# On the Determination of Series or a new Method to find the general Terms of Series*

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**§1** Since there are infinitely many laws of progression the terms of a series follow, it seems that not only all different species of series but not even all classes, no matter how far they extend, can actually be enumerated. For, two or more series exist, which, even though they have arbitrarily many terms in common, nevertheless differ and are described by very different laws. Everyone who has studied the very broad field of series even only for a short time, will easily understand that the nature of a series is not determined, no matter how many of its terms are exhibited. So if it is in question, what the series is, whose initial terms are

$$1, \quad 3, \quad 5, \quad 7, \quad 9, \quad 11, \quad 13, \quad 15,$$

this question is most undetermined; and except for the series of the odd natural numbers proceeding in natural order innumerable other series can be assigned, which have the same initial terms; and this lack of determination is not restricted to a certain number of given terms, but, no matter how large that number was, all infinite series have it in common.

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**§2** But this will be seen more clearly, if we represent the nature of series geometrically. For, each arbitrary series can be represented by means of a curved line, whose ordinate is expressed by appropriate terms of the series, while the abscissas denote their indices or numbers, which represent the order of each term. This way any arbitrary term of the series defines a point, which corresponds to the given abscissa, on the curved line. Hence, if a series is required, which has arbitrarily many given terms, the question reduces to this that a curved line is found, which passes through as many given points. But it is perspicuous that always innumerable curved lines can be assigned which pass through the single points at the same time. Although Newton showed this only for parabolic curves, there is no doubt, if not only all algebraic curves but also transcendental ones are admitted, that the number of curves additionally enjoying that property becomes infinitely larger.

**§3** It will be even more remarkable, if I say that a series is not even determined, even though innumerable of its terms are given. So if I define this series

$$1 + 2 + 3 + 4 + 5 + 6 + \text{etc.}$$

in such a way that I say that all integer numbers in natural order are contained in it, who will then not believe that this series is completely determined, since for each index the corresponding term of the series was assigned? For, the term corresponding to the index, which is $x$ units away from the beginning, will be $=$ the number $x$ itself, or the term, whose index is $= x$, will also be $= x$. But in the way the series was defined here, nothing more is known than that the term $x$ corresponds to the index $x$, if $x$ was an integer number; but if a fractional number is assumed for the index $x$, there is now reason, why the term corresponding to that index $x$ should also be $= x$. And I will show, if for this series the term corresponding to the index $x$ is put $= y$, that it can happen in infinitely many ways that, if $x$ is an integer number, it always is $y = x$, even though the value of $y$ differs from $x$, if rational numbers are taken for $x$. Hence, even though all terms corresponding to integer indices of the series are determined, the intermediate ones corresponding to rational indices, can nevertheless be determined in infinitely many ways, so that the interpolation of this series remains undetermined.

**§4** In order to see this more clearly, let us consider the example of circular arcs; for, since having put the half of the circumference of the circle, whose radius is $= 1$, $= \pi$, the sine of the arc $n\pi$ is $= 0$, if $n$ is an integer, it is manifest, if one puts $y = x + P \sin \pi x$ while $P$ denotes either a constant quantity or an arbitrary function of $x$ and for $x$ one successively puts the integer numbers 1, 2, 3, 4, 5 etc., that then the values of $y$ will be $= 1, 2, 3, 4, 5$, etc., as if it was $P = 0$. And nevertheless the intermediate values corresponding to rational indices will not be equal to these indices. For, for the sake of an example, let $P = xx$ and put $x = \frac{1}{2}$; because of $\sin \frac{1}{2}\pi = 1$ the term corresponding to the index $\frac{1}{2}$ will become

$$\frac{1}{2} + \frac{1}{4} \cdot 1 = \frac{3}{4}.$$

But one can think of infinitely many other expressions of this kind, which fulfill the condition in like manner; for example

$$y = x + P \sin \pi x + Q \sin 2\pi x + R \sin 3\pi x + S \sin 4\pi x + \text{etc.},$$

hence the interpolation is clearly seen to be undermined.

**§5** I already exhibited a similar example of a series, which could seem determined, some time ago[1]; for, I had found an expression or a function of $x$, which, if for $x$ any power of 10 is substituted, becomes equal to this power, if this exponent is a positive integer, of course. That function of $x$, which I will indicate by the letter $y$, was of such a nature that having put $x = 1$ it becomes $y = 0$ and, if one puts $x = 10^n$ while $n$ is a positive integer number, always becomes $y = n$; hence it seemed to follow that the function $y$ will always be the common logarithm of $x$. Nevertheless, I showed, if for $x$ not a certain power of ten is substituted, that the value of $y$ often differs a lot from the logarithm of ten. Therefore, having set up the series, for which we have

$$\begin{array}{lcccccccc} \text{Indices} & 1, & 10^1, & 10^2, & 10^3, & 10^4, & 10^5, & 10^6 & \text{etc.} \\ \text{and} \\ \text{Terms} & 0, & 1, & 2, & 3, & 4, & 5, & 6 & \text{etc.} \end{array}$$

[1] Euler refers to his paper "Consideratio quarumdam serierum, quae singularibus proprietatibus sunt praeditae". This is E190 in the Eneström-Index.

it does not suffice for the description of logarithms, if someone says that the logarithms are the middle terms, which correspond to the indices assumed in the upper series, of the lower series.

**§6** Therefore, since the nature of a series is not determined by some of its terms, even though their number is infinite, since the interpolation nevertheless remains undetermined and can be done in infinitely many ways, it is easily seen, how uncertain all these interpolation methods are, which teach to complete the task using only the terms corresponding to integer indices. For, the interpolation can only be considered as certain, if the nature of the series is taken into account in the operation. But the nature of a series is seen perfectly, if its general term or a formula, which for each index $x$, whether integer or rational or even surdic[2], exhibits the corresponding terms, was known. For, this way not only all terms of the series, which correspond to integer indices, are determined, but also the terms, which correspond to arbitrary non-integer indices, are defined without any ambiguity; and so the task of interpolation is no longer uncertain.

**§7** But except for the general term one has innumerable other ways to form series; nevertheless all these ways can be conveniently reduced to three classes. To the first class I count these ways of forming series, in which each term of the series is only determined by the corresponding index; since this is achieved by certain operations to be done for this aim, the formula containing these operations in general will be the general term of the series itself; I already noted that by it the series is determined perfectly and absolutely. To the second class I count all these ways of forming series, in which the general term of the series is determined by some of the preceding terms according to a certain rule, which way is usually especially applied in recurring series. But whenever not only the preceding terms are to be taken into account, but also the index itself must be used, to find a certain term of a series, I hence constitute the third class of determination.

**§8** If an arbitrary term of the series is determined only by the index, then, no matter whether an integer or a rational number is assumed for the index, the corresponding term is equally defined and so the interpolation of the series will have neither any difficulty nor uncertainty. But if, as we put in

[2]Euler uses this word to describe formulas involving roots.

the second class, an arbitrary term is determined by the preceding one or several preceding ones, then, having assumed the first or some first terms arbitrarily, the single terms corresponding to integer indices, will be found, but it is not possible to define the intermediate terms corresponding to rational indices from this, which is also be said about the third class. But although this way in the second and third class not only all terms, which correspond to integer indices, are assigned, but also the relation among one term and its preceding ones is prescribed, which equally extends to terms of fractional indices, nevertheless not even this way the series is completely determined, but for any arbitrary series of this class infinitely many general terms can be exhibited, which, while they yield the same terms for integer indices, nevertheless deviate for the rational ones.

**§9** Since this might seem to be paradoxical, it will be worth one's while to consider this lack of determination in the series, in which each term is determined by the preceding ones, more diligently. Therefore, let us take the simplest case and assume that the series is defined in such a way that each term is equal to the preceding one itself. If now the first term of the series is set $= 1$, the second will also be $= 1$ and all following ones, which correspond to integer indices, will become equal to the unity and this series will result:

$$\begin{array}{rcccccccccc l} \text{Indices:} & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10 & \text{etc.} \\ \text{Terms:} & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1 & \text{etc.} \end{array}$$

and it is manifest that the term $= 1$ corresponds to each integer index $x$. But the nature of the terms corresponding to fractional indices will not be clear from this; it is only known, if the term corresponding to the index $\frac{1}{2}$ was $= a$, that also all terms, which correspond to the indices

$$\frac{3}{2}, \quad \frac{5}{2}, \quad \frac{7}{2}, \quad \frac{9}{2} \quad \text{etc.}$$

will also be $= a$. For, all terms, whose indices differ by one unit or several units, must be equal according to the prescribed law, since each preceding term is understood as the one, whose index is the one smaller by one unit.

**§10** Therefore, this series is defined in such a way that, if the term corresponding to the index $x$ is put $= y$ and the following term corresponding to the index $x + 1$ is $= y'$, one has $y' = y$; but then furthermore it is assumed, if it was $x = 1$, that it will also be $y = 1$. Hence, if the general term of this series is in question, that term must be a function of $x$ of such a kind, which function we want to set $= y$, that, if instead of $x$ one puts $x + 1$, the resulting value $y'$ of the function $y$ will be equal to $y$ itself, and that having put $x = 1$ it also is $y = 1$. But it is manifest, if in general one puts $y = 1$, that this condition is fulfilled and in this case not only the terms corresponding to integer indices, but also those corresponding to rational ones, will be smaller than 1. But on the other hand these conditions can also be fulfilled in infinitely many other ways; for, if one puts

$$y = 1 + \alpha \sin 2\pi x,$$

while $\pi$ denotes the half of the circumference of the circle, whose radius is $= 1$, it will be

$$y' = 1 + \alpha \sin 2\pi x;$$

but it is

$$\sin 2\pi(x + 1) = \sin 2\pi x$$

and hence $y' = y$, but then for $x = 1$ it will also be $y = 1$. In this case the intermediate terms or the ones corresponding to fractional indices are no longer equal to 1; for, having put $x = \frac{1}{4}$ it will be $y = 1 + \alpha$.

**§11** Since here not only $\alpha$ can be assumed arbitrarily, but one can also think of innumerable other formulas of this kind, which fulfill the prescribed conditions, e.g.,

$$y = 1 + \alpha \sin 2\pi x + \beta \sin 4\pi x + \gamma \sin 6\pi x + \text{etc.},$$

it is perspicuous that the interpolation even of this most simple series $1 + 1 + 1 + \text{etc.}$, if it is only defined in such a way that each term is equal to the preceding one, but the first is said to be expressed by the unity, is highly undetermined, since the intermediate terms corresponding to rational indices can be equal to any numbers. Nevertheless, even though innumerable general terms can be exhibited for this series, they are all described by the same general rule and can be found by means of Analysis without exception. Of course, a very-far extending method can be given, by means of which it is possible to define the general terms of all series, whose terms are determined by the preceding ones, whether including the index or excluding the index, in a very general way; and I decided to develop this method here diligently, since it does not only lead to a more complete understanding of series, but also contains remarkable insights which will be helpful to expand the whole field of Analysis even further; for this aim, I will consider the following problems.

## PROBLEM 1

**§12** *To find the general term of the series, in which any arbitrary term is equal to the preceding one, but whose first term* $= 1$.

### SOLUTION

Let the general term or the one corresponding to the index $x$ be $= y$ and put the following term (whose index is $= x+1) = y'$ and it has to be $y' = y$; and having put $x = 1$ is must be $y = 1$. Since now $y$ is a certain function of $x$, by the nature of differential calculus, if instead of $x$ one puts $x+1$, it will be

$$y' = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.},$$

having assumed the differential $dx$ to be constant. Therefore, it must be

$$0 = \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.},$$

And this equation contains completely all appropriate values of $y$, if the integration is done in such a way that having put $x = 1$ it is $y = 1$ or, what reduces to the same, that having put $x = 0$ it is $y = 1$. Therefore, the question was reduced to the resolution of this differential equation, which not only consists of an infinite number of terms, but also contains all orders of differentials. But since the variable $y$ does not have more than one dimension anywhere and only the differential $dx$, which is assumed to be constant, of the other variable $x$ occurs, this equation can be treated in the same way which I explained in MISCELLANEA BEROLOLIN. Volume 7[3]. Therefore, by putting $z$ instead of $\frac{dy}{dx}$, $z^2$ instead of $\frac{ddy}{dx^2}$ and in general $z^n$ instead of $\frac{d^ny}{dx^n}$ form the algebraic equation

[3]Euler refers to his paper E62 "De integratione aequationum differentialium altiorum gradu- um".

$$0 = \frac{z}{1} + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.},$$

which, having taken $e$ for the number, whose hyperbolic logarithm is $= 1$, goes over into this finite form $0 = e^z - 1$. Now one has to investigate all roots, whose number is infinite, of this equation or one has to assign all factors of the formula $e^z - 1$. But it is

$$e^z = \left(1 + \frac{z}{n}\right)^n,$$

having put $n$ to be an infinite number; if this value is substituted, one will have to resolve this expression (into factors)

$$\left(1 + \frac{z}{n}\right)^n - 1;$$

one simple factor of it is $= \frac{z}{n}$ or $z$, of course, which the infinite equation immediately reveals. To find the remaining factors one has to recall the theorem stating that a factor of the binomial form $a^n - b^n$ is given by

$$aa - 2ab\cos\frac{2k\pi}{n} + bb,$$

while $k$ denotes an integer number. Therefore, in the present case it is

$$a = 1 + \frac{z}{n} \quad \text{and} \quad b = 1,$$

whence all factors of the propounded formula $e^z - 1$ are contained in this general form

$$1 + \frac{2z}{n} + \frac{zz}{nn} - 2\left(1 + \frac{z}{n}\right)\cos\frac{2k\pi}{n} + 1$$

or

$$2\left(1 + \frac{z}{n}\right)\text{versin}\,\frac{2k\pi}{n} + \frac{zz}{nn};$$

hence by dividing this factor by the constant quantity $2\,\text{versin}\,\frac{2k\pi}{n}$ the general factor will be

$$= 1 + \frac{z}{n} + \frac{zz}{2nn\ \text{versin}\frac{2k\pi}{n}}.$$

Since now $n$ is an infinite number, it will be

$$\cos\frac{2k\pi}{n} = 1 - \frac{2kk\pi\pi}{nn} \quad \text{and} \quad \text{versin}\frac{2k\pi}{n} = \frac{2kk\pi\pi}{nn};$$

having substituted this value the general factor of the formula $e^z - 1$ will be

$$= 1 + \frac{z}{n} + \frac{zz}{4kknn},$$

and by successively substituting the integer numbers 1, 2, 3, 4 etc. for $k$ completely all factors of the formula $e^z - 1$ will result. But the first factor $z$ gives the constant part of the integral, which we want to put $= C$; but if the remaining factors, which are reduced to this form

$$4kk\pi\pi + \frac{4kk\pi\pi}{n}z + zz,$$

are compared to the form of the factors, which I expanded in the dissertation mentioned before[4],

$$ff - 2fz\cos\varphi + zz,$$

it will be

$$f = 2k\pi \quad \text{and} \quad \cos\varphi = -\frac{k\pi}{n}$$

and $\sin\varphi = 1$ because of the infinite number $n$, in which case it is $\cos\varphi = 0$. Therefore, the part of the integral to result from this will be

$$\alpha e^{\frac{-2kk\pi\pi}{n}x}\sin 2k\pi x + \mathfrak{A}e^{\frac{-2kk\pi\pi}{n}x}\cos 2k\pi x$$

or because of $n = \infty$

$$\alpha\sin 2k\pi x + \mathfrak{A}\cos 2k\pi x,$$

Therefore, having successively substituted all integer numbers 1, 2, 3, 4 etc. for $k$ the integral of the found equation

$$0 = \frac{dy}{1\cdot dx} + \frac{ddy}{1\cdot 2\cdot dx^2} + \frac{d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \frac{d^4y}{1\cdot 2\cdot 3\cdot 4\cdot x^4} + \text{etc.}$$

will emerge expressed in the following form

$$\begin{aligned} y = C + \alpha\sin 2\pi x &+ \mathfrak{A}\cos 2\pi x \\ + \beta\sin 4\pi x &+ \mathfrak{B}\cos 4\pi x \\ + \gamma\sin 6\pi x &+ \mathfrak{C}\cos 6\pi x + \text{etc.} \end{aligned}$$

[4]Euler refers to E62 again.

Now define the constant $C$ in such a way that having put $x = 0$ it is $y = 1$, and one will find the general term of the propounded series as

$$\begin{aligned} y = 1 + \alpha \sin 2\pi x &+ \mathfrak{A} (\cos 2\pi x - 1) \\ + \beta \sin 4\pi x &+ \mathfrak{B} (\cos 4\pi x - 1) \\ + \gamma \sin 6\pi x &+ \mathfrak{C} (\cos 6\pi x - 1) + \text{etc.} \end{aligned}$$

Therefore, whatever values are substituted for $\alpha$, $\beta$, $\gamma$, $\delta$ etc., $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., in every case a formula will result, which exhibits the general term of the propounded series. Q. E. I.

## Corollary 1

**§13** If the first term, to which all remaining ones having integer exponents are equal, must not be the 1 but an arbitrary quantity, the general term of the series $y$ or the term corresponding to the index $x$, is found to be

$$\begin{aligned} y = a + \alpha \sin 2\pi x + \beta \sin 4\pi x + \gamma \sin 6\pi x + \delta \sin 8\pi x + \text{etc.} \\ + \mathfrak{A} \cos 2\pi x + \mathfrak{B} \cos 4\pi x + \mathfrak{C} \cos 6\pi x + \mathfrak{D} \cos 8\pi x + \text{etc.} \end{aligned}$$

and an arbitrary term corresponding to an integer index will be

$$= a + \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \text{etc.}$$

## Corollary 2

**§14** Since sines and cosines of the arcs $4\pi x$, $6\pi x$, $8\pi x$ etc. can be expressed by means of powers of $\sin 2\pi x$ and $\cos 2\pi x$ and vice versa all rational functions or functions which do not share the ambiguity of the square root sign, can be exhibited by series of this kind we found for $y$, we will be able to define the general term $y$ in such a way that we say that $y$ is an arbitrary function of $\sin 2\pi x$ and $\cos 2\pi x$, as long as no formulas of this kind

$$\sqrt{1 \pm \cos 2\pi x}$$

and other similar ones occur, which involve the sines and cosines of submultiple angles of $2\pi x$.

## COROLLARY 3

**§15** Therefore, having excluded these cases, if we put $\sin 2\pi x = p$ and $\cos 2\pi x = q$, $y$ will be equal to an arbitrary function of $p$ and $q$; hence this differential equation of infinite order

$$0 = \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

will be integrated in general in such a way that $y$ is an arbitrary function of $p$ and $q$.

## COROLLARY 4

**§16** But if we call $\sin \pi x = r$ and $\cos \pi x = s$, it will be $p = 2s$ and $q = ss - rr$ and functions of $p$ and $q$ will be functions of even dimensions of $r$ and $s$. Hence using the infinite differential equation the value of $y$ will in general become equal to an arbitrary function of even dimensions of $r$ and $s$, where it is to be noted that, because of the whole sine $= 1$, it is $rr + ss = 1$.

## COROLLARY 5

**§17** Put $\frac{x}{a}$ instead of $x$ that one has this equation

$$0 = \frac{ady}{1 \cdot dx} + \frac{a^2ddy}{1 \cdot 2 \cdot dx^2} + \frac{a^3d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{a^4d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

If we now put

$$\sin \frac{\pi x}{a} = r \quad \text{and} \quad \cos \frac{\pi x}{a} = s.$$

the integral of this equation will be described in such a way that $y =$ an arbitrary function of even dimensions of $r$ and $s$.

## COROLLARY 6

**§18** Therefore, two formulas for the value of this integral can be exhibited, the one of which is

$$y=\frac{A+Br^2+Crs+Ds^2+Er^4+Fr^3s+Gr^2s^2+Hrs^3+Is^4+\text{etc.}}{\alpha+\beta r^2+\gamma rs+\delta s^2+\varepsilon r^4+\zeta r^3s+\eta r^2s^2+\theta rs^3+\iota s^4+\text{etc.}}$$

The other form will be

$$y=\frac{Ar+Bs+Cr^3+Dr^2s+Ers^2+Fs^3+Gr^5+\text{etc.}}{\alpha r+\beta s+\gamma r^3+\delta r^2s+\varepsilon rs^2+\zeta s^3+\eta r^5+\text{etc.}}$$

## COROLLARY 7

**§19** Therefore, whatever value of this kind is substituted for $y$ in the equation

$$0=\frac{ady}{1\cdot dx}+\frac{a^2ddy}{1\cdot 2\cdot dx^2}+\frac{a^3d^3y}{1\cdot 2\cdot 3\cdot dx^3}+\frac{a^4d^4y}{1\cdot 2\cdot 3\cdot 4\cdot dx^4}+\text{etc.},$$

the identical equation will result or an infinite series will result, whose sum will be $=0$. But for the iterated differentiations it is to be noted that it is

$$\frac{dr}{dx}=\frac{\pi s}{a}\quad\text{and}\quad\frac{ds}{dx}=-\frac{\pi r}{a}$$

and hence by means of the substitution the differentials $dx$ will cancel each other everywhere.

## SCHOLIUM 1

**§20** But the factors, into which this infinite algebraic expression

$$\frac{z}{1}+\frac{z^2}{1\cdot 2}+\frac{z^3}{1\cdot 2\cdot 3}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4}+\frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.},$$

was resolved above, deserve it to be mentioned. For, since the first simple factor is $=z$ and the remaining trinomial ones are contained in this general form

$$1+\frac{z}{n}+\frac{zz}{4kk\pi\pi},$$

if the numbers 1, 2, 3, 4 etc. are successively substituted for $k$, let us, for the sake of brevity, put

$$Z=\frac{z}{1}+\frac{z^2}{1\cdot 2}+\frac{z^3}{1\cdot 2\cdot 3}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.},$$

and using the infinitely many factors it will be

$$Z = z\left(1+\frac{z}{n}+\frac{zz}{4\pi\pi}\right)\left(1+\frac{z}{n}+\frac{zz}{16\pi\pi}\right)\left(1+\frac{z}{n}+\frac{zz}{36\pi\pi}\right)\left(1+\frac{z}{n}+\frac{zz}{64\pi\pi}\right)\text{etc.};$$

the total amount of these factors, having excluded the first, is infinite and $= \frac{1}{2}n$. Therefore, let $\frac{1}{2}n = m$ or $n = 2m$ and put $z = 2v$;

$$\frac{2v}{1}+\frac{2^2v^2}{1\cdot 2}+\frac{2^3v^3}{1\cdot 2\cdot 3}+\frac{2^4v^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}$$

$$= 2v\left(1+\frac{v}{m}+\frac{vv}{\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{4\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{9\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{16\pi\pi}\right)\text{etc.}$$

and hence the following product of infinitely many factors, whose number is $= m$, will be

$$\left(1+\frac{v}{m}+\frac{vv}{\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{4\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{9\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{16\pi\pi}\right)\text{etc.}$$

$$= 1+\frac{2}{1\cdot 2}v+\frac{4}{1\cdot 2\cdot 3}v^2+\frac{8}{1\cdot 2\cdot 3\cdot 4}v^3+\frac{16}{1\cdot 2\cdot 3\cdot 4\cdot 5}v^5+\text{etc.}$$

If now this product is actually expanded, since the number of factors is $= m$ while $m$ is an infinite number, this expression will result

$$\begin{aligned} 1+v+\frac{m(m-1)}{1\cdot 2}\cdot\frac{vv}{mm}+\frac{vv}{\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}\right)\\ +\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cdot\frac{v^3}{m^3}+\frac{(m-1)v^3}{m\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}\right) \end{aligned}$$

$$\text{etc.},$$

which terms compared to the series already found will give

$$1=\frac{2}{1\cdot 2},\qquad \frac{1}{1\cdot 2}+\frac{1}{\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}\right)=\frac{4}{1\cdot 2\cdot 3},$$

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}\right)=\frac{8}{1\cdot 2\cdot 3\cdot 4}$$

Hence in each of both one has

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\text{etc.}=\frac{\pi\pi}{6},$$

which is the same summation, I had found already first many years before and have confirmed having given several proofs[5]. Furthermore, hence it is perspicuous, even though in these factors the number $m$ is infinite that it is nevertheless not possible to omit the other term $\frac{v}{m}$, since in the expansion, because of the infinite repetition, finite terms result from the infinitely small terms $\frac{v}{m}$. But whenever an arbitrary term is considered separately, as we did it in the formation of the integral, then it is possible to omit these infinitely small terms without error.

## SCHOLIUM 2

But it is also possible to sum higher powers of the terms of the series

$$1+\frac{1}{4}+\frac{1}{9}+\text{etc.}$$

using this approach and the same progressions I had found once[6] will result. But in order to shorten this calculation, proceed as follows. Put

$$V=1+\frac{2v}{1\cdot 2}+\frac{2^2v^2}{1\cdot 2\cdot 3}+\frac{2^3v^3}{1\cdot 2\cdot 3\cdot 4}+\frac{2^4v^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.};$$

it will be

$$V=\frac{e^{2v}-1}{2v}$$

and

$$\frac{dV}{Vdv}=\frac{2e^{2v}}{e^{2v}-1}-\frac{1}{v},$$

which equation is reduced to this more convenient form

$$\frac{dV}{Vdv}=\frac{2e^{v}}{e^{v}-e^{-v}}-\frac{1}{v}=\frac{1+\frac{v}{1}+\frac{v^2}{1\cdot 2}+\frac{v^3}{1\cdot 2\cdot 3}+\frac{v^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}}{\frac{v}{1}+\frac{v^3}{1\cdot 2\cdot 3}+\frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{v^7}{1\cdot 2\cdots 7}+\text{etc.}}-\frac{1}{v},$$

[5] Euler finds this sum in "De summis serierum reciprocarum" for the first time. This is paper E41 in the Eneström-Index. But there he uses the infinite product for the sine he had not proved rigorously at that point. In his paper "Demonstration de la somme de cette suite $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$", E63, he gives the first completely rigorous proof using the series expansion of $\arcsin^2 x$.

[6] Confer also E41.

such that it is

$$\frac{dV}{Vdv} - 1 = \frac{1 + \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} + \frac{v^6}{1\cdot 2\cdots 6} + \text{etc.}}{\frac{v}{1} + \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{v^7}{1\cdot 2\cdots 7}\text{etc.}} - \frac{1}{v}$$

or

$$\frac{dV}{Vdv} - 1 = \frac{\frac{2v}{1\cdot 2\cdot 3} + \frac{4v^3}{1\cdot 2\cdots 5} + \frac{4v^5}{1\cdot 2\cdots 7} + \frac{8v^7}{1\cdot 2\cdots 9} + \text{etc.}}{1 + \frac{v^2}{1\cdot 2\cdot 3} + \frac{v^4}{1\cdot 2\cdots 5} + \frac{v^6}{1\cdot 2\cdots 7} + + \frac{v^8}{1\cdot 2\cdots 9} + \text{etc.}} - \frac{1}{v}.$$

Put

$$\frac{dV}{Vdv} = 1 + \mathfrak{A}v - \mathfrak{B}v^3 + \mathfrak{C}v^5 - \mathfrak{D}v^7 + \mathfrak{E}v^9 - \text{etc.};$$

it will be

$$\begin{aligned}
\mathfrak{A} &= \frac{2}{1\cdot 2\cdot 3}, \\
\mathfrak{B} &= \frac{\mathfrak{A}}{1\cdot 2\cdot 3} - \frac{4}{1\cdot 2\cdots 5}, \\
\mathfrak{C} &= \frac{\mathfrak{B}}{1\cdot 2\cdot 3} - \frac{\mathfrak{A}}{1\cdot 2\cdots 5} + \frac{6}{1\cdot 2\cdots 7}, \\
\mathfrak{D} &= \frac{\mathfrak{C}}{1\cdot 2\cdot 3} - \frac{\mathfrak{B}}{1\cdot 2\cdots 5} + \frac{\mathfrak{A}}{1\cdot 2\cdots 7} - \frac{8}{1\cdot 2\cdots 9} \\
&\text{etc.}
\end{aligned}$$

Having found these values consider this other form of the quantity $V$ expressed by means of factors

$$V = \left(1 + \frac{v}{m} + \frac{vv}{1\pi\pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{4\pi\pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{9\pi\pi}\right)\text{etc.};$$

using this form by means of differentiation one finds this equation

$$\frac{dV}{Vdv} = \frac{\frac{1}{m} + \frac{2v}{1\pi\pi}}{1 + \frac{v}{m} + \frac{vv}{1\pi\pi}} + \frac{\frac{1}{m} + \frac{2v}{4\pi\pi}}{1 + \frac{v}{m} + \frac{vv}{4\pi\pi}} + \frac{\frac{1}{m} + \frac{2v}{9\pi\pi}}{1 + \frac{v}{m} + \frac{vv}{9\pi\pi}} + \text{etc.}$$

But in general it is

$$\begin{aligned}
\frac{\frac{1}{m} + \frac{2v}{\lambda\pi\pi}}{1 + \frac{v}{m} + \frac{vv}{\lambda\pi\pi}} = \frac{1}{m} &+ \frac{2}{\lambda\pi\pi}v - \frac{3}{m\lambda\pi\pi}v^2 + \frac{4}{m^2\lambda\pi\pi}v^3 - \text{etc.} \\
&- \frac{1}{mm} \quad + \frac{1}{m^3} \qquad - \frac{1}{m^4} \\
&\qquad\qquad\qquad\qquad - \frac{2}{\lambda\lambda\pi^4}.
\end{aligned}$$

But since $m$ is an infinite number and is equal to the number of factors, having excluded the first terms one will be able to omit the remaining ones divided by $m$ without any error

$$\frac{\frac{1}{m}+\frac{2v}{\lambda\pi\pi}}{1+\frac{v}{m}+\frac{vv}{\lambda\pi\pi}}=\frac{1}{m}+\frac{2v}{\lambda\pi\pi}-\frac{2v^3}{\lambda^2\pi^4}+\frac{2v^5}{\lambda^3\pi^6}-\frac{2v^7}{\lambda^4\pi^8}+\text{etc.};$$

therefore, having successively substituted the square numbers 1, 4, 9, 16 etc. and having combined these series, whose total amount is $m$, into one single sum one will find

$$\begin{aligned}\frac{dV}{Vdv}=1+\frac{2v}{\pi\pi}&\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}\right)\\ -\frac{2v^3}{\pi^4}&\left(1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}+\text{etc.}\right)\\ +\frac{2v^5}{\pi^6}&\left(1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\frac{1}{25^3}+\text{etc.}\right)\\ -\frac{2v^7}{\pi^8}&\left(1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\frac{1}{25^4}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

If now this series is compared to the one found first, one will find

$$\begin{aligned}1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}&=\frac{1}{2}\mathfrak{A}\pi^2=\frac{1}{6}\cdot\pi^2,\\ 1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\text{etc.}&=\frac{1}{2}\mathfrak{B}\pi^4=\frac{1}{90}\cdot\pi^4,\\ 1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\text{etc.}&=\frac{1}{2}\mathfrak{C}\pi^6=\frac{1}{945}\cdot\pi^6,\\ 1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\text{etc.}&=\frac{1}{2}\mathfrak{D}\pi^8=\frac{1}{9450}\cdot\pi^8\\ &\text{etc.}\end{aligned}$$

And this way all summations I already exhibited some time ago will be confirmed more, since the principle, which I had used then, could seem erroneous to some people.

# PROBLEM 2

**§22** *To find the general term of the series, in which any arbitrary term exceeds the preceding one by a given quantity and whose first term is given.*

## SOLUTION

Let the first term be $= a$ and the excess of each term over the preceding be $= g$; the terms corresponding to the integer indices will of course be these

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a, & a+g, & a+2g, & a+3g, & a+4g, & a+5g, & a+6g \quad \text{etc.,} \end{array}$$

so that the term $y = a + (x-1)g$ corresponds to the integer index $x$. But while $x$ is an arbitrary number infinitely many other formulas can be taken for $y$. For let $y'$ be the term corresponding to the index $x+1$; it will be

$$y' = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

Since now by assumption it must be $y' = y + g$, it will be

$$g = \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

Although I treated the resolution of equations of this kind, where except for the terms, which contain the differentials of $y$, a either a constant term or an arbitrary function is present, some time ago[7], it will nevertheless be helpful to get rid of this term $g$ by means of the substitution $y = gx + u$; for, it will be

$$dy = gdx + du, \quad ddy = ddu, \quad d^3y = d^3u \quad \text{etc.,}$$

because of the constant $dx$. Therefore, it will be

$$0 = \frac{du}{1 \cdot dx} + \frac{ddu}{1 \cdot 2 \cdot dx^2} + \frac{d^3u}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4u}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

Since this equation agrees with the one which we found in the preceding problem, if we put $\sin \pi x = r$ and $\cos \pi x = s$, $u$ will be an arbitrary function of even dimensions of $r$ and $s$, of which kind we exhibited one in § 18; and having found this the general term in question will be $y = A + gx + u$, if the constant $A$ is defined in such a way that having put $x = 1$ it is $y = a$. Q.E.I.

[7] Euler refers to his paper " Methodus aequationes differentiales altiorum graduum integrandi ulterius promota". This is E188 in the Eneström-Index.

# Problem 3

**§23** *To find the general term of the series, in which any arbitrary term results, if the preceding is multiplied by a given number m, and whose first term is = a.*

## Solution

Therefore, the terms of this series corresponding to integer indices, will constitute the following geometric progression

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ a, & ma, & m^2s, & m^3a, & m^4a, & m^5a \quad \text{etc.,} \end{array}$$

such that the term $am^{x-1}$ corresponds to the integer index $x$. Therefore, let in general $y$ be the term corresponding to the index $x$ and $y'$ the term corresponding to the index $x+1$ and it will be $y' = my$. But it is

$$y' = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.} = my.$$

In order to resolve this equation, according to the prescriptions put 1 for $y$, $z$ for $\frac{dy}{dx}$, $z^2$ for $\frac{ddy}{dx^2}$ etc. that the following algebraic equation results

$$m = 1 + \frac{z}{1} + \frac{zz}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.},$$

whose single roots must be investigated. But it will be $m = e^z$; let the hyperbolic logarithm of $m$ be $= \lambda$ that it is $m = e^\lambda$ and hence $e^\lambda - e^m = 0$. But since having taken an infinite number for $n$ it is

$$e^\lambda = \left(1 + \frac{\lambda}{n}\right)^n \quad \text{and} \quad e^z = \left(1 + \frac{z}{n}\right)^n,$$

one will have this equation, whose roots are to be investigated

$$\left(1 + \frac{\lambda}{n}\right)^n - \left(1 + \frac{z}{n}\right)^n = 0;$$

the one root $z - \lambda = 0$ of this equation is certainly clear immediately, whence, because of $e^\lambda = m$, the part $y = \alpha e^{\lambda x} = \alpha m^x$ of the integral is obtained. The remaining roots are imaginary and are contained in this trinomial factor

$$\left(1 + \frac{\lambda}{n}\right)^2 - 2\left(1 + \frac{\lambda}{n}\right)\left(1 + \frac{z}{n}\right)\cos\frac{2k\pi}{n} + \left(1 + \frac{z}{n}\right)^2,$$

while $k$ is an arbitrary integer number; this form goes over into this one

$$2 + \frac{2\lambda}{n} - 2\left(1 + \frac{\lambda}{n}\right)\cos\frac{2k\pi}{n} + \frac{\lambda\lambda}{nn} - \frac{2z}{n} - \frac{2z}{n}\left(1 + \frac{\lambda}{n}\right)\cos\frac{2k\pi}{n} + \frac{zz}{nn}.$$

But because of the infinite number $n$ it is

$$\cos\frac{2k\pi}{n} = 1 - \frac{2kk\pi\pi}{nn}.$$

Therefore, having multiplied that form by $nn$ the general factor will be

$$= 2n(n+\lambda)\left(1 - \cos\frac{2k\pi}{n}\right) + \lambda\lambda + 2nz\left(1 - \cos\frac{2k\pi}{n}\right) - 2\lambda z\cos\frac{2k\pi}{n} + zz$$

$$= \lambda\lambda + 4kk\pi\pi + \frac{4kk\pi\pi z}{n} - 2\lambda z + zz,$$

having neglected the vanishing terms; in like manner even the term $\frac{4kk\pi\pi z}{n}$ can be omitted so that the general factor is

$$\lambda\lambda + 4kk\pi\pi - 2\lambda z + zz,$$

and the number of these factors, if for $k$ successively the numbers 1, 2, 3, 4 etc. are substituted, will be $= \frac{n}{2}$. But this compared to the general form given in my dissertation printed in Volume 7 of the Miscellanea Berolin. [8]

$$ff - 2fz\cos\varphi + zz$$

will give

$$f = \sqrt{\lambda\lambda + 4kk\pi\pi} \quad \text{and} \quad \cos\varphi = \frac{\lambda}{\sqrt{\lambda\lambda + 4kk\pi\pi}}$$

and hence

$$\sin\varphi = \frac{2k\pi}{\sqrt{\lambda\lambda + 4kk\pi\pi}}.$$

Hence this part of the integral $y$ results

$$y = e^{\lambda x}(\alpha\sin 2k\pi x + \mathfrak{A}\cos 2k\pi x).$$

[8] Euler refers to his paper E62 again.

Therefore, having successively substituted the values for $k$, because of $e^{\lambda} = m$, one will find

$$y = m^x \left\{ \begin{array}{l} C + \alpha \sin 2\pi x + \beta \sin 4\pi x + \gamma \sin 6\pi x + \text{etc.} \\ \quad + \mathfrak{A} \cos 2\pi x + \mathfrak{B} \cos 4\pi x + \mathfrak{C} \cos 6\pi x + \text{etc.} \end{array} \right\}.$$

Therefore, since having put $x = 1$ it must be $y = a$, it will be

$$a = m(C + \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \text{etc.}),$$

whence the constant $C$ is defined. Or if having put $\sin \pi x = r$ and $\cos \pi x = s$ $Q$ was an arbitrary function of even dimension of $r$ and $s$, the general term in question will be $y = m^x$. Q. E. I.

## COROLLARY 1

**§24** Therefore, in the geometric progression, insofar it is only described in such a way that each term is said to have a constant ratio to the preceding one, the interpolation is not determined, since the intermediate terms can be expressed in infinitely many different ways, they can even receive any value.

## COROLLARY 2

**§25** Therefore, the complete integral of the following infinite differential equation can be expressed in general

$$(m-1)y = \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

For, having put $\sin \pi x = r$ and $\cos \pi x = s$, if $Q$ denotes an even function of $r$ and $s$, it will be $y = m^x Q$ and hence $m^{-x}y$ becomes equal to an arbitrary function of even dimension of $r$ and $s$.

## COROLLARY 3

**§26** If one writes $\frac{x}{a}$ for $x$, this equation will result

$$(m-1)y = \frac{ady}{1 \cdot dx} + \frac{aaddy}{1 \cdot 2 \cdot dx^2} + \frac{a^3d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{a^4d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

To integrate equation this put

$$\sin\frac{\pi x}{a} = r \quad \text{and} \quad \cos\frac{\pi x}{a} = s$$

and let $Q$ denote an arbitrary function of even dimensions of $r$ and $s$ so that $Q$ retains the same value, even though for $r$ and $s$ one writes $-r$ and $-s$. Having done this it will be $y = m^{x:a}Q$.

## COROLLARY 4

**§27** And the solution of this problem can even be reduced to the solution of the first problem. For, since it must be $y' = my$, it will be $\log y' = \log y + \log m$. Put $\log y = v$ that it is $\log y' = v'$, and let it be $\log m = \lambda$; it must be $v' = v + \lambda$, whence because of

$$v' = v + \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.}$$

it is

$$\lambda = \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4v}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

and having put $v = u + \lambda x$ one will have

$$0 = \frac{du}{1 \cdot dx} + \frac{ddu}{1 \cdot 2 \cdot dx^2} + \frac{d^3u}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4u}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.},$$

which is the equation, we obtained in the first problem. Therefore, if one puts $\sin \pi x = r$ and $\cos \pi x = s$ and $Q$ denotes a function of even dimensions of $r$ and $s$, it will be $u = Q$ and hence

$$v = \lambda x + Q = \log y = x \log m + Q.$$

Therefore, by going back to numbers one has $y = m^x e^Q$; since there $e^Q$ also is a function of even dimensions of $r$ and $s$, if for it one writes $Q$, it will be, as we found before, be $y = m^x Q$.

## SCHOLIUM

**§28** Since we found all roots of the algebraic equation

$$m = 1 + \frac{z}{1} + \frac{zz}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.},$$

we will hence be able to exhibit all factors of this infinite expression

$$Z = 1 + \frac{z}{1(1-m)} + \frac{zz}{1 \cdot 2(1-m)} + \frac{z^3}{1 \cdot 2 \cdot 3(1-m)} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4(1-m)} + \text{etc.}$$

For, having put $\log m = \lambda$ the first simple factor will be $1 - \frac{z}{\lambda}$ and the remaining trinomial factors will be contained in this general form

$$1 + \frac{4kk\pi\pi}{n(\lambda\lambda + 4kk\pi\pi)} - \frac{2\lambda z - zz}{\lambda\lambda + 4kk\pi\pi},$$

which is transformed into this one

$$1 + \frac{z}{n} - \frac{\lambda\lambda z}{n(\lambda\lambda + 4kk\pi\pi)} - \frac{2\lambda z - zz}{\lambda\lambda + 4kk\pi\pi},$$

if the numbers 1, 2, 3, 4 etc. are successively substituted for $k$ and $n$ is an infinitely large number, whose half $\frac{n}{2}$ exhibits the number of factors itself. For the sake of brevity, let it be

$$\lambda\lambda + 4kk\pi\pi = \Phi$$

and it will be

$$Z = \left(1 - \frac{z}{\lambda}\right)\left(1 + \frac{z}{n} - \frac{\lambda\lambda z}{n\Phi} - \frac{2\lambda z}{\Phi} + \frac{zz}{\Phi}\right),$$

where the second factor represents all infinitely many factors, which result from the variation of the quantity $\Phi$. Therefore, having taken logarithms and differentiated them one will obtain

$$\frac{dZ}{Zdz} = \frac{-1}{\lambda - z} + \frac{\frac{1}{n} - \frac{\lambda\lambda}{n\Phi} - \frac{2\lambda}{\Phi} + \frac{2z}{\Phi}}{1 + \frac{z}{n} - \frac{\lambda\lambda z}{n\Phi} - \frac{2\lambda z}{\Phi} + \frac{zz}{\Phi}}.$$

And having resolved these terms into infinite series

$$
\begin{array}{rcccccccc}
\dfrac{dZ}{Zdz} = & -\dfrac{1}{\lambda} & -\dfrac{z}{\lambda^2} & -\dfrac{zz}{\lambda^3} & -\dfrac{z^3}{\lambda^4} & -\dfrac{z^4}{\lambda^5} & -\dfrac{z^5}{\lambda^6} & -\text{etc.} \\
& -\dfrac{1}{n} & -\dfrac{4\lambda^2 z}{\Phi^2} & -\dfrac{8\lambda^3 zz}{\Phi^3} & -\dfrac{16\lambda^4 z^3}{\Phi^4} & -\dfrac{32\lambda^5 z^4}{\Phi^5} & -\dfrac{64\lambda^6 z^5}{\Phi^6} & \\
& -\dfrac{\lambda\lambda}{n\Phi} & +\dfrac{2z}{\Phi} & +\dfrac{6\lambda zz}{\Phi\Phi} & +\dfrac{16\lambda^2 z^3}{\Phi^3} & +\dfrac{40\lambda^3 z^4}{\Phi^4} & +\dfrac{96\lambda^4 z^5}{\Phi^5} & \\
& -\dfrac{2\lambda}{\Phi} & & & -\dfrac{-2z^3}{\Phi^3} & -\dfrac{10\lambda z^4}{\Phi^3} & -\dfrac{36\lambda^2 z^5}{\Phi^4}, &
\end{array}
$$

put

$$\frac{dZ}{Zdz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.},$$

and because it is $\Phi = \lambda\lambda + 4kk\pi\pi$, where it is to be understood that successively all numbers 1, 2, 3, 4 etc. up to $\frac{1}{2}n$ are substituted for $k$, it will be

$$A = \frac{1}{2} - \frac{1}{\lambda} - 2\lambda\left(\frac{1}{\lambda\lambda + 4\pi\pi} + \frac{1}{\lambda\lambda + 16\pi\pi} + \frac{1}{\lambda\lambda + 36\pi\pi} + \text{etc.}\right)$$

If, for the sake of brevity, one sets

$$
\begin{aligned}
&\frac{1}{(\lambda\lambda + 4\pi\pi)} + \frac{1}{(\lambda\lambda + 16\pi\pi)} + \frac{1}{(\lambda\lambda + 36\pi\pi)} + \text{etc.} = \mathfrak{A}, \\
&\frac{1}{(\lambda\lambda + 4\pi\pi)^2} + \frac{1}{(\lambda\lambda + 16\pi\pi)^2} + \frac{1}{(\lambda\lambda + 36\pi\pi)^2} + \text{etc.} = \mathfrak{B}, \\
&\frac{1}{(\lambda\lambda + 4\pi\pi)^3} + \frac{1}{(\lambda\lambda + 16\pi\pi)^3} + \frac{1}{(\lambda\lambda + 36\pi\pi)^3} + \text{etc.} = \mathfrak{C}, \\
&\frac{1}{(\lambda\lambda + 4\pi\pi)^4} + \frac{1}{(\lambda\lambda + 16\pi\pi)^4} + \frac{1}{(\lambda\lambda + 36\pi\pi)^4} + \text{etc.} = \mathfrak{D}
\end{aligned}
$$

etc.,

it will be

$$\begin{aligned}
A &= \quad \frac{1}{2} \quad - \frac{1}{\lambda} \quad\quad - 2\lambda\mathfrak{A},\\
B &= -\frac{1}{\lambda\lambda} + 2\mathfrak{A} \quad - 4\lambda^2\mathfrak{B}\\
C &= -\frac{1}{\lambda^3} + 6\lambda\mathfrak{B} - 8\lambda^3\mathfrak{C},\\
D &= -\frac{1}{\lambda^4} - 2\mathfrak{B} \quad + 16\lambda^2\mathfrak{C} - 16\lambda^4\mathfrak{D},\\
E &= -\frac{1}{\lambda^5} - 10\lambda\mathfrak{C} + 40\lambda^3\mathfrak{D} - 32\lambda^5\mathfrak{E},\\
F &= -\frac{1}{\lambda^6} + 2\mathfrak{C} \quad - 36\lambda^2\mathfrak{D} + 96\lambda^4\mathfrak{E} - 64\lambda\mathfrak{F}
\end{aligned}$$

etc.

Since now it is

$$Z = 1 + \frac{z}{1(1-m)} + \frac{zz}{1\cdot 2(1-m)} + \frac{z^3}{1\cdot 2\cdot 3(1-m)} + \text{etc.},$$

it will be

$$Z = \frac{e^z - m}{1-m} = \frac{e^z - e^\lambda}{1 - e^\lambda} \quad \text{und} \quad \frac{dZ}{dz} = \frac{e^z}{1-e^z};$$

hence

$$\frac{dZ}{Zdz} = \frac{e^z}{e^z - e^\lambda} = \frac{1}{1 - e^\lambda e^{-z}} = \frac{1}{1 - me^{-m}}$$

and from this

$$\frac{dZ}{Zdz} = \frac{1}{1 - m + mz - \frac{mzz}{1\cdot 2} + \frac{mz^3}{1\cdot 2\cdot 3} - \frac{mz^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}}.$$

Now because of

$$\frac{dZ}{Zdz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$$

it will become

$$
\begin{array}{rccccccccc}
1 = (1-m)A & + & (1-m)Bz & + & (1-m)Cz^2 & + & (1-m)Dz^3 & + & (1-m)Ez^4 & + \text{etc.}, \\
 & + & mA & + & mB & + & mC & + & mD & \\
 & & & - & \frac{1}{2}mA & - & \frac{1}{2}mB & - & \frac{1}{2}mC & \\
 & & & & & + & \frac{1}{6}mA & + & \frac{1}{6}mB & \\
 & & & & & & & - & \frac{1}{24}mA &
\end{array}
$$

whence the following relations will result

$$
\begin{aligned}
A &= \frac{1}{1-m}, \\
B &= \frac{-mA}{1-m} = \frac{-m}{(1-m)^2}, \\
C &= \frac{-mB + \frac{1}{2}mA}{1-m} = \frac{mm}{(1-m)^3} + \frac{m}{2(1-m)^2}, \\
D &= \frac{-mC + \frac{1}{2}mB - \frac{1}{6}mA}{1-m} = \frac{-m^3}{(1-m)^4} - \frac{mm}{(1-m)^3} - \frac{m}{6(1-m)^2}, \\
E &= \frac{-mD + \frac{1}{2}mC - \frac{1}{6}mB + \frac{1}{24}mA}{1-m} = \frac{m^4}{(1-m)^5} + \frac{3m^3}{2(1-m)^4} + \frac{7mm}{12(1-m)^3} + \frac{m}{24(1-m)^2}
\end{aligned}
$$

etc.

Therefore, the following summations of the series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{B}$, $\mathfrak{D}$ etc. will result:

$$\text{I.} \quad \frac{1}{1-m} = \frac{1}{2} - \frac{1}{\lambda} - 2\lambda\mathfrak{A}$$

or

$$\mathfrak{A} = \frac{1}{4\lambda} - \frac{1}{2\lambda\lambda} - \frac{1}{2\lambda(1-m)};$$

$$\text{II.} \quad \frac{-m}{(1-m)^2} = -\frac{1}{\lambda\lambda} + 2\mathfrak{A} - 4\lambda\lambda\mathfrak{B} = \frac{1}{2\lambda} - \frac{2}{\lambda\lambda} - \frac{1}{\lambda(1-m)} - 4\lambda\lambda\mathfrak{B},$$

whence it is

$$\mathfrak{B} = \frac{1}{8\lambda^3} - \frac{1}{2\lambda^4} - \frac{1}{4\lambda^3(1-m)} + \frac{m}{4\lambda\lambda(1-m)^2};$$

$$\text{III.}\quad \frac{mm}{(1-m)^3}+\frac{m}{2(1-m)^2}=-\frac{1}{\lambda^3}+6\lambda\mathfrak{B}-8\lambda^3\mathfrak{C}=\frac{3}{4\lambda\lambda}-\frac{4}{\lambda^3}-\frac{3}{2\lambda^2(1-m)}+\frac{3m}{2\lambda(1-m)^2}-8\lambda\mathfrak{C},$$

therefore

$$\mathfrak{C}=\frac{3}{32\lambda^5}-\frac{1}{2\lambda^6}-\frac{3}{16\lambda^5(1-m)}+\frac{3m}{16\lambda^4(1-m)^2}-\frac{m}{16\lambda^3(1-m)^2}-\frac{mm}{8\lambda^3(1-m)^3}.$$

And so the following sums of the propounded series $\mathfrak{D}$, $\mathfrak{E}$ etc. will be found.

## COROLLARY 1

**§29** Therefore, because it is $m=e^\lambda$, it will be

$$\frac{1}{\lambda\lambda+4\pi\pi}+\frac{1}{\lambda\lambda+16\pi\pi}+\frac{1}{\lambda\lambda+36\pi\pi}+\text{etc.}=\frac{1}{4\lambda}-\frac{1}{2\lambda\lambda}-\frac{1}{2\lambda(1-e^\lambda)};$$

let $\lambda=\frac{2\pi a}{b}$; it will be

$$\frac{bb}{4(aa+bb)\pi^2}+\frac{bb}{4(aa+4bb)\pi^2}+\frac{bb}{4(aa+9bb)\pi^2}+\text{etc.}=\frac{b}{8\pi a}-\frac{bb}{8\pi\pi aa}-\frac{b}{4\pi a(1-e^{2\pi a:b})}$$

and hence by multiplying by $\frac{4\pi\pi}{bb}$ one will have

$$\frac{1}{aa+bb}+\frac{1}{aa+4bb}+\frac{1}{aa+9bb}+\text{etc.}=\frac{\pi}{2ab}-\frac{1}{2aa}+\frac{\pi}{ab(e^{2\pi a:b}-1)},$$

which sum I exhibited already elsewhere[9] but deduced it from another source.

[9] Euler derives this sum in his paper "De seriebus quibusdam considerationes". This is E130 in the Eneström-Index.

## COROLLARY 2

**§30** Therefore, if one sets $b = 1$, one has this sum

$$\frac{1}{aa+1} + \frac{1}{aa+4} + \frac{1}{aa+9} + \text{etc.} = \frac{\pi}{2a} - \frac{1}{2aa} + \frac{\pi}{a(e^{2\pi a}-1)},$$

and if furthermore one sets $a = 0$ that this series results

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.};$$

the sum of this series because of the terms growing to infinity has to be derived this way from the given formula: Assume $a$ to be infinitely small; it will be

$$e^{2\pi a} = 1 + 2\pi a + 2\pi\pi aa + \frac{4}{3}\pi^3 a^3$$

and hence the sum will be

$$= \frac{\pi}{2a} - \frac{1}{2aa} + \frac{1}{2aa + 2\pi a^3 + \frac{4}{3}\pi^2 a^4}$$

$$= \frac{\pi a + \pi\pi aa + \frac{2}{3}\pi^3 a^3 - 1 - \pi a - \frac{2}{3}\pi^2 aa + 1}{2aa\left(1 + \pi a + \frac{2}{3}\pi\pi a^2\right)} = \frac{1}{6}\pi^2,$$

which, as it is known, is the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$

## COROLLARY 3

**§31** If in the series found before

$$\frac{1}{aa+bb} + \frac{1}{aa+4bb} + \frac{1}{aa+9bb} + \text{etc.} = \frac{\pi}{2ab} - \frac{1}{2aa} + \frac{\pi}{ab(e^{2\pi a:b}-1)}$$

the quantity $a$ is considered as a variable and the series is differentiated with respect to this variable, the sum of the series $\mathfrak{B}$ will result; and so forth by means of iterated differentiation, starting from the series $\mathfrak{A}$ one will find the sums of the following series $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc.

## COROLLARY 4

**§32** The sum of this series can be expressed more conveniently this way

$$\frac{1}{aa+bb}+\frac{1}{aa+4bb}+\frac{1}{aa+9bb}+\text{etc.}=\frac{-1}{2aa}+\frac{\pi(e^{2\pi a:b}+1)}{2ab(e^{2\pi a:b}-1)}$$
$$=\frac{-1}{2aa}+\frac{\pi(e^{\pi a:b}+e^{-\pi a:b})}{2ab(e^{\pi a:b}-e^{-\pi a:b})}.$$

The value of the series, if $b$ is imaginary, is easily calculated from this form; for, let $b=\frac{c}{\sqrt{-1}}$; it will be

$$\frac{1}{aa-cc}+\frac{1}{aa-4cc}+\frac{1}{aa-9cc}+\text{etc.}=\frac{-1}{2aa}+\frac{\pi\left(e^{\frac{\pi a\sqrt{-1}}{c}}+e^{\frac{-\pi a\sqrt{-1}}{c}}\right)\sqrt{-1}}{2ac\left(e^{\frac{\pi a\sqrt{-1}}{c}}-e^{\frac{-\pi a\sqrt{-1}}{c}}\right)}.$$

But it is

$$e^{\frac{\pi a\sqrt{-1}}{c}}+e^{\frac{-\pi a\sqrt{-1}}{c}}=2\cos\frac{\pi a}{c}$$

and

$$e^{\frac{\pi a\sqrt{-1}}{c}}-e^{\frac{-\pi a\sqrt{-1}}{c}}=2\sqrt{-1}\cdot\sin\frac{\pi a}{c},$$

whence it is

$$\frac{1}{aa-cc}+\frac{1}{aa-4cc}+\frac{1}{aa-9cc}+\text{etc.}=\frac{-1}{2aa}+\frac{\pi\cos\pi a:c}{2ac\sin\pi a:c}.$$

## COROLLARY 5

**§33** Since it is $\cos\frac{(2k+1)\pi}{2}=0$, in the cases, in which it is $a=2k+1$ and $c=2$, the sum of the series is

$$=\frac{-1}{2aa}=-\frac{1}{2(2k+1)^2},$$

while $k$ is an arbitrary integer number. Hence it will be

$$\frac{1}{(2k+1)^2-4}+\frac{1}{(2k+1)^2-16}+\frac{1}{(2k+1)^2-36}+\frac{1}{(2k+1)^2-64}+\text{etc.}=\frac{-1}{2(2k+1)^2},$$

which summation I demonstrated elsewhere[10]. For, if the single fractions are resolved into partial fractions, it results

[10]Euler did so in E130.

$$\frac{-1}{2k+1} = \frac{1}{2k-1} + \frac{1}{2k-3} + \frac{1}{2k-5} + \frac{1}{2k-7} + \frac{1}{2k-9} + \text{etc.}$$
$$+ \frac{1}{2k+3} + \frac{1}{2k+5} + \frac{1}{2k+7} + \frac{1}{2k+9} + \text{etc.}$$

### COROLLARY 6

**§34** Having brought the term $\frac{-1}{2k+1}$ to the other side and having collected each two terms a new series will result, whose sum is $= 0$. Of course, having divided the single terms by $4k$ it will be

$$0 = \frac{1}{4kk-1} + \frac{1}{4kk-9} + \frac{1}{4kk-25} + \frac{1}{4kk-49} + \frac{1}{4kk-81} + \text{etc.},$$

whose truth will easily be seen in each case.

## PROBLEM 4

**§35** *To find the general term of the series, in which each arbitrary term results, if the preceding term is multiplied by a given number m and a given number c is added to that product, and the first term of that series is equally given and is = a.*

### SOLUTION

Therefore, the terms corresponding to integer indices, will be as follows

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ a, & ma+c, & m^2a+mc+c, & m^3a+m^2c+mc+c & \text{etc.}; \end{array}$$

hence, if the index $x$ is an integer number, the corresponding term will be

$$= m^{x-1}a + \frac{m^{x-1}-1}{m-1}c.$$

But if $x$ is not an integer, infinitely many other formulas, except for this one, will equally satisfy this equation; in order to find them let $y$ be the term corresponding to the index $x$ and $y'$ the following or the one corresponding to $x+1$; it will be

$$y' = my + c,$$

whence it will be

$$my + c = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

Put

$$y = v - \frac{c}{m-1}$$

and it will be

$$mv = v + \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.};$$

since this equation agrees with the one we found in the preceding problem, if one puts $\sin \pi x = r$ and $\cos \pi x = s$ and $Q$ is taken for an arbitrary function of even dimensions of $r$ and $s$, it will be

$$v = m^x Q$$

and hence

$$y = m^x Q - \frac{c}{m-1}.$$

Put $x = 1$, in which case it is $r = 0$ and $s = -1$, and let $Q$ go over into $C$; it must be

$$a = mC - \frac{c}{m-1}$$

and hence it will be

$$C = \frac{a}{m} + \frac{c}{m(m-1)}.$$

Hence, if for $Q$ the constant quantity $C$ itself is taken, it will be

$$y = m^{x-1}a + \frac{(m^{x-1}-1)c}{m-1}$$

for the simplest case. And if $P$ is a function of even dimensions of $r$ and $s$, which vanishes for $x = 1$, one will be able to put $Q = C + P$ and the form of the general term in question will be this one extending very far

$$y = m^{x-1}a + \frac{(m^{x-1}-1)c}{m-1} + m^x P.$$

Q. E. I.

# Problem 5

**§36** *To find the general term of recurring series of second order, in which each term becomes equal to the aggregate of the two preceding terms multiplied by any arbitrary numbers.*

## Solution

Let

| | | | |
|---|---|---|---|
| the term to which the index | $x$ | corresponds | $=y$, |
| the term to which the index | $x-1$ | corresponds | $='y$, |
| the term to which the index | $x-2$ | corresponds | $=''y$, |

and let this law of the recurring series be propounded that it is

$$y = \alpha' y + \beta'' y.$$

Therefore, since it is

$$\begin{aligned} 'y &= y - \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} - \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.}, \\ ''y &= y - \frac{2dy}{1 \cdot dx} + \frac{4ddy}{1 \cdot 2 \cdot dx^2} - \frac{8d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{16d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.}, \end{aligned}$$

having substituted these formulas it will be

$$\begin{aligned} y = &+ \alpha\left(y - \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} - \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.}\right) \\ &+ \beta\left(y - \frac{2dy}{1 \cdot dx} + \frac{4ddy}{1 \cdot 2 \cdot dx^2} - \frac{8d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{16d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.}\right) \end{aligned}$$

To resolve this equation according to the general prescription put 1 for $y$, $z$ for $\frac{dy}{dx}$, $z^2$ for $\frac{ddy}{dx^2}$ etc. and it will be

$$1 = +\alpha\left(1 - \frac{z}{1} + \frac{zz}{1\cdot 2} - \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}\right)$$
$$1 = +\alpha\left(1 - \frac{2z}{1} + \frac{4zz}{1\cdot 2} - \frac{8z^3}{1\cdot 2\cdot 3} + \frac{16z^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}\right),$$

which equation is reduced to this finite form

$$1 = \alpha e^{-z} + \beta e^{-2z},$$

whose factors must be investigated. Therefore, having put $e^{+z} = u$ resolve this equation

$$uu = \alpha u + \beta;$$

Of this equation either both roots are real or both are imaginary or finally both are equal to each other. These three cases must be expanded separately.

I. Therefore, first let the two roots be real and different to each other, or let

$$uu - \alpha u - \beta = (u - A)(u - B)$$

and hence by putting $e^z$ for $u$ we will have the two general factors $e^z - A$ and $e^z - B$. But we saw above that the formula $e^z - m$ gave this integral

$$y = +A^x \begin{Bmatrix} C + \alpha\sin 2\pi x + \beta\sin 4\pi x + \gamma\sin 6\pi x + \text{etc.} \\ + \mathfrak{A}\cos 2\pi x + \mathfrak{B}\cos 4\pi x + \mathfrak{C}\cos 6\pi x + \text{etc.} \end{Bmatrix}.$$

Therefore, both factors $e^z - A$ and $e^z - B$ taken together will give this value for the general term $y$

$$y = +A^x \begin{Bmatrix} C + \alpha\sin 2\pi x + \beta\sin 4\pi x + \gamma\sin 6\pi x + \text{etc.} \\ + \mathfrak{A}\cos 2\pi x + \mathfrak{B}\cos 4\pi x + \mathfrak{C}\cos 6\pi x + \text{etc.} \end{Bmatrix}$$

$$+B^x \begin{Bmatrix} C + \alpha'\sin 2\pi x + \beta'\sin 4\pi x + \gamma'\sin 6\pi x + \text{etc.} \\ + \mathfrak{A}'\cos 2\pi x + \mathfrak{B}'\cos 4\pi x + \mathfrak{C}'\cos 6\pi x + \text{etc.} \end{Bmatrix}.$$

Or put $\sin\pi x = r$ and $\cos\pi x = s$ and let $P$ and $Q$ be arbitrary functions of even dimensions of $r$ and $s$ and, if it was

$$uu - \alpha u - \beta = (u - A)(u - B)$$

or if $A$ and $B$ are the roots of the equation $uu - \alpha u - \beta = 0$, in this case it will be

$$y = A^x P + B^x Q.$$

II. If both roots were imaginary, then certainly the same formula already found can be used, since in each case the imaginary quantities cancel each other; nevertheless, one can exhibit a formula for $y$ not containing any imaginary quantities. For, in this case the equation $uu - \alpha u - \beta = 0$ will obtain a form of such a kind

$$uu - 2fu\cos\omega + ff = 0,$$

whose roots are

$$u = f\cos\omega \pm f\sqrt{-1} \cdot \sin\omega,$$

such that it is

$$A = f\cos\omega + f\sqrt{-1} \cdot \sin\omega \quad \text{und} \quad B = f\cos\omega - f\sqrt{-1} \cdot \sin\omega.$$

But hence it will be

$$A^x = f^x \cos\omega x + f^x\sqrt{-1} \cdot \sin\omega x$$

and

$$B^x = f^x \cos\omega x - f^x\sqrt{-1} \cdot \sin\omega x.$$

Therefore, if these values are substituted for $A^x$ and $B^x$, it will be

$$y = (P + Q)f^x \cos\omega x + (P - Q)\sqrt{-1} \cdot f^x \sin\omega x.$$

Since now $P$ and $Q$ are arbitrary functions of $r$ and $s$, if they have even dimensions, write $P$ instead of $P + Q$ and put $Q$ instead of $(P - Q)\sqrt{-1}$ and using the equation

$$uu - \alpha u - \beta = uu - 2fu\cos\omega + ff = 0$$

the general term in question will be

$$y = f^x P \cos \omega x + f^x Q \sin \omega x.$$

III. If both roots $A$ and $B$ of the equation $uu - \alpha u - \beta = 0$ were equal, say $A = B = m$, one will have the equation

$$(e^z - m)^2 = 0.$$

As in § 23 put $m = e^\lambda$; the first factor of the formula $(e^z - e^\lambda)^2$ will be the square $= (z - \lambda)^2$, whence the this part of the integral results

$$(\mathfrak{A} + \mathfrak{B}x)e^{\lambda x} = (\mathfrak{A} + \mathfrak{B}x)m^x = (\mathfrak{A} + \mathfrak{B}x)A^x.$$

All remaining ones will equally be quadratic and will be contained in this general form

$$(\lambda\lambda + 4kk\pi\pi - 2\lambda z + zz)^2;$$

hence according to the prescriptions I gave once[11] this part of the integral results

$$A^x(\mathfrak{A} + \mathfrak{B}x) \sin 2k\pi x + A^x(\mathfrak{C} + \mathfrak{D}x) \cos 2k\pi x.$$

By collecting all these it follows, if it was

$$uu - \alpha u - \beta = (u - A)^2 = uu - 2Au + AA,$$

that the general term in question will be

$$y = A^x \begin{Bmatrix} \mathfrak{A} + \mathfrak{B}x(\mathfrak{C} + \mathfrak{D}x)\sin 2\pi x + (\mathfrak{G} + \mathfrak{H}x)\sin 4\pi x + \text{etc.} \\ (\mathfrak{E} + \mathfrak{F}x)\cos 2\pi x + (\mathfrak{I} + \mathfrak{K}x)\cos 4\pi x + \text{etc.} \end{Bmatrix}.$$

Put $\sin \pi x = r$ and $\cos \pi x = s$ again and let $P$ and $Q$ be arbitrary even functions of $r$ and $s$ and one will be able to express the general term in such a way that it is

$$y = A^x(P + Qx).$$

Q. E. I.

[11] Euler refers to E62 again.

## COROLLARY 1

**§37** Therefore, if in a recurring series the arbitrary term $y$ is determined by the two preceding ones $'y$ and $''y$ in such a way that it is $y = \alpha' y + \beta'' y$, or if according to de Moivre $+\alpha$, $+\beta$ was the scale of relation, and if $x$ was the index of the term $y$, $y$ will be a highly undetermined function of $x$, since innumerable formulas can be exhibited, which yield satisfying values for $y$.

## COROLLARY 2

**§38** But in order to find all expressions for $y$ using only the scale of relation $+\alpha$, $+\beta$, form this equation $uu - \alpha u - \beta = 0$; for, from its resolution the form of the general term $y$ will be found in the following way.

## COROLLARY 3

**§39** Let the roots of the equation

$$uu - \alpha u - \beta = 0$$

be $A$ and $B$ such that it is

$$A = \frac{1}{2}\alpha + \sqrt{\frac{1}{4}\alpha\alpha + \beta} \quad \text{and} \quad B = \frac{1}{2}\alpha - \sqrt{\frac{1}{4}\alpha\alpha + \beta},$$

and having put $\sin \pi x = r$ and $\cos \pi x = s$ take any arbitrary functions of $r$ and $s$, which we want to put $P$ and $Q$; it will be

$$y = A^x P + B^x Q = \left(\frac{1}{2}\alpha + \sqrt{\frac{1}{4}\alpha\alpha + \beta}\right)^x P + \left(\frac{1}{2}\alpha - \sqrt{\frac{1}{4}\alpha\alpha + \beta}\right)^x Q.$$

## COROLLARY 4

**§40** But if both roots of the equation $uu = \alpha u + \beta$ were equal, that formula, because of $\beta + \frac{1}{4}\alpha\alpha = 0$, is useless. But in this case, since both roots are $\frac{1}{2}\alpha$, if one puts $\frac{1}{2}\alpha = A$, the general term will be

$$y = A^x (P + Qx).$$

## COROLLARY 5

**§41** But if $\frac{1}{4}\alpha\alpha + \beta$ is a negative quantity, the parts found before will be imaginary. Therefore, to find the imaginary form compare the equation

$$uu - \alpha u - \beta = 0$$

to this one

$$uu - 2fu\cos\omega + ff = 0;$$

it will be

$$f = \sqrt{-\beta} \quad \text{and} \quad \alpha = 2\sqrt{-\beta}\cdot\cos\omega$$

or

$$\cos\omega = \frac{\alpha}{2\sqrt{-\beta}} \quad \text{and} \quad \sin\omega = \frac{\sqrt{-4\beta - \alpha\alpha}}{2\sqrt{-\beta}} = \sqrt{1 + \frac{\alpha\alpha}{4\beta}},$$

whence the angle $\omega$ will be found, whence it will be

$$y = f^x(P\cos\omega x + Q\sin\omega x).$$

## COROLLARY 6

**§42** If constant quantities are assumed for $P$ and $Q$, the same form of the general term results, which is usually exhibited and is considered as the only one satisfying the equation. But having propounded an arbitrary determined series one has to define these two constant quantities from the first two terms, which are assumed to be given. But in general, since the two arbitrary functions $P$ and $Q$ enter, which, as often as $x$ is an integer number, will obtain the same constant values, it is plain that two terms of the series corresponding to integer indices can be assumed arbitrarily.

## SCHOLIUM

**§43** This method to find the general terms of recurring series is mainly remarkable for that reason that it not only exhibits all possible forms but also proceeds a priori and completes the task using only analytical principles, whereas other authors, which treated those series, all got to the special form of the general term in an indirect way. For, it is the principal property and quasi a criterion of a direct method that it not only finds the nature of the subject in consideration from first principles, but also contains all solutions at the same time. But an indirect method, even though they often yield short and elegant solutions of problems, nevertheless very rarely exhaust the nature of the question in consideration. An extraordinary example of this difference is seen in the preceding problem, but will even occur more clearly in the following problem, where the general terms of all recurring series will be investigated in general.

## PROBLEM 6

**§44** *To find the general term of recurring series of arbitrary order, in which any arbitrary term becomes equal to an aggregate of several of the preceding terms multiplied by arbitrary numbers.*

### SOLUTION

Let the term corresponding to the index $x$ be $= y$, but denote the preceding terms corresponding to the indices $x-1$, $x-2$, $x-3$, $x-4$ etc., by $'y$, $''y$, $'''y$, ${}^{\mathrm{IV}}y$ and let this law of the series be propounded that one everywhere has

$$y = \alpha\, 'y + \beta\, ''y + \gamma\, '''y + \delta\, {}^{\mathrm{IV}}y + \text{etc.}$$

Since now from the nature of differentials it is

$$\begin{aligned}
'y &= y - \frac{dy}{1\cdot dx} + \frac{ddy}{1\cdot 2\cdot dx^2} - \frac{d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \text{etc.},\\
''y &= y - \frac{2dy}{1\cdot dx} + \frac{2^2ddy}{1\cdot 2\cdot dx^2} - \frac{2^3d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \text{etc.},\\
'''y &= y - \frac{3dy}{1\cdot dx} + \frac{3^2ddy}{1\cdot 2\cdot dx^2} - \frac{3^3d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \text{etc.}\\
&\text{etc.},
\end{aligned}$$

if these values are substituted there, an equation will result, in whose single terms one dimension of the variable $y$ occurs, but only the differential $dx$, which is assumed to be constant, of the other variable $x$ enters. Hence, if everywhere 1 is set instead of $y$, $z$ instead of $\frac{dy}{dx}$ and generally $z^m$ instead of $\frac{d^my}{dx^m}$, after some simplification this equation will emerge

$$1 = \alpha e^{-z} + \beta e^{-2z} + \gamma e^{-3z} + \delta e^{-4z} + \text{etc.}$$

Now let $e^z = u$, and having got rid of the fractions an algebraic equation of this kind will result

$$u^n = \alpha u^{n-1} + \beta u^{n-2} + \gamma u^{n-3} + \delta u^{n-4} + \text{etc.},$$

which will be of so many dimensions as preceding terms are required for the determination of the term $y$ or of what order the recurring series itself was. Now the form of the general term $y$ will be concluded from the roots of this equation or from the factors of this formula

$$u^n - \alpha u^{n-1} - \beta u^{n-2} - \gamma u^{n-3} - \delta u^{n-4} - \text{etc.} = U$$

in the same way we did it in the solutions of the problems propounded up to now: Of course, if it is $\sin \pi x = r$ and $\cos \pi x = s$ and $P$, $Q$, $R$, $S$, $T$ etc. denote arbitrary functions of even dimensions of $r$ and $s$, further, investigate all real so simple as trinomial factors of the formula $U$ and, if some of them were equal, express them combined by means of powers. But these single factors will yield as many parts of the general term $y$, which parts will be formed by means of the following rules:

I. If one factor is $u - A$, the part of the integral will be

$$y = A^x P.$$

II. If the factor is $(u - A)^2$, the part of the integral will be

$$y = A^x (P + Qx).$$

III. If the factor is $(u - A)^3$, the part of the integral will be

$$y = A^x (P + Qx + Rx^2).$$

IV. If the factor is $(u - A)^4$, the part of the integral will be

$$y = A^x (P + Qx + Rx^2 + Sx^3).$$

etc.

1. If the factors is $u - 2Au\cos\omega + AA$, it will be

$$y = A^x(P\cos\omega x + Q\sin\omega x).$$

2. If the factor is $(u - 2Au\cos\omega + AA)^2$, it will be

$$y = A^x(P + Qx)\cos\omega x + A^x(R + Sx)\sin\omega x$$

3. If the factor is $(u - 2Au\cos\omega + AA)^3$, the part will be

$$\begin{aligned} y = &+ A^x(P+Qx+Rxx)\cos\omega x \\ &+ A^x(S+Tx+Vxx)\sin\omega x \end{aligned}$$

etc.

Therefore, if hence the parts of the integral are found for the single factors of the formula $U$ and these parts are combined into one sum, one will have the complete value for the general term in question. Q. E. I.

## COROLLARY 1

**§45** Therefore, this way one obtains the complete integral of the following differential equation of infinite order

$$\begin{aligned}
y = {} & y \, (\alpha + \beta + \gamma + \delta + \text{etc.}) \\
- {} & \frac{dy}{1 \cdot dx} (\alpha + 2\beta + 3\gamma + 4\delta + \text{etc.}) \\
+ {} & \frac{ddy}{1 \cdot 2 \cdot dx^2} (\alpha + 2^2\beta + 3^2\gamma + 4^2\delta + \text{etc.}) \\
- {} & \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} (\alpha + 2^3\beta + 3^3\gamma + 4^3\delta + \text{etc.}) \\
+ {} & \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} (\alpha + 2^4\beta + 3^4\gamma + 4^4\delta + \text{etc.}) \\
& \text{etc.}
\end{aligned}$$

or the value of $y$ will be expressed by means of a function of $x$.

### Corollary 2

**§46** Therefore, the complete difficulty is reduced to the resolution of the algebraic equation

$$u^n = \alpha u^{n-1} + \beta u^{n-2} + \gamma u^{n-3} + \delta u^{n-4} + \text{etc.}$$

For, having found its roots or factors it is easy to determine the value of $y$ using the rules given before.

### Corollary 3

**§47** Since by integration so many arbitrary quantities $P$, $Q$, $R$, $S$ etc. are introduced as the exponent $n$ contains unities or as preceding terms enter into the determination of the following, it is manifest that as many terms can be taken arbitrary, whence all remaining ones, whose indices are integer numbers, are determined. This is nevertheless no obstruction that the terms of the non integer indices stay most undetermined, as it was already noted in the preceding problems.

## Problem 7

**§48** *If any arbitrary term of the series becomes equal to a certain constant quantity c together with an aggregate of several preceding terms multiplied by given numbers (as in the preceding problem), to find the general term of this series.*

### SOLUTION

Having as before put the term corresponding to the undetermined index $x = y$ let the preceding ones corresponding to the indices $x-1$, $x-2$, $x-3$ etc. be $'y$, $''y$, $'''y$ etc. and let this law of progression be propounded

$$y = c + \alpha\, 'y + \beta\, ''y + +\gamma\, '''y + \delta\, ^{\mathrm{IV}}y + \text{etc.};$$

therefore, having propounded the values exhibited above for $'y$, $''y$, $'''y$, $^{\mathrm{IV}}y$ etc. it will be

$$\begin{aligned} y = c + {} & y(\alpha + \beta + \gamma + \delta + \text{etc.}) \\ & - \frac{dy}{1 \cdot dx}(\alpha + 2\beta + 3\gamma + 4\delta + \text{etc.}) \\ & + \frac{ddy}{1 \cdot 2 \cdot dx^2}(\alpha + 2^2\beta + 3^2\gamma + 4^2\delta + \text{etc.}) \\ & - \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3}(\alpha + 2^3\beta + 3^3\gamma + 4^3\delta + \text{etc.}) \\ & \text{etc.} \end{aligned}$$

Now, to get rid of the constant term $c$ in the equation put $y = v + g$ and let

$$g = c + g(\alpha + \beta + \gamma + \delta + \text{etc.})$$

and hence

$$g = \frac{c}{1 - \alpha - \beta - \gamma - \delta - \text{etc.}}.$$

Having done this because of $dy = dv$, $ddy = ddv$ etc. one will have this equation:

$$\begin{aligned} v = {} & v(\alpha + \beta + \gamma + \delta + \text{etc.}) \\ & - \frac{dv}{1 \cdot dx}(\alpha + 2\beta + 3\gamma + 4\delta + \text{etc.}) \\ & + \frac{ddv}{1 \cdot 2 \cdot dx^2}(\alpha + 2^2\beta + 3^2\gamma + 4^2\delta + \text{etc.}) \\ & \text{etc.} \end{aligned}$$

Since this equation is similar to the one we resolved in the preceding problem, the value of $v$ will be found by means of the rules given there. Having found this one will have the general term in question

$$y = v + \frac{c}{1 - \alpha - \beta - \gamma - \delta - \text{etc.}},$$

whence the nature of the propounded series will be known. Q. E. I.

### COROLLARY 1

**§49** Therefore, the constant quantity $c$, which is added to the formula

$$\alpha\, 'y + \beta\, ''y + \gamma\, '''y + \text{etc.},$$

does only affect the general term $y$ in that regard that it adds a constant to it. Therefore, find the general term for the pure recurring series, whose relation scale is $+\alpha$, $+\beta$, $+\gamma$, $+\delta$ etc., and add the number $\frac{c}{1-\alpha-\beta-\gamma-\text{etc.}}$ to it.

### COROLLARY 2

**§50** But this constant quantity to be added $\frac{c}{1-\alpha-\beta-\gamma-\text{etc.}}$ becomes infinite and hence uncertain, if the denominator vanishes or if it is

$$1-\alpha-\beta-\gamma-\delta-\text{etc.}=0.$$

But in this case the equation

$$u^n - \alpha u^{n-1} - \beta u^{n-2} - \gamma u^{n-3} - \text{etc.} = 0$$

will have the root $u-1=0$, whence the part $y=P$ of the integral results; this quantity $P$, that not all terms become infinite, must be infinite in such a way that it together with that infinite constant yields a finite quantity, which will be $= P + Qx$.

### SCHOLIUM 1

**§51** That this becomes more clear, it is to be observed that series of this kind, as we considered here, can always be reduced to pure recurring series higher by one degree. For, if it is

$$y = c + \alpha\, 'y + \beta\, ''y + \gamma\, '''y + \delta\, ^{\mathrm{IV}}y,$$

it will be

$$'y = c + \alpha\, ''y + \beta\, '''y + \gamma\, ^{\mathrm{IV}}y + \delta\, ^{\mathrm{V}}y,$$

whose difference gives

$$y = (\alpha+1)\, 'y + (\beta-\alpha)\, ''y + (\gamma-\beta)\, '''y + (\delta-\gamma)\, ^{\mathrm{IV}}y - \delta\, ^{\mathrm{V}}y,$$

which is the law for a pure recurring series, whose general term will be formed from the resolution of this equation:

$$u^{n+1} - (\alpha+1)u^n - (\beta-\alpha)u^{n-1} - (\gamma-\beta)u^{n-2} - \text{etc.} = 0.$$

But one factor of this equation is already known, namely $u-1$, because it is

$$(u-1)(u^n - \alpha u^{n-1} - \beta u^{n-2} - \gamma u^{n-3} - \text{etc.}) = 0.$$

But the factor $u-1$ gives the part $1^x P$ of the integral only then, whenever it is not a factor of the other form $u^n - \alpha u^{n-1} - $ etc. at the same time; but if this expression also has the factor $u-1$ or its power as a factor, the exponent of this factor must be augmented by 1 and hence the corresponding part of the integral must be investigated. But having found the general term $y$ this way, because the quantity $c$ is not contained in it, it will be too general; therefore, it must be restricted to the propounded case. Of course, using the value of $y$ find the values of the preceding terms ${}'y$, ${}''y$, ${}'''y$ etc. by putting $x-1$, $x-2$, $x-3$ etc. instead of $x$ etc., where it is to be noted that the functions $P$, $Q$, $R$ etc. retain the same values and hence are not changed. Further, substitute these values in the equation

$$y = c + \alpha\,{}'y + \beta\,{}''y + \gamma\,{}'''y + \delta\,{}^{\mathrm{IV}}y + \text{etc.}$$

and this way one of the those functions $P$, $Q$, $R$ etc. will be determined. So, if this law of the series is propounded

$$y = c + 3'y - 2''y,$$

hence this equation will result

$$(u-1)(u^2 - 3u - 2)) = 0,$$

whose factors are

$$(u-1)^2(u-2) = 0;$$

hence this general term is concluded

$$y = P + Qx + 2^x R;$$

therefore, it will be

$${}'y = P + Qx - Q + 2^{x-1}R$$

and

$$''y = P + Qx - 2Q + 2^{x-2}R;$$

Having substituted this will give this equality

$$P + Qx + 4 \cdot 2^{x-2}R = c + Qx + Q + 4 \cdot 2^{x-2}R,$$

whence one finds $Q = -c$; and so the general term corresponding to the propounded law will be

$$y = P - cx + 2^x R,$$

where for $P$ and $R$ arbitrary functions of even dimensions of $r$ and $s$ can be assumed.

### SCHOLIUM 2

**§52** Therefore, since we gave a universal method to find the general term of series and each term this series is determined by the preceding ones, if no powers of the preceding terms occur, let us accommodate this same method to series, each term of which is not only determined using the preceding ones but also the index itself; hence let us consider the third class of the formation of series now. But if squares or higher powers enter into the determination of the following term, as if it was

$$y' = yy + ay,$$

then the infinite differential equation, containing the general term in question as a solution, is certainly easily exhibited, which in this case will be

$$yy + ay = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.};$$

but since until now no artifice is known to resolve equations of this kind, we are forced to omit the treatment of this class of series here.

## PROBLEM 8

**§53** *To find the general term of the series, in which any arbitrary term corresponding to the index x becomes equal to a multiple of the preceding term together with a multiple of the index and a certain constant quantity.*

SOLUTION

Let $y$ be the term corresponding to the index $x$ and let $y'$ denote the following term and let this law of the series be propounded

$$y' = my + a + bx,$$

whence the value of $y$ must be defined. Therefore, if we substitute its value for $y'$, we will have this equation:

$$a + bx + my = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.}$$

Even though I taught to solve equations of this kind in general[12], it will nevertheless be helpful to resolve this equation into another one, in which all terms contain only one dimension of $y$. Therefore, put

$$y = A + Bx + v;$$

it will be

$$dy = Bdx + dv, \qquad ddy = ddv \quad \text{etc.}$$

and it will be

$$\begin{array}{l} a + \quad b\,x = \quad A + Bx + v + \dfrac{dv}{1 \cdot dx} + \dfrac{ddv}{1 \cdot 2 \cdot dx^2} + \dfrac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.} \\ +mA + mBx \quad +B \end{array}$$

Now let

$$A + B = a + mA \quad \text{and} \quad B = b + mB$$

and one will find

$$B = \frac{-b}{m-1} \quad \text{and} \quad A = \frac{-b}{(m-1)^2} - \frac{a}{m-1}.$$

Therefore, this equation will remain

$$mv = v + \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.};$$

[12] He did so in E188, but only for the case of finite order.

since this is reduced to $e^z - m = u - m = 0$, it will be

$$v = m^x P$$

and hence the general term in question is

$$y = \frac{-b}{(m-1)^2} - \frac{a+bx}{m-1} + m^x P.$$

Here, the one case is excluded, in which it is $m = 1$, because of the vanishing denominator $m - 1$. For, since in this case one will have

$$a + bx = v + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.},$$

to get rid of the term $bx$ one has to take a value of this kind for $y$

$$y = A + Bx + Cxx + v,$$

whence it is

$$\frac{dy}{dx} = B + 2Cx + \frac{dv}{dx} \quad \text{und} \quad \frac{ddy}{dx^2} = 2C + \frac{ddv}{dx^2};$$

and so one will have

$$\begin{aligned} a + bx = \quad & B + 2Cx + \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.} \\ +C & \end{aligned}$$

Therefore, let it be

$$C = \frac{1}{2}b \quad \text{and} \quad B = a - \frac{1}{2}b$$

and it will be $v = P$ and the general term

$$y = A + \left(a - \frac{1}{2}b\right)x + \frac{1}{2}bxx + P,$$

or since $A$ can be contained in the function $P$, it will be

$$y = \left(a - \frac{1}{2}\right)x + \frac{1}{2}bxx + P.$$

Q. E. I.

## SCHOLIUM

**§54** But series of this kind can be reduced to the law of simple recurring series. For, because it is

$$y' = a + bx + my,$$

it will be

$$y'' = a + b(x+1) + my',$$

whence by subtracting it will be

$$y'' - y' = b + my' - my;$$

in like manner it will be

$$y''' - y'' = b + my'' - my'$$

and by subtracting again

$$y''' - 2y'' + y' = my'' - 2my' + my$$

or

$$y''' = (m+2)y'' - (2m+1)y' + my$$

or for the preceding terms

$$y = (m+2)\,{}'y - (2m+1)\,{}''y + m\,{}'''y.$$

Therefore, hence according to § 51 this equation will be formed

$$u^3 - (m+2)u^2 + (2m+1)u - m = 0,$$

which has these factors

$$(u-1)^2(u-m) = 0,$$

whence this general term results

$$y = P + Qx + m^x R.$$

Now to accommodate this too far-extending formula to the propounded case $y' = a + bx + my$, because of

$$y' = P + Qx + Q + m \cdot m^x R,$$

it will be

$$P + Q + Qx + m \cdot m^x R = a + bx + mP + mQx + m \cdot m^x R$$

and hence

$$P + Q = a + mP \quad \text{and} \quad Q = b + mQ,$$

whence it is found

$$Q = \frac{-b}{m-1} \quad \text{and} \quad P = \frac{-b}{(m-1)^2} - \frac{a}{m-1},$$

such that the general term, as it was found before, is

$$y = \frac{-b}{(m-1)^2} - \frac{a+bx}{m-1} + m^x R.$$

But if it is $m = 1$, it is immediately plain that the three factors of the equation $(u-1)^2(u-m) = 0$ will be equal and that it is $(u-1)^3 = 0$, whence the general term is

$$y = P + Qx + Rxx$$

and hence

$$\begin{array}{llll} y' = & P + & Qx + Rxx & = P + Qx + Rxx \\ & + Q + & 2Rx & \quad a + b\;x \\ & + R & & \end{array}$$

Therefore, it results

$$R = \frac{1}{2}b \quad \text{und} \quad Q = a - \frac{1}{2}b,$$

such that the general term is

$$y = P + \left(a - \frac{1}{2}b\right)x + \frac{1}{2}bxx,$$

as before. In like manner it is clear, if the law of progression in general is

$$y = X + \alpha\,{}'y + \beta\,{}''y + \gamma\,{}'''y + +\delta\,{}^{\mathrm{IV}}y + \text{etc.}$$

and $X$ is a polynomial function of $x$, as

$$X = a + bx + cxx + dx^3 + \text{etc.},$$

that by continued subtraction one finally gets to a law determining the single terms taking into account only the preceding ones, and so the series will always be recurring, whose general term can be defined by the prescriptions given before. But this term will extend too far and therefore, by finding the values of the terms $'y$, $''y$, $'''y$ etc., must be accommodated to the propounded law; having done this always so many functions $P$, $Q$, $R$ etc. will be determined as letters $a$, $b$, $c$, $d$ etc. were eliminated by subtraction. Therefore, because series of this kind do no longer cause any difficulties, let us consider others, in which $X$ is neither a rational nor a polynomial function of $x$.

## PROBLEM 9

**§55** *To find the general term of the series, in which any arbitrary term becomes equal to the preceding one together with an arbitrary function of the index itself.*

### SOLUTION

Let the term corresponding to the index $x$ be $= y$ and its preceding one

$$'y = y - \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} - \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.}$$

But let the law of progression be

$$y = ' y + X,$$

whence it will be

$$X = \frac{dy}{1 \cdot dx} - \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} - \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.},$$

which equation is resolved by means of the rules I gave some time ago[13]. Of course, by putting $z^n$ for $\frac{d^ny}{dx^n}$ form this expression

$$Z = z - \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} - \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.} = 1 - e^{-z},$$

[13] Euler refers to E188 again.

and all factors of it must be found, the first of which will be $z$; the remaining ones are contained in this general form $zz + 4kk\pi\pi$. But from the factor $z - 0$ this part of the integral will result

$$y = \int X dx + \text{etc.}$$

But from the factor $zz + 4kk\pi\pi$, if it is compared to the formula $zz - 2kz\cos\varphi + kk$, it will be $k = 2k\pi$ and $\cos\varphi = 0$, whence it is $\varphi = 90^\circ$, and hence the letters $\mathfrak{M}$ and $\mathfrak{N}$ because of

$$A = 0, \quad B = 1, \quad C = \frac{-1}{1\cdot 2}, \quad D = \frac{1}{1\cdot 2\cdot 3} \quad \text{etc.}$$

will be determined in such a way

$$\begin{aligned} \mathfrak{M} &= 1 - \frac{4k^2\pi^2}{1\cdot 2} + \frac{16k^4\pi^4}{1\cdot 2\cdot 3\cdot 4} - \frac{64k^6\pi^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.},\\ \mathfrak{N} &= -\frac{2k\pi}{1} + \frac{8k^3\pi^3}{1\cdot 2\cdot 3} - \frac{32k^5\pi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.,} \end{aligned}$$

that it is

$$\mathfrak{M} = \cos 2k\pi \quad \text{and} \quad \mathfrak{N} = -\sin 2k\pi.$$

Having found these values the part of the integral to result from the factor $zz + 4kk\pi\pi$ will be

$$y = 2\left\{\begin{matrix} (\cos 2k\pi \cos 2k\pi x - \sin 2k\pi \sin 2k\pi x)\int X dx \cos 2k\pi x \\ (\cos 2k\pi \sin 2k\pi x + \sin 2k\pi \cos 2k\pi x)\int X dx \sin 2k\pi x \end{matrix}\right\};$$

but it is $\sin 2k\pi = 0$, $\cos 2k\pi = 1$, whence it will be

$$v = 2\cos 2k\pi x \int X dx \cos 2k\pi x + 2\sin 2k\pi x \int X dx \sin 2k\pi x.$$

If now all these values which have to result from the variability of the number $k$ are collected into one sum, the general term in question will turn out to be:

$$
\begin{aligned}
y = \int Xdx + 2\cos 2\pi x \int Xdx \cos 2\pi x + 2\cos 4\pi x \int Xdx \cos 4\pi x& \\
+ 2\cos 6\pi x \int Xdx \cos 6\pi x + \text{etc.}& \\
+ 2\sin 2\pi x \int Xdx \sin 2\pi x + 2\sin 4\pi x \int Xdx \sin 4\pi x& \\
+ 2\sin 6\pi x \int Xdx \sin 6\pi x + \text{etc.}&
\end{aligned}
$$

Q.E.I.

## COROLLARY 1

**§56** Since it is $y = {}'y + X$, it is manifest that $y$ expresses the summatory term of the series, whose general term is $= X$. For, if the sum of all terms from the first to this one $X$, whose index is $= x$, is put $= y$, it will be the sum of all except for the last $= {}'y$ and hence $y = {}'y + X$.

## COROLLARY 2

**§57** Therefore, the expression found for $y$ or the general term of the propounded series at the same time is the summatory term of the series, whose general term is $= X$; and so we obtained a new expression for the sum of a series, whose general term is given; but because of the infinite amount of integrals it will very rarely be of any use.

## SCHOLIUM

**§58** If except for the arbitrary function of the index $x$ not only the closet preceding term but also more of the preceding terms are used for the formation of the following term of the series, in like manner one will get to the resolution of an infinite differential equation, which can be treated by means of the method I propounded[14]. Therefore, not only the series, whose law of formation extends to the second class, by the method explained here can be reduced to a calculation and their general terms can be found, but it also equally extends to the third class and shows the true nature of those series even more clearly.

[14]Euler refers to E188 again.

# PROBLEM 10

**§59** *To find the general term of the series, in which any arbitrary term is equal to the preceding term multiplied by its index.*

## SOLUTION

If the first term is put equal to unity, Wallis's hypergeometric series will result

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Indices: | 1, | 2, | 3, | 4, | 5, | 6, | 7, | 8, | 9 | etc. |
| Terms: | 1, | 1, | 2, | 6, | 24, | 120, | 720, | 5040, | 40320 | etc. |

Put the term corresponding to the index $x = y$ and the one following it $= y'$; it will be

$$y' = yx,$$

whence this equation results

$$yx = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.};$$

but no general rule is known to solve such an equation. But this equation is easily transformed into another one, which can be solved. Of course, put $y = e^v$; it will be $y' = e^{v'}$ and hence it will be $e^{v'} = e^v x$ and having taken logarithms

$$v' = v + \log x,$$

whence one will have

$$\log x = \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4v}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + + \frac{d^4v}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot dx^5} + \text{etc.},$$

which equation is contained in the preceding one by taking $X = \log x$; therefore, the integral will be

$$\begin{aligned} v = \int dx \log x + 2\cos 2\pi x \int dx \log x \cos 2\pi x + 2\cos 4\pi x \int dx \log x \cos 4\pi x + \text{etc.} \\ + 2\sin 2\pi x \int dx \log x \sin 2\pi x + 2\sin 4\pi x \int dx \log x \sin 4\pi x + \text{etc.} \end{aligned}$$

But having found the value of $v$ the general term in question will be $e^v$ while $e$ denotes a number, whose hyperbolic logarithm is $= 1$. Q. E. I.

**§60** The first term of this expression is $\int dx \log x = x\log x - x$, the remaining single terms can be integrated by means of infinite series. For, it is

$$\begin{aligned}\int dx \log x \cos mx =& +\frac{1}{m}\sin mx\left(\log x + \frac{1}{m^2x^2} - \frac{1\cdot 2\cdot 3}{m^4x^4} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{m^6x^6} - \text{etc.}\right)\\ & +\frac{1}{m}\cos mx\left(\frac{1}{mx} - \frac{1\cdot 2}{m^3x^3} + \frac{1\cdot 2\cdot 3\cdot 4}{m^5x^5} - \text{etc.}\right)\\ \int dx \log x \sin mx =& -\frac{1}{m}\cos mx\left(\log x + \frac{1}{m^2x^2} - \frac{1\cdot 2\cdot 3}{m^4x^4} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{m^6x^6} - \text{etc.}\right)\\ & +\frac{1}{m}\sin mx\left(\frac{1}{mx} - \frac{1\cdot 2}{m^3x^3} + \frac{1\cdot 2\cdot 3\cdot 4}{m^5x^5} - \text{etc.}\right)\end{aligned}$$

Hence it is concluded that it will be

$$2\cos mx \int dx \log x \cos mx + 2\sin mx \int dx \log x \sin mx$$

$$= \frac{2}{mmx}\left(1 - \frac{1\cdot 2}{m^2x^2} + \frac{1\cdot 2\cdot 3\cdot 4}{m^4x^4} - \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{m^6x^6} + \text{etc.}\right) + \alpha\cos mx + \mathfrak{A}\sin mx.$$

Now having successively substituted the values $2\pi$, $4\pi$, $6\pi$ etc. for $m$ and having collected all these expressions one will find

$$\begin{aligned}v = C + x\log x - x +& \frac{1}{2\pi^2 x}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}\right) + \alpha\cos 2\pi x + \mathfrak{A}\sin 2\pi x\\ -& \frac{1\cdot 2}{8\pi^4x^3}\left(1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \text{etc.}\right) + \beta\cos 4\pi x + \mathfrak{B}\sin 4\pi x\\ -& \frac{1\cdot 2\cdot 3\cdot 4}{32\pi^6x^5}\left(1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \text{etc.}\right) + \gamma\cos 6\pi x + \mathfrak{C}\sin 6\pi x\end{aligned}$$

etc.

If now for the these series of powers the sums found be me a long time ago[15] are substituted, one will have

[15] See, e.g., E41 and E130 again.

$$v = C + x\log x - x + \alpha\ \cos 2\pi x + \beta\ \cos 4\pi x + \gamma \cos 6\pi x + \text{etc.}$$
$$+ \mathfrak{A} \sin 2\pi x + \mathfrak{B} \sin 4\pi x + \mathfrak{C} \sin 6\pi x + \text{etc.}$$

$$+\frac{1}{1\cdot 2\cdot 3}\cdot\frac{1}{2x} - \frac{1}{3\cdot 4\cdot 5}\cdot\frac{1}{6x^3} + \frac{1}{5\cdot 6\cdot 7}\cdot\frac{1}{6x^5} - \frac{1}{7\cdot 8\cdot 9}\cdot\frac{3}{10x^7} + \frac{1}{9\cdot 10\cdot 11}\cdot\frac{5}{6x^9} - \text{etc.},$$

or if $P$ is a function of even dimensions of $r = \sin \pi x$ and $s = \cos \pi x$, it will be

$$v = P + x\log x - x + \frac{1}{1\cdot 2\cdot 3}\cdot\frac{1}{2x} - \frac{1}{3\cdot 4\cdot 5}\cdot\frac{1}{6x^3} + \frac{1}{5\cdot 6\cdot 7}\cdot\frac{1}{6x^5} - \text{etc.}$$

Since now having put $x = 1$ it is $y = 1$ and $v = 0$, in this case it must be

$$P = 1 - \frac{1}{1\cdot 2\cdot 3\cdot 2} + \frac{1}{3\cdot 4\cdot 5\cdot 6} - \frac{1}{5\cdot 6\cdot 7\cdot 6} + \frac{3}{7\cdot 8\cdot 9\cdot 10} - \text{etc.},$$

whose value I showed elsewhere[16] to be

$$P = \frac{1}{2}\log 2\pi.$$

and it will have this value, as often as $x$ is an arbitrary integer. Hence by going back to numbers, one will find the general term in question

$$y = \frac{x^x}{e^x}\cdot e^{\frac{1}{1\cdot 2\cdot 3}\cdot\frac{1}{2x} - \frac{1}{3\cdot 4\cdot 5}\cdot\frac{1}{6x^3} + \frac{1}{5\cdot 6\cdot 6}\cdot\frac{1}{x^5} - \text{etc.}}\sqrt{2\pi}$$

or

$$y = \frac{x^x}{e^x}\cdot e^{\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}.\text{etc.}}\sqrt{2\pi}.$$

Hence, if $x$ is a very large number, it will approximately be

$$y = \frac{x^x}{e^x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} + \text{etc.}\right)\sqrt{2\pi}$$

and so the magnitude of each term moved away from the beginning very far is easily approximately assigned.

[16]Euler did so in his book on differential calculus "Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum". This is E212 in the Eneström-Index.