Unliftable Calabi-Yau threefolds

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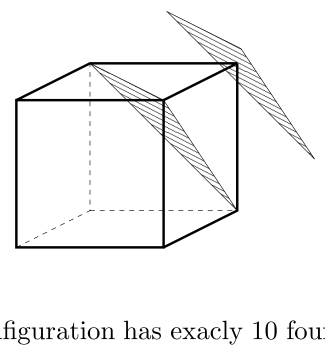
R. Vakil paraphrased *Murphys Law* in algebraic geometry as the statement that deformation spaces can and will be arbitrarily bad, unless there is a good reason that prevents this, [8]. It is well-known that curves, abelian varieties and K3 surfaces in characteristic p can be lifted to characteristic zero and in fact in each of these cases there are good cohomological reasons for the result to hold. The question arises what is the case for Calabi-Yau threefolds. By the theorem of Bogomolov-Tian-Todorov, a Calabi-Yau threefold in characteristic zero has unobstructed deformations. Somewhat surprisingly, M. Hirokado [4] and S. Schroer [7] have constructed examples of Calabi-Yau 3-folds in characteristic 2 and 3 that do not lift to characteristic zero. It is an interesting problem to construct further examples, especially in higher characterstic, [3].
We propose a general strategy to obtain examples using the peculiar properties of small resolutions of rigid Calabi-Yau spaces with nodes, [2]. The following theorem follows from general deformation theoretical considerations.
Theorem
Let $\mathcal{X}$ be a scheme over $S = Spec(A)$, A a complete domain with residue field k and fraction field $K = Q(A)$. Assume that:
1) The generic fibre $X_\eta := \mathcal{X} \otimes_A K$ is smooth.
2) The special fibre $X := \mathcal{X} \otimes_A k$ is rigid with nodes as singularities.
Let $\pi : Y \longrightarrow X$ be a small resolution. Then Y does not lift to S.

Many interesting examples can be constructed by considering so called *double octics*, two-fold coverings of $\mathbb{P}^3$ ramified over a surface of degree 8. If the singularities of the octic are not too bad, the resulting singular double cover will still have a Calabi-Yau resolution, whose infinitesimal deformations can be identified with the infinitesimal *equisingular* deformations of the ramification octic, [1].
As an example, consider the double octic ramified over a union of eigth planes.

$$u^2 = (x-t)(x+t)(y-t)(y+t)(z-t)(z+t)(x+y+z-t)(x+y+z-3t)$$



In charactersitic zero, the configuration has exacly 10 fourfold points. The Hodge numbers of a small resolution can be seen to be $h^{11} = 42$, $h^{12} = 0$. However, in characterisitc 3 something special happens: the plane $x+y+z-3t$ passes through the points $(1,1,1,1)$ and $(-1,-1,-1,1)$ of the cube and the configuration has 11 fourfold points. In charactertic 3 the double cover thus has two extra nodes, which can be resolved to give a non-liftable smooth projective Calabi-Yau threefold with $h^{1,1}=41$, $h^{1,2}=0$.

In a similar way, an example can be given of a smooth projective Calabi-Yau space over $\mathbb{F}_5$ which has an obstructed deformation, but which lifts to characteristic zero. In fact, it has a smooth model over $Spec(\mathbb{Z}[x]/(x^2+x-1))$.

A further nice example arises from the resolution of a double octic with equation

$$v^2 = (x^3+y^3+z^3+t^3+u^3)(x+y)(y+z)(z+t)(t+u)(u+x)$$

where

$$x+y+z+t+u=0$$

The branch-octic consists of the Clebsch cubic, together with tangent planes at 5 of its 10 Eckart points. This configuration is rigid and has a Calabi-Yau resolution whose Hodge numbers are $h^{1,1}=52$, $h^{1,2}=0$. In characteristic 5 the Clebsch cubic aquires an extra node, as to be expected from its interpretation as Hilbert-modular surface for $\mathbb{Q}(\sqrt{5})$, [5]. The resulting Calabi-Yau space has an extra node in characteristic 5. Upon taking a small resolution of this node, we obtain a non-liftable Calabi-Yau in charactersitic 5, which however only exists in the category of algebraic spaces and can not be realised as a projective variety.

In fact, it is quite easy to obtain such non-projective examples. Consider for example the fibre product of $E_1 \times_{\mathbb{P}^1} E_2 \longrightarrow \mathbb{P}^1$ of two semi-stable rational elliptic fibred surfaces. For example, one can pair up the singular fibres as follows:

Value	8/9	0	1	∞	$\frac{1}{2}(55\sqrt{5}-123)$
E_1	I_1	I_3	I_6	I_2	$-$
E_2	$-$	I_5	I_1	I_5	I_1

This singular threefold admits a Calabi-Yau resolution. As

$$\frac{8}{9} = \frac{1}{2}(55\sqrt{5}-123) \mod 9001,$$

in characteristic 9001 the two 'free' I_1 fibres match up and so the space aquires an extra A_1 singularties, the small resolution of which does not have a lifting to characteristic zero. These examples were also found independently by C. Schoen, [6].

References

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