Conifold Period Expansions

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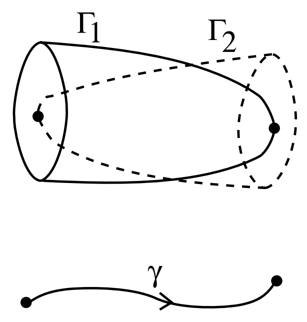
(joint work with Slawek Cynk)

In the talk two different 1-parameter families of Calabi-Yau 3-folds with $h^{12} = 1, h^{11} = 54$ were described. Both arise as small resolutions of fibre products of rational elliptic surfaces of the type described originally by C. Schoen. Their Picard-Fuchs differential operators are determinded by calculating a conifold power series expansion: the integral of the holomorphic 3-form over a 3-sphere that vanishes at the point of expansion.

The operator of the first family has a point of maximal unipotent monodromy (MUM-point) as is usually encountered in mirror symmetry, whereas the operator of the second family has no such point. W. Zudilin remarked that such operators should be considered as "orphans", as "every Calabi-Yau equation needs a MUM" in order to be listed in the *Table of Calabi-Yau operators of* [AESZ]. Examples of such families without MUM-points were described earlier by J. Rohde [R] and studied further by A. Garbagnati and B. van Geemen [GG]. But in our example the Picard-Fuchs operator has the full Sp_4 as differential Galois group. As such it seems to be the first example and thereby answers a question posed by J. Rohde.

The two families in the talk are constructed as follows. A rational elliptic surface with singular fibres I_1, I_1, I_1, I_1, I_8 is the pull-back of a surfaces with fibres I_1, I_1, II^* under a quadratic map of $\mathbf{P}^1$, which ramifies at the II^* fibre and a variable point, which is the modulus of the surface. Two I_1 fibres are said to be 'in involution' if they are pull-pack of the same I_1-fibre. As a second rational elliptic surface we take the Beauville surface with singular fibres I_1, I_2, I_3, I_6. We use two using different "matchings" of the singular fibres of these surfaces. For both we match the I_8 and I_6, but for the first family we match the I_2 and I_3 fibre with two I_1-fibres 'in involution', for the second with two I_1 fibres that are *not* 'in involution'. A conifold degeneration occurs, when for a special choice of the modulus the I_1-fibre of the Beauville-surface collides with an I_1 fibre of the other surface.



In general, we obtain a vanishing sphere S^3 as 'fibre product' of Lefschetz thimbles Γ_1 and Γ_2 in the two factors of fibre product, taken over a path γ connecting nearby critical points of the two fibrations. The conifold expansion then takes the form

$$\Phi(x) = \int_{S^3} \omega = \int_\gamma F(t,x)G(t,x)dt$$

where F and G are the period integrals of the two factors. After a tedious computation the Picard-Fuchs operator for the second family is found to be the following

$$\begin{aligned}
&\theta^2(\theta-1)(\theta+1) - \tfrac{1}{45}x\theta(\theta+1)(1687\theta^2 - 1121\theta + 10) \\
&+\tfrac{4}{75}x^2(\theta+1)(10645\theta^3 - 4079\theta^2 + 304\theta + 570) \\
&-x^3\left(\tfrac{544}{45} + \tfrac{11856}{25}\theta - \tfrac{748864}{1125}\theta^2 + \tfrac{269152}{75}\theta^3 + \tfrac{1654928}{375}\theta^4\right) \\
&-x^4\left(\tfrac{429952}{225} + \tfrac{1118272}{1125}\theta + \tfrac{395008}{125}\theta^2 - \tfrac{17839232}{1125}\theta^3 - \tfrac{20686912}{1125}\theta^4\right) \\
&+x^5\left(\tfrac{14239232}{1125} + \tfrac{2029312}{75}\theta + \tfrac{11358464}{375}\theta^2 - \tfrac{8363008}{375}\theta^3 - \tfrac{4775424}{125}\theta^4\right) \\
&-x^6\left(\tfrac{221184}{5} + \tfrac{9739264}{75}\theta + \tfrac{161364992}{1125}\theta^2 + \tfrac{15122432}{375}\theta^3 - \tfrac{9969664}{375}\theta^4\right) \\
&+x^7\left(\tfrac{14106624}{125} + \tfrac{127442944}{375}\theta + \tfrac{426057728}{1125}\theta^2 + \tfrac{8126464}{45}\theta^3 + \tfrac{29229056}{1125}\theta^4\right) \\
&-\tfrac{65536}{375}x^8(\theta+1)(260\theta^3 + 1092\theta^2 + 1627\theta + 834) + \tfrac{524288}{125}x^9(\theta+2)(\theta+1)(2\theta+3)^2
\end{aligned}$$

We thank M. Bogner for checking that the differential Galois group of above operator is the full symplectic group Sp_4. It has the following **Riemann scheme**

$$\left\{\begin{array}{ccccccc}
-1 & 0 & \frac{1}{8} & \frac{5}{12} & \frac{9}{8} & \frac{5}{4} & \infty \\
\hline
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\
1 & 0 & \frac{1}{2} & 1 & 1 & 1 & \frac{3}{2} \\
1 & -1 & \frac{3}{2} & 3 & 1 & 1 & \frac{3}{2} \\
2 & 1 & \frac{3}{2} & 4 & 2 & 2 & 2
\end{array}\right\}$$

One can see that the operator has 4 singular points of 'conifold type' $(-1, 0, 9/8, 5/4)$ where the local monodromy is a symplectic reflection, one apparent singularity $(5/12)$ and two further singular points $(1/8, \infty)$ with Jordan blocks of size two, so there are no MUM-points.
We thank J. Hofmann for computing the following tuple of **monodromy matrices**, which in an appropriate basis of H^3 with integral coefficients take the form:

$$(\tfrac{5}{4}, \tfrac{9}{8}, \tfrac{5}{12}, \tfrac{1}{8}, 0, -1) \sim \left(\begin{pmatrix} 14 & 0 & -143 & 26 \\ -37 & 1 & 407 & -74 \\ 1 & 0 & -10 & 2 \\ -1 & 0 & 11 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 17 & -1 \\ -1 & 0 & -17 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right.$$

$$
,\begin{pmatrix} 20 & 9 & 51 & 9 \\ -42 & -19 & -96 & -18 \\ 0 & 0 & -1 & 0 \\ -7 & -3 & -23 & -4 \end{pmatrix}, \begin{pmatrix} 9 & 4 & 22 & 2 \\ -16 & -7 & -44 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 218 & 93 & 372 & 124 \\ -469 & -200 & -804 & -268 \\ 7 & 3 & 13 & 4 \\ -49 & -21 & -84 & -27 \end{pmatrix} \Bigg)
$$

We remark that our varieties are non-projective small resolutions of nodal projective varieties. If we replace the I_8, I_1, I_1, I_1, I_1 surface by the isogenous I_2, I_4, I_4, I_1, I_1 surface (where the two I_4-fibres are in involution), the families are replaced by projective varieties with Hodge numbers $h^{1,2} = 1, h^{1,1} = 33$ for the first, but non-projective varieties with $h^{1,2} = 4, h^{1,1} = 30$ for the second family. There are other operators of this type arising from various fibre products of rational elliptic surfaces and their systematic determination is part of work in progress with S. Cynk.

References

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