^{m+n+1}$, with $Y \subset V$ and put $X := V \setminus Y$ and $X_{reg}$ its nonsingular part. Let $G$ denote the graph in $\mathbb{C}^{m+n+1} \times \mathbb{P}^n \times \text{Grass}(d, m + n + 1)$ of the map

$$X_{reg} \longrightarrow \mathbb{P}^n \times \text{Grass}(d, m + n + 1), \quad x \mapsto \Big(\overline{x\pi(x)}, T_x X\Big).$$

Let $p_1, p_2$ denote the projections from $\mathbb{C}^{m+n+1} \times \mathbb{P}^n \times \text{Grass}(d, m+n+1)$ to $\mathbb{C}^{m+n+1}$ and $\mathbb{P}^n$ respectively. We put $E := p_1^{-1}(0) \cap \overline{G}$. The set

$$\Lambda^\pi(X,Y) := \{\lambda \in \mathbb{P}^n \mid \exists \text{ a sequence } x_i \in X \text{ such that } \lambda = \lim \overline{x_i \pi(x_i)} \notin \lim T_{x_i} X\},$$

is the *bad limit set for* $(b^\pi)$*-regularity*. It is related to $E$ by $\Lambda^\pi(X,Y) = p_2(E \cap B)$, where $B := \{(0, \lambda, T) \in \{0\} \times \mathbb{P}^n \times \text{Grass}(d, m+n+1) \mid \lambda \notin T\}$, hence it is a constructible subset of $\mathbb{P}^n$. Properties of $\Lambda^\pi(X,Y)$ were studied in [NT].

## 4. MAIN THEOREM.

Let $F$ be an analytic function germ defined on a neighbourhood of $0$

$$\begin{array}{rccc} F: & (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \times \mathbb{C} & \to & (\mathbb{C}, 0) \\ & (z,t) & \mapsto & F(z,t) \end{array}$$

We denote by $\pi$ the projection on the second factor, $V := F^{-1}(0)$, $Y := \{0\} \times \mathbb{C}$ and

$$V_t := \{z \in \mathbb{C}^{n+1} : F(z,t) = 0\}.$$

Suppose $V_t$ has an isolated singularity at $(0,t)$, i.e. the critical set of the restriction of $\pi$ to $V$ is $Y$. Then $X := V \setminus Y$ is a complex analytic manifold of dimension $n+1$, and for each point $(z,t) \in X$ we have $T_{(x,t)}X = \{(u,v) \in \mathbb{C}^{n+1} \times \mathbb{C} \mid \sum_{i=0}^n u_i \frac{\partial F}{\partial z_i}(z,t) + v \frac{\partial F}{\partial t}(z,t) = 0\}$. We set

$$gradF = (\frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n}, \frac{\partial F}{\partial t}), \;\; grad_z F = (\frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n}) \text{ and } \|grad_z F\|^2 = \sum_{i=0}^n \|\frac{\partial F}{\partial z_i}\|^2.$$

**Characterisation of condition $(a)$.**

*The pair $(X,Y)$ satisfies Whitney's condition $(a)$ at $0$ iff*

$$\lim_{\substack{(z,t)\to 0 \\ (z,t)\in X}} \left( \frac{\frac{\partial F}{\partial t}(z,t)}{\|gradF(z,t)\|} \right) = 0.$$

**Characterisation of condition $(b^\pi)$.**

*The pair $(X,Y)$ satisfies Whitney's condition $(b^\pi)$ at $0$ iff*

$$\lim_{\substack{(z,t)\to 0 \\ (z,t)\in X}} \left( \frac{\sum_{i=0}^n z_i \frac{\partial F}{\partial z_i}(z,t)}{\|z\|\|gradF(z,t)\|} \right) = 0.$$

**Characterisation of condition $(\delta^\pi)$.**

*The pair $(X,Y)$ satisfies condition $(\delta^\pi)$ at $0$ iff there exists a real number $0 \le \delta < 1$ such that*

$$\lim_{\substack{(z,t)\to 0 \\ (z,t)\in X}} \frac{|\sum_{i=0}^n z_i \frac{\partial F}{\partial z_i}(z,t)|}{\|z\|\|gradF(z,t)\|} \le \delta.$$

We shall prove our main theorem using this characterisation of $(\delta^\pi)$, recalling that weak Whitney regularity is $(a+\delta)$, which is equivalent to $(a+\delta^\pi)$ by Lemma 2.8.

**Theorem 4.1.** *Weak Whitney regularity implies equimultiplicity for a family of complex hypersurfaces with isolated singularities defined by* $F : (\mathbb{C}^{n+1} \times \mathbb{C}, 0 \times \mathbb{C}) \to (\mathbb{C}, 0)$.

**Proof.** Suppose that the multiplicity at $z = 0$ of the function $f_t(z) = F(z, t)$ varies with $t$ at $t = 0$. Then there exists an open set $U$ of the $\mathbb{P}^n$ of complex 2-planes $P$ containing $Y = 0 \times \mathbb{C}$ such that the germ at $(0, 0)$ of $P \cap X$ is a nonempty curve $C$, where $V = F^{-1}(0)$ and $X = V - Y$. It follows from Theorem 3.12 of [NT] or the Proposition proved in [HM] that we may suppose that each plane $P$ of the open set $U$ is transverse to the limit at 0 of tangent planes to $X$ given by $C$ and thus will provide a distinct bad limit for Whitney $(b^\pi)$-regularity, i.e. an element of $\Lambda^\pi(X, Y)$ as defined above (this uses the hypothesis of weak Whitney regularity, which implies that Whitney's condition $(a)$ holds, and uses also that $(b)$ is equivalent to $(a + b^\pi)$). It follows in particular that dim $\Lambda^\pi(X, Y) = n$.

Let now $E_1$ be the fibre over $(0, 0)$ of the closure of the graph of the map taking a point $(z, t)$ in $X$ to $(\frac{z}{\|z\|}, \frac{grad_z F}{\|grad_z F\|})$ in $\mathbb{P}^n \times \mathbb{P}^n$, as for $E$ in the previous section. Then $\Lambda^\pi(X, Y) \subset p_1(E) = p_1(E_1)$, and because dim $\Lambda^\pi(X, Y) = n$, it follows that dim $E_1 \geq n$.

By proposition 5.1 below, this implies that $E_1$ intersects the diagonal $\Delta_{\mathbb{P}^n}$ in $\mathbb{P}^n \times \mathbb{P}^n$.

**Lemma 4.2.** *Suppose that* $(a)$*-regularity holds for* $(X, Y)$ *at* $(0, 0)$*. Then* $E_1$ *intersects the diagonal* $\Delta_{\mathbb{P}^n}$ *in* $\mathbb{P}^n \times \mathbb{P}^n$ *if and only if* $(\delta)$ *fails to hold at* $(0, 0)$ *for* $(X, Y)$.

**Proof.** Suppose $(\lambda, \lambda) \in E_1 \cap \Delta_{\mathbb{P}^n}$. Then by definition of $E_1$ there exists a sequence of points $(z_i, t_i) \in X$ such that $(z_i, t_i) \to (0, 0)$ as $i \to \infty$, and both $\frac{z_i}{||z_i||}$ and $\frac{grad_z F(z_i,t_i)}{||grad_z F(z_i,t_i)||}$ tend to $\lambda$. This means that the limit as $i$ tends to $\infty$ of the scalar product of $\frac{z_i}{||z_i||}$ and $\frac{grad_z F(z_i,t_i)}{||grad_z F(z_i,t_i)||}$ is 1, i.e. the angle between these two unit vectors tends to 0 as $i$ tends to $\infty$.

The hypothesis of $(a)$-regularity implies that $\frac{\partial F/\partial t}{||grad F(z_i,t_i)||}$ tends to 0, so that the two sequences of unit vectors $\frac{grad F(z_i,t_i)}{||grad F(z_i,t_i)||}$ and $\frac{grad_z F(z_i,t_i)}{||grad_z F(z_i,t_i)||}$ have the same limit $\lambda$. Hence

$$\lim_{\substack{i \to \infty \\ (z_i,t_i) \in X}} \frac{|\sum_{i=0}^{n} z_i \frac{\partial F}{\partial z_i}(z, t)|}{\|z\| \|grad F(z, t)\|} = 1.$$

By the characterisation of $(\delta^\pi)$ above, this implies that $(\delta^\pi)$ fails to hold for $(X, Y)$ at $(0, 0)$, and so by Lemma 2.8 $(\delta)$ fails to hold at $(0, 0)$ for $(X, Y)$.

The converse, which we do not use, is proved similarly. This completes the proof of lemma 4.2.

By lemma 4.2, the existence of points in $E_1 \cap \Delta_{\mathbb{P}^n}$ implies the failure of $(\delta)$, and hence the failure of weak Whitney regularity, a contradiction. Thus we have proved that weak Whitney regularity implies equimultiplicity of the family of hypersurfaces along the $t$-axis. □

Now we give an application.

**Corollary 4.3.** *For families of plane curves defined by* $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \times \mathbb{C} \to (\mathbb{C}, 0)$*, with* $Y = 0 \times \mathbb{C}$*, and* $X = F^{-1}(0) \setminus Y$*, weak Whitney regularity of* $(X, Y, 0)$ *is equivalent to Whitney regularity of* $(X, Y, 0)$.

**Proof.** Suppose that weak Whitney regularity holds. Then Bekka's $(C)$-regularity holds [B2] and by a theorem of Bekka, we can apply the Thom-Mather isotopy theorem to prove local topological triviality [B1, B2]. But locally topologically trivial families of hypersurfaces have Milnor number

$\mu$ constant [Te1]. As shown in [NT], Milnor number constant families cannot have $\Lambda^\pi(X,Y)$ of dimension zero, so either Whitney regularity holds and $\Lambda^\pi(X,Y) = \emptyset$, or Whitney regularity fails and $\dim_{\mathbb{C}}\Lambda^\pi(X,Y) \geq 1$. For families of plane curves, 1 is the maximum dimension, so that $\dim_{\mathbb{C}}\Lambda^\pi(X,Y) = 1$. But $\dim_{\mathbb{C}}\Lambda^\pi(X,Y) = 1$ is equivalent to $b_{cod1}$ failing by a theorem of Navarro-Aznar and the first author [NT], and this in turn is equivalent to equimultiplicity failing in this case. But this is excluded by our theorem, because weak Whitney regularity is assumed to hold. Therefore $\Lambda^\pi(X,Y) = \emptyset$, and because $(a)$-regularity is a consequence of constant Milnor number [LS], we obtain that weak Whitney regularity implies Whitney regularity. □

This can also be proved by showing that the multiplicity of plane curves is a topological invariant, and thus weak Whitney regularity, implying topological triviality, implies that $\mu^\star$ is constant, and this implies Whitney regularity (by Teissier [Te1]).

That Whitney regularity implies equimultiplicity has been known since Hironaka's general theorem of 1969 [Hi]. This fundamental fact also follows for hypersurfaces from the theorem of Briançon and Speder [BS2], that says that Whitney regularity implies the constancy of $\mu^* = (\mu^{n+1}, \mu^n, \ldots, \mu^1)$, as the multiplicity is just $\mu^1$; Navarro Aznar extended their proof to the general complex case [N].

## 5. Calculation of homology classes

The intersection result used in the proof of Theorem 4.1 is probably well-known and can be found in [Fu] ((a) in the Theorem on page 28), where it is presented in the more general context of connectedness results of the type obtained by Fulton and Hansen in [FH]. We thank W. Kucharz for pointing out this reference. For the convenience of the reader we include a proof.

**Proposition 5.1.** *If $V \subset \mathbb{P}^n \times \mathbb{P}^n$ is an algebraic subset such that dim $V \geq n$, then $V \cap \Delta \neq \emptyset$.*

**Proof.** Replacing $V$ by an algebraic subset, it is clearly sufficient to show the result for algebraic subsets $V \subset \mathbb{P}^n \times \mathbb{P}^n$ of dimension $n$.

Recall that the homology groups of $\mathbb{P}^n$ are generated by the classes $[\mathbb{P}^k]$, where $\mathbb{P}^k \subset \mathbb{P}^n$ is a linear sub-space:

$$H_{2k}(\mathbb{P}^n, \mathbf{Z}) = \mathbf{Z}.[\mathbb{P}^k], \quad k = 0, 1, \ldots, n$$

It follows from the Künneth theorem, that

$$H_{2n}(\mathbb{P}^n \times \mathbb{P}^n, \mathbf{Z}) = \bigoplus_{k=0}^{n} \mathbf{Z}[\mathbb{P}^k \times \mathbb{P}^{n-k}]$$

The homology class $[V]$ of any algebraic subset $V \subset \mathbb{P}^n \times \mathbb{P}^n$ thus can be written in a unique way as linear combination of the classes $[\mathbb{P}^k \times \mathbb{P}^{n-k}]$:

$$[V] = \sum_{k=0}^{n} a_k [\mathbb{P}^k \times \mathbb{P}^{n-k}]$$

where the coefficients $a_k \in \mathbb{Z}$. As the class $[\mathbb{P}^k \times \mathbb{P}^{n-k}]$ has intersection product $= 1$ with $[\mathbb{P}^{n-k} \times \mathbb{P}^k]$ and $= 0$ with all other classes, the coefficients $a_k$ can be found by intersecting $[V]$ with the classes $[\mathbb{P}^{n-k} \times \mathbb{P}^k]$:

$$a_k = [V] \bullet [\mathbb{P}^{n-k} \times \mathbb{P}^k]$$

It follows that $a_k \geq 0$, as it is an intersection number of two algebraic varieties. Note that not all $a_k$ can be zero, as $[V] \neq 0$. A particular class is the class of the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$. As in projective space $\mathbb{P}^n$ a $\mathbb{P}^k \subset \mathbb{P}^n$ and a $\mathbb{P}^{n-k} \subset \mathbb{P}^n$ intersect in a single point, it follows that

$$[\Delta] \bullet [\mathbb{P}^{n-k} \times \mathbb{P}^k] = 1$$

and hence

$$[\Delta] = \sum_{k=0}^{n} [\mathbb{P}^k \times \mathbb{P}^{n-k}].$$

It follows that

$$[\Delta] \cdot [V] = \sum_{k=0}^{n} a_k > 0,$$

as all $a_k \geq 0$ and at least one $a_k \neq 0$. It follows that the set $V$ has to intersect the diagonal, $V \cap \Delta \neq \emptyset$. □

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