# On Products arising from infinitely many factors*

Leonhard Euler

**§1** If in Analysis one gets to quantities of such a kind, which can neither be expressed by rational nor irrational numbers, usually infinite expressions are used to denote the quantities; they are to be considered the more suitable the faster by means of them one gets to cognition and estimation of the quantities expressed by them. Therefore, the use of expressions of this kind is greatest and broadest to represent the values of transcendental quantities, of which kind logarithms, circular arcs and other quantities determined by quadratures of curves are, and by means of them we get so to an exact cognition both of logarithms and circular arcs and even of other transcendental quantities. Yes, infinite expressions of this kind even have an extraordinary use to define irrational quantities and roots of algebraic equations approximately; they, if their use is considered, in most cases are to be preferred to the true expressions by far.

**§2** But of the infinite expressions of this kind several species very different to each other are to be constituted, the first of which contains all infinite series, consisting of infinitely many terms affected with the signs $+$ and $-$; this doctrine is now certainly developed that far that one not only has many methods to express so algebraic as transcendental quantities of this kind by infinite series, but also having propounded an infinite series one has

*Original Title: „De Productis ex infinitis Factoribus ortis", first published in „*Commentarii academiae scientiarum Petropolitanae* 11 (1739), 1750, p. 3-31", reprinted in „*Opera Omnia*: Series 1, Volume 14, pp. 260 - 290", Eneström-Number E122, translated by Alexander Aycock for the project „Euler-Kreis Mainz" methods to investigate a quantity of which kind is indicated by it. For, infinite expressions of each species must be treated in two ways, the one of which consists in the conversion of either algebraic or transcendental equations into infinite expressions, the other on the other hand in the investigation of that quantity, which the propounded infinite expression denotes.

**§3** It is convenient to refer those expression to another other species of infinite expressions, which consist of innumerable factors; although many expressions of this kind are already found and known, nevertheless still neither a way to get to them nor to discover their values was ever explained. But the expressions of this species seem to be equally worthy to be developed as the first consisting of an infinite number of terms, and their treatment will bring a lot of advantages for whole Analysis with them. For, furthermore, since expressions of this kind show the nature of the quantities, which they describe, very plainly and are often more than accommodated to find approximate values, they have a tremendous use to form the logarithms of the quantities themselves, which is very often immensely useful in calculations. So, if any arbitrary quantity $X$ was transformed into an expression of this kind

$$\frac{a}{\alpha} \cdot \frac{b}{\beta} \cdot \frac{c}{\gamma} \cdot \frac{d}{\delta} \cdot \frac{e}{\varepsilon} \cdot \text{etc.},$$

one will immediately have the logarithm of the quantity $X$

$$\ln\left(\frac{a}{\alpha}\right) + \ln\left(\frac{b}{\beta}\right) + \ln\left(\frac{c}{\gamma}\right) + \ln\left(\frac{d}{\delta}\right) + \text{etc}$$

which series converges the more, the closer those factors are inclined to the unity. Therefore, I decided to start the theory of infinite expressions of this kind, insofar as my observations provide some help, in this dissertation, that it is easier for other to expand it further some time.

**§4** At first Wallis published an expression containing infinitely many factors of this kind in his book *Arithmetica infinitorum*, where he showed, if the diameter of the circle is $=1$ that the area of the circle will be

$$\frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11} \text{etc.},$$

which expression he deduced from the interpolation of this series

$$\frac{2}{3} + \frac{2 \cdot 4}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} + \text{etc.},$$

whose intermediate terms he demonstrated to depend on the quadrature of the circle.

Therefore, since these expressions must have their origin in the interpolation of series, it seemed to be appropriate to begin this treatise on products consisting of infinitely many factors with interpolations. For, after in the fifth tome of our COMMENTARII I had given a method to perform the interpolations by means of quadratures of curves, it was known at the same time, transcendental quantities of which kind infinite products arising this way exhibit.

**§5** Therefore, I consider the following progression

$$\overset{1}{(f+g)}+\overset{2}{(f+g)(f+2g)}+\overset{3}{(f+g)(f+2g)(f+3g)}+\overset{4}{(f+g)(f+2g)(f+3g)(f+4g)}+\text{etc}$$

whose arbitrary term, whose index is $n$, is found from the preceding by multiplying this one by $f+ng$; but I showed in the mentioned dissertation that the term, whose index is $n$, of this series is

$$=\frac{g^{n+1}\int dx(-\ln(x))^n}{(f+(n+1)g)\int x^{f:g}dx(1-x)^n}$$

having performed each of both integrations in such a way that the integrals vanish having put $x=0$ and then $x=1$. Therefore, this expression will at the same time indicate, on which quadrature the single intermediate terms depend. For, although, if $n$ is a fractional number, it is not clear that easy, which quadrature the quantity $\int dx(-\ln x)^n$ contains, I nevertheless at the same place showed that having put $\frac{p}{q}$ instead of $n$ the formula $\int dx(-\ln x)^{\frac{p}{q}}$ agrees with

$$\sqrt[q]{1\cdot 2\cdot 3\cdots p\left(\frac{2p}{q}+1\right)\left(\frac{3p}{q}+1\right)\left(\frac{4p}{q}+1\right)\cdots\left(\frac{qp}{q}+1\right)}$$
$$\times\int dx(x^2-x^3)^{\frac{p}{q}}\cdot\int dx(x^3-x^4)^{\frac{p}{q}}\cdot\int dx(x^4-x^5)^{\frac{p}{q}}\cdots\int dx(x^{q-1}-x^q)^{\frac{p}{q}}$$

by means of which reduction the value of $\int dx(-\ln(x))^{\frac{p}{q}}$ can be expressed by means of algebraic curves.

**§6** If now in the assumed series the term, whose index is $=\frac{1}{2}$, is put $z$, from the law of the series the terms, whose indices are $\frac{3}{2},\frac{5}{2},\frac{7}{2}$ etc. will behave as follows:

$$z+z\left(f+\frac{3}{2}g\right)+z\left(f+\frac{3}{2}g\right)\left(f+\frac{5}{2}g\right)+z\left(f+\frac{3}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{7}{2}g\right)+\text{etc.}$$

But since the assumed progression is finally confounded with the geometric progression, these interpolated terms will finally become the arithmetic means between two contiguous terms of the series. Hence, if the single interpolated terms are considered as the arithmetic means from the beginning, the following will arise as approximations to the term, whose index is $z$.

$$\begin{aligned}
I.\quad z &= \sqrt{f+g}\\
II.\quad z &= \sqrt{\frac{(f+g)(f+g)(f+2g)}{1\left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)}}\\
III.\quad z &= \sqrt{\frac{(f+g)(f+g)(f+2g)(f+2g)(f+3g)}{1\left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{5}{2}g\right)}}\\
&\textit{etc.}
\end{aligned}$$

from which law of progression it is understood that the term of the index $\frac{1}{2}$ almost is

$$=(f+g)^{\frac{1}{2}}\sqrt{\frac{(f+g)(f+2g)(f+2g)(f+3g)(f+3g)(f+4g)(f+4g)(f+5g)(f+5g)(f+6g)}{\left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{7}{2}g\right)\left(f+\frac{7}{2}g\right)\left(f+\frac{9}{2}g\right)\left(f+\frac{9}{2}g\right)\left(f+\frac{11}{2}g\right)\left(f+\frac{11}{2}g\right)}}\,\text{etc.}$$

§7 Therefore, now it is not only certain that by this infinite expression the term of the assumed series

$$\overset{1}{(f+g)}+\overset{2}{(f+g)(f+2g)}+\overset{3}{(f+g)(f+2g)(f+3g)}+\text{ etc}$$

whose index is $=\frac{1}{2}$ is exhibited, but also the same found expression is reduced to quadratures of curves. For, having put $n=\frac{1}{2}$ because of $p=1$ and $q=2$ it is

$$\int dx(-\ln(x))^{\frac{1}{2}}=\sqrt{1\cdot 2\int dx\sqrt{x-xx}};$$

this expression integrated in the corrected way gives the square root of the area of the circle, whose diameter is $=1$; or having put the ratio of the diameter to the circumference $1:\pi$ it will be

$$\int dx(-\ln(x))^{\frac{1}{2}}=\frac{\sqrt{\pi}}{2}$$

Therefore, the same term, whose index is $=\frac{1}{2}$, which we put $z$, is found as

$$=\frac{h\sqrt{\pi g}}{(2f+3g)\int x^{f:g}dx\sqrt{1-x}}=\frac{\sqrt{\pi g}}{(2f+3g)\int y^{f+g-1}dy\sqrt{1-y^g}}$$

having treated the integral in the same way as it was prescribed before. But by means of the reduction of integral formulas of this kind it is

$$\int y^{f+g-1}dy\sqrt{1-y^g}=\frac{2fg}{(2f+g)(2f+3g)}\int\frac{y^{f-1}dy}{\sqrt{1-y^g}}=\frac{2f}{2f+3g}\int y^{f-1}dy\sqrt{1-y^g}$$

Having substituted these one finds

$$\frac{(2f+g)(2f+3g)(2f+3g)(2f+5g)(3f+5g)(2f+7g)}{(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+4g)(2f+6g)}\text{ etc.}$$

$$=\frac{2ff(2f+g)}{\pi g}\left(\int y^{f-1}dy\sqrt{1-y^g}\right)^2=\frac{2ffg}{\pi(2f+g)}\left(\frac{y^{f-1}dy}{\sqrt{1-y^g}}\right)^2$$

Therefore, by means of this equation innumerable quadratures can be transformed into infinite products and vice versa the value of infinite products of this kind can be transformed in quadratures of curves.

**§8** To illustrate this equality by examples, let $g=1$ and it will be

$$\int y^{f-1}dy\sqrt{1-y}=\frac{2\cdot 2\cdot 4\cdot 6\cdot 8\cdot 10\cdots(2f-2)}{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdots(2f-1)}\text{etc.}$$

Hence it will be

$$\frac{2ff(2f+1)\cdot 2\cdot 2\cdot 2\cdot 2\cdot 4\cdot 4\cdots(2f-2)(2f-2)}{\pi\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdots(2f+1)(2f+1)}\text{ etc}=\frac{(2f+1)(2f+3)(2f+3)}{(2f+2)(2f+2)(2f+4)}\text{etc.}$$

which expression ordered or reduced to continuity gives

$$\pi=4\cdot\frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11}\text{ etc.,}$$

which is the Wallis formula itself and arises, whatever positive integer number is substituted for $f$. This same expression arises, if one puts $g=2$ and $f=$ an arbitrary odd integer.

**§9** Therefore, because it is

$$\frac{fg}{\pi}\left(\int\frac{y^{f-1}dy}{\sqrt{1-y^g}}\right)^2=\frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+3g)(2f+6g)}\text{ etc.,}$$

in the same way it will be

$$\frac{hk}{\pi}\left(\frac{y^{h-1}dy}{\sqrt{1-y^k}}\right)^2=\frac{(2h+k)(2h+k)(2h+3k)(2h+3k)(2h+5k)(2h+5k)}{2h(2h+2k)(2h+2k)(2h+4k)(2h+4k)(2h+6k)}\text{ etc.}$$

Hence having divided this expression by that one will obtain the following equation free from the circumference of the circle $\pi$

$$\frac{dg(\int y^{f-1}dy:\sqrt{1-y^g})^2}{hk(\int y^{h-1}dy:\sqrt{1-y^k})^2}=\frac{2h(2f+g)^2(2h+2k)^2(2f+3g)^2(2h+4k)^2(2f+5g)^2}{2f(2h+k)^2(2f+2g)^2(2f+3h)^2(2f+4g)^2(2h+5k)^2}\text{ etc.}$$

which having extracted the square root yields this equation

$$\frac{\int y^{f-1}dy:\sqrt{1-y^g}}{\int y^{h-1}dy:\sqrt{1-y^k}}\cdot\sqrt{\frac{g}{k}}=\frac{2h(2f+g)(2f+2k)(2f+3g)(2h+4k)(2f+5g)}{2f(2h+k)(2f+2g)(2h+3k)(2f+4g)(2h+5k)}\text{ etc.}$$

**§10** But this infinite expression does not have a constant value; for, even though it is continued to infinity, it nevertheless has one value, if an even number of factors is taken, another, if an odd number is taken. Therefore, if it is not $k=g$, in which case is does not matter, where the multiplication is interrupted, two factors are to be taken together, having done which one will obtain two equations, depending on whether an even or an odd number of factors is taken. But first having expanded the general expression accurately one will obtain

$$\frac{g\int y^{f-1}dy:\sqrt{1-y^g}}{k\int y^{h-1}dy:\sqrt{1-y^k}}$$

$$=\frac{2h(2f+g)}{2f(2h+k)}\cdot\frac{(2f+2g)(2h+3k)}{(2h+2k)(2f+3g)}\cdot\frac{(2f+4g)(2h+5k)}{(2h+4k)(2f+5g)}\cdot\frac{(2f+6g)(2h+7k)}{(2h+6k)(2f+7g)}\cdot\text{etc.}$$

But by taking the other pairs of terms it will be

$$\frac{f\int y^{f-1}dy:\sqrt{1-y^g}}{h\int y^{h-1}dy:\sqrt{1-y^k}}$$

$$= \frac{(2f+g)(2h+2k)}{(2h+k)(2f+2g)} \cdot \frac{(2f+3g)(2h+4k)}{(2h+3k)(2f+4g)} \cdot \frac{(2f+5g)(2h+6k)}{(2h+5k)(2f+6g)} \cdot \frac{(2f+7g)(2h+8k)}{(2h+7k)(2f+8g)} \cdot \text{etc.}$$

in which expressions the spots, where it is possible to interrupt the operation, are marked by points.

**§11** But let us consider the case with more attention, in which it is $k = g$, in which the infinite expression can certainly be imagined as consisting of simple factors, and it will be

$$\frac{\int y^{f-1}dy : \sqrt{1-y^g}}{\int y^{h-1}dy : \sqrt{1-y^g}} = \frac{2h(2f+g)(2h+2g)(2f+3g)(2h+4g)}{2f(2h+g)(2f+2g)(2h+3g)(2f+4g)} \text{ etc.;}$$

that this expression is less confounded with the preceding because of the same letters, let us put $2f = a$ and $2h = b$ and $y = x^2$ here, having substituted what it will arise

$$\frac{\int x^{a-1}dx : \sqrt{1-x^{2g}}}{\int x^{b-1}dx : \sqrt{1-x^{2g}}} = \frac{b(a+g)(b+2g)(a+3g)(b+4g)(a+5g)}{a(b+g)(a+2g)(b+3g)(a+4g)(b+5g)} \text{ etc.,}$$

which expression compared to the first given in § 9, which having equally put $y = x^2$ goes over into this one

$$\frac{4fg}{\pi}\left(\frac{x^{2f-1}dx}{\sqrt{1-x^{2g}}}\right)^2 = \frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)} \text{ etc.,}$$

will manifest extraordinary properties, whose truth can otherwise hardly be demonstrated.

**§12** For, it is immediately plain, if one puts $a = 3f$, $b = 2f + g$, that that infinite expression is transformed into this one; therefore, also the expressions equal to those and containing the quadratures of curves will become equal in this case, from which the following equality emerges

$$\frac{\int x^{2f-1}dx : \sqrt{1-y^{2g}}}{\int x^{2f+g-1}dx : \sqrt{1-x^{2g}}} = \frac{4fg}{\pi}\left(\int x^{2f-1}dx : \sqrt{1-x^{2g}}\right)^2,$$

if after the integration one puts $x = 1$, of course. Therefore, hence it follows that it will be

$$\pi = 4fg \int \frac{x^{2f-1}dx}{\sqrt{1-x^{2g}}} \cdot \int \frac{x^{2f+g-1}dx}{\sqrt{1-x^{2g}}};$$

or having put $2f = a$ it will be

$$\pi = 2ag \int \frac{x^{a-1}dx}{\sqrt{1-x^{2g}}} \cdot \int \frac{x^{a+g-1}dx}{\sqrt{1-x^{2g}}},$$

which certainly is a most remarkable theorem, since by means of it one can assign the product of two integral formulas, of which often none can be exhibited.

**§13** The truth of this theorem is certainly easily demonstrated in the cases, in which the one integral formula either admits an integration absolutely or depends on the quadrature of the circle. For, let us put $g = 1$ and $a = 1$; of course, it will be

$$\pi = 2\int \frac{dx}{\sqrt{1-x^2}} \cdot \int \frac{xdx}{\sqrt{1-x^2}},$$

for,

$$2\int \frac{dx}{\sqrt{1-x^2}}$$

having put $x = 1$ after the integration gives the quantity $\pi$ itself and

$$\int \frac{xdx}{\sqrt{1-xx}} = 1 - \sqrt{1-xx}$$

for $x = 1$ becomes $= 1$. In similar manner, if it is $a = 2$ while still $g = 1$, it is understood that it will be

$$\pi = 4\int \frac{xdx}{\sqrt{1-xx}} \cdot \int \frac{xxdx}{\sqrt{1-xx}};$$

for, it is

$$\int \frac{xdx}{\sqrt{1-xx}} = 1 \quad \text{und} \quad \int \frac{xxdx}{1-xx} = \frac{\pi}{4};$$

in these cases the truth of the theorem is confirmed from elsewhere.

**§14** But the remaining cases, in which none of both integral formulas can exhibited either actually or by means of the quadrature of the circle, yield as many strange and remarkable theorems. So, having put $g = 2$ and $a = 1$ it will be

$$\pi = 4\int \frac{dx}{\sqrt{1-x^4}} \cdot \int \frac{xxdx}{\sqrt{1-x^4}},$$

where

$$\int \frac{xxdx}{\sqrt{1-x^4}}$$

exhibits the ordinate in the *curva elastica rectangula*[1],

$$\int \frac{dx}{\sqrt{1-x^4}}$$

on the other hand the arc corresponding to the abscissa $x$ of the *elastica*. Therefore, the rectangle of the arc corresponding to the abscissa 1 of the *elastica* and the corresponding ordinate will become equal to the area of the circle, whose diameter is that abscissa 1; this property of the *elastica* can maybe hardly or not even hardly be seen and demonstrated by another method.

**§15** But before I leave this case of the *elastica*, it will be helpful to have expressed the integral by means of an ordinary series at least in the case, in which it is $x = 1$. For, since it is

$$\frac{1}{\sqrt{1-x^4}} = \frac{(1+x^2)^{-\frac{1}{2}}}{\sqrt{1-x^2}}$$

and

$$(1+xx)^{-\frac{1}{2}} = 1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \text{etc.},$$

the single terms will depend on the quadrature of the circle. But having done both integrations for the case $x = 1$ it will be

$$\int \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{2}\left(1 - \frac{1}{4} + \frac{1\cdot 9}{4\cdot 16} - \frac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36} + \text{etc}\right)$$

and

$$\int \frac{x^2}{\sqrt{1-x^4}} = \frac{\pi}{2}\left(\frac{1}{2} - \frac{1\cdot 3}{4\cdot 4} + \frac{1\cdot 9\cdot 5}{4\cdot 16\cdot 6} - \frac{1\cdot 9\cdot 25\cdot 7}{4\cdot 16\cdot 36\cdot 8} + \text{etc}\right)$$

But hence by approximation it arises

$$\int \frac{dx}{\sqrt{1-x^4}} = \frac{5}{6}\cdot\frac{\pi}{2} \quad \text{and} \quad \int \frac{xxdx}{\sqrt{1-x^4}} = \frac{3}{5}.$$

[1] rectangular elastic curve

**§16** If it was $a = 1$, it will be

$$\pi = 2g \int \frac{dx}{\sqrt{1-x^{2g}}} \cdot \int \frac{x^g dx}{\sqrt{1-x^{2g}}}$$

which two integral expressions are of such a nature that, if

$$\int \frac{x^g dx}{\sqrt{1-x^{2g}}}$$

the ordinate corresponding to the curve of a certain abscissa $x$, that

$$\int \frac{dx}{\sqrt{1-x^{2g}}}$$

will be the length of the same curve. Therefore, if in this curve the abscissa is taken as $x = 1$, the product or the rectangle of the ordinate by the length of the curve to the area of the circle, whose diameter is the abscissa $x = 1$, will be in the ratio of 2 to the number $g$; this proposition holds, as long as $g$ was a positive number; for, negative values are excluded automatically.

**§17** If $a - 1$ is assumed smaller than $g$, such that $a$ and $g$ are prime to each other, one will have the following remarkable theorems; for, if it is

$$a + g - 1 > 2g,$$

then the integration could be reduced to a simpler formula.

$$
\begin{array}{l|l}
\pi = 2\int \frac{dx}{\sqrt{1-x^2}} \cdot \int \frac{xdx}{\sqrt{1-x^2}} & \pi = 24\int \frac{x^2dx}{\sqrt{1-x^8}} \cdot \int \frac{x^6dx}{\sqrt{1-x^8}} \\
\pi = 4\int \frac{dx}{\sqrt{1-x^4}} \cdot \int \frac{x^2dx}{\sqrt{1-x^4}} & \pi = 10\int \frac{dx}{\sqrt{1-x^{10}}} \cdot \int \frac{x^5dx}{\sqrt{1-x^{10}}} \\
\pi = 6\int \frac{dx}{\sqrt{1-x^6}} \cdot \int \frac{x^3dx}{\sqrt{1-x^6}} & \pi = 20\int \frac{xdx}{\sqrt{1-x^{10}}} \cdot \int \frac{x^6dx}{\sqrt{1-x^{10}}} \\
\pi = 12\int \frac{xdx}{\sqrt{1-x^6}} \cdot \int \frac{x^4dx}{\sqrt{1-x^6}} & \pi = 30\int \frac{x^2dx}{\sqrt{1-x^{10}}} \cdot \int \frac{x^7dx}{\sqrt{1-x^{10}}} \\
\pi = 8\int \frac{dx}{\sqrt{1-x^8}} \cdot \int \frac{x^4dx}{\sqrt{1-x^8}} & \pi = 40\int \frac{x^3dx}{\sqrt{1-x^{10}}} \cdot \int \frac{x^8dx}{\sqrt{1-x^{10}}} \\
\pi = 12\int \frac{dx}{\sqrt{1-x^{12}}} \cdot \int \frac{x^6dx}{\sqrt{1-x^{12}}} & \pi = 28\int \frac{xdx}{\sqrt{1-x^{14}}} \cdot \int \frac{x^8dx}{\sqrt{1-x^{14}}} \\
\pi = 60\int \frac{x^4dx}{\sqrt{1-x^{12}}} \cdot \int \frac{x^{10}dx}{\sqrt{1-x^{12}}} & \pi = 42\int \frac{x^2dx}{\sqrt{1-x^{14}}} \cdot \int \frac{x^9dx}{\sqrt{1-x^{14}}} \\
\pi = 14\int \frac{dx}{\sqrt{1-x^{14}}} \cdot \int \frac{x^7dx}{\sqrt{1-x^{14}}} & \pi = 56\int \frac{x^3dx}{\sqrt{1-x^{14}}} \cdot \int \frac{x^{10}dx}{\sqrt{1-x^{14}}}
\end{array}
$$

$$\pi = 70\int \frac{x^4dx}{\sqrt{1-x^{14}}} \cdot \int \frac{x^{11}dx}{\sqrt{1-x^{14}}}$$

**§18** Therefore, having found this the reduction of integral formulas to simpler ones is also significantly promoted. For, since these two integral formulas

$$\int \frac{x^m dx}{\sqrt{1-x^{2g}}} \quad \text{and} \quad \int \frac{x^{m+n}}{\sqrt{1-x^{2g}}}$$

could have still only be reduced to each other, if $n$ was a multiple of the exponent $2g$, so this reduction now also succeeds, if $n$ was only a multiple of $g$, in the case $x = 1$ only, of course. But as, if $n$ is a product of the exponent $g$ by an even number, the quotient, which results from the division of the one formula by the other, is easily assigned, so on the contrary, if $n$ is the product of $g$ by an odd number, then the product of these formulas will be assigned very easily.

**§19** Therefore, all these things reduce to this, that, if the integral of this formula was known

$$\int \frac{x^m dx}{\sqrt{1-x^{2g}}}$$

in the case, in which it is $x=1$, that in the same case also the integral of this formula

$$\int \frac{x^{m+n}dx}{\sqrt{1-x^{2g}}},$$

if $n$ is a multiple of $g$, can be exhibited. For, let $A$ is the integral of the formula

$$\int \frac{x^{m}dx}{\sqrt{1-x^{2g}}}$$

in the case, in which it is $x=1$; the integral of the other formula by putting $g$, $2g$, $3g$ etc. successively for $n$ will behave in the following way

$$\begin{aligned}
\int \frac{x^{m}dx}{\sqrt{1-x^{2g}}} &= A \\
\int \frac{x^{m+g}dx}{\sqrt{1-x^{2g}}} &= \frac{\pi}{2(m+1)gA} \\
\int \frac{x^{m+2g}dx}{\sqrt{1-x^{2g}}} &= \frac{(m+1)A}{m+g+1} \\
\int \frac{x^{m+3g}dx}{\sqrt{1-x^{2g}}} &= \frac{(m+g+1)\pi}{2(m+1)(m+2g+1)gA} \\
\int \frac{x^{m+4g}dx}{\sqrt{1-x^{2g}}} &= \frac{(m+1)(m+2g+1)A}{(m+g+1)(m+3g+1)} \\
\int \frac{x^{m+5g}dx}{\sqrt{1-x^{2g}}} &= \frac{(m+g+1)(m+3g+1)\pi}{2(m+1)(m+2g+1)(m+4g+1)gA} \\
&\text{etc}
\end{aligned}$$

**§20** Further, since this general integral formula

$$\int x^{m+ig}dx(1+x^2)^{k-\frac{1}{2}}$$

while $i$ and $k$ denote arbitrary integer numbers can be reduced to this formula

$$\int \frac{x^{m+ig}dx}{\sqrt{1-x^{2g}}},$$

it is understood that the integral of this very far extending formula $\int x^{m+ig}dx(1-x^{2g})^{k-\frac{1}{2}}$ can be assigned from the integral

$$\int \frac{x^{m}dx}{\sqrt{1-x^{2g}}},$$

known at least in the case, in which it is $x = 1$ after the integration. But these cases, in which $i$ is an odd number, except for this integral also require the quadrature of the circle $p$.

**§21** Therefore, as by the term of the index $\frac{1}{2}$ of the series assumed above in § 5 I was led to the nature of these integral formulas, so it will be worth one's while to investigate the other intermediate terms in similar manner. Therefore, let the term be in question, whose index is $\frac{p}{q}$, which shall be put $= z$, from which the following will behave this way:

$$\begin{array}{cccc}
\frac{p}{q} & \frac{p+q}{q} & \frac{p+2q}{q} & \\
z+ & \frac{z(fq+(p+q))}{q} + & \frac{z(fq+(p+q)g)(fq+(p+2q)g)}{q^2} & +\text{etc.}
\end{array}$$

Now, by considering in the same way that this progression finally goes over into the geometric one, the following series of approximations to the term $z$ will arise:

$$I. \quad z = 1(f+g)^{\frac{p}{q}}$$

$$II. \quad \frac{z(fq+(p+q)g}{q} = (f+g)^{\frac{q-p}{q}}(f+g)^{\frac{p}{q}}(f+2g)^{\frac{p}{q}}$$

$$III. \quad z\left(f+\frac{p+q}{q}g\right)\left(f+\frac{p+2q}{q}g\right) = (f+g)^{\frac{p-q}{q}}(f+g)^{\frac{p}{q}}(f+2g)^{\frac{q-p}{q}}(f+2g)^{\frac{p}{q}}(f+3g)^{\frac{p}{q}}$$

therefore, hence the true value of $z$ will be found

$$\frac{(f+g)^{\frac{p}{q}}(f+g)^{\frac{q-p}{q}}(f+2g)^{\frac{p}{q}}(f+2g)^{\frac{q-p}{q}}(f+3g)^{\frac{p}{q}}(f+2g)^{\frac{q-p}{q}}}{1\cdot\left(f+\frac{p+q}{q}g\right)^{\frac{p}{q}}\left(f+\frac{p+q}{q}g\right)^{\frac{q-p}{q}}\left(f+\frac{p+2q}{q}g\right)^{\frac{p}{q}}\left(f+\frac{p+2q}{q}g\right)^{\frac{q-p}{q}}\left(f+\frac{p+3q}{q}g\right)^{\frac{p}{q}}}\text{etc.}$$

Or having made some small changes that the infinitesimal factors become $= 1$ and the expression can be interrupted at any arbitrary place, it will be

$$\frac{z}{\left(f+\frac{p}{q}g\right)^{\frac{p}{q}}} = \frac{(f+g)^{\frac{p}{q}}}{\left(f+\frac{p}{q}g\right)^{\frac{p}{q}}}\cdot\frac{(f+g)^{\frac{q-p}{q}}}{\left(f+\frac{p+q}{q}g\right)^{\frac{q-p}{q}}}\cdot\frac{(f+2g)^{\frac{p}{q}}}{\left(f+\frac{p+q}{q}g\right)^{\frac{p}{q}}}$$

$$\cdot\frac{(f+2g)^{\frac{q-p}{q}}}{\left(f+\frac{p+2q}{q}g\right)^{\frac{q-p}{q}}}\cdot\frac{(f+3g)^{\frac{p}{q}}}{\left(f+\frac{p+2q}{q}g\right)^{\frac{p}{q}}}\cdot\text{etc.},$$

the law of progression, according to which the factors proceed, of which expression is immediately clear.

**§22** But the value of the same intermediate term $z$ can be expressed by means of the general term of this series

$$z = \frac{g^{\frac{p+q}{q}}\int dx(-\ln(x))^{\frac{p}{q}}}{\left(f+\frac{p+q}{q}g\right)\int x^{f:g}dx(1-x)^{\frac{p}{q}}}.$$

Hence, if one puts

$$\int dx(-\ln(x))^{\frac{p}{q}} = \sqrt[q]{1\cdot 2\cdot 3\cdots p\left(\frac{2p}{q}+1\right)\left(\frac{3p}{q}+1\right)\left(\frac{4p}{q}+1\right)\cdots\left(\frac{qp}{q}+1\right)}$$

$$\times\int dx(x-x^2)^{\frac{p}{q}}\cdot\int dx(x^2-x^3)^{\frac{p}{q}}\cdot\int dx(x^3-x^4)^{\frac{p}{q}}\cdots\int dx(x^{q-1}-x^q)^{\frac{p}{q}} = \sqrt[q]{P}$$

and $x = y^g$, in which case it is

$$\int x^{f:g}dx(1-x)^{\frac{p}{q}} = g\int y^{f+g-1}dx(1-y^g)^{\frac{p}{q}},$$

$$= \frac{ggp}{fq+(p+q)g}\int\frac{y^{f+g-1}dy}{(1-y^g)^{\frac{q-p}{q}}} = \frac{pfqq}{q\left(f+\frac{p}{q}g\right)\left(f+\frac{p+q}{q}g\right)}\int\frac{y^{f-1}dy}{(1-y^g)^{\frac{q-p}{q}}}$$

and further it is put

$$\int\frac{y^{f-1}dy}{(1-y^g)^{\frac{q-p}{q}}} = Q,$$

it will be

$$z = \frac{q\left(f+\frac{p}{q}g\right)p^{\frac{1}{q}}}{pfg^{\frac{q-p}{q}}Q}$$

**§23** Now, having substituted the infinite expression for $z$ and having taken the power of the exponent $g$, this equation will arise

$$\frac{q^qP}{p^qf^pg^{q-p}Q^q} = \frac{f^{q-p}}{\left(f+\frac{p}{q}g\right)^{q-p}}\cdot\frac{(f+g)^p}{\left(f+\frac{p}{q}g\right)^p}\cdot\frac{(f+g)^{q-p}}{\left(f+\frac{p+q}{q}\right)^{q-p}}\cdot\frac{(f+2g)^p}{\left(f+\frac{p+q}{q}g\right)^p}\cdot\frac{(f+2g)^{q-p}}{\left(f+\frac{p+2q}{q}g\right)^{q-p}}\cdot\text{etc.}$$

Therefore, if in similar manner it is put

$$\int \frac{y^{h-1}dy}{(1-y^g)^{\frac{q-p}{q}}} = R,$$

it will be

$$\frac{p^q h^p g^{q-p} R^q}{q^q P} = \frac{\left(h+\frac{p}{q}g\right)^{q-p}}{h^{q-p}} \cdot \frac{\left(h+\frac{p}{q}g\right)^{p}}{(h+g)^{p}} \cdot \frac{\left(h+\frac{p+q}{q}g\right)^{q-p}}{(h+g)^{q-p}} \cdot \text{etc.},$$

which two expressions multiplied by each other will give

$$\frac{h^p R^q}{f^p Q^q} = \frac{f^{q-p}(h+\frac{p}{q}g)(f+g)^q(h+\frac{p+q}{q}g)^q(f+2g)^q(h+\frac{p+sq}{q})^q}{h^{q-p}(f+\frac{p}{q}g)(h+g)^q(f+\frac{p+q}{q}g)^q(h+2g)^q(f+\frac{p+sq}{q})^q} \text{ etc.}$$

**§24** Therefore, if both sides are multiplied by $\frac{f^p}{h^p}$ and the root of the power $q$ is taken, one will find

$$\frac{R}{Q} = \frac{f\left(h+\frac{p}{q}g\right)(f+g)\left(h+\frac{p+q}{q}g\right)(f+2g)\left(h+\frac{p+2q}{q}g\right)}{h\left(f+\frac{p}{q}g\right)(h+g)\left(f+\frac{p+q}{q}g\right)(h+2g)\left(f+\frac{p+2q}{q}g\right)} \text{ etc}$$

$$= \frac{\int y^{h-1}dy(1-y^g)^{\frac{p-q}{q}}}{\int y^{f-1}dy(1-y^g)^{\frac{p-q}{q}}},$$

in which integrals, since they were taken in such a way that they vanish having put $y=0$, it must be $y=1$, having done which one will have the value of the propounded infinite expression by means of quadratures. Therefore, by means of this infinite expression the one quadrature can be reduced to the other, if one puts $y=1$, of course.

**§25** But that we are hence led to comparisons of integral formulas of this kind, as from the first case, in which it was $p=1$ and $q=2$, let us put $p=1$ and $q=3$ here and it will be

$$P = \frac{10}{3}\int dx(x-x^2)^{\frac{1}{3}} \cdot \int dx(x^2-x^3)^{\frac{1}{3}}$$

and

$$Q = \int \frac{y^{h-1}dy}{(1-y^g)^{\frac{2}{3}}},$$

Therefore, it will be

$$\frac{27P}{fg^2Q^3} = \frac{ff(f+g)(f+g)(f+g)(f+2g)}{\left(f+\frac{1}{3}g\right)\left(f+\frac{1}{3}g\right)\left(f+\frac{1}{3}g\right)\left(f+\frac{4}{3}g\right)\left(f+\frac{4}{3}g\right)\left(f+\frac{4}{3}g\right)} \text{ etc}$$

and

$$\frac{R}{Q} = \frac{f\left(h+\frac{1}{3}g\right)(f+g)\left(h+\frac{4}{3}g\right)(f+2g)\left(h+\frac{7}{3}g\right)}{h\left(f+\frac{1}{3}g\right)(h+g)\left(f+\frac{4}{3}g\right)(h+2g)\left(f+\frac{7}{3}g\right)} \text{ etc.},$$

which two expressions, since in that one a revolution consists of three factors, but here of two factors, cannot be transformed into each other, whatever is substituted for $h$.

**§26** Therefore, let it be

$$S = \int \frac{y^{k-1}dy}{(1-y^g)^{\frac{2}{3}}};$$

it will be

$$\frac{S}{Q} = \frac{f\left(k+\frac{1}{3}g\right)(f+g)\left(k+\frac{4}{3}g\right)(f+2g)\left(k+\frac{7}{3}g\right)}{k\left(f+\frac{1}{3}g\right)(k+g)\left(f+\frac{4}{3}g\right)(k+2g)\left(f+\frac{7}{3}g\right)} \text{ etc.},$$

which expression combined with the preceding will give

$$\frac{RS}{Q^2} = \frac{ff\left(h+\frac{1}{3}g\right)\left(k+\frac{1}{3}g\right)(f+g)(f+g)\left(h+\frac{4}{3}g\right)}{hk\left(f+\frac{1}{3}g\right)\left(f+\frac{1}{3}g\right)(h+g)(k+g)\left(f+\frac{4}{3}g\right)} \text{ etc.},$$

which expression will be converted into that one equal to $\frac{27P}{fg^2Q^3}$ by putting

$$h = f + \frac{1}{3}g \quad \text{and} \quad k = f + \frac{2}{3}g.$$

Therefore, one will have this equation

$$\frac{27P}{fg^2} = QRS$$

or having substituted the true values it will be

$$90\int dx(x-x^2)^{\frac{1}{3}} \cdot \int dx(x^2-x^3)^{\frac{1}{3}} = fg^2 \int \frac{y^{f-1}dy}{(1-y^g)^{\frac{2}{3}}} \cdot \int \frac{y^{f+\frac{1}{3}g-1}dy}{(1-y^g)^{\frac{2}{3}}} \cdot \int \frac{y^{f+\frac{2}{3}g-1}dy}{(1-y^g)^{\frac{2}{3}}}.$$

**§27** But before we prosecute this any further, it will be convenient to attribute a more beautiful form to the value of $P$ in general. But because having put $x = z^q$ it is

$$\int dx(x^n - x^{n+1})^{\frac{p}{q}} = \frac{npq}{(n+1)((n+1)p+q)} \int \frac{z^{np-1}dz}{(1-z^q)^{\frac{q-p}{q}}},$$

after the substitution it will arise

$$P = 1 \cdot 2 \cdot 3 \cdots p \cdot \frac{p^{q-1}}{q} \int \frac{z^{p-1}dz}{(1-z^p)^{\frac{q-p}{q}}} \cdot \int \frac{z^{3p-1}dz}{(1-z^q)^{\frac{q-p}{q}}} \cdots \int \frac{z^{(q-1)p-1}dz}{(1-z^q)^{\frac{q-p}{q}}}.$$

If the root of the power $q$ is extracted from this expression, the value of $\int dx(-\ln x)^{\frac{p}{q}}$ will arise.

**§28** Now having put $p = 1$ and $q = 3$ it will arise

$$P = \frac{1}{3} \int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \cdot \int \frac{zdz}{(1-z^3)^{\frac{2}{3}}}.$$

But having put $y = z^3$ one will obtain the following equation

$$\int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \cdot \int \frac{zdz}{(1-z^3)^{\frac{2}{3}}} = 3fg^2 \int \frac{z^{3f-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{3f+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{3f+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}.$$

If one now puts $3f = a$, the following remarkable equation will arise

$$\int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \int \frac{zdz}{(1-z^3)^{\frac{2}{3}}} = ag^2 \int \frac{z^{a-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \int \frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}.$$

This one compared to the superior one

$$\int \frac{dz}{\sqrt{1-z^2}} = ag \int \frac{z^{a-1}dz}{\sqrt{1-z^{2g}}} \cdot \int \frac{z^{a+g-1}dz}{\sqrt{1-z^{2g}}}$$

already indicates clearly enough, how the following equations of this kind will behave.

**§29** But before I risk it to conclude anything by induction, I want to actually expand some cases. Therefore, let $p=2$ and $q=3$ and hence it will be found

$$P=\frac{8}{3}\int\frac{zdz}{(1-z^3)^{\frac{1}{3}}}\cdot\int\frac{z^3dz}{(1-z^3)^{\frac{1}{3}}}=\frac{8}{9}\int\frac{dz}{(1-z^3)^{\frac{1}{3}}}\cdot\int\frac{zdz}{(1-z^3)^{\frac{1}{3}}},$$

$$Q=\int\frac{y^{f-1}dy}{(1-y^g)^{\frac{1}{3}}},\quad R=\int\frac{y^{h-1}dy}{(1-y^g)^{\frac{1}{3}}}.$$

But the infinite expression will behave this way:

$$\frac{27P}{8f^2gQ^3}=\frac{f(f+g)(f+g)(f+g)(f+2g)(f+2g)}{\left(f+\frac{2}{3}g\right)\left(f+\frac{2}{3}g\right)\left(f+\frac{2}{3}g\right)\left(f+\frac{5}{3}g\right)\left(f+\frac{5}{3}g\right)\left(f+\frac{5}{3}g\right)}\text{ etc.}$$

and

$$\frac{R}{Q}=\frac{f\left(h+\frac{2}{3}g\right)(f+g)\left(h+\frac{5}{3}g\right)(f+2g)\left(h+\frac{8}{3}g\right)}{h\left(f+\frac{2}{3}g\right)(h+g)\left(f+\frac{5}{3}g\right)(h+2g)\left(f+\frac{8}{3}g\right)}\text{ etc.}$$

Furthermore, let it be

$$S=\int\frac{y^{m-1}dy}{(1-y^g)^{\frac{1}{3}}}\quad\text{and}\quad T=\int\frac{y^{n-1}dy}{(1-y^g)^{\frac{1}{3}}}$$

it will be

$$\frac{T}{S}=\frac{m\left(n+\frac{2}{3}g\right)(m+g)\left(n+\frac{5}{3}g\right)(m+2g)}{n\left(m+\frac{2}{3}g\right)(n+g)\left(m+\frac{5}{3}g\right)(n+2g)}\text{ etc.}$$

which two expression multiplied by each other give

$$\frac{RT}{QS}=\frac{fm\left(h+\frac{2}{3}g\right)\left(n+\frac{2}{3}g\right)(f+g)(m+g)\left(h+\frac{5}{3}g\right)\left(n+\frac{5}{3}g\right)}{hn\left(f+\frac{2}{3}g\right)\left(m+\frac{2}{3}g\right)(h+g)(n+g)\left(f+\frac{5}{3}g\right)\left(m+\frac{5}{3}g\right)}\text{ etc}$$

**§30** But this expression cannot be reduced to that one, to which $\frac{27P}{8f^2gQ^3}$ was found to be equal, if that one is not multiplied by $\frac{f}{f-\frac{1}{3}g}$, such that it is

$$\frac{27P}{8fg\left(f-\frac{1}{3}g\right)Q^3}=\frac{ff(f+g)(f+g)(f+g)(f+2g)}{\left(f-\frac{1}{3}g\right)\left(f+\frac{2}{3}g\right)\left(f+\frac{2}{3}g\right)\left(f+\frac{2}{3}g\right)\left(f+\frac{5}{3}g\right)\left(f+\frac{5}{3}g\right)}\text{ etc.;}$$

for, now the reduction will happen by putting

$$m=f,\quad h=f-\frac{1}{3}g\quad\text{and}\quad n=f+\frac{1}{3}g.$$

Therefore, having substituted these values it will be

$$\frac{27P}{8fg\left(f-\frac{1}{3}g\right)Q^3}=\frac{RT}{QS}.$$

But since it is $S=Q$ and

$$R=\int\frac{y^{f-\frac{1}{3}g-1}dy}{(1-y^g)^{\frac{1}{3}}}=\frac{f+\frac{1}{3}g}{f-\frac{1}{3}g}\int\frac{y^{f+\frac{2}{3}g-1}dy}{(1-y^g)^{\frac{1}{3}}}$$

and

$$T=\int\frac{y^{f+\frac{1}{3}g-1}dy}{(1-y^g)^{\frac{1}{3}}},$$

one will obtain this equation by putting $y=z^3$

$$\int\frac{dz}{(1-z^3)^{\frac{1}{3}}}\cdot\int\frac{zdz}{(1-z^3)^{\frac{1}{3}}}=3fg(3f+g)\int\frac{z^{3f-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{3f+g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{3f+2g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}.$$

And if one puts $3f=a$, it will be

$$\int\frac{dz}{(1-z^3)^{\frac{1}{3}}}\cdot\int\frac{zdz}{(1-z^3)^{\frac{1}{3}}}=ag(a+g)\int\frac{z^{a-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}.$$

**§31** Let us put $p=1$ and $q=4$ and one will have

$$\frac{4^4P}{fg^3Q^4}=\frac{fff\,(f+g)\,(f+g)\,(f+g)}{\left(f+\frac{1}{4}g\right)\left(f+\frac{1}{4}g\right)\left(f+\frac{1}{4}g\right)\left(f+\frac{1}{4}g\right)\left(f+\frac{5}{4}g\right)\left(f+\frac{5}{4}g\right)}\text{ etc.}$$

and

$$\frac{R}{Q}=\frac{f\left(h+\frac{1}{4}g\right)(f+g)\left(h+\frac{5}{4}g\right)(f+2g)}{h\left(f+\frac{1}{4}g\right)(h+g)\left(f+\frac{5}{4}g\right)(h+2g)}\text{ etc.}$$

But let as before

$$S=\int\frac{y^{m-1}dy}{(1-y^g)^{\frac{q-p}{q}}},\quad T=\int\frac{y^{n-1}dy}{(1-y^g)^{\frac{q-p}{q}}};$$

it will be

$$\frac{RST}{Q^3}=\frac{fff\left(h+\frac{1}{4}g\right)\left(m+\frac{1}{4}g\right)\left(n+\frac{1}{4}g\right)(f+g)}{hmn\left(f+\frac{1}{4}g\right)\left(f+\frac{1}{4}g\right)\left(f+\frac{1}{4}g\right)(h+g)}\text{ etc}$$

6 factors of which expression are to be transformed into four of that one, what will happen by putting

$$h = f + \frac{1}{4}\,, \quad m = f + \frac{2}{4}g \quad \text{and} \quad n = f + \frac{3}{4}g,$$

having done which one will have

$$4^4 P = fg^3 QRST.$$

Hence, because it is

$$P = \frac{1}{4}\int \frac{dz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zzdz}{(1-z^4)^{\frac{3}{4}}},$$

if on puts $y = z^4$ and $4f = a$, this equation will arise

$$\int \frac{dz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{3}{4}}} \int \frac{zzdz}{(1-z^4)^{\frac{3}{4}}}$$

$$= ag^3 \int \frac{z^{a-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{3}{4}}},$$

whose connection with the preceding cases, in which it was $p = 1$, $q = 2$ and $p = 1$, $q = 3$, is easily seen.

**§32** From these it will therefore be possible to form all equations of this kind, which will arise, if one puts $p = 1$ and $q =$ an arbitrary positive integer; Of course, it will be

$$I. \int \frac{dz}{\sqrt{1-z^2}}$$

$$= ag \int \frac{z^{a-1}dz}{\sqrt{1-z^{2g}}} \cdot \int \frac{z^{a+g-1}dz}{\sqrt{1-z^{2g}}}$$

$$II. \int \frac{dz}{(1-z^3)^{\frac{2}{3}}} \cdot \int \frac{zdz}{(1-z^3)^{\frac{2}{3}}}$$

$$= ag^2 \int \frac{z^{a-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}$$

$$III. \int \frac{dz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{3}{4}}} \cdot \int \frac{z^2dz}{(1-z^4)^{\frac{3}{4}}}$$

$$= ag^3 \int \frac{z^{a-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{3}{4}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{3}{4}}}$$

$$IV. \int \frac{dz}{(1-z^5)^{\frac{4}{5}}} \cdot \int \frac{zdz}{(1-z^5)^{\frac{4}{5}}} \cdot \int \frac{z^2dz}{(1-z^5)^{\frac{4}{5}}} \cdot \int \frac{z^3dz}{(1-z^5)^{\frac{4}{5}}}$$

$$= ag^4 \int \frac{z^{a-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{5g})^{\frac{4}{5}}} \cdot \int \frac{z^{a+4g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}$$

etc.

**§33** But that we can also calculate the equations, which arise, if $p$ is not $= 1$, let us put $p = 2$ and $q = 4$; having done this and while everything else remains the same as above it will be

$$\frac{4^4 P}{3^4 f^3 g Q^4} = \frac{f\,(f+g)\,(f+g)\,(f+g)}{\left(f+\frac{3}{4}g\right)\left(f+\frac{3}{4}g\right)\left(f+\frac{3}{4}g\right)\left(f+\frac{3}{4}g\right)} \text{ etc.,}$$

where the remaining terms consisting of four factors are formed from these by augmenting the single factors by the quantity $g$. In similar manner, it will on the other hand be

$$\frac{RST}{Q^3} = \frac{fff\left(h+\frac{3}{4}g\right)\left(m+\frac{3}{4}g\right)\left(n+\frac{3}{4}g\right)}{hmn\left(f+\frac{3}{4}g\right)\left(f+\frac{3}{4}g\right)\left(f+\frac{3}{4}g\right)} \text{ etc.,}$$

where six factors each constitute one revolution or period. But to make the comparison it is necessary that both series are contemplated this way

$$\frac{4^4P}{3^4f^2g\left(f-\frac{1}{4}g\right)Q^4}=\frac{ff\,(f+g)\,(f+g)}{\left(f-\frac{1}{4}g\right)\left(f+\frac{3}{4}g\right)\left(f+\frac{3}{4}g\right)\left(f+\frac{3}{4}g\right)}\text{ etc.}$$

$$\frac{hRST}{fQ^3}=\frac{ff\left(h+\frac{3}{4}g\right)\left(m+\frac{3}{4}g\right)\left(n+\frac{3}{4}g\right)(f+g)}{mn\left(f+\frac{3}{4}g\right)\left(f+\frac{3}{4}g\right)\left(f+\frac{3}{4}g\right)(h+g)}\text{ etc.,}$$

of which this one is transformed in that one, such that it is

$$\frac{4^4P}{3^4fgh\left(f-\frac{1}{4}g\right)Q^4}=QRST,$$

if it is

$$h=f+\frac{1}{4}g\,,\quad m=f-\frac{1}{4}g\quad\text{and}\quad n=f+\frac{2}{4}g.$$

**§34** Therefore, because it is

$$P=\frac{3^4}{2}\int\frac{z^2dz}{(1-z^4)^{\frac{1}{4}}}\cdot\int\frac{z^5dz}{(1-z^4)^{\frac{1}{4}}}\int\frac{z^8dz}{(1-z^4)^{\frac{1}{4}}}$$
$$=\frac{3^4}{32}\int\frac{dz}{(1-z^4)^{\frac{1}{4}}}\cdot\int\frac{zdz}{(1-z^4)^{\frac{1}{4}}}\cdot\int\frac{zzdz}{(1-z^4)^{\frac{1}{4}}}$$

and

$$Q=\int\frac{y^{f-1}dy}{(1-y^g)^{\frac{1}{4}}}\,,\quad R=\int\frac{y^{f+\frac{1}{4}g-1}dy}{(1-y^g)^{\frac{1}{4}}}$$
$$S=\int\frac{y^{f-\frac{1}{4}g-1}dy}{(1-y^g)^{\frac{1}{4}}}=\frac{f+\frac{2}{4}g}{f-\frac{1}{4}g}\int\frac{y^{f+\frac{3}{4}g-1}dy}{(1-y^g)^{\frac{1}{4}}}$$

and

$$T=\int\frac{y^{f+\frac{2}{4}g-1}dy}{(1-y^g)^{\frac{1}{4}}},$$

from these having put $y=z^4$ and $4f=a$ the following equation is set up

$$\int\frac{dz}{(1-z^4)^{\frac{1}{4}}}\cdot\int\frac{zdz}{(1-z^4)^{\frac{1}{4}}}\cdot\int\frac{zzdz}{(1-z^4)^{\frac{1}{4}}}$$
$$=ag\frac{(a+g)(a+2g)}{1\cdot 2}\int\frac{z^{a-1}dz}{(1-z^{4g})^{\frac{1}{4}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{1}{4}}}\int\frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{1}{4}}}\cdot\int\frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{1}{4}}}$$

§35 By proceeding this way one will find all following equations, whenever $p$ is not $=1$; and, if $p=2$, it will be found

$$I. \int \frac{dz}{(1-z^3)^{\frac{1}{3}}} \cdot \int \frac{zdz}{(1-z^3)^{\frac{1}{3}}} = ag(a+g) \int \frac{z^{a-1}dz}{(1-z^{3g})^{\frac{1}{3}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{1}{3}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}$$

$$II. \int \frac{dz}{(1-z^4)^{\frac{2}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{2}{4}}} \cdot \int \frac{zzdz}{(1-z^4)^{\frac{2}{4}}}$$
$$= ag^2(a+g) \int \frac{z^{a-1}dz}{(1-z^{4g})^{\frac{2}{4}}} \cdot \int \frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{2}{4}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{2}{4}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{2}{4}}}$$

But in general, whatever $q$ is, if one puts

$$\frac{dz}{(1-z^q)^{\frac{q-2}{q}}} = Xdz \quad \text{and} \quad \frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-2}{q}}} = Ydz,$$

it will be

$$\int Xdz \cdot \int zXdz \cdot \int z^2Xdz \cdots \int z^{q-2}Xdz$$
$$= ag^{q-2}(a+g) \int Ydz \cdot \int z^gYdz \cdot \int z^{2g}Ydz \cdots \int z^{(q-1)g}Ydz.$$

§36 In similar manner, if it is $p=3$ and one puts

$$\frac{dz}{(1-z^q)^{\frac{q-3}{q}}} = Xdz \quad \text{und} \quad \frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-3}{q}}} = Ydz$$

the following general equation will arise

$$\int Xdz \cdot \int zXdz \cdot \int z^2Xdz \cdots \int z^{q-2}Xdz$$
$$= ag^{q-3}\frac{(a+g)(a+2g)}{1\cdot 2} \int Ydz \cdot \int z^gYdz \int z^{2g}Ydz \cdots \int z^{(q-1)g}Ydz.$$

And hence it is possible to collect all these formulas into one very far extending one. For, let $p$ and $q$ be arbitrary positive numbers and put

$$\frac{dz}{(1-z^q)^{\frac{q-p}{q}}} = Xdz \quad \text{and} \quad \frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-p}{q}}} = Ydz,$$

one will have

$$\int Xdz \cdot zXdz \cdot \int z^2Xdz \cdots \int z^{q-2}Xdz$$
$$= ag^{q-p}\frac{(a+g)(a+2g)(a+3g)\cdots(a+(p-1)q)}{1\cdot 2\cdot 3\cdots(p-1)}\int Ydz \cdot \int z^gYdz \cdot \int z^{2g}Ydz \cdots \int z^{(q-1)g}Ydz.$$

**§37** But because it is

$$\int z^{q-1}Xdz = \frac{1}{p},$$

if both sides are multiplied by this factors, the following elegant equation will arise

$$\frac{a(a+g)(a+2g)(a+3g)\cdots(a+(p-1)g)}{1\cdot 2\cdot 3\cdot 4\cdots p}g^{q-p}$$
$$= \frac{\int Xdz}{\int Ydz}\cdot\frac{\int zXdz}{\int z^gYdz}\cdot\frac{\int z^2Xdz}{\int z^{2g}Ydz}\cdot\frac{\int z^3Xdz}{\int z^{3g}Ydz}\cdots\frac{\int z^{q-1}Xdz}{\int z^{(q-1)g}Ydz},$$

which expression contains all the ones found until this point and because of the extraordinary structure is remarkable.

**§38** Now I will proceed to another method, by means of which it is possible to get to expressions of this kind consisting of innumerable factors, which method is more accommodated to analysis. For, I observed that from the reduction of integral formulas to others one can obtain expressions of this kind. For, let this integral formula be propounded

$$\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}},$$

which is easily transformed in this expression

$$\frac{x^m(1-x^{nq})^{\frac{p+q}{q}}}{m} + \frac{m+(p+q)n}{m}\int x^{m+nq-1}dx(1-x^{nq})^{\frac{p}{q}}.$$

Therefore, if $m$ and $\frac{p+q}{q}$ are positive numbers and the integrals are taken in such a way, that they vanish for $x=0$, and then one puts $x=1$, it will be

$$\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{m+(p+q)n}{m}\int x^{n+nq-1}dx(1-x^{nq})^{\frac{p}{q}}.$$

**§39** Further, since it similar manner it is

$$\int x^{m+nq-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{m+(p+2q)n}{m+nq}\int x^{n+2nq-1}dx(1-x^{nq})^{\frac{p}{q}},$$

it will also be

$$\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{(m+(p+q)n)(m+(p+2q)n)}{m(m+nq)}\int x^{m+2nq-1}dx(1-x^{nq})^{\frac{p}{q}}.$$

Therefore, having continued this reduction to infinity it will arise

$$\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}}$$
$$= \frac{(m+(p+q)n)(m+(p+2q)n)(m+(p+3q)n)\cdots(m+(p+\infty q)n)}{m(m+nq)(m+2nq)\cdots(m+\infty nq)}\int x^{m+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}}.$$

and in similar manner it is

$$\int x^{\mu-1}dx(1-x^{nq})^{\frac{p}{q}}$$
$$= \frac{(\mu+(p+q)n)(\mu+(p+2q)n)(\mu+(p+3q)n)\cdots(\mu+(p+\infty q)n)}{\mu(\mu+nq)(\mu+2nq)\cdots(\mu+\infty nq)}\int x^{\mu+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}},$$

as long as $m$ and $\mu$ and $nq$ and $\frac{p+q}{q}$ are positive integer numbers or greater than zero.

**§40** But since, if $m$ is infinite, it is

$$\int x^{m}dx(1-x^{nq})^{\frac{p}{q}} = \int x^{m+\alpha}dx(1-x^{nq})^{\frac{p}{q}},$$

whatever finite number is assumed for $\alpha$, as it was concluded in paragraph 38, it will also be

$$\int x^{m+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}} = \int x^{\mu+\infty nq-1}dx(1-x^{nq})^{\frac{p}{q}}.$$

Therefore, if the one of the preceding expressions is divided be the other one, this equation will arise

$$\frac{\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}}}{\int x^{\mu-1}dx(1-x^{nq})^{\frac{p}{q}}}$$
$$= \frac{\mu(m+(p+q)n)(\mu+nq)(m+(p+2q)n)(\mu+2nq)(m+(p+3q)n)(\mu+3nq)}{m(\mu+(p+q)n)(m+nq)(\mu+(p+2q)n)(m+2nq)(\mu+(p+3q)n)(m+3nq)} \text{ etc. to infinity,}$$

by means of which expressions innumerable products consisting of infinitely many factors, whose values can be assigned by means of quadratures of curves.

**§41** If the one integral formula admits an integration, then one will have a nice infinite expression for the other integral formula. For, let is be $\mu = nq$; it will be

$$\int x^{\mu-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{1}{(p+q)q},$$

having substituted this value it will arise

$$\int x^{m-1}dx(1-x^{nq})^{\frac{p}{q}} = \frac{1}{(p+q)n} \cdot \frac{nq(m+(p+q)n)2nq(m+(p+2q)n)3nq}{m(p+2q)n(m+nq)(p+3q)n(m+3nq)} \text{ etc.},$$

by means of which for innumerable integrals expressions by means infinite products can be found; at least in the case, in which $x=1$, which is mainly desired in most cases, of course.

**§42** Put $n$ instead of $nq$ and it will arise

$$\int x^{m-1}dx(1-x^{n})^{\frac{p}{q}} = \frac{q}{(p+q)n} \cdot \frac{n(mq+(p+q)n)2n(mq+(p+2q)n)3n(mq+(p+3q)n)}{m(p+2q)n(m+n)(p+3q)n(m+2n)(p+4q)n} \text{ etc.},$$

which resolved into two factors becomes simpler and it is

$$\int x^{m-1}dx(1-x^{n})^{\frac{p}{q}}$$

$$= \frac{q}{(p+q)n} \cdot \frac{1(mq+(p+q)n)}{m(p+2q)} \cdot \frac{2(mq+(p+2q)n)}{(m+n)(p+3q)} \cdot \frac{3(mq+(p+3q)n)}{(m+2n)(p+4q)} \cdot \text{etc.},$$

whence the following more notable examples are deduced:

$$\int \frac{dx}{\sqrt{1-xx}} = 1 \cdot \frac{1\cdot 4}{1\cdot 3} \cdot \frac{2\cdot 8}{3\cdot 5} \cdot \frac{3\cdot 12}{5\cdot 7} \cdot \text{ etc} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}\text{ etc.,}$$

$$\int \frac{xdx}{\sqrt{1-xx}} = 1 \cdot \frac{1\cdot 6}{2\cdot 3} \cdot \frac{2\cdot 10}{4\cdot 5} \cdot \frac{3\cdot 14}{6\cdot 7} \cdot \text{ etc.} = 1,$$

$$\int \frac{x^2dx}{\sqrt{1-xx}} = 1 \cdot \frac{1\cdot 8}{3\cdot 3} \cdot \frac{2\cdot 12}{5\cdot 5} \cdot \frac{3\cdot 16}{7\cdot 7} \cdot \text{ etc} = \frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7}\text{ etc.,}$$

$$\int \frac{dx}{\sqrt{1-x^3}} = \frac{2}{3} \cdot \frac{1\cdot 5\cdot 2\cdot 11\cdot 3\cdot 17\cdot 4\cdot 23\cdot 5\cdot 29}{1\cdot 3\cdot 4\cdot 5\cdot 7\cdot 7\cdot 10\cdot 9\cdot 13\cdot 11}\text{ etc.,}$$

$$\int \frac{xdx}{\sqrt{1-x^3}} = \frac{2}{3} \cdot \frac{1\cdot 7\cdot 2\cdot 13\cdot 3\cdot 19\cdot 4\cdot 25\cdot 5\cdot 31}{2\cdot 3\cdot 5\cdot 5\cdot 8\cdot 7\cdot 11\cdot 9\cdot 14\cdot 11}\text{ etc}$$

$$\int \frac{dx}{\sqrt{1-x^4}} = \frac{1}{2} \cdot \frac{1\cdot 6\cdot 2\cdot 14\cdot 3\cdot 22\cdot 4\cdot 30}{1\cdot 3\cdot 5\cdot 5\cdot 9\cdot 7\cdot 13\cdot 9}\text{ etc} = \frac{1}{2} \cdot \frac{2\cdot 3\cdot 4\cdot 7\cdot 6\cdot 11\cdot 8\cdot 15}{1\cdot 3\cdot 5\cdot 5\cdot 9\cdot 7\cdot 13\cdot 9}\text{ etc.,}$$

$$\int \frac{xxdx}{\sqrt{1-x^4}} = \frac{1}{2} \cdot \frac{1\cdot 10\cdot 2\cdot 18\cdot 3\cdot 26\cdot 4\cdot 34}{3\cdot 3\cdot 7\cdot 5\cdot 11\cdot 7\cdot 15\cdot 9}\text{ etc.,}$$

$$\int \frac{dx}{\sqrt[3]{1-x^3}} = \frac{1}{2} \cdot \frac{3\cdot 3\cdot 6\cdot 6\cdot 9\cdot 9\cdot 12\cdot 12}{1\cdot 5\cdot 4\cdot 8\cdot 7\cdot 11\cdot 10\cdot 14}\text{ etc.,}$$

$$\int \frac{dx}{\sqrt[4]{1-x^4}} = \frac{1}{3} \cdot \frac{4\cdot 4\cdot 8\cdot 8\cdot 12\cdot 12\cdot 16\cdot 16}{1\cdot 7\cdot 5\cdot 11\cdot 9\cdot 15\cdot 13\cdot 19}\text{ etc.}$$

Furthermore, these expressions deserve it to be noted

$$\int x^{m-1}dx(1-x^n)^{-\frac{m}{n}} = \frac{1}{n-m} \cdot \frac{n\cdot n\cdot 2n\cdot 2n\cdot 3n\cdot 3n}{m(2n-m)(m+n)(2n-m)(m+2n)(4n-m)}\text{ etc.,}$$

$$\int x^{m-1}dx(1-x^n)^{\frac{m-n}{n}}$$

$$= \frac{1}{m} \cdot \frac{n\cdot 2m\cdot 2n(2m+n)3n(2m+2n)4n(2m+3n)}{m(m+n)(m+n)(m+2n)(m+2n)(m+3n)(m+3n)(m+4n)}\text{ etc.}$$

**§43** But because in the same way it is

$$\int x^{\mu-1}dx(1-x^\vartheta)^{\frac{r}{s}} = \frac{s}{(r+s)\vartheta} \cdot \frac{1(\mu s+(r+s)\vartheta)2(\mu s+(r+2s)\vartheta)3(\mu s+(r+3s)\vartheta)}{\mu(r+2s)(\mu+\vartheta)(r+3s)(\mu+2\vartheta)(r+4s)}\text{ etc.,}$$

by dividing the first expression by this one it will be

$$\frac{\int x^{m-1}dx(1-x^n)^{\frac{p}{q}}}{\int x^{\mu-1}dx(1-x^{\vartheta})^{\frac{r}{s}}}$$
$$= \frac{(r+s)q\vartheta}{(p+q)sn}\cdot\frac{\mu(r+2s)(mq+(p+q)n)}{m(p+2q)(\mu s+(r+s)\vartheta)}\cdot\frac{(\mu+\vartheta)(r+3s)(mq+(p+2q)n)}{(m+n)(p+3q)(\mu s+(r+2s)\vartheta)}\cdot \text{ etc.}$$

Therefore, as often as this infinite expression has a finite value, so often the summation of the one integral formula can be reduced to the other one. But cases of this kind exist, whenever the factors of the numerator cancel the factors of the denominator, such that after the cancellation a finite number of factors remains. For, in this expression completely all reductions of integral formulas to other ones are contained.

**§44** But that more expressions of this kind can be compared to each other, it is advisable to assume them this way

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{f-1}dx(1-x^g)^h} = \frac{(h+1)g}{(c+1)b}\cdot\frac{f(h+2)(a+(c+1)b)}{a(c+2)(f+(h+1)g)}\cdot\frac{(f+g)(h+3)(a+(c+2)b)}{(a+b)(c+3)(f+(h+2)g)}\cdot \text{ etc.,}$$

In similar manner, it will be

$$\frac{\int x^{\alpha-1}dx(1-x^{\beta})^{\gamma}}{\int x^{\xi-1}dx(1-x^{\eta})^{\Theta}} = \frac{(\Theta+1)\eta}{(\gamma+1)\beta}\cdot\frac{\xi(\Theta+2)(\alpha+(\gamma+1)\beta)}{\alpha(\gamma+2)(\xi+(\Theta+1)\eta)}\cdot\frac{(\xi+\eta)(\Theta+3)(\alpha+(\gamma+2)\beta)}{(\alpha+\beta)(\gamma+3)(\xi+(\Theta+2)\eta)}\cdot \text{ etc.,}$$

which expressions, even though they do not differ in principle, nevertheless, since they have a different form, can be compared to each other.

**§45** But that we now from these expressions find the same theorems, which found above, let it be

$$\theta=\gamma=h=c, \qquad \eta=\beta=g=b;$$

it will be

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{f-1}dx(1-x^b)^c} = \frac{f(a+c+1)b)(f+b)(a+(c+2)b)(f+2b)(a+(c+3b))}{a(f+c+1)b)(a+b)(f+(c+2)b)(a+2b)(f+(c+3b))} \text{ etc.}$$

and the other form

$$\frac{\int x^{\alpha-1}dx(1-x^b)^c}{\int x^{\zeta-1}dx(1-x^b)^c} = \frac{\zeta(\alpha+c+1)b)(\zeta+b)(\alpha+(c+2)b)(\zeta+2b)(\alpha+(c+3b))}{\alpha(\zeta+c+1)b)(\alpha+b)(\zeta+(c+2)b)(\alpha+2b)(\zeta+(c+3b))} \text{ etc.}$$

If the product of these expressions is put $= \frac{f}{a}$, it must be

$$\frac{(a+(c+1)b)(f+b)\xi(a+(c+1)b)}{(f+(c+1)b)(a+b)\alpha(\xi+(c+1)b)} = 1;$$

for, if this was the case, the product of the whole infinite expressions will become $= \frac{f}{a}$. But this will be obtained by putting

$$\alpha = a+(c+1)b\ , \quad \xi = f+(c+1)b$$

and it will be

$$c = -\frac{1}{2},$$

such that it is

$$\alpha = a+\frac{1}{2}b\ , \quad \xi = f+\frac{1}{2}b,$$

and it will hence be

$$\int \frac{x^{a-1}dx}{\sqrt{1-x^b}} \cdot \int \frac{x^{a+\frac{1}{2}b-1}dx}{\sqrt{1-x^b}} = \frac{f}{a}\int \frac{x^{f-1}dx}{\sqrt{1-x^b}} \cdot \int \frac{x^{a\frac{1}{2}b-1}dx}{\sqrt{1-x^b}};$$

or, if one puts $x = z^2$, it will be

$$\int \frac{z^{a-1}dz}{\sqrt{1-z^{2b}}} \cdot \int \frac{z^{a+b-1}dz}{\sqrt{1-z^{2b}}} = \frac{f}{a}\int \frac{z^{f-1}dz}{\sqrt{1-z^{2b}}} \cdot \int \frac{z^{f+b-1}dz}{\sqrt{1-z^{2b}}}$$

having put $a$ and $f$ instead of $2a$ and $2f$. But this equation is nothing else but the theorem found above in § 12; for, having put $f = b$ it is

$$\int \frac{z^{2b-1}dz}{\sqrt{1-z^{2b}}} = \frac{1}{b} \quad \text{and} \quad \int \frac{z^{b-1}dz}{\sqrt{1-z^{2b}}} = \frac{\pi}{2b},$$

whence it will be

$$\pi = 2ab\int \frac{z^{a-1}dz}{\sqrt{1-z^{2b}}} \cdot \int \frac{z^{a+b-1}dz}{\sqrt{1-z^{2b}}}.$$

**§46** In similar manner other theorems of this kind can be found; for, let it be

$$g = b\,, \quad h = c\,, \quad \eta = \beta = b \quad \text{and} \quad \Theta = \gamma$$

and let the case be in question, in which the product of the two expressions becomes $= 1$. But this will be obtained, if it is

$$\frac{f(a+(c+1)b)\xi(\alpha+(\gamma+1)b)}{a(f+(c+1)b)\alpha(\xi+(\gamma+1)b)} = 1,$$

what will happen by taking

$$\alpha = a+(c+1)b\,, \quad f = a+(\gamma+1)b\,, \quad \xi = a.$$

Therefore, having substituted these values the following rather elegant theorem will arise.

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{a-1}dx(1-x^b)^\gamma} \cdot \frac{\int x^{a+(c+1)b-1}dx(1-x^b)^\gamma}{\int x^{a+(\gamma+1)b-1}dx(1-x^b)^c} = 1;$$

or, if one puts

$$c+1 = m \quad \text{and} \quad \gamma+1 = n,$$

one will have

$$\int \frac{x^{a-1}dx}{(1-x^b)^{1-m}} \cdot \int \frac{x^{a+mb-1}dx}{(1-x^b)^{1-n}} = \int \frac{x^{a-1}dx}{(1-x^b)^{1-n}} \cdot \int \frac{x^{a+nb-1}dx}{(1-x^b)^{1-m}}.$$

**§47** Additionally, in another way one can find a nice theorem by putting $\gamma = h$ and $\theta = c$ while it remains $\eta = \beta = g = h$ and by causing that the product of the integral expressions becomes $= \frac{f}{a}$; that this happens, it must necessarily be

$$\frac{(f+(c+1)b)(f+b)\xi(\alpha+(h+1)b)}{(f+(h+1)b)(a+b)\alpha(\xi+(c+1)b)} = 1.$$

But this will be caused by taking

$$\alpha = a+(c+1)b\,, \quad \xi = f+(h+1)b,$$

whence one will find

$$c+h+1 = 0 \quad \text{oder} \quad h = -1-c;$$

hence take

$$c = -\frac{1}{2} + n \quad \text{und} \quad h = -\frac{1}{2} - n,$$

and the following theorem will arise

$$\frac{f}{a} = \frac{\int x^{a-1}dx(1-x^b)^{-\frac{1}{2}+n} \cdot \int x^{a+\left(\frac{1}{2}+n\right)b-1}dx(1-x^b)^{-\frac{1}{2}-n}}{\int x^{f-1}dx(1-x^b)^{-\frac{1}{2}-n} \cdot \int x^{f+\left(\frac{1}{2}-n\right)b-1}dx(1-x^b)^{-\frac{1}{2}+n}}$$

**§48** Now let all exponents $c$, $h$, $\gamma$ and $\theta$ be different, but $g = \beta = \eta = b$, and let the cases be in question, in which the product of both expressions becomes $= \frac{(h+1)(\theta+1)}{(c+1)(\gamma+1)}$. But this will happen, if it is

$$\frac{f(bh+2b)(a+(c+1)b)\zeta(b\Theta+2b)(\alpha+(\gamma+1)b)}{a(bc+2b)(f+(h+1)b)\alpha(b\gamma+2b)(\zeta+(\Theta+1)b)} = 1,$$

which factors I expressed in such a way, that the single one grow by the quantity $b$ in the following terms. Now put

$$\zeta + (\Theta+1)b = bh + 2b \quad \text{or} \quad \zeta = b(1+h-\Theta)$$

and

$$\alpha + (\gamma+1)b = bc + 2b \quad \text{or} \quad \alpha = b(1+c-\gamma).$$

Further, let

$$f + (h+1)b = b\Theta + 2b \quad \text{or} \quad f = b(1+\Theta-h)$$

and

$$a + (c+1)b = b\gamma + 2b \quad \text{or} \quad a = b(1+\gamma-c).$$

Finally, it must be $\alpha = f$ and $\zeta = a$, which two equations require that it is

$$c - \gamma = \Theta - h \quad \text{or} \quad c + h = \gamma + \Theta.$$

Hence the following theorem will arise

$$\frac{(h+1)(\Theta+1)}{(c+1)(\gamma+1)} = \frac{\int x^{b(1+\gamma-c)-1}dx(1-x^b)^c \cdot \int x^{b(1+c-\gamma)-1}dx(1-x^b)^\gamma}{\int x^{b(1+\theta-h)-1}dx(1-x^b)^h \cdot \int x^{b(1+h+\Theta)-1}dx(1-x^b)^\Theta},$$

as long as it is $c + h = \gamma + \Theta$.

**§49** But additionally another way the rendered $= 1$, by putting

$$\alpha = a + (c+1)b \quad \text{and} \zeta = f + (h+1)b, \quad f = b(\gamma + 2), \quad a = b(\theta + 2),$$

such that it is

$$\alpha = b(3 + c + \Theta) \quad \text{and} \quad \xi = b(3 + h + \gamma).$$

But further it must be

$$\xi + (\Theta + 1)b = bh + 2b \quad \text{and} \quad \alpha + (\gamma + 1)b = bc + 2b,$$

by which it is postulated that it is

$$\gamma + \Theta + 2 = 0.$$

Therefore, put

$$\gamma = -1 + n \quad \text{and} \quad \Theta = -1 - n$$

But if it is required that the product of both expressions is $= \frac{f(h+1)(\theta+1)}{a(c+1)(\gamma+1)}$, it will be obtained by putting

$$\alpha = a + (c+1)b\ , \quad \xi = f + (h+1)b\ , \quad f = b(\gamma + 1)\ , \quad a = b(\Theta + 1),$$

whence it will be

$$\alpha = b(2 + c + \Theta) \quad \text{and} \quad \xi = b(2 + h + \gamma).$$

Finally, it must be

$$\gamma + \Theta + 1 = 0.$$

Put

$$\gamma = -\frac{1}{2} + n \quad \text{und} \quad \Theta = -\frac{1}{2} + n$$

and one will have this theorem

$$\frac{h+1}{c+1} = \frac{\int x^{b\left(\frac{1}{2}-n\right)-1}dx(1-x^b)^c}{\int x^{b\left(\frac{1}{2}+n\right)-1}dx(1-x^b)^h} \cdot \frac{\int x^{b\left(\frac{3}{2}+c-n\right)-1}dx(1-x^b)^{-\frac{1}{2}+n}}{\int x^{b\left(\frac{3}{2}+h+n\right)-1}dx(1-x^b)^{-\frac{1}{2}-n}};$$

in this it is to be noted that the exponents $c$, $h$, $-\frac{1}{2}+n$, $-\frac{1}{2}-n$ can certainly be negative numbers, but such one, that together with the unity the go over into positive ones; for, otherwise the integrals would not obtain a finite value in the case $x = 1$.

**§50** Therefore, as I not only detected the theorem found above on products of two integrals formulas by this more direct method, but also found new not less remarkable ones, so, if in similar manner three expressions of this kind are multiplied by each other, many theorems on the products of three integral formulas will arise and it will be possible to proceed further to an arbitrary number of factors; but since this investigation requires very cumbersome calculations, that even the letters hardly suffice, I will be contented both with the main theorems given and the way of proving them.