Various Consideration on hypergeometric Series*

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§1 Having propounded this infinite product

$$\frac{P}{Q} = \frac{a(a+2b)}{(a+b)(a+b)} \cdot \frac{(a+2b)(a+4b)}{(a+3b)(a+3b)} \cdot \frac{(a+4b)(a+6b)}{(a+5b)(a+5b)} \cdot \frac{(a+6b)(a+8b)}{(a+7b)(a+7b)} \cdot \text{etc.},$$

it is known to be

$$P = \int \frac{x^{a+b-1}\partial x}{\sqrt{1-x^{2b}}} \quad \text{and} \quad Q = \int \frac{x^{a-1}\partial x}{\sqrt{1-x^{2b}}},$$

having extended these integrals form the lower limit $x = 0$ to the upper limit of integration $x = 1$; here, note that the term corresponding to the index i of that product is

$$\frac{(a+(2i-2)b)(a+2ib)}{(a+(2i-1)b)(a+(2i-1)b)}.$$

§2 On the occasion of this infinite product let us now consider the following indefinite product, in which we want the number of factors to be $= n$, and put

$$\Delta : n = a(a+2b)(a+4b)(a+6b) \cdot (a+(2n-2)b),$$

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$$\Delta : (n+1) = \Delta : n \cdot (a+2nb)$$

and in like manner

$$\Delta : (n+2) = \Delta : (n+1) \cdot (a+(2n+2)b)$$

and so forth.

Hence, if i denotes an infinitely large number, it will be

$$\Delta : i = a(a+2b)(a+4b)(a+6b)\cdots(a+(2i-2)b),$$

whence it is similarly concluded that it will be

$$\begin{aligned}
&\Delta : (i+1) = \Delta : i(a+2ib)\\
&\Delta : (i+2) = \Delta : i(a+2ib)(a+(2i+2)b)\\
&\Delta : (i+3) = \Delta : i(a+2ib)(a+(2i+2)b)(a+(2i+4)b)\\
&\qquad\qquad \text{etc.,}
\end{aligned}$$

where the additionally added factors can be considered as equal to each other; therefore, in general one will be able to put

$$\Delta : (i+n) = \Delta : i(a+2ib)^n,$$

where, because $(a+2ib)$ is the next following factor, likewise an arbitrary one of the following could have been taken, whence we will be able to set in even more generality

$$\Delta : (i+n) = (\alpha+2ib)^n \Delta : i,$$

while α denotes an arbitrary finite number, which vanishes with respect to $2ib$, of course.

§3 Now, let us also consider the case, in which it is $n = \frac{1}{2}$, of the infinite product and let us put $\Delta : \frac{1}{2} = k$, which value using interpolations can always assigned approximately. Therefore, hence applying the formulas given above it will be

$$\Delta : \left(1 + \frac{1}{2}\right) = k(a+b)$$
$$\Delta : \left(2 + \frac{1}{2}\right) = k(a+b)(a+3b)$$
$$\Delta : \left(3 + \frac{1}{2}\right) = k(a+b)(a+3b)(a+5b)$$
etc.,

whence, by proceeding to infinity, it will be

$$\Delta : \left(i + \tfrac{1}{2}\right) = k(a+b)(a+3b)(a+5b)\cdots(a+(2i-1)b).$$

§4 Therefore, since we already gave the formula for $\Delta : (i+n)$ above, having put $n = \frac{1}{2}$ we will now also have

$$\Delta : \left(i + \tfrac{1}{2}\right) = \Delta : i\sqrt{\alpha + 2ib};$$

and so we obtained two different expressions for the same formula $\Delta : \left(i + \frac{1}{2}\right)$ and comparing them this equation is derived

$$\Delta : i\sqrt{\alpha + 2ib} = k(a+b)(a+3b)(a+5b)\cdots(a+(2i-1)b)$$

and hence we will be able to conclude the value of this infinite product

$$(a+b)(a+3b)(a+5b)\cdots(a+(2i-1)b) = \frac{\Delta : i\sqrt{\alpha + 2ib}}{k}$$

and so the relation among this product and the one we expressed by $\Delta : i$ above becomes known. But here it is to be carefully noted that the factors of this product are those constituting the denominator of the product mentioned at the beginning. Therefore, we will be able to express as the numerators as the denominator by means of the values just found, namely

$$\Delta : i \quad \text{and} \quad \frac{\Delta : i\sqrt{\alpha + 2ib}}{k}.$$

§5 But the numerator of the propounded product expanded into infinity can be represented this way

$$a(a+2b)^2(a+4b)^2\cdots(a+(2i-2)b)^2(a+2ib),$$

where the first and the last factor are separated, the remaining ones on the other hand are all quadratic. Therefore, because it is

$$(\Delta:i)^2=(a)^2(a+2b)^2(a+4b)^2(a+6b)^2\cdots(a+(2i-2)b)^2,$$

it is evident that the numerator is $\frac{(\Delta:i)^2}{a}(a+2ib)$. But for the denominator it is obvious that it is equal to the square of the other product $(a+b)(a+3b)$ etc.; because its value was found to be

$$\frac{\Delta:i\sqrt{\alpha+2ib}}{k},$$

the denominator will be

$$\frac{(\Delta:i)^2(\alpha+2ib)}{kk};$$

therefore, having substituted these values for the fraction $\frac{P}{Q}$ given above we obtain this expression

$$\frac{P}{Q}=\frac{\frac{(\Delta:i)^2(a+2ib)}{a}}{\frac{(\Delta:i)^2(\alpha+2ib)}{kk}}=\frac{kk(a+2ib)}{a(\alpha+2ib)}=\frac{kk}{a}.$$

Therefore, using this equation immediately the true value of the interpolated formula $k=\Delta:\frac{1}{2}$ can be found, since it will be

$$\Delta:\tfrac{1}{2}=\sqrt{\frac{aP}{Q}}$$

and hence further for the following ones

$$\Delta:(1+\tfrac{1}{2})=(a+b)\sqrt{\frac{aP}{Q}}$$

$$\Delta:(2+\tfrac{1}{2})=(a+b)(a+3b)\sqrt{\frac{aP}{Q}}$$

$$\Delta:(3+\tfrac{1}{2})=(a+b)(a+3b)(a+5b)\sqrt{\frac{aP}{Q}}$$

etc.

and this interpolation is even more remarkable, since without any approximations it immediately yields the true value of these interpolated terms.

§6 If we contemplate this infinite product, in which each two factors are combined, and set

$$a(a+b)(a+2b)(a+3b)\cdots(a+(i-1)b) = \Gamma : i;$$

it will be

$$\Gamma : 2i = a(a+b)(a+2b)(a+3b)\cdots(a+(2i-1)b),$$

which manifestly is the product of the two products above so that it is

$$\Gamma : 2i = \frac{(\Delta : i)^n \sqrt{\alpha + 2ib}}{k};$$

hence, if we wanted to use the form $\Gamma : 2i$, we will be able to assign the values of the two preceding ones from it, since it is

$$\Delta : i = \frac{k \cdot \Gamma : 2i}{\sqrt{\alpha + 2ib}},$$

which itself is the value of this first product

$$a(a+2b)(a+4b)(a+6b) \quad \text{etc.};$$

the value of the other product

$$(a+b)(a+3b)(a+5b) \quad \text{etc.}$$

will be

$$\frac{\sqrt{\Gamma : 2i \sqrt{\alpha + 2ib}}}{k}.$$

§7 Therefore, up to this point we have contemplated three infinite and related products, which, since we will investigate them more accurately, we want to show here plainly again

$$\begin{array}{ll} \text{I.} & a(a+b)(a+2b)(a+3b)\cdots(a+(i-1)b) = \Gamma : i \\ \text{II.} & a(a+2b)(a+4b)(a+6b)\cdots(a+(2i-2)b) = \Delta : i \\ \text{III.} & (a+b)(a+3b)(a+5b)\cdots(a+(2i-1)b) = \Theta : i \end{array}$$

and we already found that it is

$$\Theta : i = \frac{\Delta : i\sqrt{\alpha + 2ib}}{k};$$

but then we expressed so $\Delta : i$ as $\Theta : i$ by the function $\Gamma : 2i$ in the following way

$$\Delta : i = \sqrt{\frac{k \cdot \Gamma : 2i}{\sqrt{\alpha + 2ib}}} \quad \text{and} \quad \Theta : i = \sqrt{\frac{\Gamma : 2i\sqrt{\alpha + 2ib}}{k}},$$

since it is manifest that it is

$$\Gamma : 2i = \Delta : i \cdot \Theta : i;$$

here one has to recall that it is $k = \Delta : \frac{1}{2}$, which, of course, has to be defined from the second form by considering the series

$$a, \quad a(a+2b), \quad a(a+2b)(a+4b), \quad a(a+2b)(a+4b)(a+6b), \quad \text{etc.},$$

whose term corresponding to the index $\frac{1}{2}$ we denoted by the letter k.

§8 Now let us apply the general method to sum progressions of every kind using only their general term to these forms; this method works this way that having propounded an arbitrary series A, B, C, D, E etc., whose term corresponding to the indefinite index x we want to put $= X$, its sum

$$A + B + C + D + \cdots + X,$$

which we want to call $= S$, is

$$S = \int X \partial x + \frac{1}{2}X + \frac{1}{1\cdot 2\cdot 3}\frac{1}{2}\frac{\partial X}{\partial x} - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\frac{1}{6}\frac{\partial^3 X}{\partial x^3} + \frac{1}{1\cdot 2\cdots 7}\frac{1}{6}\frac{\partial^5 X}{\partial x^5} - \text{etc.},$$

where the fractions $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{3}{10}$, $\frac{5}{6}$, etc. are the Bernoulli numbers.

Expansion of the first Form $(a+b)(a+2b)(a+3b)\cdots(a+(i-1)b) = \Gamma : i$

§9 Since here the number of factors is considered to be infinite, in order to be able to apply the summation method to it, let us consider the same form consisting of a finite number of terms $= x$ and let us in the same way set

$$a(a+b)(a+2b)(a+3b)\cdots(a+(x-1)b) = \Gamma : x.$$

But now, in order to obtain a series to be summed instead of this product, let us take logarithms and it will be

$$\log \Gamma : x = \log a + \log (a+b) + \log (a+2b) + \log (a+3b) + \cdots + \log (a+(x-1)b);$$

therefore, having found its sum, it will give the logarithm of the formula $\Gamma : x$ and hence the formula $\Gamma : x$ itself; if afterwards one sets $x = i$ in that formula, one will obtain the formula $\Gamma : i$, whose value we considered in the preceding paragraphs. Therefore, hence after the comparison to the most general series it will be

$$X = \log (a + (x-1)b)$$

and the sum itself

$$S = \log \Gamma : x,$$

or it will be

$$X = \log (a - b + bx),$$

whence one deduces

$$\int X \partial x = \int \partial x \log (a - b + bx).$$

§10 Therefore, because it is

$$\int \partial z \log z = z \log z - z$$

and

$$\int \partial y \log (a+y) = (a+y) \log (a+y) - (a+y),$$

now, by writing bx instead of y, it will be

$$\int b \partial x \log (a+bx) = (a+bx) \log (a+bx) - a - bx$$

and hence

$$\int \partial x \log (a+bx) = \frac{a+bx}{b} \log (a+bx) - \frac{a}{b} - x,$$

whence one concludes that for our case it will be

$$\int X \partial x = \frac{(a-b+bx)}{b} \log (a-b+bx) - \frac{a}{b} + 1 - x,$$

where on the right-hand side the constant term $\frac{a}{b} - 1$ can be omitted, since the expression per se postulates an indefinite constant quantity; this constant must be defined from the nature of the series itself afterwards. Further, it will be

$$\frac{\partial X}{\partial x} = \frac{b}{a - b + bx},$$

but then further

$$\frac{\partial^3 X}{\partial x^3} = \frac{2b^3}{(a-b+bx)^3}, \quad \frac{\partial^5 X}{\partial x^5} = \frac{2 \cdot 3 \cdot 4 b^5}{(a-b+bx)^5} \quad \text{etc.;}$$

having used these values it will be

$$\begin{aligned} \log \Gamma : x = & A + \left(\frac{a}{b} - \frac{1}{2} + x\right) \log (a - b + bx) - x + \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2} \cdot \frac{b}{a - b + bx} \\ & - \frac{1}{3 \cdot 4 \cdot 5} \cdot \frac{1}{6} \cdot \frac{b^3}{(a-b+bx)^3} + \frac{1}{5 \cdot 6 \cdot 7} \cdot \frac{1}{6} \cdot \frac{b^5}{(a-b+bx)^5} \\ & - \frac{1}{7 \cdot 8 \cdot 9} \cdot \frac{3}{10} \cdot \frac{b^7}{(a-b+bx)^7} + \frac{1}{9 \cdot 10 \cdot 11} \cdot \frac{5}{6} \cdot \frac{b^9}{(a-b+bx)^9} - \text{etc.,} \end{aligned}$$

where the letter A denotes the constant to be defined from the nature of the series itself.

§11 But that constant A must be determined from a case, in which the sum of the series is known, which could therefore be done from the case $x = 0$, in which the sum has to be zero, of course; therefore, it would hence be

$$\begin{aligned} -A = & \left(\frac{a}{b} - \frac{1}{2}\right) \log (a - b) + \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2} \cdot \frac{b}{a - b} - \frac{1}{3 \cdot 4 \cdot 5} \cdot \frac{1}{6} \cdot \frac{b^3}{(a-b)^3} \\ & + \frac{1}{5 \cdot 6 \cdot 7} \cdot \frac{1}{6} \cdot \frac{b^5}{(a-b)^5} - \text{etc.} \end{aligned}$$

But since this series hardly converges and in the case $b = a$ all terms would even become infinite, hence completely nothing can be gained. But if we would want to take $x = 1$, this sum would have to yield $\log a$, whence in like manner hardly anything for our undertaking could be concluded, since we would always get to an infinite series, whose sum would have to found at first; for this task maybe those things, I once commented on series involving the Bernoulli numbers[1], could be of some use. But it is not the time to consider these things in more detail here.

[1] Euler refers to his paper "De summis serierum numeros Bernoullianos involventium". This is paper E393 in the Eneström-Index.

§12 For, since in the present undertaking we mainly consider the value $\Gamma : i$, it will be sufficient to put an infinite number for x immediately. Therefore, let $x = i$, while i denotes an infinitely large number, and our equation will obtain this form

$$\log \Gamma : i = A + \left(\frac{a}{b} - \frac{1}{2} + i\right) \log\left(a - b + bi\right) - i,$$

whence the constant A is immediately determined, which we will therefore consider as known. Therefore, hence by going back to numbers, where we understand $\log A$ to be written instead of A, of course, we will get to this expression

$$\Gamma : i = A(a - b + bi)^{\frac{a}{b} - \frac{1}{2} + i} e^{-i}.$$

Here, it will certainly be convenient to represent the power of the exponent i this way

$$\Gamma : i = A(a - b + bi)^{\frac{a}{b} - \frac{1}{2}} (a - b + bi)^{i} e^{-i}.$$

Expansion of the two remaining formulas

§13 The second form differs from the first only in that regard that instead of b one has to write $2b$ here, whence we do not need a new expansion; but instead of the constant A let us write B here, since it is not known, how this letter b enters into the constant A. Therefore, this way we will immediately have

$$\Delta : i = B(a - 2b + 2bi)^{\frac{a}{2b} - \frac{1}{2}} (a - 2b + 2bi)^{i} e^{-1}.$$

In like manner it is evident that from this second form the third results, if only one writes $a + b$ instead of a, whence by introducing the constant C instead of B we will immediately have

$$\Theta : i = C(a - b + 2bi)^{\frac{a}{2b}} (a - b + 2bi)^{i} e^{-i}.$$

Here, note that the letter e is put for the number whose hyperbolic logarithm is $= 1$.

Conclusions following from this

§14 Now let us see, how these new equations look compared to the relations found above; hence, because from these new values it is

$$\Gamma : 2i = A(a - b + 2bi)^{\frac{a}{b} - \frac{1}{2}} (a - b + 2bi)^{2i} e^{-2i},$$

since we found

$$\Gamma : 2i = \Delta : i \cdot \Theta : i,$$

if we here substitute the values just found everywhere, we will at first have this product for this equation

$$\Delta : i \cdot \Theta : i = BC(a-2b+2bi)^{\frac{a}{2b}-\frac{1}{2}}(a-b+2bi)^{\frac{a}{2b}}(a-b+2bi)^{i}(a-b+2bi)^{i}e^{-2i};$$

since this product has to be equal to that value $\Gamma : 2i$, if we divide by the factors both have in common on both sides, this equation will result

$$A(a-b+2bi)^{\frac{a}{2b}-\frac{1}{2}}(a-b+2bi)^{i} = BC(a-2b+2bi)^{\frac{a}{2b}-\frac{1}{2}}(a-2b+2bi)^{i}.$$

§15 Now let us divide this equation by $(a-2b+2bi)^i$ on both sides, and because it is

$$\frac{a-b+2bi}{a-2b+2bi} = 1 + \frac{b}{a-2b+2bi} = 1 + \frac{1}{2i}$$

because of the infinite number i, by means of the ordinary resolution it will be

$$\left(1+\frac{1}{2i}\right)^{i} = e^{\frac{1}{2}},$$

whence our equation is reduced to this form

$$A(a-b+2bi)^{\frac{a}{2b}-\frac{1}{2}}e^{\frac{1}{2}} = BC(a-2b+2bi)^{\frac{a}{2b}-\frac{1}{2}},$$

where also the last factors cancel each other, because it is

$$\left(\frac{a-b+2bi}{a-2b+2bi}\right)^{\frac{a}{2b}-\frac{1}{2}} = \left(1+\frac{1}{2i}\right)^{\frac{a}{2b}-\frac{1}{2}} = 1,$$

so that we get to this simple equality

$$Ae^{\frac{1}{2}} = BC.$$

§16 Further, because we found above that it is

$$\Theta : i = \frac{\Delta : i\sqrt{\alpha+2ib}}{k}$$

or

$$\frac{\Theta : i}{\Delta : i} = \frac{\sqrt{\alpha+2ib}}{k},$$

let us divide the value found for $\Theta : i$ by $\Delta : i$ and we will find

$$\frac{\Theta : i}{\Delta : i} = \frac{C}{B}\sqrt{a-2b+2bi}\left(\frac{a-b+2bi}{a-2b+2bi}\right)^i = \frac{C}{B}\sqrt{e(a-2b+2bi)}.$$

Therefore, it will be

$$\frac{\sqrt{\alpha+2ib}}{k} = \frac{C}{B}\sqrt{e(a-2b+2bi)}$$

or

$$\frac{1}{k} = \frac{C}{B}\sqrt{\frac{e(a-2b+2bi)}{\alpha+2ib}} = \frac{C}{B}\sqrt{e}$$

or it will be

$$B = Ck\sqrt{e}.$$

§17 Therefore, we obtained two relations among those three constant letters A, B, C of such a kind that, if one of them would be known, from it the two remaining ones could be defined. For, because it is

$$A = \frac{BC}{\sqrt{e}} \quad \text{and} \quad B = Ck\sqrt{e},$$

if we consider the constant A to be known, the two remaining ones will be determined the following way. Because it is $B = Ck\sqrt{e}$, this value substituted in the first equation gives $A = CCk$, whence one finds $C = \sqrt{\frac{A}{k}}$ and hence further $B = \sqrt{kAe}$. Nevertheless, it is hence not clear, how this constant A can be determined independently, and hence one will have to go back to that summation of the logarithmic series we denoted by the letter A above; but there one will have to write $\log A$ instead of A. And hence we have only gained that, if the two remaining forms are in like manner expanded by logarithmic series, the constants to be used there, of course $\log B$ and $\log C$, become known at the same time.

§18 It remains that we add a few things on the value of the letter k; we mentioned above already that it has to be found by interpolation. Nevertheless, this letter can also be determined absolutely from the comparison of the formulas $\Delta : i$ and $\Theta : i$ by means of certain quadratures. For, because it is

$$k = \frac{\Delta : i}{\Theta : i}\sqrt{\alpha+2ib}$$

and hence

$$kk = \frac{(\Delta : i)^2(\alpha + 2ib)}{(\Theta : i)^2},$$

if we substitute the infinite products for $\Delta : i$ and $\Theta : i$ and, since both of them consist of i factors, but here in the numerator the one single factor $\alpha + 2ib$ enters the expression additionally, let us express the first factor of the numerator separately; this way we will get to the following determined product:

$$kk = a \cdot \frac{a(a+2b)(a+2b)(a+4b)(a+4b)(a+6b)}{(a+b)(a+b)(a+3b)(a+3b)(a+5b)(a+5b)} \cdot \text{etc.}$$

§19 But in order to find the true value of this infinite product, one has to note, if the letters P and Q denote the following integral formulas

$$P = \int \frac{x^{p-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}} \quad \text{and} \quad Q = \int \frac{x^{q-1}\partial x}{(1-x^n)^{1-\frac{m}{n}}},$$

which integrals are to be understood to be extended from $x = 0$ to $x = 1$, that then by means of an infinite product it will be

$$\frac{P}{Q} = \frac{q(m+p)}{p(m+q)} \cdot \frac{(q+n)(m+p+n)}{(p+n)(m+q+n)} \cdot \frac{(q+2n)(m+p+2n)}{(p+2n)(m+q+2n)} \cdot \text{etc.},$$

which product is easily reduced to our form by taking

$$q = a, \quad p = a+b, \quad m = b, \quad n = 2b,$$

so that for our case it is

$$P = \int \frac{x^{a+b-1}\partial x}{\sqrt{1-x^{2b}}} \quad \text{and} \quad Q = \int \frac{x^{a-1}\partial x}{\sqrt{1-x^{2b}}};$$

but then it will be

$$kk = \frac{aP}{Q}$$

and hence

$$k = \sqrt{\frac{aP}{Q}}$$

and so we found the same value k in another way than above.

§20 But as it is $k = \Delta : \frac{1}{2}$, in like manner we will be able to assign the values $\Gamma : \frac{1}{2}$ and $\Theta : \frac{1}{2}$ for the two remaining forms. For, because the form Γ results from the form Δ, if in this form one writes $\frac{1}{2}b$ instead of b, but the form Θ results from Δ, if one writes $a+b$ for a, having observed these things it will be

$$\Gamma : \frac{1}{2} = \sqrt{a\frac{\int \frac{x^{a+\frac{1}{2}b-1}\partial x}{\sqrt{1-x^b}}}{\int \frac{x^{a-1}\partial x}{\sqrt{1-x^b}}}}$$

and

$$\Theta : \frac{1}{2} = \sqrt{(a+b)\frac{\int \frac{x^{a+2b-1}\partial x}{\sqrt{1-x^{2b}}}}{\int \frac{x^{a+b-1}\partial x}{\sqrt{1-x^{2b}}}}}.$$

But it is easily understood that the value $\Theta : \frac{1}{2}$ can equally be introduced into our calculations as $\Delta : \frac{1}{2} = k$, because it is

$$\Delta : \frac{1}{2} \cdot \Theta : \frac{1}{2} = a.$$

For, having multiplied those integral values by each other this expression results

$$\Delta : \frac{1}{2} \cdot \Theta : \frac{1}{2} = \sqrt{\frac{a(a+b)\int \frac{x^{a+2b-1}\partial x}{\sqrt{1-x^{2b}}}}{\int \frac{x^{a-1}\partial x}{(1-x^{2b})}}};$$

but from a very well known reduction of such integrals it is known to be

$$\int \frac{x^{a+2b-1}\partial x}{\sqrt{1-x^{2b}}} = \frac{a}{a+b}\int \frac{x^{a-1}\partial x}{\sqrt{1-x^{2b}}},$$

for the limits of integration $x = 0$ and $x = 1$, of course, and so it is perspicuous that it will be

$$\Delta : \frac{1}{2} \cdot \Theta : \frac{1}{2} = a.$$

But it not possible to determine by any means, how the value $\Gamma : \frac{1}{2}$ is related to the two remaining ones.