Observations on continued fractions *

Leonhard Euler

§1 After in the last year I had begun, to investigate continued fractions and to develop this almost new branch of analysis, I have made several observations in the mean time, which will maybe not incongruous, to expand this theory. Therefore, because the exploration of this theory seems to provide not less of help for analysis in general, I will take on this subject again, and explain all the things, that refer to here, in more detail. So let this continued fraction be propounded

$$A+\cfrac{B}{C+\cfrac{D}{E+\cfrac{F}{G+\cfrac{H}{I+\text{etc.}}}}}$$

whose true value will be found by continuing the following infinite series

$$A+\frac{B}{1P}-\frac{BD}{PQ}+\frac{BDF}{QR}-\frac{BDFH}{RS}-\text{etc.},$$

in which series the letters P, Q, R, S etc. obtain the following values

$$P=C, \quad Q=EP+D, \quad R=GQ+FP, \quad S=IR+HQ \quad \text{etc.}$$

*Original title: " De fractionibus continuis Observationes", first published in „*Commentarii academiae scientiarum Petropolitanae 11*, 1750, pp. 32-81", reprinted in „*Opera Omnia*: Series 1, Volume 14, pp. 291 - 349 ", Eneström-Number E123, translated by: Alexander Aycock, for the „Euler-Kreis Mainz"

But this series is always convergent, no matter how the letters B, C, D, E, F etc. either grow or decrease, as long they are all positive; for, any arbitrary term is smaller than the preceding, but greater than the following, which the law, according to which the values P, Q, R, S etc. are formed, immediately reveals.

§2 If vice versa this infinite series was propounded

$$\frac{B}{P}-\frac{BD}{PQ}+\frac{BDF}{QR}-\frac{BDFH}{RS}+\text{etc.},$$

its sum can conveniently be expressed by the following continued fraction. Because it is

$$C=P,\quad E=\frac{Q-D}{P},\quad G=\frac{R-FP}{Q},\quad I=\frac{S-HQ}{R}\quad \text{etc.},$$

one will have this continued fraction equal to that series

$$\cfrac{B}{P+\cfrac{D}{\cfrac{Q-D}{P}+\cfrac{F}{\cfrac{R-FP}{Q}+\cfrac{H}{\cfrac{S-HQ}{R}+\cfrac{K}{\text{etc.}}}}}}$$

or

$$\cfrac{B}{P+\cfrac{DP}{Q-D+\cfrac{FPQ}{R-FP+\cfrac{HQR}{S-HQ+\cfrac{KRS}{\text{etc.}}}}}}$$

If this series was given

$$\frac{a}{p}-\frac{b}{q}+\frac{c}{r}-\frac{d}{s}+\frac{e}{t}-\text{etc.},$$

because of

$$B = a, \quad D = b : a, \quad F = c : b, \quad H = d : c, \quad K = e : d \quad \text{etc.}$$

and

$$P = p, \quad Q = q : p, \quad R = pr : q, \quad S = qs : pr, \quad T = prt : qs \quad \text{etc.}$$

the sum of this series

$$\frac{a}{p} - \frac{b}{q} + \frac{c}{r} - \frac{d}{s} + \frac{e}{t} - \text{etc.}$$

will be equal to the following continued fraction

$$\cfrac{a}{p + \cfrac{b : a}{\cfrac{aq-bp}{app} + \cfrac{c : b}{\cfrac{p^2(br-cq)}{bqq} + \cfrac{d : c}{\cfrac{q^2(cs-dr)}{cp^2r^2} + \cfrac{e : d}{\cfrac{p^2r^2(dt-es)}{dq^2s^2} + \text{etc.}}}}}}$$

$$= \cfrac{a}{p + \cfrac{bp^2}{aq - bp + \cfrac{acqq}{br - cq + \cfrac{bdrr}{cs - dr + \cfrac{cess}{dt - es + \text{etc.}}}}}}$$

§3 To illustrate these things with some examples, let us take this series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.},$$

whose sum is $= \log 2$ or $\int \frac{dx}{1+x}$, if after the integration x is put $= 1$; it will therefore be

$$a = b = c = d = \text{etc.} = 1, \quad p = 1, \quad q = 2, \quad r = 3, \quad s = 4 \quad \text{etc.}$$

and

$$p = 1, \quad aq - bp = 1, \quad br - cq = 1, \quad cs - dr = 1 \quad \text{etc.}$$

Hence it becomes

$$\int \frac{dx}{1+x} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \text{etc.}}}}}}$$

or the value of this continued fraction is $= \log 2$.

§4 Let us consider this series now

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.},$$

whose sum is the area of the circle having diameter $= 1$ or $\int \frac{dx}{1+x^2}$, having put $x = 1$ after the integration. It will therefore be

$$a = b = c = d = \text{etc.} = 1 \quad \text{and} \quad p = 1, \quad q = 3, \quad r = 5, \quad s = 7 \quad \text{etc.},$$

whence it becomes

$$\int \frac{dx}{1+xx} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \text{etc.}}}}}}$$

which is Brouncker's continued fraction itself, which he exhibited for the quadrature of the circle.

§5 In similar manner by taking other series of this kind, the following conversions of integral formulas to continued fractions will arise, after having put $x = 1$ after the integration:

$$\int \frac{dx}{1+x^3} = \cfrac{1}{1+\cfrac{1^2}{3+\cfrac{4^2}{3+\cfrac{7^2}{3+\cfrac{10^2}{3+\text{etc.}}}}}} \qquad \int \frac{dx}{1+x^4} = \cfrac{1}{1+\cfrac{1^2}{4+\cfrac{5^2}{4+\cfrac{9^2}{4+\cfrac{13^2}{4+\text{etc.}}}}}}$$

$$\int \frac{dx}{1+x^5} = \cfrac{1}{1+\cfrac{1^2}{5+\cfrac{6^2}{5+\cfrac{11^2}{5+\cfrac{16^2}{5+\text{etc.}}}}}} \qquad \int \frac{dx}{1+x^6} = \cfrac{1}{1+\cfrac{1^2}{6+\cfrac{7^2}{6+\cfrac{13^2}{6+\cfrac{19^2}{6+\text{etc.}}}}}}$$

§6 Hence it follows, that it will be in general

$$\int \frac{dx}{1+x^m} = \cfrac{1}{1+\cfrac{1^2}{m+\cfrac{(m+1)^2}{m+\cfrac{(2m+1)^2}{m+\cfrac{(3m+1)^2}{m+\text{etc.}}}}}}$$

having put $x = 1$ after the integration. And if m was a fractional number, one will have

$$\int \frac{dx}{1+x^{\frac{m}{n}}} = \cfrac{1}{1+\cfrac{n}{m+\cfrac{(m+n)^2}{m+\cfrac{(2m+n)^2}{m+\cfrac{(3m+n)^2}{m+\text{etc.}}}}}}$$

§7 Now let us consider the formula $\int \frac{x^{n-1}dx}{1+x^m}$, which integrated and having put $x = 1$ afterwards, yields this series

$$\frac{1}{n} - \frac{1}{m+n} + \frac{1}{2m+n} - \frac{1}{3m+n} + \text{etc.}$$

Hence it will be

$$a = b = c = d = \text{etc.} = 1 \quad \text{and} \quad p = n, \quad q = m+n, \quad r = 2m+n, \quad s = 3m+n \quad \text{etc.}$$

Hence one will have

$$\int \frac{x^{n-1}dx}{1+x^m} = \cfrac{1}{n + \cfrac{n^2}{m + \cfrac{(m+n)^2}{m + \cfrac{(2m+n)^2}{m + \text{etc.}}}}}$$

which continued fraction coincides with the last one found.

§8 Now let this formula be propounded $\int \frac{x^{n-1}dx}{(1+x^m)^{\frac{\mu}{\nu}}}$, which integrated and having put $x = 1$ afterwards yields this series

$$\frac{1}{n} - \frac{\mu}{\nu(m+n)} + \frac{\mu(\mu+\nu)}{1 \cdot 2\nu^2(2m+n)} - \frac{\mu(\mu+\nu)(\mu+2\nu)}{1 \cdot 2 \cdot 3\nu^3(3m+n)} + \text{etc.},$$

which compared to the general one yields

$$\begin{gathered} a = 1, \quad b = \mu, \quad c = \mu(\mu+\nu), \quad d = \mu(\mu+\nu)(\mu+2\nu) \quad \text{etc.}, \\ p = n, \quad q = \nu(m+n), \quad R = 2\nu^2(2m+n), \quad s = 6\nu^3(3m+n), \\ t = 24\nu^4(4m+n) \quad \text{etc.} \end{gathered}$$

and

$$\begin{aligned} aq - bp &= \nu m + (\nu - \mu)n, \\ br - cq &= \mu\nu(3\nu - \mu)m + \mu\nu(\nu - \mu)n, \\ cs - dr &= 2\mu\nu^3(\mu+\nu)(m(5\nu - 2\mu) + n(\nu - \mu)), \\ dt - es &= 6\mu\nu^3(\mu+\nu)(\mu+2\nu)(m(7\nu - 3\mu) + n(\nu - \mu)) \\ &\text{etc.}, \end{aligned}$$

having substituted which values and after a reduction one will have

$$\int \frac{x^{n-1}dx}{(1+x^m)^{\frac{\mu}{\nu}}}$$

$$= \cfrac{1}{n + \cfrac{\mu n^2}{\nu m + (\nu - \mu)n + \cfrac{\nu(\mu+\nu)(m+n)^2}{(3\nu - \mu) + (\nu - \mu)n + \cfrac{2\nu(\mu+2\nu)(2m+n)^2}{(5\nu - 2\mu)m + (\nu - \mu)n + \cfrac{3\nu(\mu+3\nu)(3m+n)^2}{(7\nu - 3\mu)m + (\nu - \mu)n + \text{etc.}}}}}}$$

Let $\mu = 1$ and $\nu = 2$; it will be

$$\int \frac{x^{n-1}dx}{\sqrt{1+x^m}} = \cfrac{1}{n + \cfrac{n^2}{2m + n + \cfrac{6(m+n)^2}{5m + n + \cfrac{20(2m+n)^2}{8m + n + \cfrac{42(3m+n)^2}{11m + n + \cfrac{72(4m+n)^2}{14m + n + \text{etc.}}}}}}}$$

§9 But if $\nu = 1$ and μ was an integer, the following continued fractions will arise

$$\int \frac{x^{n-1}dx}{(1+x^m)^2} = \cfrac{1}{n + \cfrac{2n^2}{m - n + \cfrac{1 \cdot 3(m+n)^2}{m - n + \cfrac{2 \cdot 4(2m+n)^2}{m - n + \cfrac{3 \cdot 5(3m+n)^2}{m - n + \cfrac{4 \cdot 6(4m+n)^2}{m - n + \text{etc.}}}}}}}$$

$$\int \frac{x^{n-1}dx}{(1+x^m)^3} = \cfrac{1}{n+\cfrac{3n^3}{m-2n+\cfrac{1\cdot 4(m+n)^2}{-2n+\cfrac{2\cdot 5(2m+n)^2}{-m-2n+\cfrac{3\cdot 6(3m+n)^2}{-2m-2n+\cfrac{4\cdot 7(4m+n)^2}{-3m-2n+\text{etc.}}}}}}}$$

which expressions in the same way as the following ones, because of the negative quantities, do not converge, but diverge.

§10 All these things follow from the conversion of the general continued fraction given in § 1 into an infinite series

$$A+\frac{B}{1P}-\frac{BD}{PQ}+\frac{BDF}{QR}-\frac{BDFH}{RS}+\text{etc.}$$

But this same series by adding each two terms is transformed into this one

$$A+\frac{BE}{1Q}+\frac{BDFI}{QS}+\frac{BDFHKN}{SV}+\text{etc.}$$

But it is

$$C=P=\frac{Q-D}{E},\quad G=\frac{S-HQ}{IQ}-\frac{F(Q-D)}{EQ},\quad L=\frac{V-MS}{NS}-\frac{K(S-HQ)}{IS}\quad \text{etc.}$$

Hence this infinite series

$$A+\frac{BE}{Q}+\frac{BDFI}{QS}+\frac{BDFHKN}{SV}+\text{etc.}$$

is converted into the following continued fraction

$$A+\cfrac{B}{\cfrac{Q-D}{E}+\cfrac{D}{E+\cfrac{F}{\cfrac{E(S-HQ)-FI(Q-D)}{EIQ}+\cfrac{H}{I+\cfrac{K}{\cfrac{I(V-MS)-KN(S-HQ)}{INS}+\text{etc.}}}}}}$$

which liberated from the fractions becomes this one

$$A+\cfrac{BE}{Q-D+\cfrac{D}{1+\cfrac{FIQ}{E(S-HQ)-FI(Q-D)+\cfrac{EHQ}{1+\cfrac{KNS}{I(V-MS)-KN(S-HQ)+\cfrac{IMS}{1+\text{etc.}}}}}}}$$

§11 If now vice versa this infinite series is

$$\frac{a}{p}+\frac{b}{q}+\frac{c}{r}+\frac{d}{s}+\frac{e}{t}+\text{etc.}$$

and one compares it to the preceding, it will be

$$Q=p,\quad S=\frac{q}{p},\quad V=\frac{pr}{q},\quad X=\frac{qs}{pr},\quad Z=\frac{prt}{qs}\quad \text{etc.}$$

and in the same way

$$E=\frac{a}{B},\quad I=\frac{b}{BDF},\quad N=\frac{c}{BDFHK}\quad \text{etc.},$$

by means of which values the propounded series is converted into this continued fraction

$$\cfrac{a}{p-D+\cfrac{D}{1+\cfrac{bp:1}{Da\left(\frac{q}{p}-Hp\right)-b(p-D)+\cfrac{DHap:1}{1+\cfrac{cq:p}{Hb\left(\frac{pr}{q}-\frac{Mq}{p}\right)-c\left(\frac{q}{p}-Hp\right)+\cfrac{HMbq:p}{1+\cfrac{dp:q}{Mc\left(\frac{qs}{pr}-\text{etc.}\right.}}}}}}}$$

in which continued fraction innumerable new quantities are included, which were not in the propounded series.

§12 But because from § 2 this series

$$\frac{b}{p}-\frac{bd}{pq}+\frac{bdf}{qr}-\frac{bdfh}{rs}+\text{etc.}$$

is equal to this continued fraction

$$\cfrac{b}{p+\cfrac{dp}{q-d+\cfrac{fpq}{r-fp+\cfrac{hqr}{s-hq+\cfrac{krs}{\text{etc.}}}}}}$$

if this series is reduced to the preceding, it will be

$$b=BE,\quad d=\frac{-DFI}{E},\quad f=\frac{-HKN}{I}\quad \text{etc.,}$$
$$p=Q,\quad q=S,\quad r=V,\quad s=X\quad \text{etc.}$$

Hence the continued fraction given in the preceding paragraph is transformed into this one

$$A+\cfrac{BE}{Q-\cfrac{DFI\cdot Q}{ES+DFI-\cfrac{EHKN\cdot QS}{IV+HKNQ-\cfrac{IMOR\cdot SV}{NX+MORS+\text{etc.}}}}}$$

whose progression law is easily perceived.

§13 But that series

$$A+\frac{B}{P}-\frac{BD}{PQ}+\frac{BDF}{QR}-\text{etc.,}$$

which we first found from the general continued fraction, is easily transformed into this form

$$A + \frac{B}{2P} + \frac{BE}{2Q} - \frac{BDG}{2PR} + \frac{BDFI}{2QS} - \frac{BDFHL}{2RT} + \text{etc.},$$

which, if the letters C, E, G, I etc. are expressed by means of the given equations, merges into this one

$$A + \frac{B}{2P} + \frac{B(Q-D)}{2PQ} - \frac{BD(R-FP)}{2PQR} + \frac{BDF(S-HQ)}{2QRS} - \text{etc.},$$

to which therefore this continued fraction is equal to

$$A + \cfrac{B}{P + \cfrac{DP}{Q - D + \cfrac{FPQ}{R - FP + \cfrac{HQR}{A - HQ + \text{etc.}}}}}$$

§14 So all these things immediately follow from the consideration of continued fractions and I mentioned the most observations of this kind already in the preceding dissertation. But now having treated these things I come to other ones and will explain several ways so to get to continued fractions as to assign the values of continued fractions of that kind by means of integrations. Therefore at first, because the BROUNCKERIAN expression for the quadrature of the cirlce is not only demonstrated, but was also found a priori, I want to investigate other similar expressions, either found by BROUNCKER himself or by WALLIS; for, the were listed up by WALLIS, but it was not indicated clear enough, whether BROUNCKER found all of them or merely that one, that was exhibited for the quadrature of the circle. But afterwards I will also prove those remaining continued fractions, which seem to be of more profoundity, from most different priciples and will teach to find a lot more of this kind.

§15 But what is written in WALLIS's book, reduces to that, that the product of the following two continued fractions is $= a^2$:

$$a - 1 + \cfrac{1}{2(a-1) + \cfrac{9}{2(a-1) + \cfrac{25}{2(a-1) + \text{etc.}}}}$$

and

$$a+1+\cfrac{1}{2(a+1)+\cfrac{9}{2(a+1)+\cfrac{25}{2(a+1)+\text{etc.}}}}$$

But because it is in the same way

$$(a+2)^2 = a+1+\cfrac{1}{2(a+1)+\cfrac{9}{2(a+1)+\text{etc.}}} \quad \times \quad a+3+\cfrac{1}{2(a+3)+\cfrac{9}{2(a+3)+\text{etc.}}}$$

by proceeding in this way to infinity one will find

$$\begin{aligned} &a\cdot\frac{a(a+4)}{(a+2)(a+2)}\,\frac{(a+8)(a+\ \ 8)}{(a+6)(a+10)}\,\frac{(a+12)(a+12)}{(a+10)(a+14)}\ \ \text{etc.} \\ &= a-1+\cfrac{1}{2(a-1)+\cfrac{9}{2(a-1)+\cfrac{25}{2(a-1)+\text{etc.}}}} \end{aligned}$$

§16 If this product consisting of an infinite number of factors is examined by the method given in the preceding paper, one will find that it will be

$$\frac{a(a+4)(a+4)(a+8)\text{etc.}}{(a+2)(a+2)(a+6)(a+6)\text{etc.}} = \frac{\int x^{a+1}dx : \sqrt{1-x^4}}{\int x^{a-1}dx : \sqrt{1-x^4}}.$$

Therefore the value of this continued fraction

$$= a-1+\cfrac{1}{2(a-1)+\cfrac{9}{2(a-1)+\cfrac{25}{2(a-1)+\text{etc.}}}}$$

will become equal to this expression

$$a\frac{\int x^{a+1}dx : \sqrt{1-x^4}}{\int x^{a-1}dx : \sqrt{1-x^4}}$$

having put $x = 1$ after each integration.

§17 This theorem, by which the value of a fairly far extending continued fraction is expressed by integrals, is even more remarkable, because its validity is far away from obvious. Although that case, in which $a = 2$, was already found before and its value was exposed by the quadrature of the circle, the remaining cases do not follow from there. Hence if this continued fraction in converted into a series in the way described at the beginning, one gets to such an intricate series, that its sum can in no way be calculated, except in the case $a = 2$. Therefore I have already worked quite hard before, to demonstrate the validity of this theorem and detect a way, on which it would be possible to get to this continued fraction a priori; this investigation seemed the more difficult to me, the greater utility I believed to arise from it. But as long as all effort was without success concerning this task, I highly regret that the method used by Brouncker was never exposed and is maybe completely lost.

§18 As far it is clear from Wallis's treatment, Brouncker was led to this form by interpolation of this series

$$\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \text{etc.},$$

whose intermediate terms to give the quadrature of the cirlce itself Wallis had demonstrated. And even the beginning of this interpolation, done by Brouncker, was indicated. Then it is said, that it was propounded to him, to resolve the single fractions $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$ etc. into two factors, which all constitute a continous progression. So if it was

$$AB = \frac{1}{2}, \quad CD = \frac{2}{3}, \quad EF = \frac{5}{6}, \quad GH = \frac{7}{8} \quad \text{etc.}$$

and the quantities A, B, C, D, E etc. constitute a continuous progression, that series merges into this one

$$AB + ABCD + ABCDEF + \text{etc.},$$

which is reduced to this form is interpolated by itself; hence the term, whose index is $\frac{1}{2}$, $= A$ and the term having index $\frac{3}{2}$ is $= ABC$ and so on. Hence this whole interpolation is reduced to the resolution of the single fractions into two factors each.

§19 But from the law of continuity it will be

$$BC = \frac{2}{3}, \quad DE = \frac{4}{5}, \quad FG = \frac{6}{7} \quad \text{etc.}$$

Because therefore it is

$$A = \frac{1}{2B}, \quad B = \frac{2}{3C}, \quad C = \frac{3}{4D}, \quad D = \frac{4}{5E} \quad \text{etc.},$$

one immediately obtains

$$A = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} \quad \text{etc.},$$

which is indeed the formula given by Wallis first itself, by which he expressed the quadrature of the circle, and that diviates highly from the Brounckerian expression. Because therefore this formula in this kind of investigation proves itself to be so useful, one has to wonder even more, that Brouncker having taken the same way, got to such a different expression; hence there seems to remain no way, that would lead that continued fraction. And it is indeed not to be thought, that Brouncker wanted to express the value A by a continued fraction on purpose, but rather following a certain peculiar method stumbled upon it by accident, because at that time continued fractions were completely unknown and were introduced at first on this occasion. From this it is possible to conclude certain enough, that there is an obvious method, leading to those continued fractions, no matter how concealed it might appear now.

§20 Although I was occupied a long time by rediscovering this method with no success, I nevertheless hit upon another way to do interpolations of series of this kind by continued fractions; but it gave me expressions, very different from the Brounckerian ones. Nevertheless I hope, that it will not be completely useless to expose this method, because by its means one finds continued fractions, whose values are known from elsewhere and can be exhibited by quadratures. Because then I will give another method, to express the values of any continued fractions by quadratures, from this extraordinary comparisons of integral formulas will arise, at least in that case, in which a definite value is assigned to the variable after the integration; I exhibited many comparisons of such kind in the preceding dissertation on infinite products consisting of constant factors.

§21 To explain this way of interpolation found by me, let this far reaching series be propounded

$$\frac{p}{p+2q} + \frac{p(p+2r)}{(p+2q)(p+2q+2r)} + + \frac{p(p+2r)(p+4r)}{(p+2q)(p+2q+2r)(p+2q+4r)} + \text{etc.},$$

whose term with index $\frac{1}{2}$ is $= A$, the term with index $\frac{3}{2}$ is $= ABC$, the term with index $\frac{5}{2}$ is $= ABCDE$ etc. Hence it will be

$$AB = \frac{p}{p+2q}, \quad CD = \frac{p+2r}{p+2q+2r}, \quad EF = \frac{p+4r}{p+2q+4r} \quad \text{etc.}$$

and from the law of continuity

$$BC = \frac{p+r}{p+2q+r}, \quad DE = \frac{p+3r}{p+2q+3r}, \quad FG = \frac{p+5r}{p+2q+5r}$$

and so on.

§22 To get rid of the fractions put

$$A = \frac{a}{p+2q-r}, \quad B = \frac{b}{p+2q}, \quad C = \frac{c}{p+2q+r}, \quad D = \frac{d}{p+2q+2r} \quad \text{etc.}$$

and it will be

$$ab = (p+2q-r)p, \quad bc = (p+2q)(p+r), \quad cd = (p+2q+r)(p+2r),$$
$$de = (p+2q+2r)(p+3r) \quad \text{etc.}$$

Now let be

$$a = m-r+\frac{1}{\alpha}, \quad b = m+\frac{1}{\beta}, \quad c = m+r+\frac{1}{\gamma},$$
$$d = m-2r+\frac{1}{\delta}, \quad e = m+3r+\frac{1}{\epsilon} \quad \text{etc.},$$

in which substitutions the whole parts constitute an arithmetic progression, whose constant difference is r, what the progression of those factors itself demands. Having subsituted these values therefore by putting for the sake of brevity

$$p^2 + 2pq - pr - m^2 + mr = P$$

and

$$2r(p + q - m) = Q$$

the following equations will arise

$$\begin{aligned}
P\ \ \alpha\beta - (m-r)\ \alpha &= \qquad m\ \ \beta + 1, \\
(P+\ \ Q)\beta\gamma - \qquad m\beta &= (m+r)\ \ \gamma + 1, \\
(P+2Q)\gamma\delta - (m+\ \ r)\gamma &= (m+2r)\delta + 1, \\
(P+3Q)\delta\varepsilon - (m+2r)\delta &= (m+3r)\varepsilon + 1, \\
&\text{etc.}
\end{aligned}$$

§23 From these equations therefore the following comparisons between the letters α, β, γ, δ etc. arise:

$$\begin{aligned}
\alpha &= \frac{m\beta + 1}{P\beta - (m-r)} = \frac{m}{P} + \frac{p(p+2q-r) : P^2}{-\frac{m-r}{P} + \beta} \\
\beta &= \frac{(m+r)\gamma + 1}{(P+Q)\gamma - m} = \frac{m+r}{P+Q} + \frac{(p+r)(p+2q) : (P+Q)^2}{-\frac{m}{P+Q} + \gamma} \\
\gamma &= \frac{(m+2r)\delta + 1}{(P+2Q)\delta - (m+r)} = \frac{m+2r}{P+2Q} + \frac{(p+2r)(p+2q+r) : (P+2Q)^2}{-\frac{m-r}{P+2Q} + \delta} \\
&\text{etc.}
\end{aligned}$$

So if for the sake of brevity one puts

$$p^2 + 2pq - mp - mq + qr = R$$

and

$$pr + qr - mr = S$$

and the values of the assumed letters are continuously substituted into one another, the followng continued fraction will arise

$$\alpha = \frac{m}{P} + \cfrac{p(p+2q-r):P^2}{\cfrac{2rR}{P(P+Q)} + \cfrac{(p+r)(p+2q):(P+Q)^2}{\cfrac{2r(R+S)}{(P+Q)(P+2Q)} + \cfrac{(p+2r)(p+2q+r):(P+2Q)^2}{\cfrac{2r(R+2S)}{(P+2Q)(P+3Q)} + \text{etc.}}}}$$

§24 But because it is $a = m - r + \frac{1}{\alpha}$, one will have

$$a = m - r + \cfrac{P}{m + \cfrac{p(p+2q-r)(P+Q)}{2rR + \cfrac{(p+r)(p+2q)P(P+2Q)}{2r(R+S) + \cfrac{(p+2r)(p+2q+r)(P+Q)(P+3Q)}{2r(R+2S) + \text{etc.}}}}}$$

Hence the term of the propounded series

$$\frac{p}{p+2q} + \frac{p(p+2r)}{(p+2q)(p+2q+2r)} + \frac{p(p+2r)(p+4r)}{(p+2q)(p+2q+2r)(p+2q+4r)} + \text{etc.}$$

with index $\frac{1}{2}$ will be

$$= A = \frac{a}{p+2q-r}.$$

But because the general term of this series having the index n is

$$= \frac{\int y^{p+2q-1}dy(1-y^{2r})^{n-1}}{\int y^{p-1}dy(1-y^{2r})^{n-1}}$$

the found continued fraction or the value of the letter a

$$= (p+2q-r)\frac{\int y^{p+2q-1}dy : \sqrt{1-y^{2r}}}{\int y^{p-1}dy : \sqrt{1-y^{2r}}}$$

after having put $y = 1$ after both integrations.

§25 But because there is the arbitrary letter m in our continued fraction, one will have innumerable continued fractions, whose value is the same and known; so it will be helpful, to consider the principal ones. So at first let be

$$m-r=p \quad \text{or} \quad m=p+r;$$

it will be

$$P=2p(q-r), \quad Q=2r(q-r), \quad R=p(q-r) \quad \text{and} \quad S=r(q-r),$$

whence it will be

$$a=p+\cfrac{2p(q-r)}{p+r+\cfrac{(p+2q-r)(p+r)}{r+\cfrac{(p+2q)(p+2r)}{r+\cfrac{(p+2q+r)(p+3r)}{r+\text{etc.}}}}}$$

But if $r>q$, that the fraction does not become negative, it will be

$$a=\cfrac{p}{1+\cfrac{2(r-q)}{p+2q-r+\cfrac{(p+2q-r)(p+r)}{r+\cfrac{(p+2q)(p+2r)}{r+\cfrac{(p+2q+r)(p+3r)}{r+\text{etc.}}}}}}$$

§26 Now let m be $=p+q$, in which case Q and S vanish; but it will be

$$P=q(r-q) \quad \text{and} \quad R=q(r-q)$$

and hence it will arise

$$a=p+q-r+\cfrac{q(r-q)}{p+q+\cfrac{p(p+2q-r)}{2r+\cfrac{(p+r)(p+2q)}{2r+\cfrac{(p+2r)(p+2q+r)}{2r+\text{etc.}}}}}$$

which continued fraction is therefore equal to the preceding ones, even though the forms itself are different.

§27 Now put $m = p + 2q$ and it will be

$$\begin{aligned} P &= 2q(r - p - 2q) = -2q(p + 2q - r), \\ Q &= -2qr, \\ R &= -q(p + 2q - r) \end{aligned}$$

and

$$S = -qr.$$

From this one will therefore obtain the following continued fraction:

$$a = p + 2q - r - \cfrac{2q(p+2q-r)}{p + 2q + \cfrac{p(p+2q)}{r + \cfrac{(p+r)(p+2q+r)}{r + \cfrac{(p+2r)(p+2q+2r)}{r + \text{etc.}}}}}$$

So innumerable continued fractions arise, whose all value is the same a, which by means of integral formulas was found to be

$$= (p + 2q - r)\frac{\int y^{p+2q-1}dy : \sqrt{1 - y^{2r}}}{\int y^{p-1}dy : \sqrt{1 - y^{2r}}} = (p + 2q - 2r)\frac{\int y^{p+2q-2r-1}dy : \sqrt{1 - y^{2r}}}{\int y^{p-1}dy : \sqrt{1 - y^{2r}}}.$$

§28 Before we proceed, let us consider some cases. Let $r = 2q$ and it will be

$$a = p\frac{\int y^{p+2q-1} : \sqrt{1 - y^{4q}}}{\int y^{p-1}dy : \sqrt{1 - y^{4q}}}.$$

Because therefore it is

$$\begin{aligned} P &= p^2 + 2mq - m^2, \\ Q &= 4q(p + q - m), \\ R &= p^2 + 2pq + 2qq - mp - mq \end{aligned}$$

and

$$S = 2q(p+q-m),$$

it will be in general

$$a = m - 2q + \cfrac{P}{m + \cfrac{p^2(P+Q)}{4qR + \cfrac{(p+2q)^2P(P+2Q)}{4q(R+S) + \cfrac{(p+4q)^2(P+Q)(P+3Q)}{4q(R+2S) + \text{etc.}}}}}$$

§29 But if we substitute those different values for m, the following determined continued fraction will arise

$$a = p - \cfrac{2pq}{p + 2q + \cfrac{p(p+2q)}{2q + \cfrac{(p+2q)(p+4q)}{2q + \cfrac{(p+4q)(p+6q)}{2q + \text{etc.}}}}}$$

or instead of this continued fraction because of $r > q$

$$a = \cfrac{p}{1 + \cfrac{2q}{p + \cfrac{p(p+2q)}{2q + \cfrac{(p+2q)(p+4q)}{2q + \cfrac{(p+4q)(p+6q)}{2q + \text{etc.}}}}}}$$

Further from § 26 one obtains this fraction for this case

$$a = p - q + \cfrac{qq}{p + q + \cfrac{pp}{4q + \cfrac{(p+2q)^2}{4q + \cfrac{(p+4q)^2}{4q + \text{etc.}}}}}$$

But thirdly § 27 will give this continued fraction

$$a = p - \cfrac{2pq}{p + 2q + \cfrac{p(p+2q)}{2q + \cfrac{(p+2q)(p+4q)}{2q + \cfrac{(p+4q)(p+6q)}{2q + \text{etc.}}}}}$$

which agrees with this first one exhibited here, so that one has only two simpler continued fractions for this case, in which $r = 2q$.

§30 Now futher put $q = p = 1$, that it is

$$a = \frac{\int yydy : \sqrt{1-y^4}}{\int dy : \sqrt{1-y^4}};$$

at first it will be

$$a = -\cfrac{2}{3 + \cfrac{1 \cdot 3}{2 + \cfrac{3 \cdot 5}{2 + \cfrac{5 \cdot 7}{2 + \text{etc.}}}}}$$

Then one will have

$$a = \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{9}{4 + \cfrac{25}{4 + \cfrac{49}{4 + \text{etc.}}}}}}$$

Hence it follows that it will be

$$\frac{\int dy : \sqrt{1-y^4}}{\int yydy : \sqrt{1-y^4}} = 2 + \cfrac{1}{4 + \cfrac{9}{4 + \cfrac{25}{4 + \cfrac{49}{4 + \text{etc.}}}}}$$

which case in contained in the expresion given in § 16, whence that formula, not rigorously proven, is confirmed even more. Then having put $a = 3$ there it will be

$$3\frac{\int x^4dx : \sqrt{1-x^4}}{\int xxdx : \sqrt{1-x^4}} = \frac{\int dx : \sqrt{1-x^4}}{\int xxdx : \sqrt{1-x^4}} = 2 + \cfrac{1}{4 + \cfrac{9}{4 + \cfrac{25}{4 + \cfrac{49}{4 + \text{etc.}}}}}$$

so that it is now certain, that the formula exhibited in § 16 is true in the cases, in which either $a = 2$ or $a = 3$; but soon its validity in the broadest sense will be shown.

§31 Let $q = \frac{1}{2}$ and $p = 1$; with still $r = 2q = 1$ it will be

$$a = \frac{\int ydy : \sqrt{1-y^2}}{\int dy : \sqrt{1-y^2}} = \frac{2}{\pi}$$

while π denotes the circumference of the circle, whose diameter is $= 1$. Therefore it will be in general

$$P = 1 + m - m^2, \quad Q = 2 - 2m,$$
$$R = \frac{5-3m}{2} \quad \text{and} \quad S = \frac{3-2m}{2}$$

and therefore

$$a = m - 1 + \cfrac{1 + m - m^2}{m + \cfrac{1^2(4 - m + m^2)}{5 - 3m + \cfrac{2^2(1 + m - m^2)(7 - 3m - m^2)}{8 - 5m + \cfrac{3^2(4 - m - m^2)(10 - 5m - m^2)}{11 - 7m + \text{etc.}}}}}$$

But in the exposed special cases it will be

$$\frac{\pi}{2} = \cfrac{1}{1 - \cfrac{1}{2 + \cfrac{1 \cdot 2}{1 + \cfrac{2 \cdot 3}{1 + \cfrac{3 \cdot 4}{1 + \text{etc.}}}}}} \quad = 1 + \cfrac{1}{1 + \cfrac{1 \cdot 2}{1 + \cfrac{2 \cdot 3}{1 + \cfrac{3 \cdot 4}{1 + \text{etc.}}}}}$$

and

$$\frac{\pi}{2} = \cfrac{1}{\frac{1}{2} + \cfrac{1 : 4}{\frac{3}{2} + \cfrac{1^2}{2 + \cfrac{2^2}{2 + \cfrac{3^2}{2 + \text{etc.}}}}}} \quad = 2 - \cfrac{1}{2 + \cfrac{1^2}{2 + \cfrac{2^2}{2 + \cfrac{3^2}{2 + \text{etc.}}}}}$$

§32 To understand the use of these formualas concerning interpolations, let this series be propounded

$$\frac{2}{1} + \frac{2 \cdot 4}{1 \cdot 3} + \frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5} + \text{etc.}$$

whose term with index $\frac{1}{2}$ has to be found, which shall be $= A$; it will therefore be

$$p = 2, \quad r = 1 \quad \text{and} \quad q = -\frac{1}{2}.$$

Now put

$$A = \frac{a}{p + 2q - r};$$

it will be

$$A = \frac{a}{0},$$

whence the inconvenience of the given formulas, if $p + 2q - r = 0$, is understood clear enough. This task can nevertheless be completed by searching the term with index $\frac{3}{2}$; if this one was $= Z$; it will be $A = \frac{2}{3}Z$; but $\frac{1}{2}Z$ will be the term with index $\frac{1}{2}$ of this series

$$\frac{4}{3} + \frac{4 \cdot 6}{3 \cdot 5} + \frac{4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7} + \text{etc.},$$

which compared to the general one gives

$$p = 4, \quad r = 1, \quad q = -\frac{1}{2},$$

so that it is

$$Z = \frac{2 \int y^2 dy : \sqrt{1 - y^2}}{\int y^3 dy : \sqrt{1 - y^2}} = \frac{3 \int dy : \sqrt{1 - y^2}}{2 \int dy : \sqrt{1 - y^2}} = \frac{3}{4}\pi$$

and $A = \frac{\pi}{2}$. But because it is by § 24

$$Z = a \quad \text{and} \quad A = \frac{2}{3}Z = \frac{2}{3}a,$$

at first because of

$$P = 8 + m - m^2, \quad Q = 7 - 2m$$
$$R = \frac{23 - 7m}{2} \quad \text{and} \quad S = \frac{7 - 2m}{2}$$

it will be in general

$$A = \frac{3}{2}a = \frac{\pi}{2}$$

$$= \frac{2(m-1)}{3} + \cfrac{2(8+m-m^2)}{3m + \cfrac{2\cdot 4\cdot 3(15-m-m^2)}{23-7m + \cfrac{3\cdot 5(8+m-m^2)(22-3m-m^2)}{30-9m+\cfrac{4\cdot 6(15-m-m^2)(29-5m-^2)}{37-11m+\text{etc.}}}}}$$

§33 But by expanding the particular cases it will be

$$a = \frac{3}{4}\pi = 4 - \cfrac{12}{5+\cfrac{2\cdot 5}{1+\cfrac{3\cdot 6}{1+\cfrac{4\cdot 7}{1+\text{etc.}}}}} = \cfrac{4}{1+\cfrac{1\cdot 3}{2+\cfrac{2\cdot 5}{1+\cfrac{3\cdot 6}{1+\cfrac{4\cdot 7}{1+\text{etc.}}}}}}$$

or also

$$\frac{3}{4}\pi = 1 + \cfrac{3}{1+\cfrac{1\cdot 4}{1+\cfrac{2\cdot 5}{1+\cfrac{3\cdot 6}{1+\cfrac{4\cdot 7}{1+\text{etc.}}}}}}$$

In the same way by § 26 one will have

$$a = \frac{3}{4}\pi = \frac{5}{2} - \cfrac{3:4}{\frac{7}{2}+\cfrac{2\cdot 4}{2+\cfrac{3\cdot 5}{2+\cfrac{4\cdot 6}{2+\cfrac{5\cdot 7}{2+\text{etc.}}}}}} = 2 + \cfrac{1}{2+\cfrac{1\cdot 3}{2+\cfrac{2\cdot 4}{2+\cfrac{3\cdot 5}{2+\cfrac{4\cdot 6}{2+\text{etc.}}}}}}$$

Finally the case exposed in § 27 will give

$$a=\frac{3}{4}\pi=2+\cfrac{2}{3+\cfrac{3\cdot 4}{1+\cfrac{4\cdot 5}{1+\cfrac{5\cdot 6}{1+\text{etc.}}}}}$$

or

$$\frac{\pi}{2}=1+\cfrac{1}{1+\cfrac{1\cdot 2}{1+\cfrac{2\cdot 3}{1+\cfrac{3\cdot 4}{1+\cfrac{4\cdot 5}{1+\text{etc.}}}}}}$$

which expression agrees with a certain one exhibited in § 31.

§31 Hence we obtained innumerable continued fractions from this interpolation method, whose values can be assigend by quadratures of curves or integral formulas. But because this continued fractions are irregular at the beginning, just truncate the beginnings, that contain the anomaly, that one has continued fractions, proceeding according to the same law everywhere. So from § 25 by putting

$$p+2q-r=f \quad \text{and} \quad p+r=h$$

the following equation will arise

$$r+\cfrac{fh}{r+\cfrac{(f+r)(h+r)}{r+\cfrac{(f+2r)(h+2r)}{r+\text{etc.}}}}$$
$$=\frac{h(f-r)\int y^{h+r-1}dy:\sqrt{1-y^{2r}}-f(h-r)\int y^{f+r-1}dy:\sqrt{1-y^{2r}}}{f\int y^{f+t-1}dy:\sqrt{1-y^{2r}}-h\int y^{h+r-1}dy:\sqrt{1-y^{2r}}},$$

which equation is always real, if not $f=h$. But in the case, in which $f=h$, just put $f=h+dw$ and one will find

$$\frac{\int y^{h+r+dw-1}dy:\sqrt{1-y^{2r}}}{\int y^{h+r-1}dy:\sqrt{1-y^{2r}}}=1-rdw\int\frac{dx}{x^{r+1}}\int\frac{x^{h+2r-1}dx}{1-x^{2r}}$$

after having put $x=1$ after the integration. Hence it will be

$$r+\cfrac{hh}{r+\cfrac{(h+r)^2}{r+\cfrac{(h+2r)^2}{r+\text{etc.}}}}$$

$$=\frac{r+hr(h-r)\int\frac{dx}{x^{r+1}}\int\frac{x^{h+2r-1}dx}{1-x^{2r}}}{1-hr\int\frac{dx}{x^{r+1}}\int\frac{x^{h+2r-1}dx}{1-x^{2r}}}=\frac{r(h-r)^2\int\frac{dx}{x^{r+1}}\int\frac{x^{h-1}dx}{1-x^{2r}}}{1-r(h-r)\int\frac{dx}{x^{r+1}}\int\frac{x^{h-1}dx}{1-x^{2r}}}.$$

But from the nature of the integrals one has

$$\int\frac{dx}{x^{r+1}}\int\frac{x^{h+2r-1}dx}{1-x^{2r}}=\frac{-1}{rx^r}\int\frac{x^{h+2r-1}dx}{1-x^{2r}}+\frac{1}{r}\int\frac{x^{h+r-1}dx}{1-x^{2r}}=\frac{1}{r}\int\frac{x^{h+r-1}dx}{1+x^r}$$

having put $x=1$. Therefore one will have

$$r+\cfrac{hh}{r+\cfrac{(h+r)^2}{r+\text{etc.}}}=\frac{r+h(h-r)\int\frac{x^{h+r-1}dx}{1+x^r}}{1-h\int\frac{x^{h+r-1}dx}{1+x^r}}=\frac{1-(h-r)\int\frac{x^{h-1}dx}{1+x^r}}{\int\frac{x^{h-1}dx}{1+x^r}};$$

this formula agrees with the one which was given in § 7.

§35 In the same way from § 26 by putting $p=f$ and $p+2q-r=h$ it follows, that it will be

$$2r+\cfrac{fh}{2r+\cfrac{(f+r)(h+r)}{2r+\cfrac{(f+2r)(h+2r)}{2r+\text{etc.}}}}$$

$$=\frac{2(r-f)(r-h)\int\frac{y^{f-1}dy}{\sqrt{1-y^{2r}}}-h(f+h-3r)\int\frac{y^{h+r-1}dy}{\sqrt{1-y^{2r}}}}{2h\int\frac{y^{h+r-1}dy}{\sqrt{1-y^{2r}}}-(f+h-r)\int\frac{y^{f-1}dy}{\sqrt{1-y^{2r}}}}$$

But because this formula stays unchanged, if f and h are commuted, it is manifest, that is has to be

$$\frac{h\int y^{h+r-1}dx : \sqrt{1-y^{2r}}}{y^{f-1}dy : \sqrt{1-y^{2r}}} = \frac{f\int y^{f+r-1}dy : \sqrt{1-y^{2r}}}{\int y^{h-1}dy : \sqrt{1-y^{2r}}}$$

after having put $y = 1$ after all integrations. But this theorem is already contained in those, that I exhibited in the preceding dissertation on products, consisting of infinitely many factors; hence there I produced many theorems of this kind and proved them.

§36 But here in similar way the case, in which $f = h + r$, deserves it to be noted; hence in this case so the numerator as the denominator of the found continued fraction vanishes. But having put $f = h + r + dw$ as before and having done the calculation it will arise

$$2r + \cfrac{h(h+r)}{2r + \cfrac{(h+r)(h+2r)}{2r + \cfrac{(h+2r)(h+3r)}{2r + \text{etc.}}}} = \frac{h + 2h(r-h)\int \frac{x^{h-r}dx}{1+x^r}}{-1 + 2h\int \frac{x^{h-r}dx}{1+x^r}}.$$

Hence if one puts $h = r = 1$, one will have

$$2 + \cfrac{1\cdot 2}{2 + \cfrac{2\cdot 3}{2 + \cfrac{3\cdot 4}{2 + \cfrac{4\cdot 5}{2 + \text{etc.}}}}} = \frac{1}{2\log 2 - 1}.$$

If in addition the equation in § 27 is treated in the same way, a form very similar to that one, that I found § 25, will arise.

§37 Having exposed these things, by which the interpolation of series is reduced to continued fractions, I return to BROUNCKERIAN expressions and will give a natural method not only to get to those, but also a method of such kind, that seems to had been used by BROUNCKER himself. But the continued fractions found up to now are most different from the BROUNCKERIAN ones, because the values of the letters A, B, C, D etc. by the exposed method depend on one another in such a way, that they can easily be compared to each other, but by the BROUCKERIAN method they arose different to each other, that their mutual relation is not perceived. This difference itself finally led me to the invention of another method now to be explained.

§38 But before I explain this kind of interpolation itself, it will be convenient to mention the following very far reaching lemma in advance. If they were innumerable quantities α, β, γ, δ, ε etc., which shall depend on each other in such a way, that it is

$$\begin{aligned}
&\alpha\beta - m\alpha && - n\beta && - \varkappa = 0,\\
&\beta\gamma - (m+s)\beta && - (n+s)\gamma && - \varkappa = 0,\\
&\gamma\delta - (m+2s)\gamma && - (n+2s)\delta && - \varkappa = 0,\\
&\delta\varepsilon - (m+3s)\delta && - (n+3s)\varepsilon && - \varkappa = 0\\
&&& \text{etc.,}
\end{aligned}$$

and let the following values given to the letters α, β, γ, δ etc.

$$\begin{aligned}
\alpha &= m+n-s - \frac{ss-ms+ns+\varkappa}{a}\\
,\beta &= m+n+s - \frac{ss-ms+ns+\varkappa}{b}\\
,\gamma &= m+n+3s - \frac{ss-ms+ns+\varkappa}{c}\\
,\delta &= m+n+5s - \frac{ss-ms+ns+\varkappa}{d}\\
&\text{etc.}
\end{aligned}$$

the superior equations will be transformed into the following similar ones

$$\begin{aligned}
&ab - (m-s)a && - (n+s)b && - ss + ms - ns - \varkappa = 0,\\
&bc - mb && - (n+2s)c && - ss + ms - ns - \varkappa = 0,\\
&cd - (m+s)c && - (n+3s)d && - ss + ms - ns - \varkappa = 0,\\
&de - (m+2s)d && - (n+4s)4 && - ss + ms - ns - \varkappa = 0,\\
&&& \text{etc.}
\end{aligned}$$

And from this, that similar formulas of such a kind arise, those substitutions originated.

§39 If now in similar way these last equations are transformed by means of appropriate substitutions into similar ones, one will find the following substitutions instead of a, b, c, d etc.

$$\begin{aligned}
a &= m+n-s \ \ + \frac{4ss-2ms+2ns+\varkappa}{a_1},\\
b &= m+n+s \ \ + \frac{4ss-2ms+2ns+\varkappa}{b_1},\\
c &= m+n+3s + \frac{4ss-2ms+2ns+\varkappa}{c_1},\\
d &= m+n+5s + \frac{4ss-2ms+2ns+\varkappa}{d_1}\\
&\text{etc.}
\end{aligned}$$

having done which substitutions the following equations will arise

$$\begin{aligned}
&a_1b_1-(m-2s)a_1-(n+2s)b_1-4ss+2ms-2ns-\varkappa=0,\\
&b_1c_1-(m-s)\ b_1-(n+3s)c_1-4ss+2ms-2ns-\varkappa=0,\\
&c_1d_1-m \qquad c_1-(n+4s)d_1-4ss+2ms-2ns-\varkappa=0,\\
&d_1e_1-(m+s)\ d_1-(n+5s)e_1-4ss+2ms-2ns-\varkappa=0\\
&\text{etc.}
\end{aligned}$$

§40 While proceeding further one will therefore have to put

$$\begin{aligned}
a_1 &= m+n-s \ \ + \frac{9ss-3ms-3ns+\varkappa}{a_2},\\
b_1 &= m+n+s \ \ + \frac{9ss-3ms-3ns+\varkappa}{b_2},\\
c_1 &= m+n+3s + \frac{9ss-3ms-3ns+\varkappa}{c_2},\\
&\text{etc.}
\end{aligned}$$

And from these substitutions these equations arise

$$a_2b_2-(m-3s)-(n+3s)b_2-9ss+3ms-3ns-\varkappa=0,$$
$$b_2c_2-(m-2s)-(n+4s)c_2-9ss+3ms-3ns-\varkappa=0,$$
$$c_2d_2-(m-s)\ \ -(n+5s)d_2-9ss+3ms-3ns-\varkappa=0$$
etc.

§41 If these substitutions are now continued to infinity and the following values are always substituted in the preceding, the values of the letters α, β, γ, δ etc. are expressed by the following continued fractions

$$\alpha=m+n-s\ +\cfrac{ss-ms+ns-\varkappa}{m+n-s+\cfrac{4ss-2ms+2ns+\varkappa}{m+n-s+\cfrac{9ss-3ms+3ns+\varkappa}{m+n-s+\cfrac{16ss-4ms+4ns+\varkappa}{m+n-s+\text{etc.}}}}}$$

$$\beta=m+n+s\ +\cfrac{ss-ms+ns-\varkappa}{m+n+s+\cfrac{4ss-2ms+2ns+\varkappa}{m+n+s+\cfrac{9ss-3ms+3ns+\varkappa}{m+n+s+\cfrac{16ss-4ms+4ns+\varkappa}{m+n+s+\text{etc.}}}}}$$

$$\gamma=m+n+3s+\cfrac{ss-ms+ns-\varkappa}{m+n+3s+\cfrac{4ss-2ms+2ns+\varkappa}{m+n+3s+\cfrac{9ss-3ms+3ns+\varkappa}{m+n+3s+\cfrac{16ss-4ms+4ns+\varkappa}{m+n+3s+\text{etc.}}}}}$$

etc.

which continued fractions are similar enough to those, which Brouncker gave, while the following are not contained in the preceding ones.

§42 But that the use of these formulas for interpolations becomes apparent, let this series be propounded

$$\frac{p}{p+2q}+\frac{p(p+2r)}{(p+2q)(p+2q+2r)}+\frac{p(p+2r)((p+4r)}{(p+2q)(p+2q+2r)(p+2q+4r)}+\text{etc.},$$

whose term with index $\frac{1}{2}$ shall be $= A$, the term with index $\frac{3}{2} = ABC$, the term with index $\frac{5}{2} = ABCDE$ and so on. Having put this it will be

$$AB=\frac{p}{p+2q},\quad CD=\frac{p+2r}{p+2q+2r},\quad EF=\frac{p+4r}{p+2q+4r}\quad \text{etc.}$$

and it will be

$$\begin{aligned} ab &= p(p+2q-r),\\ bc &= (p+r)(p+2q),\\ cd &= (p+2r)(p+2q+r),\\ de &= (p+3r)(p+2q+2r)\\ &\text{etc.} \end{aligned}$$

Now let further be

$$\begin{aligned} a &= p+q-r \ +\frac{g}{\alpha},\\ b &= p+q \qquad +\frac{g}{\beta},\\ c &= p+q+r \ +\frac{g}{\gamma},\\ d &= p+q+2r+\frac{g}{\delta}\\ &\text{etc.} \end{aligned}$$

having substituted which values the following equations will emerge after having put $g = q(r-q)$:

$$\begin{aligned} &\alpha\beta \ (p+q-r)\ \alpha-(p+q) \qquad \beta-q(r-q)=0,\\ &\beta\gamma \ (p+q) \qquad \beta-(p+q+r)\ \gamma-q(r-q)=0,\\ &\gamma\delta \ (p+q+r)\ \gamma-(p+q+2r)\delta-q(r-q)=0,\\ &\delta\varepsilon \ (p+q+2r)\delta-(p+q+3r)\varepsilon-q(r-q)=0,\\ &\text{etc.} \end{aligned}$$

Having substituted all these values one will obtain the following continued fractions, by which the letters a, b, c, d etc. will be expressed:

$$a = p+q-r+\cfrac{qr-qq}{2(p+q-r)+\cfrac{2rr+qr-qq}{2(p+q-r)+\cfrac{6rr+qr-qq}{2(p+q-r)+\cfrac{12rr+qr-qq}{2(p+q-r)+\text{etc.}}}}}$$

$$b = p+q+\cfrac{qr-qq}{2(p+q)+\cfrac{2rr+qr-qq}{2(p+q)+\cfrac{6rr+qr-qq}{2(p+q)+\cfrac{12rr+qr-qq}{2(p+q)+\text{etc.}}}}}$$

$$c = p+q+r+\cfrac{qr-qq}{2(p+q+r)+\cfrac{2rr+qr-qq}{2(p+q+r)+\cfrac{6rr+qr-qq}{2(p+q+r)+\cfrac{12rr+qr-qq}{2(p+q+r)+\text{etc.}}}}}$$

etc.

§44 But because the term of the propounded series, that has the index n, is

$$= \frac{\int y^{p+2q-1}dy(1-y^{2r})^{n-1}}{\int y^{p-1}dy(1-y^{2r})^{n-1}},$$

it will be

$$A = \frac{q}{P+2q-r} = \frac{\int y^{p+2q-1}dy : \sqrt{1-y^{2r}}}{\int y^{p-1}dy : \sqrt{1-y^{2r}}}$$

or

$$a = (p+2q-r)\frac{\int y^{p+2q-1}dy : \sqrt{1-y^{2r}}}{\int y^{p-1}dy : \sqrt{1-y^{2r}}}.$$

Then because of $ab = p(p+2q-r)$ it will be

$$b=\frac{b\int y^{p-1}dy:\sqrt{1-y^{2r}}}{\int y^{p+2q-1}dy:\sqrt{1-y^{2r}}}.$$

But because by a theorem exposed in the preceding dissertation it is

$$\frac{p\int y^{p-1}dy:\sqrt{1-y^{2r}}}{\int y^{f+r-1}dy:\sqrt{1-y^{2r}}}=\frac{f\int y^{f-1}dy:\sqrt{1-y^{2r}}}{\int y^{p+r-1}dy:\sqrt{1-y^{2r}}}=\frac{(f+r)\int y^{f+2r-1}dy:\sqrt{1-y^{2r}}}{\int y^{p+r-1}dy:\sqrt{1-y^{2r}}},$$

just put $f=p+2q-r$; after that it will be

$$b=\frac{(p+2q)\int y^{p+2q+r-1}dy:\sqrt{1-y^{2r}}}{\int y^{p+r-1}dy:\sqrt{1-y^{2r}}}.$$

By proceeding in the same way it will indeed be

$$c=\frac{(p+2q+r)\int y^{p+2q+2r-1}dy:\sqrt{1-y^{2r}}}{\int y^{p+2r-1}dy:\sqrt{1-y^{2r}}}$$

and

$$d=\frac{(p+2q+2r)\int y^{p+2q+3r-1}dy:\sqrt{1-y^{2r}}}{\int y^{p+3r-1}dy:\sqrt{1-y^{2r}}}$$

etc.

§45 Because therefore the law of progression of these integral formulas is known, one concluded that the value of this continued fraction

$$p+q+mr+\cfrac{qr-qq}{2(p+q+mr)+\cfrac{2rr+qr-qq}{2(p+q+mr)+\cfrac{6rr+qr-qq}{2(p+q+mr)+\text{etc.}}}}$$

is

$$=(p+2q+mr)\frac{\int y^{p+2q+(m+1)r-1}dy:\sqrt{1-y^{2r}}}{\int y^{p+(m+1)r-1}dy:\sqrt{1-y^{2r}}}.$$

If therefore one puts $p+q+mr=s$, so that $p=s-q-mr$, the following continued fraction will arise

$$s+\cfrac{qr-qq}{2s+\cfrac{2rr+qr-qq}{2s+\cfrac{6rr+qr-rr}{2s+\cfrac{12rr+qr-qq}{2s+\cfrac{20rr+qr-qq}{2s+\text{etc.}}}}}}$$

whose value will therefore be this expression

$$(q+s)\frac{\int y^{q+r+s-1}dy:\sqrt{1-y^{2r}}}{\int y^{r+s-q-1}dy:\sqrt{1-y^{2r}}}.$$

§46 Because in similar manner the value of this continued fraction

$$s+r+\cfrac{qr-qq}{2(s+r)+\cfrac{2rr+qr-qq}{2(s+r)+\cfrac{6rr+qr-qq}{2(s+r)+\text{etc.}}}}$$

is

$$=(q+r+s)\frac{\int y^{p+2r+q-1}dy:\sqrt{1-y^{2r}}}{\int y^{p+2r-q-1}dy:\sqrt{1-y^{2r}}},$$

the product of this two continued fractions will therefore be

$$=(s+q)(s+r-q),$$

as the product of the integral formulas reveals. Hence it is by the theorem given in the preceding dissertation

$$\frac{f}{a}=\frac{\int x^{a-1}dx:\sqrt{1-x^{2r}}\cdot\int x^{a+r-1}dx:\sqrt{1-x^{2r}}}{\int x^{f-1}dx:\sqrt{1-x^{2r}}\cdot\int x^{f+r-1}dx:\sqrt{1-x^{2r}}},$$

to which form the product of the integral formulas is reduced by itself.

§47 The found continued fraction can be transformed into another more convenient form, that the single numerators can be resolved into factors; so one will have this continued fraction

$$s+\cfrac{q(r-q)}{2s+\cfrac{(r+q)(2r-q)}{2s+\cfrac{(2r+q)(3r-q)}{2s+\cfrac{(3r+q)(4r-q)}{2s+\text{etc.}}}}}$$

whose value will therefore be

$$=(q+s)\frac{\int y^{r+s+q-1}dy:\sqrt{1-y^{2r}}}{\int y^{r+s-q-1}dy:\sqrt{1-y^{2r}}}.$$

If therefore s is added to the continued fraction, that the law of progression is the same everywhere, it will be

$$\frac{(q+s)\int y^{r+s+q-1}dy:\sqrt{1-y^{2r}}+\int y^{r+s-q-1}dy:\sqrt{1-y^{2r}}}{\int y^{r+s+q-1}dy:\sqrt{1-y^{2r}}}$$

$$=2s+\cfrac{q(r-q)}{2s+\cfrac{(r+q)(2r-q)}{2s+\cfrac{(2r+q)(3r-q)}{2s+\cfrac{(3r+q)((4r-q)}{2s+\text{etc.}}}}}$$

§48 If now one puts $r=2$ and $q=1$, combined all continued fractions exhibited by Brouncker will arise, those fractions, that will all be contained in this continued fraction

$$s+\cfrac{1}{2s+\cfrac{9}{2s+\cfrac{25}{2s+\cfrac{49}{2s+\cfrac{81}{2s+\text{etc.}}}}}}$$

whose value will therefore be

$$= (s+1)\frac{\int y^{s+2}dy : \sqrt{1-y^2}}{\int y^s dy : \sqrt{1-y^2}},$$

which expression agrees perfectly with that one, that we assigned above, before the validity was completely known; see § 16.

§49 Whereas up to now I gave many continued fractions, whose values can be assigned by integral formulas, I will now explain a direct method, by means of which it is possible to get vice versa from integral formulas to continued fractions. But this method is based on a reduction of one integral formula to two others, which reduction is quite similar to that usual one, by which the integration of a certain differential formula is reduced to the integration of another. Let the infinitely many integral formulas be of this kind

$$\int Pdx, \quad \int PRdx, \quad \int PR^2dx, \quad \int PR^3dx, \quad \int PR^4dx \quad \text{etc.},$$

which shall be constituted in such a way, that, if the single ones are integrated, that they vanish for $x=0$, and then x is put $=1$, it is as follows

$$\begin{array}{rlrlrl} a & \int Pdx &= b & \int PRdx &+ c & \int PR^2dx, \\ (a+\alpha) & \int PRdx &= (b+\beta) & \int PR^2dx &+ (c+\gamma) & \int PR^3dx, \\ (a+2\alpha) & \int PR^2dx &= (b+2\beta) & \int PR^3dx &+ (c+2\gamma) & \int PR^4dx, \\ (a+3\alpha) & \int PR^3dx &= (b+3\beta) & \int PR^4dx &+ (c+3\gamma) & \int PR^5dx \end{array}$$

and in general

$$(a+n\alpha)\int PR^ndx = (b+n\beta)\int PR^{n+1}dx + (c+n\gamma)\int PR^{n+2}dx.$$

§50 If therefore one has integral formulas of this kind, as an easy task continued fraction will be formed from them. Because it is

$$\begin{aligned}
\frac{\int Pdx}{\int PRdx} &= \frac{b}{a} + \frac{c\int PR^2dx}{a\int PRdx},\\
\frac{\int PRdx}{\int PR^2dx} &= \frac{b+\beta}{a+\alpha} + \frac{(c+\gamma)\int PR^3dx}{(a+\alpha)\int PR^2dx},\\
\frac{\int PR^2dx}{\int PR^3dx} &= \frac{b+2\beta}{a+2\alpha} + \frac{(c+2\gamma)\int PR^4dx}{(a+2\alpha)\int PR^3dx},\\
\frac{\int PR^3dx}{\int PR^4dx} &= \frac{b+3\beta}{a+3\alpha} + \frac{(c+3\gamma)\int PR^5dx}{(a+3\alpha)\int PR^4dx}
\end{aligned}$$

etc.,

by substituting each value in the preceding equation it will be

$$\frac{\int Pdx}{\int PRdx} = \frac{b}{a} + \cfrac{c:a}{\cfrac{b+\beta}{a+\alpha} + \cfrac{(c+\gamma):(\alpha+a)}{\cfrac{b+2\beta}{a+2\alpha} + \cfrac{(c+2\gamma):(a+2\alpha)}{\cfrac{b+3\beta}{a+3\alpha} + \cfrac{(c+3\gamma)(:(a+3\alpha)}{\cfrac{b+4\beta}{a+4\alpha} + \text{etc.}}}}}$$

But this expression inverted and freed from partial fraction merges into this one

$$\frac{\int PRdx}{\int Pdx} = \cfrac{a}{b + \cfrac{(a+\alpha)c}{b+\beta + \cfrac{(a+2\alpha)(c+\gamma)}{b+2\beta + \cfrac{(a+3\alpha)(c+2\gamma)}{b+3\beta + \cfrac{(a+4\alpha)(c+3\gamma)}{b+4\beta} + \text{etc.}}}}}$$

§51 If, while n was a negative integer, it was

$$(a+n\alpha)\int PR^n dx = (b+n\beta)\int PR^{n+1}dx + (c+n\gamma)\int PR^{n+2}dx,$$

one will have the following equations:

$$
\begin{aligned}
(a-\alpha)\int\frac{Pdx}{R} &= (b-\beta)\int Pdx + (c-\gamma)\int PRdx,\\
(a-2\alpha)\int\frac{Pdx}{R^2} &= (b-2\beta)\int\frac{Pdx}{R} + (c-2\gamma)\int Pdx,\\
(a-3\alpha)\int\frac{Pdx}{R^3} &= (b-3\beta)\int\frac{Pdx}{R^2} + (c-3\gamma)\int\frac{Pdx}{R},\\
(a-4\alpha)\int\frac{Pdx}{R^4} &= (b-4\beta)\int\frac{Pdx}{R^3} + (c-3\gamma)\int\frac{Pdx}{R^2}
\end{aligned}
$$

etc.

Hence on concludes in the same way

$$
\begin{aligned}
\frac{\int PRdx}{\int Pdx} &= \frac{-(b-\beta)}{c-\gamma} + \frac{(a-\alpha)\int Pdx:R}{(c-\gamma)\int Pdx},\\
\frac{\int Pdx}{\int Pdx:R} &= \frac{-(b-2\beta)}{c-2\gamma} + \frac{(a-2\alpha)\int Pdx:R^2}{(c-2\gamma)\int Pdx:R},\\
\frac{\int Pdx:R}{\int Pdx:R^2} &= \frac{-(b-3\beta)}{c-3\gamma} + \frac{(a-3\alpha)\int Pdx:R^3}{(c-3\gamma)\int Pdx:R^2}
\end{aligned}
$$

etc.

But from these equations one produces

$$
\frac{\int Prdx}{Pdx} = \frac{-(b-\beta)}{c-\gamma} + \cfrac{(a-\alpha):(c-\gamma)}{\cfrac{-(b-2\beta)}{c-2\gamma} + \cfrac{(a-2\alpha):(c-2\gamma)}{\cfrac{-(b-3\beta)}{c-3\gamma} + \cfrac{(a-3\alpha):(c-3\gamma)}{\cfrac{-(b-4\beta)}{c-\gamma} + \text{etc.}}}}
$$

or after having removed the partial fractions

$$
\frac{(c-\gamma)\int PRdx}{\int Pdx} = -(b-\beta) + \cfrac{(a-\alpha)(c-2\gamma)}{-(b-2\beta) + \cfrac{(a-2\alpha)(c-3\gamma)}{-(b-3\beta) + \cfrac{(a-3\alpha)(c-4\gamma)}{-(b-4\beta) + \text{etc.}}}}
$$

So one has two continued fractions, whose value is the same, $\frac{\int PRdx}{\int Pdx}$.

§52 But it is most important in this task, that appropriate functions of x to be substituted for P and R are defined, that it is

$$(a+n\alpha)\int PR^n dx = (b+n\beta)\int PR^{n+1}dx + (c+n\gamma)\int PR^{n+2}dx$$

at least in that case, in which after the single integrations one puts $x=1$. Now let us put, that it is in general

$$(a+n\alpha)\int PR^n dx + R^{n+1}S = (b+n\beta)\int PR^{n+1}dx + (c+n\gamma)\int PR^{n+2}dx$$

and that $R^{n+1}S$ is a function of such a kind of x, that vanishes so for $x=0$ as $x=1$. Having taken differentials and having divided by R^n it will be

$$(a+n\alpha Pdx + RdS + (n+1)SdR = (b+n\beta)PRdx + (c+n\gamma)PR^2dx;$$

this equation, because it has to hold in any case, no matter what n is, is resolved into these two equations

$$aPdx + RdS + SdR = bPRdx + cPR^2dx$$

and

$$\alpha Pdx + SdR = \beta PRdx + \gamma PR^2dx.$$

From these equations one finds in two ways

$$Pdx = \frac{RdS+SdR}{bR+cR^2-a} = \frac{SdR}{\beta R+\gamma R^2-\alpha},$$

whence it becomes

$$\begin{aligned}\frac{dS}{S} &= \frac{(b-\beta)RdR+(c-\gamma)R^2dR-(a-\alpha)dR}{\beta R^2+\gamma R^3-\alpha R}\\ &= \frac{(a-\alpha)dR}{\alpha R} + \frac{(\alpha b-\beta a)dR+(\alpha c-\gamma a)RdR}{\alpha(\beta R+\gamma R^2-\alpha)}.\end{aligned}$$

So from this equation S is defined by R; but having found S, it will be

$$P = \frac{SdR}{(\beta R-\gamma R^2-\alpha)dx}$$

and hence the formulas $\int Pdx$ and $\int PRdx$ will be known, by which the value of the superior continued fraction is determined.

§53 Because the quantity R is therefore not defined by x, one can take any function of x for it. But because the condition of the question demands, that $R^{n+1}S$ vanishes so for $x = 0$ as for $x = 1$, the nature of the function R is determined by this condition itself. Further one has to pay attention to that, that the integrals $\int PR^n dx$ having put $x = 1$ after the integration obtain a finite value; hence if these integrals in this case would either become 0 or ∞, then the value $\frac{\int PRdx}{\int Pdx}$ would be hard to calculate. The first inconvenience is quite surely avoided by assigning a value of such a kind to R, that PR^n never takes a negative value, as long x lies between the limits 0 and 1. But that $\int PR^n dx$ does not become infinite for $x = 1$, is harder to achieve in the most cases. But it will be convenient, to seperate the cases, in which n is either a positive or an negative number, from each other, because very often, if these conditions are satisfied, while n is a positive number, in the remaining cases they are not satisfied at the same time. But if the prescribed conditions are only fulfilled in the cases, in which n is a positive number, then only the value of the first continued fraction can be exhibited, but the second on the other hand, if the conditions were satisfied, while n is a negative number.

§54 Let us start this expansion of the method to find the values of continued fractions from the example treated before, and at first let this continued fraction be propounded

$$r + \cfrac{fh}{r + \cfrac{(f+r)(h+r)}{r + \cfrac{(f+2r)(h+2r)}{r + \text{etc.}}}}$$

whose value was above in § 34 assigned as this one

$$\frac{h(f-r)\int y^{h+r-1}dy : \sqrt{1-y^{2r}} - f(h-r)\int y^{f+r-1}dy : \sqrt{1-y^{2r}}}{f\int y^{f+r-1}dy : \sqrt{1-y^{2r}} - h\int y^{h+r-1}dy : \sqrt{1-y^{2r}}}.$$

So compare this continued fraction to this general one

$$\frac{a\int Pdx}{\int PRdx} = b + \cfrac{(a+\alpha)c}{b+\beta + \cfrac{(a+2\alpha)(c+\gamma)}{b+2\beta + \cfrac{(a+3\alpha)(c+2\gamma)}{b+3\beta + \text{etc.}}}}$$

and it will be

$$b = r, \quad \beta = 0, \quad \alpha = r, \quad \gamma = r, \quad a = f - r, \quad c = h.$$

Having substituted this value it will arise

$$\frac{ds}{S} = \frac{rRdR + (h-r)R^2dR - (f-2r)dR}{rR^3 - rR} = \frac{(f-2r)dR}{rR} + \frac{rdR + (h-f+r)RdR}{r(R^2-1)}$$

and by integrating

$$\log S = \frac{f-2r}{r}\log R + \frac{h-f}{2r}\log(R+1) + \frac{h-f+2r}{2r}\log(R-1) + \log C$$

or

$$S = CR^{\frac{f-2r}{r}}(R^2-1)^{\frac{h-f}{2r}}(R-1).$$

Hence it will therefore be

$$R^{n+1}S = CR^{\frac{f-(n-1)r}{r}}(R^2-1)^{\frac{h-f}{2r}}(R-1)$$

and

$$Pdx = \frac{CR^{\frac{f-2r}{r}}(R^2-1)^{\frac{h-f}{2r}}dR}{r(R+1)}.$$

§55 But because $R^{n+1}S$ has to vanish in two cases, so for $x = 0$ and $x = 1$, and that, no matter which positive number is substituted for n (for, to pay attention to negative values is not neccessary), let us put, that f, h and r, are positive numbers and $h > f$, what can surely be assumed, if not $f = h$, further it shall be $f > r$. Having put these things, it is manifest, that the formula $R^{n+1}S$ vanishes in two cases, of course if $R = 0$ and if $R = 1$; and this also holds, if $f = h$. As long as $f > r$, one can put $R = x$ and it will be

$$Pdx = \frac{x^{\frac{f-2r}{r}}(1-x^2)^{\frac{h-f}{2r}}dx}{1+x}$$

after having determined the constant C. From this the value of the propounded continued fraction will be

$$= (f-r)\frac{\int \frac{x^{\frac{f-2r}{r}}(1-x^2)^{\frac{h-f}{2r}}dx}{1+x}}{\int \frac{x^{\frac{f-r}{r}}(1-x^2)^{\frac{h-f}{2r}}dx}{1+x}}$$

But for $x = y^r$ the searched value will be

$$= \frac{(f-r)\int y^{f-r-1}(1-y^{2r})^{\frac{h-f}{2r}}dy : (1+y^r)}{\int y^{f-1}(1-y^{2r})^{\frac{h-f}{2r}}dy : (1+y^r)}.$$

§56 Hence we obtained another expression containing the value of this continued fraction

$$r + \cfrac{fh}{r + \cfrac{(f+r)(h+r)}{r + \text{etc.}}},$$

which expression, even though it contains integral formulas in it, nevertheless differs from the expression found before. Hence this last expression is not valid, if not $f > r$; but for h one has to take the smaller of the two quantities f and h, if there were not equal, of course. But nevertheless, if also f was was smaller than r, the value of the continued fraction can be exhibited by considering this one

$$r + \cfrac{(f+r)(h+r)}{r + \cfrac{(f+2r)(h+2r)}{r + \text{etc.}}},$$

whose value will be

$$= \frac{f\int y^{f-1}(1-y^{2r})^{\frac{h-f}{2r}}dy : (1+y^r)}{\int y^{f+r-1}(1-y^{2r})^{\frac{h-f}{2r}}dy : (1+y^r)},$$

which does not need any restriction. Hence having put this value $= V$ the value of the propounded continued fraction will be $= r + \frac{fh}{V}$.

§57 That case, in which $f = h$, which had been found before by a peculiar method and its value was detected in § 34 as

$$= \frac{1-(h-r)\int x^{h-1}dx:(1+x^r)}{\int x^{h-1}dx:(1+x^r)} = \frac{(h-r)\int x^{h-r-1}dx:(1+x^r)}{\int x^{h-1}dx:(1+x^r)},$$

flows from this last expression by itself; hence for $f = h$ the expression found in § 55 will merge into this one

$$\frac{(h-r)\int y^{h-r-1}dy:(1+y^r)}{\int y^{h-1}dy:(1+y^r)}$$

completely the same, whence the agreement of the both general expressions is seen clear enough. Here it is surely possible to assume $h > r$, because the cases, in which this does not happen, are very easily reduced to those, as it was just shown.

§58 That this agreement of both expressions is understood in any case, we have to mention this lemma in advance, which was already proved by others. If we have this series

$$1 + \frac{p}{q+s} + \frac{p(p+s)}{(q+s)(q+2s)} + \frac{p(p+s)(p+2s)}{(q+s)(q+2s)(q+3s)} + \text{etc.},$$

in which the quantities p, q and s are positive and $q > p$, the sum of this series continued to infinity will be $= \frac{q}{q-p}$. But the truth of this lemma can be shown by my general method to sum series in the following way. Just consider this series

$$x^q + \frac{p}{q+s}x^{q+s} + \frac{p(p+s)}{(q+s)(q+2s)}x^{q+2s} + \text{etc.},$$

whose sum shall be z, and it will be by differentiating

$$\frac{dz}{dx} = qx^{q-1} + px^{q+s-1} + \frac{p(p+s)}{q+s}x^{q+2s-1} + \text{etc.}$$

and

$$x^{p-q-z}dz = qx^{p-s-1}dx + px^{p-1}dx + \frac{p(p+s)}{q+s}x^{p+s-1}dx + \text{etc.},$$

which equation integrated gives

$$\int x^{p-q-z}dz = \frac{qx^{p-s}}{p-s} + x^p + \frac{px^{p+s}}{p+s} + \text{etc.} = \frac{qx^{p-s}}{p-s} + x^{p-q}z.$$

From this equation differentiated this one will arise

$$x^{p-q-s}ds = qx^{p-s-1}dx + x^{p-q}dx + (p-q)x^{p-q-1}zdx$$

or

$$dz(1-x^s) + (q-p)x^{s-1}zdx = qx^{q-1}dx$$

or

$$dz + \frac{(q-p)x^{s-1}zdx}{1-x^s} = \frac{qx^{q-1}dx}{1-x^s},$$

whose integral is

$$\frac{z}{(1-x^s)^{\frac{q-p}{s}}} = q\int\frac{x^{p-1}dx}{(1-x^s)^{\frac{q-p+s}{s}}} = \frac{qx^q}{(q-p)(1-x^s)^{\frac{q-p}{s}}} - \frac{pq}{q-p}\int\frac{x^{q-1}dx}{(1-x^s)^{\frac{q-p}{s}}},$$

whence it will be

$$z = \frac{qx^q}{q-p} - \frac{pq(1-x^s)^{\frac{q-p}{s}}}{q-p}\int\frac{x^{q-1}dx}{(1-x^s)^{\frac{q-p}{s}}}.$$

Therefore for $x=1$ it will be

$$z = \frac{q}{q-p} = 1 + \frac{p}{q+s} + \frac{p(p+s)}{(q+s)(q+2s)} + \text{etc.},$$

which is the demonstration of the given lemma, whence at the same time it is understood, that the validity of this lemma does not hold, if not $q > p$.

§59 Because we therefore have the value of the continued fraction

$$r + \cfrac{fh}{r + \cfrac{(f+r)(h+r)}{r + \cfrac{(f+2r)(h+2r)}{r + \text{etc.}}}}$$

expressed in two ways, the one of which is

$$= \frac{h(f-r)\int y^{h+r-1}dy:\sqrt{1-y^{2r}} - f(h-r)\int y^{f+r-1}dy:\sqrt{1-y^{2r}}}{f\int y^{f+r-1}dy:\sqrt{1-y^{2r}} - h\int y^{h+r-1}dy:\sqrt{1-y^{2r}}},$$

the other indeed, that was found in § 56,

$$= r + \frac{h\int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f}{2r}}:(1+y^r)}{\int y^{f-1}dy(1-y^{2r})^{\frac{h-f}{2r}}:(1+y^r)},$$

it will be worth the effort, to demonstrate the agreement of these expressions. Because therefore it is

$$\frac{1}{1+y^r} = \frac{1-y^r}{1-y^{2r}},$$

it will be

$$\int y^{f-1}dy(1-y^{2r})^{\frac{h-f}{2r}}:(1+y^r) = \int y^{f-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}} - \int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}}$$

and

$$\begin{aligned} &\int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f}{2r}}:(1+y^r) \\ &= \int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}} - \int y^{f+2r-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}} \\ &= \int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}} - \frac{f}{h}\int y^{f-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}}. \end{aligned}$$

Now just put

$$\frac{\int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}}}{\int y^{f-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}}} = V;$$

the last value of the continued fraction will be

$$= r + \frac{hV-f}{1-V}.$$

Now furthermore put

$$\frac{\int y^{h+r-1}dy:\sqrt{1-y^{2r}}}{\int y^{f+r-1}dy:\sqrt{1-y^{2r}}} = W;$$

the first value will be

$$= \frac{h(f-r)W - f(h-r)}{f-hW},$$

from whose equality it follows, that it will be

$$V = \frac{f}{hW},$$

so that it is

$$\frac{\int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}}}{\int y^{f-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}}} = \frac{f\int y^{f+r-1}dy:\sqrt{1-y^{2r}}}{h\int y^{h+r-1}dy:\sqrt{1-y^{2r}}},$$

the reason for which equality is known by the theorems exhibited in the preceding dissertation; hence it is by one from those theorems

$$\frac{\int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}}}{\int y^{f+2r-1}dy(1-y^{2r})^{\frac{h-f-2r}{2r}}} = \frac{\int y^{f+r-1}dy:\sqrt{1-y^{2r}}}{\int y^{h+r-1}dy:\sqrt{1-y^{2r}}}.$$

§60 Now let us consider this continued fraction

$$2r + \cfrac{fh}{2r + \cfrac{(f+r)(h+r)}{2r + \cfrac{(f+2r)(h+2r)}{2r + \text{etc.}}}}$$

whose value was found above in § 35

$$= \frac{2(f-r)(h-r)\int y^{f-1}dy:\sqrt{1-y^{2r}} - h(f+h-3r)\int y^{h+r-1}dy:\sqrt{1-y^{2r}}}{2h\int y^{h+r-1}dy:\sqrt{1-y^{2r}} - (f+h-r)\int y^{f-1}dy:\sqrt{1-y^{2r}}}$$

If now this continued fraction is compared to this one

$$\frac{a\int Pdx}{\int PRdx} = b + \cfrac{(a+\alpha)c}{b+\beta + \cfrac{(a+2\alpha)(c+\gamma)}{b+2\beta + \cfrac{(a+3\alpha)(c+2\gamma)}{b+2\beta + \text{etc.}}}}$$

it will be

$$b = 2r, \quad \beta = 0, \quad \alpha = r, \quad \gamma = r, \quad a = f - r \quad \text{and} \quad c = h.$$

Hence from § 52 one will have

$$\frac{dS}{S} = \frac{(f-2r)dR}{rR} + \frac{2rdR + (h-f+r)RdR}{r(R^2-1)}$$

and by integrating

$$S = CR^{\frac{f-2r}{2r}}(R^2-1)^{\frac{h-f-r}{2r}}(R-1)^2,$$

whence it becomes

$$Pdx = \frac{C}{r}R^{\frac{f-2r}{r}}(R^2-1)^{\frac{h-f-2r}{2r}}(R-1)^2dR$$

and

$$R^{n+1}S = CR^{\frac{f+(n-1)r}{r}}(R^2-1)^{\frac{h-f-r}{2r}}(R-1)^2,$$

which expression vanishes in two cases, for $R = 0$ and for $R = 1$, if only $f > r$ and $h - 3r > f$, which conditions can always be satisfied.

§61 Now let $R = x$ and having determined the constant C it will be

$$Pdx = x^{\frac{f-2r}{r}}dx(1-x^2)^{\frac{h-f-3r}{2r}}(1-x)^2$$

or having put $R = x = y^r$ it will be

$$Pdx = y^{f-r-1}dy(1-y^{2r})^{\frac{h-f-3r}{2r}}(1-y^r)^2,$$

from which the value of the propounded continued fraction will be

$$\frac{a\int Pdx}{\int PRdx} = \frac{(f-r)\int y^{f-r-1}dy(1-y^{2r})^{\frac{h-f-3r}{2r}}(1-y^r)^2}{\int y^{f-1}dy(1-y^{2r})^{\frac{h-f-3r}{2r}}(1-y^r)^2},$$

which ba the theorems of the superior dissertation will be to the first form by expanding the square $(1-y^r)^2$, whereafter each of both integral formulas will be resolved into two simpler ones. But I will show the reduction in the following further reaching example.

§62 If one has this integral formula

$$\int y^{m-1}dy(1-y^{2r})^{\varkappa}(1-y^r)^n$$

and $(1-y^r)^n$ is resolved into the series

$$1-ny^r+\frac{n(n-1)}{1\cdot 2}y^{2r}-\text{etc.},$$

by taking every second term the propounded integral formula will be reduced to the two following ones

$$\int y^{m-1}dy(1-y^{2r})^{\varkappa}\left(1+\frac{n(n-1)}{1\cdot 2}\cdot\frac{m}{p}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{m(m+2r)}{p(p+2r)}+\text{etc.}\right)$$

$$-\int y^{m+r-1}dy(1-y^{2r})^{\varkappa}\left\{\begin{array}{c} n+\dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\dfrac{m+r}{p+r} \\ +\dfrac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\dfrac{(m+r)(m+3r)}{(p+r)(p+3r)}+\text{etc.}\end{array}\right\}$$

after having put for the sake of brevity

$$m+2\varkappa r+2r=p.$$

Hence if as in the preceding case it was $n=2$, it will be

$$\int y^{m-1}dy(1-y^{2r})^{\varkappa}(1-y^r)^2=\frac{m+p}{p}\int y^{m-1}dy(1-y^{2r})^{\varkappa}-2\int y^{m+r-1}dy(1-y^{2r})^{\varkappa}.$$

From this one will have

$$\frac{a\int Pdx}{\int PRdx}=\frac{\frac{(f-r)(f+h-3r)}{h-2r}\int y^{f-r-1}dy(1-y^{2r})^{\frac{h-f-3r}{2r}}-2(f-r)\int y^{f-1}dy(1-y^{2r})^{\frac{h-f-3r}{2r}}}{\frac{f+h-r}{h-r}\int y^{f-1}dy(1-y^{2r})^{\frac{h-f-3r}{2r}}-2\int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f-3r}{2r}}}$$

$$=\frac{h(f+h-3r)\int y^{f-r-1}dy(1-y^{2r})^{\frac{h-f-r}{2r}}-2(f-r)(h-r)\int y^{f-1}dy(1-y^{2r})^{\frac{h-f-r}{2r}}}{(f+h-r)\int y^{f-1}dy(1-y^{2r})^{\frac{h-f-r}{2r}}-2h\int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f-r}{2r}}},$$

which expression, because it hast to be equal to that one, that was found above in § 35, will yield this equation

$$\frac{\int y^{f+r-1}dy(1-y^{2r})^{\frac{h-f-r}{2r}}}{\int y^{f-1}dy(1-y^{2r})^{\frac{h-f-r}{2r}}}=\frac{\int y^{h+r-1}dy:\sqrt{1-y^{2r}}}{\int y^{f-1}dy:\sqrt{1-y^{2r}}},$$

the reason for which is indeed contained in the theorems of the superior dissertation.

§63 Now let us vice versa take given values for P and R and let us form continued fractions from them and let us put

$$P=x^{m-1}(1-x^r)^n(p+qx^r)^{\varkappa}\quad\text{and}\quad R=x^r.$$

But because it has to be

$$(a+\nu\alpha)\int PR^{\nu}dx=(b+\nu\beta)\int PR^{\nu+1}dx+(c+\nu\gamma)\int PR^{\nu+2}dx$$

and hence because of the given P and R it is from § 52

$$S=\frac{1}{r}x^{m-r}(1-x^r)^n(p+qx^r)^{\varkappa}(\gamma x^{2r}+\beta x^r-\alpha),$$

it will be be

$$\begin{aligned}\frac{dS}{S}&=\frac{(m-r)dx}{x}+\frac{nrx^{r-1}dx}{-1+x^r}+\frac{\varkappa qrx^{r-1}dx}{p+qx^r}+\frac{2\gamma rx^{2r-1}dx+\beta rx^{r-1}dx}{\gamma x^{2r}+\beta x^r-\alpha}\\&=\frac{(a-\alpha)rdx}{\alpha x}+\frac{(\alpha b-\beta a)rx^{r-1}dx+(\alpha c-\gamma a)rx^{2r-1}dx}{\alpha(\gamma x^{2r}+\beta x^r-\alpha)}.\end{aligned}$$

Now let be

$$(p+qx^r)(x^r-1)=\gamma x^{2r}+\beta x^r-\alpha;$$

it will be

$$\gamma=q,\quad\beta=p-q\quad\text{and}\quad\alpha=p.$$

Let furthermore be

$$\frac{(a-\alpha)r}{\alpha}=m-r;$$

it will be

$$a = \frac{mp}{r}.$$

Hence it will further be

$$nqr + \varkappa qr + 2qr = \frac{cpr - mpq}{p}$$

or

$$c = \frac{mq}{r} + nq + (\varkappa + 2)q$$

and finally

$$b = \frac{m(p-q)}{r} + (n+1)p - (\varkappa + 1)q.$$

So as long as m and $n+1$ were positive numbers, that $R^{\nu+1}S$ vanishes so for $x =$ as for $x = 1$, the following expression will arise

$$\frac{\int x^{m+r-1}dx(1-x^r)^n(p+qx^r)^\varkappa}{\int x^{m-1}dx(1-x^r)^n(p+qx^r)^\varkappa} = \frac{\int PRdx}{\int Pdx},$$

which will therefore be equal to this continued fraction

$$\cfrac{mp}{m(p-q) + (n+1)pr - (\varkappa+1)qr + \cfrac{pq(m+r)(m+nr+(\varkappa+2)r)}{m(p-q) + (n+2)pr - (\varkappa+2)qr + \cfrac{pq(m+2r)(m+(n+1)r+(\varkappa+2)r)}{m(p-q) + (n+3)pr - (\varkappa+3)qr + \text{etc.}}}}$$

§64 That this continued fraction takes a simpler form, just put

$$m + nr + r = a, \quad m + \varkappa r + r = b \quad \text{and} \quad m + nr + \varkappa r + r = c;$$

it will be

$$\varkappa = \frac{c-a}{r}, \quad n = \frac{c-b}{r} \quad \text{and} quad m = a + b - c - r$$

and therefore it will be

$$\cfrac{p(a+b-c-r)}{ap-bq+\cfrac{pq(a+b-c)(c+r)}{(a+r)p-(b+r)q+\cfrac{pq(a+b-c+r)(c+2r)}{(a+2r)p-(b+2r)q+\cfrac{pq(a+b-c+2r)(c+3r)}{(a+3r)p-(b+3r)q+\text{etc.}}}}}$$

$$=\frac{\int x^{a+b-c-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}{\int x^{a+b-c-r-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}$$

having put $x=1$ after each integration. But it is required, that

$$a+b-c-r \quad \text{and} \quad c-b+r$$

are positive numbers. But if for the sake of brevity it is put

$$a+b-c-r=g,$$

it will be

$$\frac{\int x^{g-r-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}{\int x^{g-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}$$

$$=\cfrac{pg}{ap-bq+\cfrac{pq(c+r)(g+r)}{(a+r)p-(b+r)q+\cfrac{pq(c+2r)(g+2r)}{(a+2r)p-(b+2r)q+\text{etc.}}}}$$

which equation extends very far and contains all continued fractions found up to now.

§65 If the quantities c and g are commuted, the following continued fraction will arise

$$\cfrac{pc}{ap-bq+\cfrac{pq(c+r)(g+r)}{(a+r)p-(b+r)q+\cfrac{pq(c+2r)(g+2r)}{(a+2r)p-(b+2r)q+\text{etc.}}}},$$

whose value will therefore be

$$\frac{\int x^{c+r-1}dx(1-x^r)^{\frac{g-b}{r}}(p+qx^r)^{\frac{g-a}{r}}}{\int x^{c-1}dx(1-x^r)^{\frac{g-b}{r}}(p+qx^r)^{\frac{g-a}{r}}}.$$

Because these continued fractions have a given ratio, of course g to c, the following theorem will arise from this after having resubstituted the value of g

$$\frac{c\int x^{a+b-c-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}{\int x^{a+b-c-r-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}$$
$$=\frac{(a+b-c-r)\int x^{c+r-1}dx(1-x^r)^{\frac{a-c-r}{r}}(p+qx^r)^{\frac{b-c-r}{r}}}{\int x^{c-1}dx(1-x^r)^{\frac{a-c-r}{r}}(p+qx^r)^{\frac{b-c-r}{r}}}.$$

This very far reaching formula contains many extraordinary particular reductions. So let for example be $b=c+r$; it will be

$$\frac{a\int x^{a+r-1}dx(p+qx^r)^{\frac{c-a}{r}}:(1-x^r)}{\int x^{a-1}dx(p+qx^r)^{\frac{c-a}{r}}:(1-x^r)}=\frac{a\int x^{a+r-1}dx(1-x^r)^{\frac{a-c-r}{r}}}{\int x^{c-1}dx(1-x^r)^{\frac{c-a}{r}}}=c,$$

whence it follows, that it will be

$$\int\frac{x^{a+r-1}dx(p+qx^r)^{\frac{c-a}{r}}}{1-x^r}=\int\frac{x^{a-1}dx(p+qx^r)^{\frac{c-a}{r}}}{1-x^r}.$$

One will therefore have this further extending theorem

$$\int\frac{x^{m-1}dx(p+qx^r)^{\varkappa}}{1-x^r}=\int\frac{x^{n-1}dx(p+qx^r)^{\varkappa}}{1-x^r},$$

where, after having executed the integration in such a way, that the integrals vanish for $x=0$, x is understood to become $=1$. But alone that case is excluded, in which $q+p=0$, in which the inconvenience appears.

§66 The continued fractions, that we found up to now by interpolation, go back to this, that the partial denominators are constant. To transfer this now found general form to those, put $p=q=1$ and this continued fraction will arise

$$\cfrac{cg}{a-b+\cfrac{(c+r)(g+r)}{a-b+\cfrac{(c+2r)(g+2r)}{a-b+\cfrac{(c+3r)(g+3r)}{a-b+\text{etc.}}}}} = \frac{c\int x^{g+r-1}dx(1-x^r)^{\frac{c-b}{r}}(1+x^r)^{\frac{c-a}{r}}}{\int x^{g-1}dx(1-x^r)^{\frac{c-b}{r}}(1+x^r)^{\frac{c-a}{r}}}$$

or the value of the same will also be

$$= \frac{g\int x^{c+r-1}dx(1-x^r)^{\frac{g-b}{r}}(1+x^r)^{\frac{g-a}{r}}}{\int x^{c-1}dx(1-x^r)^{\frac{g-b}{r}}(1+x^r)^{\frac{g-a}{r}}}$$

while $g = a+b-c-r$. Just put

$$a-b=s;$$

because of

$$a+b=c+g+r$$

it will be

$$a = \frac{c+g+r+s}{2} \quad \text{and} \quad b = \frac{c+g+r-s}{2},$$

whence it will be

$$\cfrac{cg}{s+\cfrac{(c+r)(g+r)}{s+\cfrac{(c+2r)(g+2r)}{s+\text{etc.}}}}$$

$$= \frac{c\int x^{g+r-1}dx(1-x^{2r})^{\frac{c-g-r-s}{2r}}(1-x^r)^{\frac{s}{r}}}{\int x^{g-1}dx(1-x^{2r})^{\frac{c-g-r-s}{2r}}(1-x^r)^{\frac{s}{r}}} = \frac{g\int x^{c+r-1}dx(1-x^{2r})^{\frac{g-c-r-s}{2r}}(1-x^r)^{\frac{s}{r}}}{c\int x^{c-1}dx(1-x^{2r})^{\frac{g-c-r-s}{2r}}(1-x^r)^{\frac{s}{r}}}.$$

§67 Let us put, to get to the from in § 47, $2s$ instead of s and let be $c=q$ and $g=r-q$; one will have this continued fraction

$$\cfrac{q(r-q)}{2s+\cfrac{(q+r)(2r-q)}{2s+\cfrac{(q+2s)(3r-q)}{2s+\text{etc.}}}},$$

whose value will therefore be either

$$=\frac{q\int x^{2r-q-1}dx(1-x^{2r})^{\frac{q-r-s}{r}}(1-x^r)^{\frac{2s}{r}}}{\int x^{r-q-1}dx(1-x^{2r})^{\frac{q-r-s}{r}}(1-x^r)^{\frac{2s}{r}}}$$

or

$$=\frac{(r-q)\int x^{q+r-1}dx(1-x^{2r})^{\frac{-q-s}{r}}(1-x^r)^{\frac{2s}{r}}}{\int x^{q-1}dx(1-x^{2r})^{\frac{-q-s}{r}}(1-x^r)^{\frac{2s}{r}}}.$$

The value of the same continued fraction was found before

$$=\frac{(q+s)\int y^{r+s+q-1}dy:\sqrt{1-y^{2r}}}{\int y^{r+s-q-1}dy:\sqrt{1-y^{2r}}}-s.$$

Therefore these integral formulas will be equal to each other; this is a theorem, by no means to be contemned.

§68 Let, as we put in § 48, $r=2$ and $q=1$; it will be

$$\frac{(1+s)\int y^{s+2}dy:\sqrt{1-y^4}}{\int y^s dy:\sqrt{1-y^4}}-s=\frac{\int x^2dx(1-x^4)^{\frac{-s-1}{2}}(1-x^2)^s}{\int dx(1-x^4)^{-\frac{-s-1}{2}}(1-x^2)^s},$$

which equality is conspicuous, if $s=0$; but in the cases, in which s is an odd integer, the equality is not hard to show. As if $s=1$, the last formula will be

$$\frac{\int xxdx:(1+xx)}{\int dx:(1+xx)}=\frac{x-\int dx:(1+xx)}{\int dx:(1+xx)}=\frac{4-\pi}{\pi}$$

for $x=1$. The first formula on the other hand will give

$$\frac{2\int y^3dy:\sqrt{1-y^4}}{\int ydy:\sqrt{1-y^4}}-1=\frac{4}{\pi}-1=\frac{4-\pi}{\pi}$$

completely as the preceding. But if s is an even number, the agreement of the both expressions is easily perceived by expansion of $(1-xx)^s$.

§69 But except the continued fractions found up to now the discovered general formula contains innumerable others in it, from which to have expanded some will be helpful. Let therefore be $g = c$ and the value of this continued fraction

$$\cfrac{c^2}{s + \cfrac{(c+r)^2}{s + \cfrac{(c+2r)^2}{s + \text{etc.}}}}$$

will be

$$\frac{c\int x^{c+r-1}dx(1-x^r)^{\frac{s}{r}} : (1-x^{2r})^{\frac{r+s}{2r}}}{\int x^{c-1}dx(1-x^r)^{\frac{s}{r}} : (1-x^{2r})^{\frac{r+s}{2r}}}.$$

Now put $c = 1$ and $r = 1$ and it will be

$$\cfrac{1}{s + \cfrac{4}{s + \cfrac{9}{s + \cfrac{16}{s + \text{etc.}}}}} = \frac{\int xdx(1-x)^s : (1-xx)^{\frac{s+1}{2}}}{\int dx(1-x)^s : (1-xx)^{\frac{s+1}{2}}},$$

the values of which expression, which it takes for various significations of s, we want to investigate. So having put the value of this expression $= V$, it will be as follows:

if $s = 0$,

$$V = \frac{\int xdx : \sqrt{1-xx}}{\int dx : \sqrt{1-xx}} = \frac{1}{2\int dy : (1+yy)};$$

if $s = 2$,

$$V = \frac{2\int dx : \sqrt{1-xx} - 3\int xdx : \sqrt{1-xx}}{2\int xdx : \sqrt{1-xx} - \int dx : \sqrt{1-xx}} = \frac{1}{2\int y^2dy : (1+yy)} - 2;$$

if $s = 4$,

$$V = \frac{19\int xdx : \sqrt{1-xx} - 12\int dx : \sqrt{1-xx}}{3\int dx : \sqrt{1-xx} - 4\int xdx : \sqrt{1-xx}} = \frac{1}{2\int y^4dy : (1+yy)} - 4.$$

But in general it will be

$$V = \frac{1}{2\int y^s dy : (1+yy)} - s,$$

from which formula it becomes apparent, if S was an even integer, that the quadrature of the circle is involved, but otherwise, if s was odd, logarithms.

§70 Now let this continued fraction be propounded to us

$$1 + \cfrac{1}{2 + \cfrac{4}{3 + \cfrac{9}{4 + \cfrac{16}{5 + \cfrac{25}{6 + \text{etc.}}}}}}$$

Now compare this one to the form exhibited in § 64 and it will be

$$\begin{gathered} pqcg = 1, \\ pq(c+r)(g+r) = 4, \\ pq(c+2r)(g+2r) = 9, \\ ap - bq = 2 \quad \text{and} \quad (p-q)r = 1, \end{gathered}$$

whence it will be

$$\begin{gathered} p = \frac{\sqrt{5}+1}{2r}, \quad q = \frac{\sqrt{5}-1}{2r}, \\ a = \frac{r(1+3\sqrt{5})}{2\sqrt{5}} \quad \text{and} \quad b = \frac{r(3\sqrt{5}-1)}{2\sqrt{5}}, \end{gathered}$$

after having substituted which values one will have the value of the propounded continued fraction

$$= 1 + \frac{(\sqrt{5}-1)\int x^{2r-1}dx(1-x^r)^{\frac{1-\sqrt{5}}{2\sqrt{5}}}(1+\sqrt{5}+(\sqrt{5}-1)x^r)^{\frac{-\sqrt{5}-1}{2\sqrt{5}}}}{2\int x^{r-1}dx(1-x^r)^{\frac{1-\sqrt{5}}{2\sqrt{5}}}(1+\sqrt{5}+(\sqrt{5}-1)x^r)^{\frac{-\sqrt{5}-1}{2\sqrt{5}}}}.$$

From this equation because of the surdic exponent nothing remarkable can be concluded.

§71 Whereas in these continued fractions the partial numerators are composed of two factors, I now go on to continued fractions of such kind, in which these partial numerators constitute an arithmetic progression. Now therefore by recurring to § 50 let be $\gamma = 0$ and $c = 1$; it will be

$$\frac{\int PRdx}{\int Pdx} = \cfrac{a}{b + \cfrac{a+\alpha}{b+\beta+\cfrac{a+2\alpha}{b+2\beta+\cfrac{a+3\alpha}{b+3\beta+\text{etc.}}}}}$$

But one has to take

$$\frac{dS}{S} = \frac{(a-\alpha)dR}{\alpha R} + \frac{(\alpha b - \beta a)dR + \alpha R dR}{\alpha(\beta R - \alpha)} = \frac{(a-\alpha)dR}{\alpha R} + \frac{dR}{\beta} + \frac{(\alpha^2 + \alpha\beta b - \beta^2 a)dR}{\alpha\beta(\beta R - \alpha)},$$

whence it is

$$S = Ce^{\frac{R}{\beta}} R^{\frac{a-\alpha}{\alpha}} (\beta R - \alpha)^{\frac{\alpha^2 + \alpha\beta b - \beta^2 a}{\alpha\beta\beta}}.$$

Put

$$R = \frac{\alpha x}{\beta};$$

it will be

$$S = Ce^{\frac{\alpha x}{\beta\beta}} x^{\frac{a-\alpha}{\alpha}} (1-x)^{\frac{\alpha^2 + \alpha\beta b - \beta^2 a}{\alpha\beta\beta}}$$

and $R^{n+1}S$ vanishes in two cases, of course so for $x = 0$ as for $x = 1$, as long

$$\alpha^2 + \alpha\beta b > \beta^2 a.$$

Hence it will be

$$Pdx = e^{\frac{\alpha x}{\beta\beta}} x^{\frac{a-\alpha}{\alpha}} dx (1-x)^{\frac{\alpha^2 + \alpha\beta b - \alpha\beta^2 - \beta^2 a}{\alpha\beta\beta}}$$

and the value of the propounded continued fraction

$$= \frac{\int PRdx}{\int Pdx} = \frac{\alpha \int e^{\frac{\alpha x}{\beta\beta}} x^{\frac{a}{\alpha}} dx (1-x)^{\frac{\alpha^2 + \alpha\beta b - \alpha\beta^2 - \beta^2 a}{\alpha\beta\beta}}}{\beta \int e^{\frac{\alpha x}{\beta\beta}} x^{\frac{a-\alpha}{\alpha}} dx (1-x)^{\frac{\alpha^2 + \alpha\beta b - \alpha\beta^2 - \beta^2 a}{\alpha\beta\beta}}}$$

after having put $x = 1$ after integration.

§72 To illustrate this case with an example, let

$$a = 1, \quad \alpha = 1 \quad \text{and} \quad \beta = 1;$$

one will have this continued fraction

$$\cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \text{etc.}}}}}$$

whose value will be

$$\frac{\int e^x x dx}{\int e^x dx} = \frac{xe^x - e^x + 1}{e^x - 1} = \frac{1}{e - 1}$$

having put $x = 1$. Hence it will be

$$e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \text{etc.}}}}}$$

by which expression one gets to the value of the number e, whose logarithm is $= 1$, quite fast.

§73 Now let us put, that in the superior continued fraction given in § 71 $\beta = 0$, that

$$\frac{\int PRdx}{\int Pdx} = \cfrac{a}{b + \cfrac{a + \alpha}{b + \cfrac{a + 2\alpha}{b + \cfrac{a + 3\alpha}{b + \text{etc.}}}}}$$

it will be

$$\frac{dS}{S} = \frac{(a-\alpha)dR}{\alpha R} - \frac{bdR}{\alpha} - \frac{RdR}{\alpha}$$

and hence

$$S = CR^{\frac{a-\alpha}{\alpha}} e^{\frac{-2bR-RR}{2\alpha}}.$$

Now $R^{n+1}S$ vanishes in two cases, the one which is, if $R = 0$, the other, if $R = \infty$, as long a and α are positive numbers. Therefore put

$$R = \frac{x}{1-x}$$

and it will be

$$S = Cx^{\frac{a-\alpha}{\alpha}} : (1-x)^{\frac{a-\alpha}{\alpha}} e^{\frac{2bx-(2b-1)xx}{2\alpha(1-x)^2}}.$$

Because of

$$dR = \frac{dx}{(1-x)^2}$$

it will be

$$\int Pdx = \int \frac{x^{\frac{a-\alpha}{\alpha}} dx}{(1-x)^{\frac{a+\alpha}{\alpha}} e^{\frac{2bx-(2b-1)xx}{2\alpha(1-x)^2}}}$$

and

$$\int PRdx = \int \frac{x^{\frac{a}{\alpha}} dx}{(1-x)^{\frac{a+2\alpha}{\alpha}} e^{\frac{2bx-(2b-1)xx}{2\alpha(1-x)^2}}}.$$

§74 So let finally be in § 50

$$a = 1, \quad c = 1, \quad \alpha = 0 \quad \gamma = 0;$$

it will be

$$\frac{\int PRdx}{\int Pdx} = \cfrac{1}{b + \cfrac{1}{b + \beta + \cfrac{1}{b + 2\beta + \cfrac{1}{b + 3\beta + \text{etc.}}}}}$$

and

$$\frac{dS}{S} = \frac{R^2dR + (b-\beta)RdR - dR}{\beta R^2},$$

whence it will be

$$S = e^{\frac{RR+1}{\beta}} R^{\frac{b-\beta}{\beta}} \quad \text{and} \quad Pdx = e^{\frac{RR+1}{\beta}} R^{\frac{b-2\beta}{\beta}} dR$$

and

$$PRdx = e^{\frac{RR+1}{\beta}} R^{\frac{b-\beta}{\beta}} dR.$$

But R has to be such a function of x, that R^{n+1} vanishes so for $x = 0$ as for $x = 1$. But the work to assign a function of such kind is a lot harder than for the remaining cases. Hence I will not try to resolve this case by the same method, but will reserve it for another method now to be explained.

§75 I indeed metioned this method to get to continued fractions already some time ago, but because I then only treated a particular case, it will be convenient to explain it here in more detail. But it is not contained in integral formulas as the preceding, but in a resolution of a differential equation similar to that one, propounded by a certain count Riccati. I consider this differential equation of course

$$ax^m dx + bx^{m-1}ydx + cy^2dx + dy = 0,$$

which by putting

$$x^{m+3} = 0 \quad \text{and} \quad y = \frac{1}{cx} + \frac{1}{xxz}$$

merges into this one

$$\frac{-c}{m+3} t^{\frac{-m-4}{m+3}} dt - \frac{b}{m+3} t^{\frac{-1}{m+3}} zdt - \frac{ac+b}{(m+3)c} z^2 dt + dz = 0,$$

which is similar to the first one. Hence if the value of z was known by means of t, at the same time y will be known by means of x. So in the same way reduce this equation to another similar one to it by putting

$$t^{\frac{2m+5}{m+3}} = u \quad \text{and} \quad z = \frac{-(m+3)c}{(ac+b)t} + \frac{1}{ttv}$$

and just continue the reductions of this kind to infinity; having done this, if all later values are substituted in the preceding ones, y will be expressed in the following way

$$y = Ax^{-1} + \cfrac{1}{-Bx^{-m-1} + \cfrac{1}{Cx^{-1} + \cfrac{1}{-Dx^{-m-1} + \cfrac{1}{Ex^{-1} + \cfrac{1}{-Fx^{-m-1} + \text{etc.}}}}}}$$

the letters A, B, C, D etc. on the other hand will obtain the following values

$$\begin{aligned}
A &= \frac{1}{c},\\
B &= \frac{(m+3)c}{ac+b},\\
C &= \frac{(2m+5)(ac+b)}{c(ac-(m+2)b)},\\
D &= \frac{(3m+7)c(ac-(m+2)b)}{(ac+b)(ac+(m+3)b)},\\
E &= \frac{(4m+9)(ac+b)(ac+(m+3)b)}{c(ac-(m+2)b)(ac-(2m+4)b)},\\
F &= \frac{(5m+11)c(ac-(m+2)b)(ac-(2m+4)b)}{(ac+b)(ac+(m+3)b)(ac+(2m+5)b)}\\
&\text{etc.,}
\end{aligned}$$

which determination are comprehended simpler by the following equations:

$$AB = \frac{m+3}{ac+b},$$
$$BC = \frac{(m+3)(2m+5)}{ac-(m+2)b},$$
$$CD = \frac{(2m+5)(3m+7)}{ac+(m+3)b},$$
$$DE = \frac{(3m+7)(4m+9)}{ac-(2m+4)b},$$
$$EF = \frac{(4m+9)(5m+11)}{ac+(2m+5)b},$$
$$FG = \frac{(5m+11)(6m+13)}{ac-(3m+6)b}$$

etc.

§76 If now these values are substituted in the found continued fraction, one will then find

$$cxy = 1 + \cfrac{(ac+b)x^{m+2}}{-(m+3) + \cfrac{(ac-(m+2)b)x^{m+2}}{(2m+5) + \cfrac{(ac+(m+3)b)x^{m+2}}{-(3m+7) + \cfrac{(ac-(2m+4)b)x^{m+2}}{(4m+9)+\text{etc.}}}}}$$

From this expression it is clear, that the propounded equation is absolutely integrable in the cases, in which b becomes equal to a certain term of this progression

$$-ac, \quad \frac{-ac}{m+3}, \quad \frac{-ac}{2m+5}, \quad \frac{-ac}{3m+7} \quad \cdots \quad \frac{-ac}{im+2i+1},$$

further also in the cases, in which b is a term of this progression

$$\frac{ac}{m+2}, \quad \frac{ac}{2(m+2)}, \quad \frac{ac}{3(m+2)} \quad \cdots \quad \frac{ac}{im+2i}.$$

But this continued fraction of the propounded equation exhibits the integral of this condition, that for $x=0$ it is $cxy=1$, if $m+2>0$; but if $m+2<0$, then the integral will follow this law, that for $x=\infty$ it is $cxy=1$.

§77 Let us suppose, that $b = 0$ and $a = nc$ and that after the integration x is put $= 1$; from this equation

$$ncx^m dx + cy^2 dx + dy = 0$$

the following continued fraction will arise, by which the value of y will be defined in the case, in which one put $x = 1$,

$$y = \frac{1}{c} + \cfrac{n}{\frac{-(m+3)}{c} + \cfrac{n}{\frac{2m+5}{c} + \cfrac{n}{\frac{-(2m+7)}{c} + \cfrac{n}{\frac{4m+9}{c} + \text{etc.}}}}}$$

or just put $c = \frac{1}{\varkappa}$; from the equation

$$nx^m dx + y^2 dx + \varkappa dy = 0$$

the value of y in the case, in which $x = 1$, will behave as follows

$$y = \varkappa + \cfrac{n}{-(m\varkappa + 3\varkappa) + \cfrac{n}{2m\varkappa + 5\varkappa + \cfrac{n}{-(3m\varkappa + 7\varkappa) + \text{etc.}}}}$$

or

$$y = \varkappa - \cfrac{n}{m\varkappa + 3\varkappa - \cfrac{n}{2m\varkappa + 5\varkappa - \cfrac{n}{3m\varkappa + 7\varkappa - \cfrac{n}{4m\varkappa + 9\varkappa - \text{etc.}}}}}$$

§78 If therefore this continued fraction is propounded

$$b+\cfrac{1}{b+\beta+\cfrac{1}{b+2\beta+\cfrac{1}{b+3\beta+\cfrac{1}{b+4\beta+\text{etc.}}}}}$$

it will be

$$\varkappa = b, \quad n = -1, \quad (m+2)b = \beta$$

or

$$m = \frac{\beta}{b} - 2.$$

Therefore the value of this continued fraction will be the value of y in the case, in which $x = 1$, from this equation

$$x^{\frac{\beta-2b}{b}} dx = y^2 dx + b dy$$

having integrated in such a way, that for $x = 0$ xy becomes $= b$, because it is

$$m + 2 > 0,$$

if $\frac{\beta}{b}$ is a positive number, of course.