Treatise on the vibration of chords *

Leonhard Euler

1. Even though that, what has been discovered about the vibrational motion of chords first by Taylor and Bernoulli, then by others, seems to exhaust this subject, nevertheless, those findings are restricted in such a way that hardly in any case the true motion of a vibrating chord can ever be determined. For, first they assumed that the strained chord only performs infinitely small vibrations so that the chord in this motion, no matter, whether it has a straight or a curved shape, can always be considered to conserve the same length. Another restriction is that they assume all vibrations to be regular: In each vibration they assumed the whole chord to be extended straight once and at one time, and investigated its curved shape beyond this position, which they found to be an trochoid curve elongated to infinity.

2. The first restriction, i.e. that the excursions of the vibrating chord are set to be infinitely small, even though they always have a finite ratio to the length of the chord, hardly falsifies the conclusions derived from it, since in most cases the excursions are so small that the can be considered as infinitely small without any noticeable error. But on the other hand both Mechanics and Analysis have not been developed so far that the motion can be found in the case of finite vibrations. But concerning the other restriction, i.e. that they assumed all vibrations to be regular, they try to argue in favour of this assumption by saying that, even though initially the vibrations recede from this law of motion, after a short period of time they become so uniform that the chord in each vibration is extended into a straight line once and at the same time, but beyond this position it has the shape of an elongated trochoid.

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3. It has certainly been confirmed sufficiently often that, if one single vibration was conform to this rule, then all subsequent ones also have to follow the same rule. But hence it is understood at the same time, as the state of the following vibrations depends on the preceding ones and can be determined from them, that so vica versa from the state of the following the shape of the preceding ones can be concluded[1]. Thus, if the following vibrations were regular, it can not happen by any means that the preceding ones deviate from this rule; hence it is perspicuous, if the first vibration was irregular, that the following ones can never become perfectly regular. But the first vibration is completely arbitrary, while one can give any shape to the chord before it is released, and thus one can vary the vibrational motion of the same chord indefinitely, depending on which initial motion is given to it.

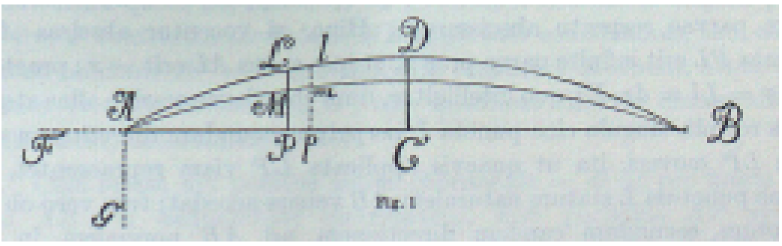
4. Therefore, the following question, containing the whole investigation, arises:

If a chord of given length and given mass is strained by a given force or weight, and it is transferred from a straight position into any arbitrary shape, which nevertheless deviates infinitely less from the straight one, and is then suddenly released, to determine the whole vibrational motion, it will then perform.

This problem, most difficult so in Mechanics as in Analysis, was first successfully tackled by d'Alembert, who communicated his most elegant solution to the Royal Academy. But since in sublime investigations of this kind always much insight is gained from the comparison of several solutions of the same problem, I do not hesitate to also present my solution of this question; even though it deviates quite a lot from d'Alembert's solution, it will nevertheless cause a immense development of this subject, so that I think many remarkable observations will be made while applying more general formulas.

5. Therefore, first I will carefully propound the problem, so that it becomes clear, which auxiliary tools so from Analysis as from Mechanics are necessary for the solution. Therefore, let (Fig. 1)

[1]This is possible since the wave equation can be reversed in time.



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a chord AB fixed at its ends A and B be propounded, and let it be strained by an arbitrary force, as it is common in musical instruments, into the direction AF. Let the chord have the same density everywhere, and call:

its length $AB = a$

its mass or its weight $= M$

and the straining force AF be equal to the weight $= F$.

Then, let this chord from its natural state AB be brought into a curved position $ALlB$ differing infinitely less from the natural straight state AB, so that the length $ALlb$ does only non-visibly exceed the length AB; and let this shape $ALlB$ given to the chord initially be known. Now it is in question, if the chord is suddenly released from this position, which motion it will undergo and vibrations of which kind it will perform.

6. Therefore, after the chord had been released from the position $ALlB$, it will immediately be forced into the natural position AB by the straining force, which acts on each of its points either simultaneously or in different moments of time: Thus, the chord will take on the one and the other shape, and each point of it will be set into vibrational motion, until each motion is eventually ended by resistance. But to understand this motion completely, no matter of what nature it was, it will suffice, to have assigned the state of the chord, i.e. its shape, at each time. For, while hence the change of the shape is defined through instantaneous successions, at the same time from this the velocity of each point of the chord can be defined, and thus the whole motion will become known: And for this reason in this investigation it will not be necessary to consider the velocities of each point of the chord, which simplifies the solution tremendously.

7. Since we assume the length of the chord to stay the same, i.e. that $ALlB = AL$, while it successively takes on all these various shapes, having drawn the ordinates PL, pl orthogonally to the axis AB, the arcs AL, Al will be equal to the corresponding abscissas AP, Ap: And the ordinates PL, pl will infinitely small with respect to the abscissas. Hence, if one calls the abscissa $AP = x$, the ordinate PL will be infinitely small with respect to x, and the arc AL will be $= x$: Furthermore, it will be $Pp = Ll = dx$. From this it is understood, while the chord successively has the one and the other shape, that each point L of it will always be moved into the direction of the ordinate LP, so that each ordinate LP represents the way how the point L of the chord moves towards the natural state AB; but then, because of the fictional motion into the same direction normal to AB, it will run to the opposite straight direction.

8. Having noted these things in advance, let us put that, after the time t has passed, the chord got into the position $AMmB$ from the initial position $ALlB$, so that the point L moved to M. Therefore, having put the abscissa $AP = x$, which at the same time exhibits the length of the arc AM, let the corresponding ordinate in this curve AMB be $PM = y$, and, since this curve AMB depends on the elapsed time $= t$, y will be a certain function of both variables x and t, so that, having put $x = 0$, the value of y exhibits the ordinate of the initial curve ALB. It is indeed perspicuous that, if the nature of this function of x and t, by which the ordinate y is expressed, was known, that from it the form of the chord can be assigned for each time, from the variability of which then further the motion of the whole chord will then easily be concluded.

9. Therefore, since y is a function of x and t, its differential will have a form of this kind $dy = pdx + qdt$, which formula contains the variability of y not only throughout the curve AMB but also also with respect to the flow of time. Of course, if the time t is set to be constant or $dt = 0$, the equation $dy = pdx$ will express the nature of the curve AMB: But, if the abscissa x is assumed to be constant, i.e. $dx = 0$, the equation $dy = qdt$ will define the motion of the point L, as long as the motion of the chord lasts, whence from it it is possible to assign the point M the point L gets to for each passed t counted from the initial point. But p and q will again be functions of x and t, the differentials of which, having put both, x and t, to be variables, are:

$$dp = r \cdot dx + s \cdot dt, \quad \text{and} \quad dq = s \cdot dx + u \cdot dt.$$

For, it is known from the nature of differentials that the element dt in dp and the element dx in dq must have a common coefficient.

10. Since now the motion of the chord must be defined from the forces acting on it, let the accelerating force, by which now the point M of the chord is moved towards the axis AB, be $= P$, and all these forces, by which each element of the chord is moved towards the axis AB, taken at the same time must be equal to the force, by which the chord is actually strained, and which we want to put $AF = F$. In other words, if in each point M of the chord we imagine these forces P opposite to the ordinates ML, then they have to be in an equilibrium with the staining force $AF = F$, and from this true property the accelerating force P, which actually acts on each element Mm of the chord, can be determined.

11. Since the mass or the weight of the whole chord is $= M$ and is uniformly distributed throughout the whole length $AB = a$, the weight of the portion AP or AM will be $= \frac{Mx}{a}$, and thus the infinitesimal weight of the element $Mm = dx$ will be $= \frac{Mdx}{a}$; since it is moved towards ML by the accelerating force $= P$, the moving force of this element will be $= \frac{Mdx}{a} \cdot P$, and the sum of all moving forces throughout the arc AM will be $= \frac{M}{a} \int Pdx$. Now, since the point A is put to be fixed, it is possible to conceive a certain force $AG = G$ acting on it in the direction AG normal to AB, which force shall be so large that the point A stays at rest. Having constituted all this, from the theory of the equilibrium of forces applied to a perfectly flexible string the following equation is deduced:

$$Fy - Gx + \frac{M}{a} \int dx \int Pdx = 0,$$

where Fy and Gx are the moments of the forces F and G with respect to the point M and $\frac{M}{a} \int dx \int Pdx$ is the sum of all moments of elementary forces with respect to the same point M.

12. Now consider the curve AMB, which the curve forms in this moment, the nature of which will be expressed by the formulas given before, if the time t is set to be constant, i.e. $dt = 0$; therefore, it will be $dy = pdx$ and $dp = rdx$. Therefore, differentiate the equation found from the state of equilibrium and, having written pdx for y, the equation, if divided by dx, will give:

$$Fp - G + \frac{M}{a} \int P dx = 0.$$

Now differentiate again, and writing rdx for dp, divide by dx, and so it will result: $Fr + \frac{M}{a} \cdot P = 0$, whence the accelerating force of the point M into the direction MP results, namely $P = -\frac{Far}{M}$. Hence, if the curve AMB would be known, from its nature the accelerating force of each element could be determined.

13. Now consider the motion of the point M only, i.e., how it gets to P acted upon by the accelerating force P, and the abscissa $AP = x$ is to be considered as invariable. Therefore, since, because of $dx = 0$, the momentous increment of the ordinate PM is $dy = q \cdot dt$ and $dq = u \cdot dt$, in the time interval dt the point M will get to P through the space $= -q \cdot dt$, the differential of which, having put the time element dt to be constant, will be $= -dq \cdot dt = -u \cdot du^2 = -ddy$. But from the acceleration resulting from the force P from principles of Mechanics one will obtain this equation: $P = -\frac{2ddy}{dt^2} = -2u$, if the time element dt is explained, as it is common practice, by the spatial element corresponding to the velocity, but the velocity on the other hand is represented by the square root of the altitude corresponding to the velocity. Therefore, since we found $P = -\frac{Far}{M}$ on the one hand, $P = -2u$ on the other hand, it will be $2u = \frac{Far}{M}$ or $u = \frac{Far}{2M}$.

14. By these two conditions, which we reduced to calculus, the whole question is answered; and hence, if after a certain time t for a point M of the chord the abscissa is put $AP = x$ and the ordinate $PM = y$, the latter will by expressed by a function of x and t of such a kind that, having put $dy = pdx + qdt$, the nature of the functions p and q is to be derived form:

$$dp = rdx + sdt \quad \text{and} \quad dq = sdx + \frac{Fa}{2M} rdt.$$

Therefore, the propounded mechanical problem is reduced to this analytical problem that functions r and s of x and t of such a kind are in question that so this differential formula $rdx + sdt$ as this one $sdx + \frac{Fa}{2M} rdt$ becomes integrable. For, having found functions of this kind for r and s one will be able to assign the values

$$p = \int (rdx + sdt) \quad \text{and} \quad q = \int \left(sdx + \frac{Fa}{2M} rdt \right),$$

whence further the value of the ordinate $y = \int (pdx + qdt)$ will be found.

15. But this analytical problem, considered by itself, is very ill-defined; thus, to accommodate it to a certain case, the following things are to be mentioned: First, in the integrations the constants are to be chosen in such a way that for $x = 0$, whatever value is just then attributed to t, always $y = 0$. Further, the same must happen in the case $x = a$, so that again, whatever t is, $y = 0$ results. Thirdly, having observed this, from infinitely many functions r and s satisfying the conditions mentioned before, for each given case those are to be chosen that for $t = 0$ the resulting value of the ordinate y exhibits the arbitrary curve, which was given to the chord initially. Having provided all this, there will not remain any undetermined constant in the solution and the true motion of the chord can be absolutely exhibited.

16. Therefore, to constitute the initial shape of the chord arbitrarily, the solution must extend very far. Thus, while one has to start the investigation from these formulas:

$$dp = rdx + sdt \quad \text{and} \quad dq = sdx + \frac{Fa}{2M}rdt,$$

in general all possible values for r and s must be found, which render both formulas integrable at the same time. To this end, let us multiply these by the constants m and n respectively and add the products, that we find:

$$mdp + ndq = dx(mr + ns) + dt\left(ms + \frac{Fa}{2M}nr\right),$$

and this formula must again be integrable, whatever constant values are attributed to the letters m and n. Therefore, let:

$$m : n = \frac{Fa}{2M}n : m \quad \text{or} \quad mm = \frac{Fa}{2M}nn, \quad \text{that} \quad m = 1 \quad \text{and} \quad n = \pm\sqrt{\frac{2M}{Fa}},$$

and it will be

$$dp \pm dq\sqrt{\frac{2M}{Fa}} = \left(dx \pm dt\sqrt{\frac{Fa}{2M}}\right)\left(r \pm s\sqrt{\frac{2M}{Fa}}\right).$$

17. For the sake of brevity, let $\frac{Fa}{2M} = b$ and one will have:

$$dp \pm dq\sqrt{\frac{1}{b}} = (dx \pm dt\sqrt{b})\left(r \pm s\sqrt{\frac{1}{b}}\right),$$

or

$$dp\sqrt{b} \pm dq = (dx \pm dt\sqrt{b})(r\sqrt{b} \pm s),$$

or even

$$dq \pm dp\sqrt{b} = (dx \pm dt\sqrt{b})(s \pm r\sqrt{b}).$$

Therefore, since this formula $(dx + \pm dt\sqrt{b})(s \pm r\sqrt{b})$ must be integrable, it is necessary that $s \pm r\sqrt{b}$ is a function of $x \pm t\sqrt{b}$. Let us, in order to take into account both signs, put:

$$x + t\sqrt{b} = v \qquad\qquad x = \frac{v+u}{2}$$

it will be

$$x - t\sqrt{b} = u \qquad\qquad t\sqrt{b} = \frac{v-u}{2}$$

and we will have these equations:

$dq + dp\sqrt{b} = dv(s + r\sqrt{b})$ and $dq - dp\sqrt{b} = du(s - r\sqrt{b})$, in which $s + r\sqrt{b}$ must be a function of v and $s - r\sqrt{b}$ must be a function of u, since otherwise the integration would not succeed.

18. Thus, having done each integration, $q + p\sqrt{b}$ will be $=$ a certain function of v and $q - p\sqrt{b}$ will be $=$ a certain function of u. Therefore, for the solution to extend as far as possible, let:

$$V \text{ be a certain function of } V = x + t\sqrt{b}$$
$$U \text{ be a certain function of } u = x - t\sqrt{b}$$

and the mentioned conditions will be satisfied by putting:

$$q + p\sqrt{b} = V \qquad\qquad q = \frac{V+U}{2}$$

whence

$$q - p\sqrt{b} = U \qquad\qquad p = \frac{V-U}{2\sqrt{b}}.$$

Therefore, since $dy = pdx + qdt$, having substituted the values for p, q, dx and dy, it will be:

$$dy = \frac{(dv+du)(V-U)}{4\sqrt{b}} + \frac{(dv-du)(V+U)}{4\sqrt{b}},$$

which after expansion yields:

$$dy = \frac{Vdv - Udu}{2\sqrt{b}} \quad \text{and} \quad y = \frac{1}{2\sqrt{b}}\left(\int Vdv - \int Udu\right).$$

19. But $\int Vdv$ will be a function of $v = x + t\sqrt{b}$ and $\int Udu$ will be a function of $u = x - t\sqrt{b}$, while $b = \frac{Fa}{2M}$. Hence, if one uses the characters f and φ to indicate arbitrary functions of the quantities they are written in front of, then we will have the following general expression for the ordinate y, by which its quantity at each time t and for each abscissa x is exhibited:

$$y = f : (x + t\sqrt{b}) + \varphi : (x - t\sqrt{b}).$$

For, trying to go backwards, in the formula $dy = pdx + qdt$ the values p and q will be as follows:

$$p = f' : (x + t\sqrt{b}) + \varphi' : (x - t\sqrt{b})$$
$$q = \sqrt{b}\left[f' : (x + t\sqrt{b}) - \varphi' : (x - t\sqrt{b})\right]$$

and for the formula $dp = rdx + sdt$ and $dq = sdx + brdt$, as the nature of the question requires it, it will be

$$r = f'' : (x + t\sqrt{b}) + \varphi'' : (x - t\sqrt{b})$$
$$s = \sqrt{b}[f'' : (x + t\sqrt{b}) - \varphi'' : (x - t\sqrt{b})],$$

if we denote the differential of the function $f : z$ by $dzf' : z$ and the differential of the function $f' : z$ by $dzf'' : z$.

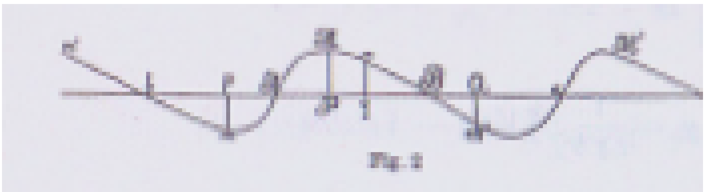
20. Up to this point the characters f and φ in the equation:

$$y = f : (x + t\sqrt{b}) + \varphi : (x - t\sqrt{b})$$

denote arbitrary functions in regard of the composition, but their mutual relation is more determined by the remaining conditions. For, since, having put $x = 0$, it always has to be $y = 0$, it will be $f : +t\sqrt{b} + \varphi : -t\sqrt{b} = 0$, and thus $\varphi : -t\sqrt{b} = -f : t\sqrt{b}$. But on the other hand, since, having put $x = a$, the value of y likewise has to vanish, it will also be $f : (a + t\sqrt{b}) + \varphi : (a - t\sqrt{b}) = 0$, and so the nature of the functions f and φ must be determined in such a way that these conditions are satisfied:

$$\begin{aligned} &\varphi : -t\sqrt{b} = -f : t\sqrt{b} \\ &\varphi : (a - t\sqrt{b}) = -f : (a + t\sqrt{b}). \end{aligned}$$

21. Since in general $f : z$ can be represented by an ordinate of a curve, the abscissa of which is z, let (Fig. 2) AMB be a curve, the ordinates PM of which exhibit functions of the abscissas AP, which ordinates are denoted by the character f :.



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so that $PM = f : AP$. If now one takes $AP = t\sqrt{b}$, it will be $PM = f : t\sqrt{b}$, since to which taken negatively $\varphi : -t\sqrt{b}$ is equal, take $Ap = AP$, that $AP = -t\sqrt{b}$, and, having positioned the curve Amb below the axis similar to the curve AMB, it will be $pm = -f : t\sqrt{b} = \varphi : -t\sqrt{b}$. Therefore, the curve Amb similar to the curve AMB represents the nature of the other function φ. But then continue the curve AMB in like manner beyond B, while $AB = a$, below the axis, that the portion BNa is similar and equal to the curve BnA, and, having taken $BQ = Bq = t\sqrt{b}$, it will be $AQ = a + t\sqrt{b}$ and $QN = -f : (a + t\sqrt{b}) = \varphi : (a - t\sqrt{b})$, and likewise, because of $Aq = a - t\sqrt{b}$, it will be $qn = f : (a - t\sqrt{b})$: Hence it is plain that the curve of this form

AMB, which repeats periodically similar and equal to Amb over and BNa under the axis to both directions, is apt to represent the nature of each of the functions f and φ.

22. Therefore, having described a snake-shaped or mechanical curve of this kind, contained in a either regular or irregular equation, each ordinate PM will yield the functions we need to solve the problem; for, if to a certain abscissa z the values $x + t\sqrt{b}$ and $x - t\sqrt{b}$ are attributed, it will be $y = f : (x + t\sqrt{b}) + f : (x - t\sqrt{b})$, whence for each time in the vibrating chord for each abscissa the corresponding ordinate y can be assigned. But let us put $t = 0$, so that we obtain the initial curve of the chord, and, having taken $AP = x$, the ordinate in the vibrating chord will be $y = f : x + f : x = 2PM$; or, since it is possible to take the halves of the above functions that:

$$y = \frac{1}{2}f : (x + t\sqrt{b}) + \frac{1}{2}f : (x - t\sqrt{b}).$$

The curve AMB will exhibit the shape of the chord which it has at the beginning of the motion.

23. Therefore, vice versa, if a curve or the shape the chord has at the beginning of the motion is given, hence one will be able to determine the shape of the chord after each time t. For, having described the initial shape AMB of the chord over the axis $AB = a$, which is equal to the length of the chord, and this shape is iterated to both directions in inverse position, that $Amb = AMB$ and $BNa = BnA$, and in like manner imagine this iteration to be continued to both directions to infinity. Then, if this curve is applied to express the found functions, after a time $= t$, to the abscissa x in the vibrating chord this ordinate will correspond:

$$y = \frac{1}{2}f : (x + t\sqrt{b}) + \frac{1}{2}f : (x - t\sqrt{b}),$$

whence a simple construction of the curve, which the chord describes at a certain time, can be deduced.

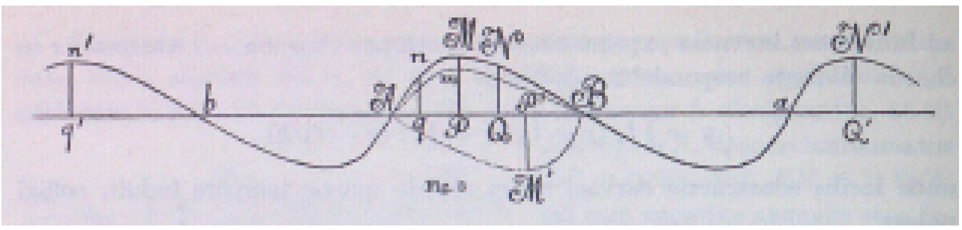
24. But for this formula not to involve heterogeneous quantities, it is to be noted that $t\sqrt{b}$ is represented by a straight line and hence is homogeneous to x. For, let z be the altitude, from which a heavy body falls down in the time t, and, if the expression of time is treated in the way explained above, it will be $t = 2\sqrt{z}$, and hence one can write $2\sqrt{z}$ for t and from the altitude z vice versa the time t passed since the beginning of the fall will be found. Therefore, it will be $t\sqrt{b} = 2\sqrt{bz} = 2\sqrt{\frac{Faz}{2M}} = \sqrt{\frac{2Faz}{M}}$, and will hence be expressed by a straight line. For the sake of brevity, let us put $\sqrt{\frac{2Faz}{M}} = v$, so that the value of v can be assigned for each time, and after the time, in which the heavy body has fallen down from the altitude $= z$, it will be:

$$y = \frac{1}{2}f : (x+v) + \frac{1}{2}f : (x-v).$$

25. Therefore, if the initial shape (Fig. 3) AMN had been given to chord of the length $AB = a$ and hence by repetition of that shape the snake-shaped curve $n'AMBaN'$ had been formed, the shape the chord will have after a time t, in which a heavy body falls down from an altitude $= z$, will be defined this way. From the known altitude z find the value $v = \sqrt{\frac{2Faz}{M}}$, and, having propounded a certain abscissa $AP = x$, take $PQ = Pq = v$ to both directions, and having drawn the ordinates QN and qn to the points Q and q, because of $QN = f : (x+v)$ and $qn = f : (x-v)$, the ordinate corresponding to the abscissa $AP = x$ will be $y = \frac{1}{2}QN + \frac{1}{2}qn$, or take $Pm = \frac{QN+qn}{2}$, and m will be the location of the point M, and if this construction is repeated for each point of the axis AB, the points m will give the present shape of the chord AmB. And this way for each moment in time the shape of the chord it has while vibrating will easily be described.

26. Let us (Fig. 3) find the shape of the chord, after only some time has passed, more precisely, that $v = a$ or $z = \frac{Ma}{2F}$, and it will be

$$y = \frac{1}{2}f : (x+a) + \frac{1}{2}f : (x-a).$$



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But from the nature of the described curve it will be

$$f : (x - a) = -f : (a - x) \quad \text{and} \quad f : (a + x) = -f : (a - x),$$

whence it will be

$$y = -f : (a - x).$$

Thus, it is seen that at this time the whole chord will be bent below the axis and it will have the shape $AM'B$, equal to the given shape AMB, but just inverse position, so that, having taken the abscissa $BP' = AP$, the ordinate will be $P'M' = PM$. And hence vice versa, if again the same time t, whence $v = a$, passes, the whole chord will return to the shape AMB it had initially; this is also clear considering that, after a certain time has passed from the initial motion, whence $v = 2a$, it results:

$$y = \frac{1}{2} f : (x + 2a) + \frac{1}{2} f : (x - 2a).$$

But, having taken $PQ' = Pq' = 2a$, from the nature of the curve it will be $Q'N' = PM = q'n'$, and hence $y = PM$, as at the beginning of the motion.

27. Therefore, whatever shape was initially attributed to the chord, the chord will get into the same shape again during the vibrations, if its excursions are not decreased by resistance: From this it is perspicuous that the opinion mentioned above, according to which the vibrations of the chord, no matter how irregular they were, are believed to become uniform by itself in such a way that the shape goes over into an elongated trochoid, is not true at all. Nevertheless, it is plain, whatever the shape of the vibrating string was, that the vibrations will nevertheless be sufficiently regular; for, since, having put $v = 2a$, the chord returns to the initial state, it is nevertheless to be considered to have performed two full vibrations in between; and hence from the value $v = a$ the time of one vibration will be defined, which will be equal to the time, in which a heavy body falls down from the altitude $\frac{Ma}{2a}$, or if the length of the chord $AB = a$ is expressed in thousandth parts of Rhenanian feet, the time expressed in seconds of one vibration will be $= \frac{y}{125}\sqrt{\frac{Ma}{2F}}$, or the chord in that time will perform so many vibrations as this expression $125\sqrt{\frac{2F}{Ma}}$ will contain units, precisely as if the chord would perform its vibrations according to the law of uniformity, described by Taylor.

28. As the shape AMB given to the chord initially yields its first maximal excursion, so, having done one vibration, the chord will be found in another maximal excursion $AM'B$, which was shown to be equal to the first just inversed. Therefore, let us now see, whether in the mean time between these vibrations the chord is extended perfectly straight that it obtains its natural position or not; since from the time of one single vibration $v = a$ results, let us put $v = \frac{1}{2}a$ for the mean time, and from the general form it will be:

$$y = \frac{1}{2}f : \left(x + \frac{1}{2}a\right) + \frac{1}{2}f : \left(x - \frac{1}{2}a\right),$$

the value of which will vanish, if $f : (\frac{1}{2}a - x) = f : (\frac{1}{2}a + x)$ (Fig. 1), i.e., if the shape ADB, attributed to the chord initially, was of such a nature that to the abscissas $\frac{1}{2}a + x$ and $\frac{1}{2}a - x$ equal ordinates correspond: This happens, if in the length AB the ordinate CD drawn from the middle point C was the diameter of the curve, and the portion DB was similar and equal to the portion DA. Therefore, as often as the curve attributed to the chord initially had this property, the chord will be extended straight in the middle of each vibration; since this can happen in infinitely many ways, it is manifest that not even this condition requires that the chord always has a shape of an elongated trochoid during the vibrations.

29. Although, considering the subject in general, the times of the vibrations do not depend on the shape the chord (Fig. 3) has during the vibrations, but they are only determined by the quantities a, M and F, the first of which a denotes the length of the chord, M the weight of the chord, and F the weight equal to the straining force, nevertheless there are singular cases, in which the times of the vibrations can be contracted into half, third or fourth or any arbitrary part of the whole duration. For, if the total length of the chord was $Aa = a$, and it is initially curved in such a way, that it has the two parts AMB and Ba, which are perfectly similar and equal, then it will go through the vibrations, as if it would have only the length of the half AB, and thus the vibrations will be twice as fast. In like manner, if the initial shape of the chord had three similar and equal parts $bABa$, as they are represented in the figure, then it will perform vibrations, as if its length would be three times smaller, and each vibration will become three times as short. From this it is seen, how even four times or five times as short vibrations are possible.

30. Having given the general solution, let us expand some cases, in which the snake-shaped curve of figure 3 is connected by the law of continuity, so that its nature can be comprehended in an equation. And first it is immediately clear that these curves, since they are intersected by the axis in infinitely many points, will be transcendental. Having put the length of the chord $AB = a$, the abscissa $AP = u$, let $1 : \pi$ be the ratio of the diameter to the circumference of a circle, and it is manifest that the following equation, expressed by sines, will yield such a curve as it is required:

$$PM = \alpha \sin \frac{\pi u}{a} + \beta \sin \frac{2\pi u}{a} + \gamma \sin \frac{3\pi u}{a} + \delta \sin \frac{4\pi u}{a} + \text{etc.}$$

For, if one writes either a or $2a$ or $3a$ or $4a$ for u, the ordinate PM vanishes. And for negative u the ordinate will go over into its negative. Therefore, if the curve AMB was the initial shape of the chord, after the time t, in which a heavy body falls down from the altitude $= z$, having put $v = \sqrt{\frac{2Faz}{M}}$, to the abscissa x in the shape of the chord such an ordinate y will correspond that:

$$\begin{aligned} y = &+\frac{1}{2}\alpha \sin \frac{\pi}{a}(x+v) + \frac{1}{2}\beta \sin \frac{2\pi}{a}(x+v) + \frac{1}{2}\gamma \sin \frac{3\pi}{a}(x+v) + \text{etc.} \\ &+\frac{1}{2}\alpha \sin \frac{\pi}{a}(x-v) + \frac{1}{2}\beta \sin \frac{2\pi}{a}(x-v) + \frac{1}{2}\gamma \sin \frac{3\pi}{a}(x-v) + \text{etc.} \end{aligned}$$

31. But since $\sin(a+b) + \sin(a-b) = 2\sin a \cos b$, this equation will be transformed into this form:

$$y = \alpha \sin \frac{\pi x}{a} \cdot \cos \frac{\pi v}{a} + \beta \sin \frac{2\pi x}{a} \cos \frac{2\pi v}{a} + \gamma \sin \frac{3\pi x}{a} \cos \frac{3\pi v}{a} + \text{etc.}$$

and the initial shape of the chord will be expressed by this equation:

$$y = \alpha \sin \frac{\pi x}{a} + \beta \sin \frac{2\pi x}{a} + \gamma \sin \frac{3\pi x}{a} + \text{etc.},$$

which reduces to the same, if v becomes either $2a$ or $4a$ or $6a$ etc. But if v is either a or $3a$ or $5a$, the shape of the chord will be

$$y = -\alpha \sin \frac{\pi x}{a} + \beta \sin \frac{2\pi x}{a} - \gamma \sin \frac{3\pi x}{a} + \text{etc.}$$

There, it is to be noted, if $\beta = 0$, $\gamma = 0$, $\delta = 0$ etc., that the case arises, which is usually the only believed to occur in the vibrations of chords, namely $y = a \sin \frac{\pi x}{a} \cdot \cos \frac{\pi v}{a}$, in which the curvature of the chord is always a line of sines, or a throchoid elongated to infinity. But if only the term β or γ or δ etc. appears, one has cases, in which the duration of the vibration is rendered twice or trice or four times etc. as small.