Considerations on certain series *

Leonhard Euler

§1 After I had found out, that the sums of the reciprocal series contained in this form

$$1 \pm \frac{1}{3^n} + \frac{1}{5^n} \pm \frac{1}{7^n} + \frac{1}{9^n} \pm \frac{1}{11^n} + \text{etc.},$$

where the superior of the both signs hold, if n is an even number, the inferior on the other hand, if n is an odd number, depend on the quadrature of the circle and are determined by such a high power of the peripheria of the circle, π, whose exponent is $= n$, I have made several observations both concerning these series itself and their use for summing other series. Because they are not very obvious and can provide utility not to be hoped for for other tasks, I believed, that it will not be inappropriate to explain them here.

§2 After having constantly put the ratio of the diameter to the circumference of the circle 1 to π, I will consider the circle, whose radius or semidiameter is $= 1$, and π will denote its semicircumference or the arc of 180 degrees. If therefore now in this circle one takes the arc $= s$, whose sine shall be $= y$, the cosine $= y$ and the tangent $= t$, it will be

*Original title: "De seriebus quibusdam considerationes", first published in „*Commentarii academiae scientiarum Petropolitanae 12*, 1750, pp. 53-96", reprinted in „*Opera Omnia*: Series 1, Volume 14, pp. 407 - 462 ", Eneström-Number E130, translated by: Alexander Aycock, for the „Euler-Kreis Mainz"

$$y = s - \frac{s^3}{1\cdot 2\cdot 3} + \frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{s^7}{1\cdot 2\cdots 7} + \text{etc.},$$
$$x = 1 - \frac{s^2}{1\cdot 2} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4} - \frac{s^6}{1\cdot 2\cdots 6} + \text{etc.}$$

and therefore

$$0 = t - s - \frac{s^2 t}{1\cdot 2} + \frac{s^3}{1\cdot 2\cdot 3} + \frac{s^4 t}{1\cdot 2\cdot 3\cdot 4} - \frac{s^5}{1\cdot 2\cdots 5} - \frac{s^6 t}{1\cdot 2\cdots 6} + \text{etc.}$$

or

$$0 = 1 - \frac{s}{t} - \frac{s^2}{1\cdot 2} + \frac{s^3}{1\cdot 2\cdot 3t} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4} - \frac{s^5}{1\cdot 2\cdots 5t} - \frac{s^6}{1\cdot 2\cdots 6} + \text{etc.}$$

§3 So at first let us consider the equation, in which the relation between the sine y and the arc s is contained, and it is manifest, that the value s for given y is not constant, but denotes all the arcs, who have the common sine y. So let the smallest of these arcs be $= \frac{m}{n}\pi$; all the following arcs

$$\begin{array}{cccccc} \frac{m}{n}\pi, & \frac{n-m}{n}\pi, & \frac{2n+m}{n}\pi, & \frac{3n-m}{n}\pi, & \frac{4n+m}{n}\pi & \text{etc.} \\ \frac{-n-m}{n}\pi, & \frac{-2n+m}{n}\pi, & \frac{-3n-m}{n}\pi, & \frac{-4n+m}{n}\pi, & \frac{-5n-m}{n}\pi & \text{etc.} \end{array}$$

will have the common sine y. Therefore one will have of this equation

$$0 - \frac{s}{1y} + \frac{s^3}{1\cdot 2\cdot 3y} - \frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5y} + \frac{s^7}{1\cdot 2\cdots 7y} - \text{etc.}$$

the following innumerabe factors

$$\left(1 - \frac{ns}{m\pi}\right)\left(1 + \frac{ns}{(n+m)\pi}\right)\left(1 - \frac{ns}{(n-m)\pi}\right)\left(1 + \frac{ns}{(2n-m)\pi}\right)\left(1 - \frac{ns}{(2n+m)\pi}\right)\text{etc.}$$

§4 Hence the values of $\frac{1}{s}$ will therefore constitute the following series

$$\frac{n}{m\pi}+\frac{n}{(n-m)\pi}-\frac{n}{(n+m)\pi}-\frac{n}{(2n-m)\pi}+\frac{n}{(2n+m)\pi}+\frac{n}{(3n-m)\pi}-\text{etc.}$$

The sum of these will therefore be equal to the coefficient of $-s$ in the equation, which is

$$=\frac{1}{1y}.$$

The sum of the products of two factors each will be $=0$, the sum of three

$$=-\frac{1}{1\cdot 2\cdot 3y}\quad\text{etc.},$$

so it will behave as followns

$$\begin{aligned}
&\text{sum of the terms} && =-\frac{1}{1y},\\
&\text{sum of the products of two} && =0,\\
&\text{sum of the products of three} && =\frac{-1}{1\cdot 2\cdot 3y},\\
&\text{sum of the products of four} && =0,\\
&\text{sum of the products of five} && =\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5y},\\
&\text{sum of the products of six} && =0,\\
&\text{sum of the products of seven} && =\frac{-1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7y},\\
&\text{sum of the products of eight} && =0
\end{aligned}$$

etc.

§5 But if therefore of any series

$$a + b + c + d + e + \text{etc.}$$

it was

$$\begin{aligned} &\text{sum of the terms itself} &&= \alpha, \\ &\text{sum of the products of two} &&= \beta, \\ &\text{sum of the products of three} &&= \gamma, \\ &\text{sum of the products of four} &&= \delta, \\ &\text{sum of the products of five} &&= \varepsilon, \\ &\text{sum of the products of six} &&= \zeta, \end{aligned}$$

$$\text{etc.},$$

one can assign the sums of the squares, cubes, bisquares and of any powers of the terms of this series. If it therefore is

$$\begin{aligned} a + b + c + d + \text{etc.} &= A, \\ a^2 + b^2 + c^2 + d^2 + \text{etc.} &= B, \\ a^3 + b^3 + c^3 + d^3 + \text{etc.} &= C, \\ a^4 + b^4 + c^4 + d^4 + \text{etc.} &= D, \\ a^5 + b^5 + c^5 + d^5 + \text{etc.} &= E, \\ a^6 + b^6 + c^6 + d^6 + \text{etc.} &= F \end{aligned}$$

$$\text{etc.},$$

the values of these sums will be determined in the following way

$$\begin{aligned}
A &= \alpha,\\
B &= \alpha A - 2\beta,\\
C &= \alpha B - \beta A + 3\gamma,\\
D &= \alpha C - \beta B + \gamma A - 4\delta,\\
E &= \alpha D - \beta C + \gamma B - \delta A + 5\varepsilon,\\
F &= \alpha E - \beta D + \gamma C - \delta B + \varepsilon A - 6\zeta
\end{aligned}$$

etc.

Because this progression follows a simple law and from the preceding terms any following one is conveniently determined, we will be able to define the values of $\frac{1}{s}$ of the superior series, exhibiting the sum of any powers of the terms.

§6 But before we leave this general progression, it will be convenient to note a singular property, which connects the values of the letters A, B, C, D etc. They of course arise from the expansion of the expression

$$\frac{\alpha - 2\beta z + 3\gamma z^2 - 4\delta z^3 + 5\varepsilon z^4 - 6\zeta z^5 + 7\eta z^6 - \text{etc.}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \varepsilon z^5 + \zeta z^6 - \text{etc.}}$$

if the quotient in powers of z is found by actual division, of course. Hence having done the division in usual manner the following quotient will arise

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.},$$

so that this series is equal to that fraction. Furthermore it is to be noted, if the sum of the series

$$1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \text{etc.}$$

is put $= Z$, so that Z is the denominator of that fraction, that the numerator will be $-\frac{-dZ}{dz}$. From this the sum of the series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

will be

$$= \frac{-dZ}{Zdz}.$$

Therefore not only the sums of the powers of the propounded series $a + b + c + d +$ etc., of course the values of the letters A, B, C, D etc. can be found from the given products of two, three, four factors each, but one will also be able to assign the sum of the series, that these powers, multiplied respectively by a new geometric progression, constitute, of course the sum of this series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

And to have noted this property carefully will be helpful in the following, where we are going to investigate new series.

§7 Therefore both at first the sum of the terms of this series itself

$$\frac{n}{\pi}\left(\frac{1}{m} + \frac{1}{n-m} - \frac{1}{m+n} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} - \text{etc.}\right)$$

and the sums of the products of two, three, four and so on are given,

$$\begin{aligned}
A &= \frac{1}{1y}, \\
B &= \frac{A}{1y}, \\
C &= \frac{B}{1y} - \frac{1}{1\cdot 2y}, \\
D &= \frac{C}{1y} - \frac{A}{1\cdot 2\cdot 3y}, \\
E &= \frac{D}{1y} - \frac{B}{1\cdot 2\cdot 3y} + \frac{1}{1\cdot 2\cdot 3\cdot 4y}, \\
F &= \frac{E}{1y} - \frac{C}{1\cdot 2\cdot 3y} + \frac{A}{1\cdot 2\cdot 3\cdot 4\cdot 5y}, \\
G &= \frac{F}{1y} - \frac{D}{1\cdot 2\cdot 3y} + \frac{B}{1\cdot 2\cdot 3\cdot 4\cdot 5y} - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6y}
\end{aligned}$$

etc.,

it will be as follows

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \text{etc.} = \frac{A\pi}{n},$$

$$\frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \text{etc.} = \frac{B\pi^2}{n^2},$$

$$\frac{1}{m^3} + \frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} - \frac{1}{(2n-m)^3} + \frac{1}{(2n+m)^3} + \text{etc.} = \frac{C\pi^3}{n^3},$$

$$\frac{1}{m^4} + \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(2n-m)^4} + \frac{1}{(2n+m)^4} + \text{etc.} = \frac{D\pi^4}{n^4},$$

$$\frac{1}{m^5} + \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} - \frac{1}{(2n-m)^5} + \frac{1}{(2n+m)^5} + \text{etc.} = \frac{E\pi^5}{n^5},$$

$$\frac{1}{m^6} + \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(2n-m)^6} + \frac{1}{(2n+m)^6} + \text{etc.} = \frac{F\pi^6}{n^6}$$

etc.,

where for the even powers all terms have the sign $+$, for the odd on the other hand the signs agree with the signs of the first series.

§8 The letters A, B, C, D, E shall retain the values, the we just gave them, and let this series be propounded to us

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.},$$

whose sum we investigated from the rule given in § 6. But the sum of this series hence is

$$= \frac{-dZ}{Zdz},$$

while

$$Z = 1 - \frac{z}{1y} + \frac{z^3}{1\cdot 2\cdot 3y} - \frac{z^5}{1\cdot 2\cdots 5y} + \frac{z^7}{1\cdot 2\cdots 7y} - \text{etc.} = 1 - \frac{1}{y}\sin z.$$

From this because of the in this place constant y one will have to put

$$dZ = \frac{-dz\cos z}{y}$$

and therefore the sum of the propounded series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

will be

$$= \frac{\cos z}{y - \sin z}.$$

Therefore the sum of this series will be

$$Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.} = \frac{z\cos z}{y - \sin z}.$$

§9 Let $z = \frac{p\pi}{n}$; this series will express the sum of all these series

$$\begin{aligned}
&+\frac{p}{m} + \frac{p}{n-m} - \frac{p}{n+m} - \frac{p}{2n-m} + \frac{p}{2n+m} + \text{etc.},\\
&+\frac{p^2}{m^2} + \frac{p^2}{(n-m)^2} + \frac{p^2}{(n+m)^2} + \frac{p^2}{(2n-m)^2} + \frac{p^2}{(2n+m)^2} + \text{etc.},\\
&+\frac{p^3}{m^3} + \frac{p^3}{(n-m)^3} - \frac{p^3}{(n+m)^3} - \frac{p^3}{(2n-m)^3} + \frac{p^3}{(2n+m)^3} + \text{etc.}
\end{aligned}$$

But these series added vertically give

$$\frac{p}{m-p} + \frac{p}{n-m-p} - \frac{p}{n+m+p} - \frac{p}{2n-m+p} + \frac{p}{2n+m-p} + \text{etc.},$$

whose sum therefore is

$$= \frac{p\pi \cos \frac{p\pi}{n}}{ny - n \sin \frac{p\pi}{n}};$$

or, because y is the sine of the arc $\frac{m\pi}{n}$, one will have the sum of this series

$$= \frac{p\pi \cos \frac{p\pi}{n}}{n \sin \frac{m\pi}{n} - n \sin \frac{p\pi}{n}}.$$

If therefore one puts

$$m - p = a \quad \text{und} \quad m + p = b,$$

so that it is

$$m = \frac{a+b}{2} \quad \text{und} \quad p = \frac{b-a}{2},$$

the sum of this series

$$\frac{1}{a} + \frac{1}{n-b} - \frac{1}{n+b} - \frac{1}{2n-a} + \frac{1}{2n+a} + \frac{1}{3n-b} - \frac{1}{3n+b} - \text{etc.}$$

or of this

$$\frac{1}{a} + \frac{2b}{n^2-b^2} - \frac{2a}{4n^2-a^2} + \frac{2b}{9n^2-b^2} - \frac{2a}{16n^2-a^2} + \frac{2b}{25n^2-b^2} - \text{etc.}$$

will arise as

$$= \frac{\pi \cos \frac{(b-a)\pi}{2n}}{n \sin \frac{(b+a)\pi}{2n} - n \sin \frac{(b-a)\pi}{2n}}.$$

§10 But these things are too general, that hardly everything, what is comprehended in them, can be perceived. Therefore let us descend to more special things and let us put the sine $y =$ to the whole sine $= 1$; it will be $m = 1$ and $n = 2$. Therefore we obtain the following series

$$\frac{1}{1} + \frac{1}{1} - \frac{1}{3} - \frac{1}{3} + \frac{1}{5} + \frac{1}{5} - \frac{1}{7} - \text{etc.} = \frac{A\pi}{2},$$

$$\frac{1}{1^2} + \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = \frac{B\pi^2}{2^2},$$

$$\frac{1}{1^3} + \frac{1}{1^3} - \frac{1}{3^3} - \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{5^3} - \frac{1}{7^3} - \text{etc.} = \frac{C\pi^3}{2^3},$$

$$\frac{1}{1^4} + \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} = \frac{D\pi^4}{2^4}$$

etc.

or these

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} = \frac{A\pi}{2^2},$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.} = \frac{B\pi^2}{2^3},$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} = \frac{C\pi^3}{2^4},$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = \frac{D\pi^4}{2^5},$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.} = \frac{E\pi^5}{2^6},$$

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} = \frac{F\pi^6}{2^7},$$

$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \text{etc.} = \frac{G\pi^7}{2^8},$$

$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.} = \frac{H\pi^8}{2^9}$$

etc.

But the values of the letters A, B, C, D etc. are found from the following law

$$\begin{aligned}
A &= 1,\\
B &= \frac{A}{1},\\
C &= \frac{B}{1} - \frac{1}{1\cdot 2},\\
D &= \frac{C}{1} - \frac{A}{1\cdot 2\cdot 3},\\
E &= \frac{D}{1} - \frac{B}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4},\\
F &= \frac{E}{1} - \frac{C}{1\cdot 2\cdot 3} + \frac{A}{1\cdot 2\cdot 3\cdot 4\cdot 5},\\
G &= \frac{F}{1} - \frac{D}{1\cdot 2\cdot 3} + \frac{B}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{1}{1\cdot 2\cdots 6},\\
H &= \frac{G}{1} - \frac{E}{1\cdot 2\cdot 3} + \frac{C}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{A}{1\cdot 2\cdots 7}
\end{aligned}$$

etc.,

whence one finds the following values

$$\begin{aligned}
A &= 1 \cdot \frac{\pi}{2^2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.},\\
B &= \frac{1}{1} \cdot \frac{\pi^2}{2^3} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.},\\
C &= \frac{1}{1\cdot 2} \cdot \frac{\pi^3}{2^4} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.},\\
D &= \frac{2}{1\cdot 2\cdot 3} \cdot \frac{\pi^4}{2^5} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.},
\end{aligned}$$

$$
\begin{aligned}
E &= \frac{5}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{\pi^5}{2^6} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \text{etc.},\\
F &= \frac{16}{1\cdot 2\cdot 3\cdots 5} \cdot \frac{\pi^6}{2^7} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.},\\
G &= \frac{61}{1\cdot 2\cdot 3\cdots 6} \cdot \frac{\pi^7}{2^8} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \text{etc.},\\
H &= \frac{272}{1\cdot 2\cdot 3\cdots 7} \cdot \frac{\pi^8}{2^9} = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.},\\
I &= \frac{1385}{1\cdot 2\cdot 3\cdots 8} \cdot \frac{\pi^9}{2^{10}} = 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \text{etc.},\\
K &= \frac{7936}{1\cdot 2\cdot 3\cdots 9} \cdot \frac{\pi^{10}}{2^{11}} = 1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \text{etc.},\\
L &= \frac{50521}{1\cdot 2\cdot 3\cdots 10} \cdot \frac{\pi^{11}}{2^{12}} = 1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \text{etc.},\\
M &= \frac{353792}{1\cdot 2\cdot 3\cdots 11} \cdot \frac{\pi^{12}}{2^{13}} = 1 + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \text{etc.},\\
N &= \frac{2702765}{1\cdot 2\cdot 3\cdots 12} \cdot \frac{\pi^{13}}{2^{14}} = 1 - \frac{1}{3^{13}} + \frac{1}{5^{13}} - \frac{1}{7^{13}} + \text{etc.},\\
O &= \frac{22368256}{1\cdot 2\cdot 3\cdots 13} \cdot \frac{\pi^{14}}{2^{15}} = 1 + \frac{1}{3^{14}} + \frac{1}{5^{14}} + \frac{1}{7^{14}} + \text{etc.}
\end{aligned}
$$

etc.

§11 Here the letters A, B, C etc. denote only the numeral coefficients of the powers of π divided by the powers of two; even though the values can be defined conveniently enough from the given law, one can nevertheless exhibit another law, which seems even more appropriate for calculations. Of course I will consider the series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.},$$

whose sum, which shall constantly be denoted by the letter s, is by § 8

$$= \frac{\cos z}{1 - \sin z}$$

- because of $y = 1$. If therefore from this equation

$$s = \frac{\cos z}{1 - \sin z}$$

the value of s is expressed in the series, which proceeds in powers of z, the series itself will have to arise

$$A + Bz + Cz^2 + Dz^3 + \text{etc.}$$

For, one cannot assign another series of the same form, put it

$$P + Qz + Rz^2 + Sz^3 + \text{etc.},$$

that it equal to that one

$$A + Bz + Cz^2 + Dz^3 + \text{etc.},$$

that at the same time the coefficients of the powers of z match and it is

$$P = A, \quad Q = B, \quad R = C, \quad S = D \quad \text{etc.}$$

But $\frac{\cos z}{1-\sin z}$ on the other hand expresses the tangent of the arc $\left(\frac{\pi}{4} + \frac{z}{2}\right)$, or it will be

$$s = \tan\left(\frac{\pi}{4} + \frac{z}{2}\right)$$

and hence by converting

$$\frac{\pi}{4} + \frac{z}{2} = \arctan s = \int \frac{ds}{1 + ss}$$

and having taken differentials because of the constant $\frac{\pi}{4}$ or arc of 45 degrees one will have

$$\frac{dz}{2} = \frac{ds}{1 + ss}$$

or

$$dz + ssdz = 2ds.$$

Now just put

$$S = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.};$$

it will be

$$\begin{aligned}
\frac{2ds}{dz} &= 2B \;\; + 4C \;\; z + 6D \;\; z^2 + 8E \quad z^3 + 10F\; z^4 + \text{etc.},\\
ss \;\; &= A^2 \;\; + 2ABz + 2ACz^2 + 2ADz^3 + 2AEz^4 + \text{etc.},\\
&\qquad\qquad\qquad\quad + B^2 \quad z^2 + 2BC\, z^3 + 2BDz^4\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad + C \quad z^4\\
1 \quad &= +1.
\end{aligned}$$

Now having compared the homogenous terms to one another the values of the letters are defined, that the coefficients of the single powers of z vanish; and one will obtain the following determinations of the letters A, B, C, D,E etc., while, as we just found, $A = 1$:

$$\begin{aligned}
A &= 1,\\
B &= \frac{A^2+1}{2},\\
C &= \frac{2AB}{4},\\
D &= \frac{2AC+B^2}{6},\\
E &= \frac{2AD+2BC}{8},
\end{aligned}$$

$$
\begin{aligned}
F &= \frac{2AE+2BD+C^2}{10},\\
G &= \frac{2AF+2BE+2CD}{12},\\
H &= \frac{2AG+2BF+2CE+D^2}{14}\\
&\text{etc.}
\end{aligned}
$$

And therefore completely the same determinations of the letters A, B, C, D etc. will arise, which the other law given above in § 10 gave.

§12 Because the denominators of the fractions, to which the letters A, B, C, D etc. were found equal, proceed regular enough, from this a peculiar rule for finding the numerators can be found. Hence let us put

$$
\begin{aligned}
A &= \alpha, & F &= \frac{\zeta}{1\cdot 2\cdot 3\cdot 4\cdot 5},\\
B &= \frac{\beta}{1}, & G &= \frac{\eta}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6},\\
C &= \frac{\gamma}{1\cdot 2}, & H &= \frac{\theta}{1\cdot 2\cdot 3\cdots 7},\\
D &= \frac{\delta}{1\cdot 2\cdot 3}, & I &= \frac{\iota}{1\cdot 2\cdot 3\cdots 8},\\
E &= \frac{\gamma}{1\cdot 2\cdot 3\cdot 4}, & K &= \frac{\varkappa}{1\cdot 2\cdot 3\cdots 9}\\
& & &\text{etc.}
\end{aligned}
$$

and the law having done the substitutions will be this:

$$
\begin{aligned}
\alpha &= 1,\\
\beta &= \frac{\alpha^2+1}{2},\\
\gamma &= \alpha\beta,
\end{aligned}
$$

$$
\begin{aligned}
\delta &= \alpha\gamma + \beta^2, \\
\varepsilon &= \alpha\delta + 3\beta\gamma, \\
\zeta &= \alpha\varepsilon + 4\beta\delta \;\; + 3\gamma^2, \\
\eta &= \alpha\zeta + 5\beta\varepsilon \;\; + \frac{5\cdot 4}{1\cdot 2}\gamma\delta, \\
\theta &= \alpha\eta + 6\beta\zeta \;\; + \frac{6\cdot 5}{1\cdot 2}\gamma\varepsilon \;\; + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}\cdot\frac{\delta^2}{2}, \\
\iota &= \alpha\theta + 7\beta\eta \;\; + \frac{7\cdot 6}{1\cdot 2}\gamma\zeta \;\; + \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}\delta\varepsilon, \\
\varkappa &= \alpha\iota + 8\beta\theta \;\; + \frac{8\cdot 7}{1\cdot 2}\gamma\varepsilon \;\; + \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}\delta\zeta + \frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\varepsilon^2}{2}
\end{aligned}
$$

etc.

This law is perspicuous, if in this way it is only noted, as often the last term is a square, it has in addition to be divided by two.

§13 Now let us consider this series

$$Az + Bz^2 + C^3 + Dz^4 + Ez^5 + \text{etc.},$$

whose sum is known to be

$$= \frac{z\cos z}{1-\sin z},$$

and put

$$z = \frac{p\pi}{2};$$

it will be

$$\frac{p\pi\cos\dfrac{p\pi}{2}}{2-2\sin\dfrac{p\pi}{2}} = \frac{A\pi}{2^2}\cdot 2p + \frac{B\pi^2}{2^3}\cdot 2p^2 + \frac{C\pi^3}{2^4}\cdot 2p^3 + \frac{D\pi^4}{2^5}\cdot 2p^4 + \text{etc.}$$

or

$$\frac{\pi\cos\frac{p\pi}{2}}{4-4\sin\frac{p\pi}{2}}=\frac{A\pi}{2^2}+\frac{pB\pi^2}{2^3}+\frac{p^2C\pi^3}{2^4}+\frac{p^3D\pi^4}{2^5}+\text{etc.}$$

If therefore instead of the single terms the series from § 10 are substituted, it will arise

$$\begin{aligned}\frac{\pi\cos\frac{p\pi}{2}}{4-4\sin\frac{p\pi}{2}}=&+1\;-\frac{1}{3}\;+\frac{1}{5}\;-\frac{1}{7}\;+\frac{1}{9}\;-\text{etc.}\\&+p\;+\frac{p}{3^2}+\frac{p}{5^2}+\frac{p}{7^2}+\frac{p}{9^2}+\text{etc.}\\&+p^2-\frac{p^2}{3^3}+\frac{p^2}{5^3}-\frac{p^2}{7^3}+\frac{p^2}{9^3}-\text{etc.}\\&+p^3+\frac{p^3}{3^4}+\frac{p^3}{5^4}+\frac{p^3}{7^4}+\frac{p^3}{9^4}+\text{etc.}\\&\qquad\text{etc.}\end{aligned}$$

But all these series added coloumnwise merge to this one

$$\frac{1}{1-p}-\frac{1}{3+p}+\frac{1}{5-p}-\frac{1}{7+p}+\frac{1}{9-p}-\text{etc.}$$

whose sum therefore is

$$\frac{\pi\cos\frac{p\pi}{2}}{4-4\sin\frac{p\pi}{2}}.$$

§14 It will be possible to derive many summable series of this kind from the series exposed at the end of § 9. Let us put $a=b=m$ and we will have this

$$\frac{2m}{n^2-m^2}-\frac{2m}{4n^2-m^2}+\frac{2m}{9n^2-m^2}-\frac{2m}{16n^2-m^2}+\frac{2m}{25n^2-m^2}-\text{etc.},$$

whose sum will be

$$= \frac{\pi}{n \sin \dfrac{m\pi}{n}} - \frac{1}{m}$$

- because of $\cos 0\pi = 1$ and $\sin 0\pi = 0$. Hence, if it is divided by $2m$, one will have

$$\frac{1}{n^2 - m^2} - \frac{1}{4n^2 - m^2} + \frac{1}{9n^2 - m^2} - \frac{1}{16n^2 - m^2} + \frac{1}{25n^2 - m^2} - \text{etc.}$$
$$= \frac{\pi}{2mn \sin \dfrac{m\pi}{n}} - \frac{1}{2mm}.$$

Let us further put $a = -m$ and $b = +b$ and it will arise

$$-\frac{\pi \cos \dfrac{m\pi}{n}}{n \sin \dfrac{m\pi}{n}} + \frac{1}{m}$$
$$= \frac{2m}{n^2 - m^2} + \frac{2m}{4n^2 - m^2} + \frac{2m}{9n^2 - m^2} + \frac{2m}{16n^2 - m^2} + \frac{2m}{25n^2 - m^2} + \text{etc.}$$

As often it therefore happens, that $\cos \frac{m\pi}{n}$ vanishes, so often the sum will be assignable algebraicly, it is of course $= \frac{1}{2m^2}$. But this happens, if it was $\frac{m}{n} = \frac{2i+1}{2}$ or $m = 2i + 1$ and $n = 2$, whence it will be

$$\frac{1}{2(2i+1)^2} = \frac{1}{4 - (2i+1)^2} + \frac{1}{16 - (2i+1)^2} + \frac{1}{36 - (2i+1)^2} + \frac{1}{64 - (2i+1)^2} + \text{etc.}$$

Therefore the following paradox proposition arises, of course, that it is

$$\frac{1}{4-p} + \frac{1}{16-p} + \frac{1}{36-p} + \frac{1}{64-p} + \frac{1}{100-p} + \text{etc.} = \frac{1}{2p},$$

as often as p was an integer square and odd.

§15 Let us put $n = 1$ and $m^2 = p$; it will be

$$\frac{1}{1-p} - \frac{1}{4-p} + \frac{1}{9-p} - \frac{1}{16-p} + \frac{1}{25-p} - \text{etc.} = \frac{\pi\sqrt{p}}{2p\sin\pi\sqrt{p}} - \frac{1}{2p},$$

$$\frac{1}{1-p} + \frac{1}{4-p} + \frac{1}{9-p} + \frac{1}{16-p} + \frac{1}{25-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}\cos\pi\sqrt{p}}{2p\sin\pi\sqrt{p}};$$

if which series are added, it follows, that it will be

$$\frac{1}{1-p} + \frac{1}{-p} + \frac{1}{25-p} + \text{etc.} = \frac{\pi\sqrt{p}\ \text{versin}\ \pi\sqrt{p}}{4p\sin\pi\sqrt{p}};$$

but if the same are subtracted from each other, it will be

$$\frac{1}{4-p} + \frac{1}{16-p} + \frac{1}{36-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}(1+\cos\pi\sqrt{p})}{4p\sin\pi\sqrt{p}}.$$

But it is

$$\frac{\text{versin}\ \pi\sqrt{p}}{\sin\pi\sqrt{p}} = \tan\frac{\pi\sqrt{p}}{2} \quad \text{und} \quad \frac{1+\cos\pi\sqrt{p}}{\sin\pi\sqrt{p}} = \cot\frac{\pi\sqrt{p}}{2},$$

whence the last sums are simplyfied.

§16 We are therefore able to sum the following series

$$\frac{1}{1-p} \pm \frac{1}{4-p} + \frac{1}{9-p} \pm \frac{1}{16-p} + \text{etc.},$$

if p signifies any positive number, of course. But if instead of p a negative number is substituted, for example $-q$, then so the sine and the cosine as the arcs $\pi\sqrt{p}$ or $\pi\sqrt{-q}$ become imaginary quantities. But because the sums of the series nevertheless stay real and finite, the imaginary quantities cancel each other. Therefore it will be convenient to investigate, real quantities of which kind are contained in these forms

$$\frac{\pi\sqrt{-q}}{\sin\pi\sqrt{-q}} \quad \text{und} \quad \frac{\pi\sqrt{-q}}{\tan\pi\sqrt{-q}}.$$

Therefore let us put

$$u = \frac{\pi\sqrt{-q}}{\sin\pi\sqrt{-q}}$$

and it will be

$$\sin\pi\sqrt{-q} = \frac{\pi\sqrt{-q}}{u} \quad \text{und} \quad \pi\sqrt{-q} = \arcsin\frac{\pi\sqrt{-q}}{u};$$

now take the differentials, having pur π and u variable; one will have

$$d\pi = \frac{ud\pi - \pi du}{u\sqrt{uu + q\pi^2}}.$$

Put $u = \pi v$; it will arise

$$d\pi = \frac{-dv}{v\sqrt{q+vv}} \quad \text{und} \quad \pi = \frac{1}{\sqrt{q}}\log\frac{\sqrt{q}+\sqrt{q+vv}}{cv}.$$

Therefore it will be

$$e^{\pi\sqrt{p}}cv = \sqrt{q} + \sqrt{q+vv} \quad \text{und} \quad v = \frac{2e^{\pi\sqrt{q}}c\sqrt{q}}{e^{2\pi\sqrt{q}} - 1}$$

and

$$u = \frac{2\pi e^{\pi\sqrt{q}}c\sqrt{q}}{e^{2\pi\sqrt{q}}c^2 - 1}.$$

But the constant c has to be conditioned in such a way, that for $\pi = u$ becomes $= 1$, whence $c = 1$. Therefore it will be

$$\frac{\pi\sqrt{-q}}{\sin\pi\sqrt{-q}} = \frac{2e^{\pi\sqrt{q}}\pi\sqrt{q}}{e^{2\pi\sqrt{q}} - 1}.$$

In similar manner it will be

$$\frac{\pi\sqrt{-q}}{\tan\pi\sqrt{-q}} = \frac{\pi}{v};$$

it will be

$$\pi\sqrt{-v} = \tan\pi\sqrt{-q} \quad \text{und} \quad \pi\sqrt{-q} = \arctan v\sqrt{-q}$$

and by differentiating

$$d\pi = \frac{dv}{1 - qvv}.$$

Now integrate again; it will be

$$\pi = \frac{1}{2\sqrt{q}} \log \frac{1 + v\sqrt{q}}{1 - v\sqrt{q}} \quad \text{and} \quad e^{2\pi\sqrt{q}} - e^{2\pi\sqrt{q}} v\sqrt{q} = 1 + v\sqrt{q},$$

whence it becomes

$$v = \frac{e^{2\pi\sqrt{q}} - 1}{(e^{2\pi\sqrt{q}} + 1)\sqrt{q}}$$

and

$$\frac{\pi\sqrt{-q}}{\tan \pi\sqrt{-q}} = \frac{(e^{2\pi\sqrt{q}} + 1)\pi\sqrt{q}}{e^{2\pi\sqrt{q}} - 1}.$$

§17 Hence we obtained the following eight series, whose sums can be assigned, which we want to list up altogether:

$$\frac{1}{1-p} - \frac{1}{4-p} + \frac{1}{9-p} - \frac{1}{16-p} + \frac{1}{25-p} - \text{etc.} = \frac{\pi\sqrt{p}}{2p \sin \pi\sqrt{p}} - \frac{1}{2p},$$

$$\frac{1}{1-p} + \frac{1}{4-p} + \frac{1}{9-p} + \frac{1}{16-p} + \frac{1}{25-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}}{2p \tan \pi\sqrt{p}},$$

$$\frac{1}{1-p} + \frac{1}{9-p} + \frac{1}{25-p} + \frac{1}{49-p} + \frac{1}{81-p} + \text{etc.} = \frac{\pi\sqrt{p}}{4p \cot \frac{\pi\sqrt{p}}{2}},$$

$$\frac{1}{4-p} + \frac{1}{16-p} + \frac{1}{36-p} + \frac{1}{64-p} + \frac{1}{100-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}}{4p \tan \frac{\pi\sqrt{p}}{2}},$$

$$\frac{1}{1+q} - \frac{1}{4+q} + \frac{1}{9+q} - \frac{1}{16+q} + \frac{1}{25+q} - \text{etc.} = \frac{1}{2q} - \frac{e^{\pi\sqrt{q}}\pi\sqrt{q}}{(e^{2\pi\sqrt{q}} - 1)q},$$

$$\frac{1}{1+q}+\frac{1}{4+q}+\frac{1}{9+q}+\frac{1}{16+q}+\frac{1}{25+q}+\text{etc.}=\frac{(e^{2\pi\sqrt{q}}+1)\pi\sqrt{q}}{2(e^{2\pi\sqrt{q}}-1)q}-\frac{1}{2q},$$

$$\frac{1}{1+q}+\frac{1}{9+q}+\frac{1}{25+q}+\frac{1}{49+q}+\frac{1}{81+q}+\text{etc.}=\frac{(e^{\pi\sqrt{q}}-1)\pi\sqrt{q}}{4(e^{\pi\sqrt{q}}+1)q},$$

$$\frac{1}{4+q}+\frac{1}{16+q}+\frac{1}{36+q}+\frac{1}{64+q}+\frac{1}{100+q}+\text{etc.}=\frac{(e^{\pi\sqrt{q}}+1)\pi\sqrt{q}}{4(e^{\pi\sqrt{q}}-1)q}-\frac{1}{2q}.$$

§18 After I exhibited the law above, according to which the sums of the powers of all terms of this series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}$$

proceed, I will now investigate the law, which connects only the odd powers, that these sums without cognition of the even ones can be continued as far as desired; therefore let be

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}=A\pi,$$

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\text{etc.}=B\pi^3,$$

$$1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\text{etc.}=C\pi^5,$$

$$1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\text{etc.}=D\pi^7,$$

$$1-\frac{1}{3^9}+\frac{1}{5^9}-\frac{1}{7^9}+\frac{1}{9^9}-\text{etc.}=E\pi^9$$

etc.

and the law is to be investigated, according to which the coefficients A, B, C, D etc. proceed. For this aim I will consider this series

$$A\pi z+B\pi^3z^3+C\pi^5z^5+D\pi^7z^7+\text{etc.},$$

whose sum shall be $= s$; therefore, having multiplied these series by the corresponding powers of z respectively, it will be

$$s = \frac{z}{1-zz} - \frac{3z}{9-zz} + \frac{5z}{25-zz} - \frac{7z}{49-zz} + \text{etc.}$$

and

$$\frac{2s}{z} = \frac{1}{1-z} + \frac{1}{1+z} - \frac{1}{3-z} - \frac{1}{3+z} + \frac{1}{5-z} + \frac{1}{5+z} - \text{etc.}$$

But because it is from § 9

$$\frac{\pi \cos \frac{(b-a)\pi}{2n}}{n \sin \frac{(b+a)\pi}{2n} - n \sin \frac{(b-a)\pi}{2n}}$$
$$= \frac{1}{a} + \frac{1}{n-b} - \frac{1}{n+b} - \frac{1}{2n-a} + \frac{1}{2n+a} + \frac{1}{3n-b} - \frac{1}{3n+b} - \text{etc.},$$

let

$$a = 1-z, \quad n = 2 \quad \text{and} \quad b = 1-z,$$

and this series will merge into that one; from this it will arise

$$\frac{2s}{z} = \frac{\pi}{2 \sin \frac{(1-z)\pi}{2}} \quad \text{and} \quad s = \frac{\pi z}{4 \sin \frac{(1-z)\pi}{2}}$$

or

$$s = \frac{\pi z}{4 \cos \frac{\pi z}{2}} = \frac{\frac{\pi z}{4}}{1 - \frac{\pi^2 z^2}{1 \cdot 2 \cdot 2^2} + \frac{\pi^4 z^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2^4} - \frac{\pi^6 z^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 2^2} + \text{etc.}};$$

because this fraction, if it is actually divided, has to reproduce the assumed series

$$A\pi z + B\pi^3 z^3 + C\pi^5 z^5 + \text{etc.},$$

it will be

$$\begin{aligned}
A &= \frac{1}{4},\\
B &= \frac{A}{2\cdot 4},\\
C &= \frac{B}{2\cdot 4} - \frac{A}{2\cdot 4\cdot 6\cdot 8},\\
D &= \frac{C}{2\cdot 4} - \frac{B}{2\cdot 4\cdot 6\cdot 8} + \frac{A}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12},\\
E &= \frac{D}{2\cdot 4} - \frac{C}{2\cdot 4\cdot 6\cdot 8} + \frac{B}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} - \frac{A}{2\cdot 4\cdot 6\cdots 16}
\end{aligned}$$

etc.

§19 Or if it is put

$$\begin{aligned}
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.} &= \frac{A\pi}{2^2},\\
1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.} &= \frac{B\pi^3}{2^4},\\
1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \text{etc.} &= \frac{C\pi^5}{2^6},\\
1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \text{etc.} &= \frac{D\pi^7}{2^8}
\end{aligned}$$

etc.,

the coefficients A, B, C will follow this law:

$$
\begin{aligned}
A &= 1, \\
B &= \frac{A}{1\cdot 2}, \\
C &= \frac{B}{1\cdot 2} - \frac{A}{1\cdot 2\cdot 3\cdot 4}, \\
D &= \frac{C}{1\cdot 2} - \frac{B}{1\cdot 2\cdot 3\cdot 4} + \frac{A}{1\cdot 2\cdots 6}, \\
E &= \frac{D}{1\cdot 2} - \frac{C}{1\cdot 2\cdot 3\cdot 4} + \frac{B}{1\cdot 2\cdots 6} - \frac{A}{1\cdot 2\cdots 8}
\end{aligned}
$$

etc.

But if therefore those series are continued backwards, that one reaches positive powers, the sums of all those sums will be $= 0$; so that, even though we proceed further in these forms, there would nevertheless not arise other values. Of course it is

$$
\begin{aligned}
&1 - 3 + 5 - 7 + 9 - \text{etc.} = 0, \\
&1 - 3^3 + 5^3 - 7^3 + 9^3 - \text{etc.} = 0, \\
&1 - 3^5 + 5^5 - 7^5 + 9^5 - \text{etc.} = 0, \\
&1 - 3^7 + 5^7 - 7^7 + 9^7 - \text{etc.} = 0
\end{aligned}
$$

etc.

§20 But as the sums of the odd powers follow a certain law of progression, so also the even powers will enjoy a similar property, that all can be defined without help of the odd powers. To find this law, we will use a similar operation. Hence let be

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\text{etc.}=A\pi^2,$$
$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{etc.}=B\pi^4,$$
$$1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\text{etc.}=C\pi^6,$$
$$1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{9^8}+\text{etc.}=D\pi^8$$
$$\text{etc.}$$

and investigate the sum of this series

$$A\pi^2z^2+B\pi^4z^4+C\pi^6z^6+D\pi^8z^8+\text{etc.}=s;$$

it will be

$$s=\frac{z^2}{1-z^2}+\frac{z^2}{9-z^2}+\frac{z^2}{25-z^2}+\frac{z^2}{49-z^2}+\text{etc.},$$

whence it will be from § 17

$$s=\frac{\pi z}{4\cot\dfrac{\pi z}{2}}$$

or by a series

$$s=\frac{\dfrac{\pi^2z^2}{1\cdot 2^3}-\dfrac{\pi^4z^4}{1\cdot 2\cdot 3\cdot 2^5}+\dfrac{\pi^6z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 2^7}-\text{etc.}}{1-\dfrac{\pi^2z^2}{1\cdot 2\cdot 2^2}+\dfrac{\pi^4z^4}{1\cdot 2\cdot 3\cdot 4\cdot 2^4}-\dfrac{\pi^6z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2^6}+\text{etc.}};$$

because the assumed series itself has to arise from this division, it will be

$$A=\frac{1}{8},$$
$$B=\frac{A}{2\cdot 4}-\frac{1}{2\cdot 4\cdot 6\cdot 4},$$

$$C = \frac{B}{2 \cdot 4} - \frac{A}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{1}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 4},$$

$$D = \frac{C}{2 \cdot 4} - \frac{B}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{1}{2 \cdot 4 \cdot 6 \cdots 12} - \frac{1}{2 \cdot 4 \cdots 14 \cdot 4}$$

etc.

§21 But this law is seen easier, if one puts

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.} = A\frac{\pi^2}{2^3},$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = B\frac{\pi^4}{2^5},$$

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} = C\frac{\pi^6}{2^7},$$

$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.} = D\frac{\pi^8}{2^9}$$

etc.

For, here the coefficients A, B, C etc. will lead to the following progression:

$$A = 1,$$

$$B = \frac{A}{1 \cdot 2},$$

$$C = \frac{B}{1 \cdot 2} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5},$$

$$D = \frac{C}{1 \cdot 2} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \frac{1}{1 \cdot 2 \cdots 7}$$

etc.

If therefore one makes this ansatz as a series

$$s = Az + Bz^3 + Cz^5 + Dz^7 + Ez^9 + \text{etc.},$$

it will be

$$s = \frac{z - \dfrac{z^3}{1\cdot 2\cdot 3} + \dfrac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \dfrac{z^7}{1\cdot 2\cdots 7} + \text{etc.}}{1 - \dfrac{z^2}{1\cdot 2} + \dfrac{z^4}{1\cdot 2\cdot 3\cdot 4} - \dfrac{z^6}{1\cdot 2\cdot 3\cdots 6} + \text{etc.}}$$

and therefore

$$s = \tan z \quad \text{oder} \quad z = \arctan s.$$

We will therefore have

$$dz = \frac{ds}{1+ss} \quad \text{und} \quad dz + ssdz = ds;$$

which equations must be satisfied by this value

$$s = Az + Bz^3 + Cz^5 + Dz^7 + Ez^9 + \text{etc.},$$

substitute the values instead of ds and ss and it will be

$$\begin{aligned}
\frac{ds}{dz} &= A + 3Bz^2 + 5C\ \ z^4 + 7D\ \ z^6 + 9E\ \ \ z^8 + \text{etc.},\\
ss\ \ &= \quad + A^2z^2 + 2ABz^4 + 2ACz^6 + 2ADz^8 + \text{etc.},\\
&\qquad\qquad\qquad\qquad\quad + B^2\ \ \ z^6 + 2BC\, z^8\\
1\ \ &= 1.
\end{aligned}$$

Therefore having formed the equations the following other determinations of the letters A, B, C, D etc. will arise:

$$\begin{aligned}
A &= 1,\\
B &= \frac{A^2}{3},\\
C &= \frac{2AB}{5},
\end{aligned}$$

$$
\begin{aligned}
D &= \frac{2AC + B^2}{7}, \\
E &= \frac{2AD + 2BC}{9}, \\
F &= \frac{2AE + 2BD + C^2}{11} \\
&\text{etc.}
\end{aligned}
$$

§22 On these series of the even powers the sums of the series contained in this general form depend

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.},$$

while n denote an even number. If it therefore was

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \text{etc.} = N\pi^n,$$

it will be

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.} = \frac{2^n N\pi^n}{2^n - 1},$$

whence the sums of all these series, as long n is an even number, can be found by the quadrature of the circle and from the sums already found those of similar ones of even powers for odd numbers alone. But that these sums can be found directly, let us look for a peculiar law, according to which these sums proceed. Therefore let

$$
\begin{aligned}
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} &= A\pi^2, \\
1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} &= B\pi^4,
\end{aligned}
$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}=C\pi^6,$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\text{etc.}=D\pi^8$$

etc.

and I will consider this series

$$s=A\pi^2z^2+B\pi^4z^4+C\pi^6z^6+D\pi^8z^8+E\pi^{10}z^{10}+\text{etc.},$$

which having substituted the series, which they denote, instead of $A\pi^2$, $B\pi^4$, $C\pi^6$ etc., and added the homologuous terms it will arise

$$s=\frac{zz}{1-zz}+\frac{zz}{4-zz}+\frac{zz}{9-zz}+\frac{zz}{16-zz}+\frac{zz}{25-zz}+\text{etc.},$$

which series summed by § 17 gives

$$s=\frac{1}{2}-\frac{\pi z}{2\tan\pi z}$$

or, if the tangent of the arc πz is expressed by a series,

$$s=\frac{1}{2}-\frac{1}{2}\cdot\frac{1-\dfrac{\pi^2z^2}{1\cdot 2}+\dfrac{\pi^4z^4}{1\cdot 2\cdot 3\cdot 4}-\dfrac{\pi^6z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.}}{1-\dfrac{\pi^2z^2}{1\cdot 2\cdot 3}+\dfrac{\pi^4z^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\dfrac{\pi^6z^6}{1\cdot 2\cdots 7}+\text{etc.}}$$

$$=\frac{\dfrac{\pi^2z^2}{1\cdot 2\cdot 3}-\dfrac{2\pi^4z^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\dfrac{3\pi^6z^6}{1\cdot 2\cdots 7}-\dfrac{4\pi^8z^8}{1\cdot 2\cdots 9}}{1-\dfrac{\pi^2z^2}{1\cdot 2\cdot 3}+\dfrac{\pi^4z^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\dfrac{\pi^6z^6}{1\cdot 2\cdots 7}+\dfrac{\pi^8z^8}{1\cdot 2\cdots 9}-\text{etc.}};$$

because having expanded this expression it has to yield the assumed series itself

$$A\pi^2z^2+B\pi^4z^4+C\pi^6z^6+D\pi^8z^8+\text{etc.},$$

these determinations of the coefficients will follow:

$$A = \frac{1}{6},$$

$$B = \frac{A}{1\cdot 2\cdot 3} - \frac{2}{1\cdot 2\cdot 3\cdot 4\cdot 5},$$

$$C = \frac{B}{1\cdot 2\cdot 3} - \frac{A}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{3}{1\cdot 2\cdots 7},$$

$$D = \frac{C}{1\cdot 2\cdot 3} - \frac{B}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{A}{1\cdot 2\cdots 7} - \frac{4}{1\cdot 2\cdots 9}$$

etc.

§23 But for these same coefficients one can exhibit another law of progression, by means of which it will be possible to find them a lot easier. Because it is

$$s = \frac{1}{2} - \frac{\pi z}{2\tan \pi z},$$

it will be

$$\tan \pi z = \frac{\pi z}{1-2s} \quad \text{und} \quad \pi z = \arctan \frac{\pi z}{1-2s}.$$

Now put $\pi z = u$; it will be

$$u = \arctan \frac{u}{1-2s}$$

and by differentiating

$$du = \frac{du - 2sdu + 2uds}{1 - 4s + 4ss + uu}$$

or

$$uudu + 4ssdu = 2sdu + 2uds,$$

which equation is satisfied by this value

$$s = Au^2 + Bu^4 + Cu^6 + Du^8 + Eu^{10} + \text{etc.},$$

having substituted this value it will be

$$\begin{aligned}
uu &= uu, \\
4ss &= \qquad + 4A^2u^4 + 8ABu^6 + 8ACu^8 + 8ADu^{10} + 8AEu^{12} + \text{etc.}, \\
&\qquad\qquad\qquad\qquad + 4B^2\ u^8 + 8BC\ u^{10} + 8BDu^{12} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + 4C^2\ u^{12} \\
2s &= 2Au^2 + 2B\ \ u^4 + 2C\ \ u^6 + 2D\ \ u^8 + 2E\ \ u^{10} + 2F\ \ u^{12} + \text{etc.}, \\
\frac{2uds}{du} &= 4Au^2 + 8B\ \ u^4 + 12C\,u^6 + 16D\,u^8 + 20E\ u^{10} + 24F\ u^{12} + \text{etc.},
\end{aligned}$$

whence the following determinations follow:

$$\begin{aligned}
A &= \frac{1}{6}, \\
B &= \frac{2A^2}{5}, \\
C &= \frac{4AB}{7}, \\
D &= \frac{4AC + 2B^2}{9}, \\
E &= \frac{4AD + 4BC}{11}, \\
F &= \frac{4AE + 4BD + 2C^2}{13}, \\
G &= \frac{4AF + 4BE + 4CD}{15}, \\
A &= \frac{4AG + 4BF + 4CE + 2D^2}{17}
\end{aligned}$$

etc.

§24 But the sums of the series of this kind itself, as far I gave them, are the following:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}=\frac{2}{1\cdot 2\cdot 3}\cdot\frac{1}{2}\pi^2,$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}=\frac{2^3}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{1}{6}\pi^4,$$
$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}=\frac{2^5}{1\cdot 2\cdots 7}\cdot\frac{1}{6}\pi^6,$$
$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\text{etc.}=\frac{2^7}{1\cdot 2\cdots 9}\cdot\frac{3}{10}\pi^8,$$
$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\text{etc.}=\frac{2^9}{1\cdot 2\cdots 11}\cdot\frac{5}{6}\pi^{10},$$
$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\text{etc.}=\frac{2^{11}}{1\cdot 2\cdots 13}\cdot\frac{691}{210}\pi^{12},$$
$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\text{etc.}=\frac{2^{13}}{1\cdot 2\cdots 15}\cdot\frac{35}{2}\pi^{14},$$
$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\text{etc.}=\frac{2^{15}}{1\cdot 2\cdots 17}\cdot\frac{3617}{30}\pi^{16},$$
$$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\text{etc.}=\frac{2^{17}}{1\cdot 2\cdots 19}\cdot\frac{43867}{42}\pi^{18},$$
$$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\text{etc.}=\frac{2^{19}}{1\cdot 2\cdots 21}\cdot\frac{1222277}{110}\pi^{20},$$
$$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\text{etc.}=\frac{2^{21}}{1\cdot 2\cdots 23}\cdot\frac{854513}{6}\pi^{22},$$
$$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\text{etc.}=\frac{2^{23}}{1\cdot 2\cdots 25}\cdot\frac{1181820455}{546}\pi^{24}.$$

In these expressions only the law of the middle factions is not manifest, the remaining parts on the other hand proceed perspicuously enough. But after I had considered these middle fractions

$$\frac{1}{2},\quad \frac{1}{6},\quad \frac{1}{6},\quad \frac{3}{10},\quad \frac{5}{6}\quad \text{etc.}$$

with more attention, I discovered, that the same occur in the general expression, that I once gave for finding sum of any series from the general term, so that by means of the one expression the other can be constructed.

§25 It will therefore be worth the effort to look for the agreement of these two expression so different to each other with more attention. The one expression, that I gave for the summation of series, behaves as follows: If the general term of any series or that term, that corresponds to indefinite numerical index x, was $= X$ and the sum of the series from the first term to this term X inclusively is put $= S$, it will be

$$\begin{aligned} S = \int X dx + \frac{X}{1\cdot 2} &+ \frac{dX}{1\cdot 2\cdot 3\cdot 2dx} - \frac{d^3X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx^3} \\ &+ \frac{d^5X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 6dx^5} - \frac{d^7X}{1\cdot 2\cdot 3\cdots 9\cdot 6dx^7} \\ &+ \frac{5d^9X}{1\cdot 2\cdots 11dx^9} - \frac{691d^{11}X}{1\cdot 2\cdots 13\cdot 210dx^{11}} \\ &+ \frac{35d^{13}X}{1\cdot 2\cdots 15\cdot 2dx^{13}} - \frac{3617d^{15}X}{1\cdot 2\cdots 17\cdot 30dx^{15}} \\ &+ \frac{43867d^{17}X}{1\cdot 2\cdots 19\cdot 42dx^{17}} - \frac{1222277d^{19}X}{1\cdot 2\cdots 21\cdot 110dx^{19}} \\ &+ \frac{854513d^{21}X}{1\cdot 2\cdots 23\cdot 6dx^{21}} - \frac{1181820455d^{23}X}{1\cdot 2\cdots 25\cdot 546dx^{23}} \\ &\text{etc.,} \end{aligned}$$

in which expression it is apparent, that completely the same irregular fractions

$$\frac{1}{2}, \quad \frac{1}{6}, \quad \frac{1}{6}, \quad \frac{3}{10}, \quad \frac{5}{6} \quad \text{etc.}$$

are in it, which occured in the expression of the sums before, only with this difference, that they have alternating signs here, whereas there all were affected by the sign $+$. And this agreement itself provided me with this utility, that I was able to continue this general expression of the sum S up to this point, whereas by the law, that I had found at that time for the progression of these terms, I could do this only by a huge amount of work.

§26 But even though this mere observation of this great agreement could suffice to show the agreement in the following terms, which are not known, it will nevertheless be better, to find this same agreement from the a general principle, that is understood not to happen by chance but neccessarily. But I obtained this last expression in the following way. Because S denotes the sum of so many terms in any series, as unities are contained in the exponent x, and the last of these terms is $= X$, it is manifest, if in S $x-1$ is put in x, that then the same sum S less the last term or $S - X$ has to arise. But having put $x-1$ instead of x the quantity S will merge into this one

$$S - \frac{dS}{1dx} + \frac{ddS}{1 \cdot 2dx^2} - \frac{d^3S}{1 \cdot 2 \cdot 3dx^3} + \frac{d^4S}{1 \cdot 2 \cdot 3 \cdot 4dx^4} - \text{etc.},$$

which is therefore equal to $S - X$; hence one has this equation

$$X = \frac{dS}{1dx} - \frac{ddS}{1 \cdot 2dx^2} + \frac{d^3S}{1 \cdot 2 \cdot 3dx^3} - \frac{d^4S}{1 \cdot 2 \cdot 3 \cdot 4dx^4} + \text{etc.}$$

Now that from this equation S is expressed in X, I assume this equation

$$S = \int Xdx + \alpha X + \frac{\beta dX}{dx} + \frac{\gamma ddX}{dx^2} + \frac{\delta d^3X}{dx^3} + \text{etc.},$$

having substituted this one in the one above one will have

$$\begin{aligned} X = X &+ \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} + \frac{\gamma d^3X}{dx^3} + \frac{\delta d^4X}{dx^4} + \text{etc.} \\ &- \frac{dX}{1 \cdot 2dx} - \frac{\alpha ddX}{1 \cdot 2dx^2} - \frac{\beta d^3X}{1 \cdot 2dx^3} - \frac{\gamma d^4X}{1 \cdot 2dx^4} \\ &+ \frac{ddX}{1 \cdot 2 \cdot 3dx^2} + \frac{\alpha d^3X}{1 \cdot 2 \cdot 3dx^3} + \frac{\beta d^4X}{1 \cdot 2 \cdot 3dx^4} \\ &- \frac{d^4X}{1 \cdot 2 \cdot 3 \cdot 4dx^3} - \frac{\alpha d^4X}{1 \cdot 2 \cdot 3 \cdot 4dx^4} \\ &+ \frac{d^4X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5dx^4} \end{aligned}$$

§27 From this equality the following determinations of the coefficients α, β, γ, δ etc. arise

$$\begin{aligned}
\alpha &= \frac{1}{1\cdot 2},\\
\beta &= \frac{\alpha}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3},\\
\gamma &= \frac{\beta}{1\cdot 2} - \frac{\alpha}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4},\\
\delta &= \frac{\gamma}{1\cdot 2} - \frac{\beta}{1\cdot 2\cdot 3} + \frac{\alpha}{1\cdot 2\cdot 3\cdot 4} - \frac{1}{1\cdot 2\cdots 5},\\
\varepsilon &= \frac{\delta}{1\cdot 2} - \frac{\gamma}{1\cdot 2\cdot 3} + \frac{\beta}{1\cdot 2\cdot 3\cdot 4} - \frac{\alpha}{1\cdot 2\cdots 5} + \frac{1}{1\cdot 2\cdots 6}\\
&\text{etc.}
\end{aligned}$$

and from these formulas I then defined the values of these letters, and that with a lot of work. And not, as it happend here, different than by observation alone I recognized, that all the second values γ, ε, η vanish. But from the now formulated principles the same can be luculently shown, if another law of this progression is investigated. For this I will consider this series

$$s = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}$$

and it will be from the preceding law of the coefficients

$$s = \frac{1}{1 - \frac{z}{1\cdot 2} + \frac{zz}{1\cdot 2\cdot 3} - \frac{z^3}{1\cdot 2\cdot 3\cdot 4} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \text{etc.}},$$

which equation merges into this one

$$s = \frac{z}{1-e^{-z}} \quad \text{or} \quad s = \frac{e^z z}{e^z - 1}.$$

Hence it arises

$$e^z s - a = e^z z \quad \text{und} \quad e^z = \frac{s}{s-z} \quad \text{sowie} z = \log s - \log(s-z).$$

But by differentiating one will have

$$dz = \frac{ds}{s} - \frac{ds - dz}{s - z}$$

or

$$ssdz - szdz = sdz - zds,$$

which equation has to be satisfied by the assumed value

$$s = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.};$$

therefore substitute this value in this equation

$$\frac{zds}{dz} - s - sz + ss = 0$$

and one will obtain

$$\begin{array}{rclllllll}
\dfrac{zds}{dz} & = & & +\alpha\; z & +2\beta z^2 & +3\gamma\; z^3 & +4\delta\; z^4 & +5\varepsilon\; z^5 & +\text{etc.}, \\
-s & = & -1 & +\alpha\; z & -\beta\; z^2 & -\gamma\; z^3 & -\delta\; z^4 & -\varepsilon\; z^5 & -\text{etc.}, \\
-sz & = & & -\; z & -\alpha\; z^2 & -\beta\; z^3 & -\gamma\; z^4 & -\delta\; z^5 & -\text{etc.}, \\
+s^2 & = & 1 & +2\alpha z & +2\beta z^2 & +2\gamma\; z^3 & +2\delta\; z^4 & +2\varepsilon\; z^5 & +\text{etc.} \\
 & & & & +\alpha^2 & +2\alpha\beta & +2\alpha\gamma & +2\alpha\delta & \\
 & & & & & & +\beta^2 & +2\beta\gamma &
\end{array}$$

Therefore one concludes, that it will be

$$\begin{aligned}
\alpha &= \frac{1}{2}, \\
\beta &= \frac{\alpha - \alpha^2}{3}, \\
\gamma &= \frac{\beta - 2\alpha\beta}{4}, \\
\delta &= \frac{\gamma - 2\alpha\gamma - \beta\beta}{5},
\end{aligned}$$

$$\varepsilon = \frac{\delta - 2\alpha\delta - 2\beta\gamma}{6},$$
$$\zeta = \frac{\varepsilon - 2\alpha\varepsilon - 2\beta\delta - \gamma\gamma}{7},$$
$$\eta = \frac{\zeta - 2\alpha\zeta - 2\beta\varepsilon - 2\gamma\delta}{8}$$
etc.

§28 Because it is $\alpha = \frac{1}{2}$, it will be $1 - 2\alpha = 0$; because the value occurs in all following terms, it will be

$$\alpha = \frac{1}{2},$$
$$\beta = \frac{1}{12},$$
$$\gamma = 0,$$
$$\delta = -\frac{\beta\beta}{5},$$
$$\varepsilon = -\frac{2\beta\gamma}{6},$$
$$\zeta = \frac{-2\beta\delta - \gamma\gamma}{7},$$
$$\eta = \frac{-2\beta\varepsilon - 2\gamma\delta}{8},$$
$$\theta = \frac{-2\beta\zeta - 2\gamma\varepsilon - \delta\delta}{9},$$
$$\iota = \frac{-2\beta\eta - 2\gamma\zeta - 2\delta\varepsilon}{10}$$
etc.

Because now $\gamma = 0$, it is evident, that it also $\varepsilon = 0$ and therefore further $\eta = 0$, $\iota = 0$ etc., so that all remaining second terms, beginning from γ are $= 0$, what was clear from the preceding law only by observations, but is now understood to happend neccessarily. Hence while stell $\alpha = \frac{1}{2}$ it will be as follows,

$$\begin{aligned}
\beta &= \frac{1}{12}, \\
\delta &= -\frac{\beta^2}{5}, \\
\zeta &= -\frac{2\beta\delta}{7}, \\
\theta &= \frac{2\beta\zeta - 2\delta\delta}{9}, \\
\varkappa &= \frac{-2\beta\theta - 2\delta\zeta}{11} \\
&\text{etc.}
\end{aligned}$$

If therefore one puts

$$\beta = \frac{A}{2}, \quad \delta = -\frac{B}{2^3}, \quad \zeta = \frac{C}{2^5}, \quad -\theta = -\frac{D}{2^7}, \quad \varkappa = \frac{E}{2^9} \quad \text{etc.,}$$

so that

$$S = \int X dx = \frac{X}{2} + \frac{AdX}{2dx} - \frac{Bd^3X}{2^3dx^3} + \frac{Cd^5X}{2^5dx^5} - \frac{Dd^7X}{2^7dx^7} + \frac{Ed^9X}{2^9dx^9} - \frac{Fd^{11}X}{2^{11}dx^{11}} + \text{etc.,}$$

the coefficients A, B, C, D will give this law

$$\begin{aligned}
A &= \frac{1}{6}, \\
B &= \frac{2A^2}{5}, \\
C &= \frac{4AB}{7}, \\
D &= \frac{4AC + 2B^2}{9},
\end{aligned}$$

$$E = \frac{4AD + 4BC}{11},$$

$$F = \frac{4AE + 4BD + 2C^2}{13},$$

$$G = \frac{4AF + 4BE + 4CD}{15}$$

etc.

So the letters A, B, C, D etc. obtain the values itself, which we gave to them above in § 22 and 23. And hence were are completely certain about the agreement of the coefficients in this most different expressions and it will not be convenient to ascribe this to an accident anymore.

§29 Although we can assign the sum of this series

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.}$$

conveniently enough, if n is an even number, we are nevertheless not able to conclude anything from these same principles to find the sums, if n is an odd number. It could indeed seem, that also these series depend on the quadrature of the circle in such a way, that their sum is $= N\pi^n$, of course also in the cases, in which n is an odd number; but if we actually take these sums by approximations, we will see that the coefficient N does not become a rational number, if n is not an even number, what will be seen more clearly from this table:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \text{etc.} = 1,644934067 = \frac{\pi^2}{6},$$

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \text{etc.} = 1,202056903 = \frac{\pi^3}{26,79435},$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \text{etc.} = 1,082323234 = \frac{\pi^4}{90},$$

$$1 + \frac{1}{2^5} + \frac{1}{3^5} + \text{etc.} = 1,036927755 = \frac{\pi^5}{295,1215},$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\text{etc.}=1,017343062=\frac{\pi^6}{945},$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\text{etc.}=1,008349277=\frac{\pi^7}{2995,285},$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\text{etc.}=1,004077356=\frac{\pi^8}{9450},$$

$$1+\frac{1}{2^9}+\frac{1}{3^9}+\text{etc.}=1,002008393=\frac{\pi^9}{29749,35},$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\text{etc.}=1,000994575=\frac{\pi^{10}}{93555},$$

$$1+\frac{1}{2^{11}}+\frac{1}{3^{11}}+\text{etc.}=1,000494189=\frac{\pi^{11}}{294058,7},$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\text{etc.}=1,000246087=\frac{\pi^{12}}{924041\frac{544}{691}}.$$

And there is on the other hand no relation between the sums of the odd powers similar to that, that holds between the sums of the even powers.

§30 But it seems, that something can be concluded about the sums of the odd powers, if the signs are put to alternate. Because the first of the odd powers

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\text{etc.}$$

has a known sum, of course $\log 2$, it seems very probable, that also the sums of the following odd powers depend on the logarithm of two and maybe furthermore on the quadrature of the circle. But before we conclude anything here, let us investigate the sums of the even powers and let be

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\text{etc.}=A\pi^2,$$
$$1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\text{etc.}=B\pi^4,$$

$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\text{etc.}=C\pi^6,$$
$$1-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\text{etc.}=D\pi^8,$$
$$1-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\text{etc.}=E\pi^{10}$$
$$\text{etc.},$$

where the values of the letters A, B, C, D etc. can easily be concluded from the known values for the same series, while all terms are taken positive; but it will be better, to find an own law for these. I will therefore consider the following series

$$s=A\pi^2z^2+B\pi^4z^4+C\pi^6z^6+D\pi^8z^8+\text{etc.},$$

which having substituted the series will merge into this

$$s=\frac{zz}{1-zz}-\frac{zz}{4-zz}+\frac{zz}{9-zz}-\frac{zz}{16-zz}+\text{etc.},$$

which series summed by § 17 will give

$$s=\frac{\pi z}{2\sin\pi z}-\frac{1}{2}$$

or having expressed the sine

$$s=-\frac{1}{2}+\frac{\frac{1}{2}}{1-\frac{\pi^2z^2}{1\cdot2\cdot3}+\frac{\pi^4z^4}{1\cdot2\cdot3\cdot4\cdot5}-\frac{\pi^6z^6}{1\cdot2\cdots7}+\text{etc.}}.$$

If now the preceding term in the series of letters A, B, C, D, E etc. or the one before the first A is put $=\Delta$, it will be

$$\Delta = \frac{1}{2},$$

$$A = \frac{\Delta}{1 \cdot 2 \cdot 3} = \frac{1}{12},$$

$$B = \frac{A}{1 \cdot 2 \cdot 3} - \frac{\Delta}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5},$$

$$C = \frac{B}{1 \cdot 2 \cdot 3} - \frac{A}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{\Delta}{1 \cdot 2 \cdots 7},$$

$$D = \frac{C}{1 \cdot 2 \cdot 3} - \frac{B}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{A}{1 \cdot 2 \cdots 7} - \frac{\Delta}{1 \cdot 2 \cdots 9}$$

etc.

But the value of Δ was not mere assumed, but indeed expresses the sum of the preceding series, which is

$$1 - 1 + 1 - 1 + 1 - 1 + \text{etc.} = \Delta\pi^0 = \frac{1}{2},$$

the sums of all remaining series on the other hand, which preced this one, are $= 0$, of course

$$1 - 2^2 + 3^2 - 4^2 + \text{etc.} = 0,$$

$$1 - 2^4 + 3^4 - 4^4 + \text{etc.} = 0,$$

$$1 - 2^6 + 3^6 - 4^6 + \text{etc.} = 0$$

etc.

§31 From these things it therefore follows, that the sum of any series can be concluded completely right from the preceding ones in this way: If it was

$$1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\text{etc.}=\alpha\pi^n,$$

$$1-\frac{1}{2^{n-2}}+\frac{1}{3^{n-2}}-\frac{1}{4^{n-2}}+\text{etc.}=\beta\pi^{n-2},$$

$$1-\frac{1}{2^{n-4}}+\frac{1}{3^{n-4}}-\frac{1}{4^{n-4}}+\text{etc.}=\gamma\pi^{n-4},$$

$$1-\frac{1}{2^{n-6}}+\frac{1}{3^{n-6}}-\frac{1}{4^{n-6}}+\text{etc.}=\delta\pi^{n-6}$$

$$\text{etc.},$$

it will be

$$\alpha=\frac{\beta}{1\cdot 2\cdot 3}-\frac{\gamma}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\delta}{1\cdot 2\cdots 7}-\frac{\varepsilon}{1\cdot 2\cdots 9}+\frac{\zeta}{1\cdot 2\cdots 9}-\text{etc.}$$

So to find the sum of this series

$$1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\frac{1}{6^3}+\text{etc.},$$

one will have all series, which preced it according to this law, which are

$$1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\text{etc.}=B\pi^3,$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\text{etc.}=A\pi,$$

$$1-2+3-4+\text{etc.}=\frac{\alpha}{\pi},$$

$$1-2^3+3^3-4^3+\text{etc.}=\frac{\beta}{\pi^3},$$

$$1-2^5+3^5-4^5+\text{etc.}=\frac{\gamma}{\pi^5}$$

$$\text{etc.},$$

and it will be

$$B = \frac{A}{1\cdot 2\cdot 3} - \frac{\alpha}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{\beta}{1\cdot 2\cdots 7} - \frac{\gamma}{1\cdot 2\cdots 9} + \text{etc.}$$

But the sums of all these series can be exhibited; hence it is

$$\begin{aligned}
&1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{etc.} = \log 2,\\
&1 - 2 + 3 - 4 + \text{etc.} = \frac{1}{4} = \frac{2\cdot 1}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \text{etc.}\right),\\
&1 - 2^3 + 3^3 - 4^3 + \text{etc.} = \frac{-1}{8} = \frac{-2\cdot 1\cdot 2\cdot 3}{\pi^4}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \text{etc.}\right),\\
&1 - 2^5 + 3^5 - 4^5 + \text{etc.} = \frac{1}{4} = \frac{2\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5}{\pi^6}\left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \text{etc.}\right),\\
&1 - 2^7 + 3^7 - 4^7 + \text{etc.} = \frac{-17}{16} = \frac{2\cdot 1\cdot 2\cdots 7}{\pi^8}\left(1 + \frac{1}{3^8} + \frac{1}{5^8} + \text{etc.}\right)
\end{aligned}$$

etc.

And therefore it will be

$$\begin{aligned}
A &= \frac{\log 2}{\pi},\\
\alpha &= \frac{2\cdot 1}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.}\right),\\
\beta &= \frac{-2\cdot 1\cdot 2\cdot 3}{\pi}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.}\right),\\
\gamma &= \frac{2\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5}{\pi}\left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.}\right),\\
\delta &= \frac{-2\cdot 1\cdot 2\cdots 7}{\pi}\left(1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.}\right),\\
\varepsilon &= \frac{2\cdot 1\cdot 2\cdots 9}{\pi}\left(1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \text{etc.}\right)
\end{aligned}$$

etc.

§32 But we exhibited the sums of even powers of the fractions, whose denominators are odd numbers, above. Let

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = P\pi^2,$$
$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} = Q\pi^4,$$
$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.} = R\pi^6,$$
$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.} = S\pi^8$$

etc.:

by § 21 it will be

$$P\pi^2 + Q\pi^4 + R\pi^6 + S\pi^8 + \text{etc.} = \frac{\pi}{4}\tan\frac{\pi}{2}.$$

But the letters α, β, γ, δ will obtain the following values

$$\alpha = \frac{2 \cdot 1}{\pi} P\pi^2,$$
$$\beta = \frac{-2 \cdot 1 \cdot 2 \cdot 3}{\pi} Q\pi^4,$$
$$\gamma = \frac{2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{\pi} R\pi^6,$$
$$\delta = \frac{-2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{\pi} S\pi^8$$

etc.

Therefore from the law of progression, if one puts

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\text{etc.}=A\pi=\log 2,$$

$$1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\frac{1}{6^3}+\text{etc.}=B\pi^3,$$

$$1-\frac{1}{2^5}+\frac{1}{3^5}-\frac{1}{4^5}+\frac{1}{5^5}-\frac{1}{6^5}+\text{etc.}=C\pi^5,$$

$$1-\frac{1}{2^7}+\frac{1}{3^7}-\frac{1}{4^7}+\frac{1}{5^7}-\frac{1}{6^7}+\text{etc.}=D\pi^7$$

etc.,

we will therefore be able to determine these coefficients A, B, C, D etc. in such a way, that it is

$$A=\frac{2}{\pi}\left(\frac{P\pi^2}{2\cdot 3}+\frac{Q\pi^4}{4\cdot 5}+\frac{R\pi^6}{6\cdot 7}+\frac{S\pi^8}{8\cdot 9}+\text{etc.}\right)$$

$$B=\frac{A}{1\cdot 2\cdot 3}-\frac{2}{\pi}\left(\frac{P\pi^2}{2\cdot 3\cdot 4\cdot 5}+\frac{Q}{4\cdot 5\cdot 6\cdot 7}+\frac{R\pi^6}{6\cdot 7\cdot 8\cdot 9}+\text{etc.}\right),$$

$$C=\frac{B}{1\cdot 2\cdot 3}-\frac{A}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{2}{\pi}\left(\frac{P\pi^2}{2\cdot 3\cdots 7}+\frac{Q}{4\cdot 5\cdots 9}+\frac{R\pi^6}{6\cdot 7\cdots 11}+\text{etc.}\right),$$

$$D=\frac{C}{1\cdot 2\cdot 3}-\frac{B}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{A}{1\cdot 2\cdots 7}-\frac{2}{\pi}\left(\frac{P\pi^2}{2\cdot 3\cdots 9}+\frac{Q}{4\cdot 5\cdots 11}+\text{etc.}\right)$$

etc.

§33 But before we take it on to conclude from this, we want to teach by one example, that the rule found here behaves correctly and the true values of the letters arise from there. Now let us take the first formula, and because it is $A=\frac{\log 2}{\pi}$, one will have this equation.

$$\frac{\log 2}{2}=\frac{P\pi^2}{2\cdot 3}+\frac{Q\pi^4}{4\cdot 5}+\frac{R\pi^6}{6\cdot 7}+\frac{S\pi^8}{8\cdot 9}+\text{etc.}$$

By approximating the true values it is indeed

$$\begin{aligned}
\log 2 &= 0,693147181, \\
P\pi^2 &= 1,233700550, \\
Q\pi^4 &= 1,014678032, \\
R\pi^6 &= 1,001447077, \\
S\pi^8 &= 1,000155179, \\
T\pi^{10} &= 1,000017041, \\
V\pi^{12} &= 1,000001886, \\
W\pi^{14} &= 1,000000209, \\
X\pi^{16} &= 1,000000023, \\
Y\pi^{18} &= 1,000000003
\end{aligned}$$

etc.

At first let us take the whole unities for $P\pi^2$, $Q\pi^4$ etc.; we will have

$$\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}+\frac{1}{8\cdot 9}+\text{etc.},$$

the sum of which series is known, of course it is

$$= 1-\log 2 \quad \text{oder} \quad 0,306852819;$$

now let us take the annexed fractions of the same terms, which divided by the respective denominators will give

$$\begin{array}{r}
0{,}038950092 \\
0{,}000733902 \\
34454 \\
2155 \\
155 \\
12 \\
1 \\
\hline
0{,}039720771;
\end{array}$$

add $1 - \log 2$

$$\begin{array}{r} 0{,}306852819 \\ \hline 0{,}346573590. \end{array}$$

But on the other hand

$$\frac{\log 2}{2} = 0{,}346573590.$$

whence the equality is clearly seen.

§34 Because therefore now the validity of the proposition asserted in § 32, we have a law, according to which the sums of the series

$$1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \text{etc.},$$

while n denotes any odd number, proceed. But because it is only known to us by observation, that

$$\frac{\log 2}{2} = \frac{P\pi^2}{2 \cdot 3} + \frac{Q\pi^4}{4 \cdot 5} + \frac{R\pi^6}{6 \cdot 7} + \frac{S\pi^8}{8 \cdot 9} + \text{etc.}$$

or

$$\log 2 = \begin{Bmatrix} + & \frac{1}{3}\left(1 \quad +\frac{1}{3^2} \quad +\frac{1}{5^2} \quad +\frac{1}{7^2} \quad +\frac{1}{9^2} \quad +\text{etc.}\right) \\ + & \frac{1}{10}\left(1 \quad +\frac{1}{3^4} \quad +\frac{1}{5^4} \quad +\frac{1}{7^4} \quad +\frac{1}{9^4} \quad +\text{etc.}\right) \\ + & \frac{1}{21}\left(1 \quad +\frac{1}{3^6} \quad +\frac{1}{5^6} \quad +\frac{1}{7^6} \quad +\frac{1}{9^6} \quad +\text{etc.}\right) \\ + & \frac{1}{36}\left(1 \quad +\frac{1}{3^8} \quad +\frac{1}{5^8} \quad +\frac{1}{7^8} \quad +\frac{1}{9^8} \quad +\text{etc.}\right) \\ + & \frac{1}{55}\left(1 \quad +\frac{1}{3^{10}} \quad +\frac{1}{5^{10}} \quad +\frac{1}{7^{10}} \quad +\frac{1}{9^{10}} \quad +\text{etc.}\right) \end{Bmatrix}$$

etc.

it will be worth the effort, to look for a demonstration of this truth. Therefore let us put

$$s = \frac{P\pi^2}{2\cdot 3} + \frac{Q\pi^4}{4\cdot 5} + \frac{R\pi^6}{6\cdot 7} + \frac{S\pi^8}{8\cdot 9} + \text{etc.}$$

and perform the following transformations

$$\begin{aligned} \frac{d.\pi s}{d\pi} &= \frac{P\pi^2}{2} + \frac{Q\pi^4}{4} + \frac{R\pi^6}{6} + \text{etc.}, \\ \frac{dd.\pi s}{d\pi^2} &= P\pi^2 + Q\pi^4 + R\pi^6 + \text{etc.} \end{aligned}$$

Because this last series, if it is multiplied by π, has the sum $\frac{\pi}{4}\tan\frac{\pi}{2}$, which expression holds, even if π is a variable quantity, as we put here, and so it will be

$$dd.\pi s = \frac{d\pi^2}{4}\tan\frac{\pi}{2}$$

and therefore

$$d.\pi s = \frac{d\pi}{4}\int d\pi \tan\frac{\pi}{2}$$

and finally

$$s = \frac{1}{4\pi}\int d\pi \int d\pi \tan\frac{\pi}{2},$$

the root of which equation is already clear, of course it is

$$s = \frac{\log 2}{2}.$$

§35 Now let us at first consider this formula

$$\int d\pi \tan\frac{\pi}{2},$$

which merges into

$$\int \frac{d\pi \sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = -2\log\cos\frac{\pi}{2};$$

but having substituted this integral we will have

$$s = \frac{-1}{\pi} \int \frac{d\pi}{2} \log\cos\frac{\pi}{2}.$$

To integrate this formula I put

$$\tan\frac{\pi}{2} = t;$$

it will be

$$\cos\frac{\pi}{2} = \frac{1}{\sqrt{1+tt}}$$

and both

$$-\log\cos\frac{\pi}{2} = \log\sqrt{1+tt} = \frac{1}{2}\log(1+tt)$$

and

$$\frac{d\pi}{2} = \frac{dt}{1+tt};$$

hence it will be

$$s = \frac{1}{2\pi}\int \frac{dt}{1+tt}\log(1+tt),$$

and therefore the question was reduced to that, that one defines the integral $\int \frac{dt\log(1+tt)}{1+tt}$ having used such a constant, that the integral vanishes for $t = 0$; after that one has to resubstitute $t = \tan\frac{\pi}{2}$ and because of $\frac{\pi}{2} =$ to an arc of 90° it will be $t = \infty$. But this formula, because it is

$$\log(1+tt) = \frac{tt}{1+tt} + \frac{t^4}{2(1+tt)^4} + \frac{t^6}{3(1+tt)^3} + \frac{t^8}{4(1+tt)^4} + \text{etc.},$$

merges into the following one, so that it is

$$\int \frac{dt}{1+tt}\log(1+tt)$$

$$= \int \frac{ttdt}{(1+tt)^2} + \frac{1}{2}\int \frac{t^4dt}{(1+tt)^3} + \frac{1}{3}\int \frac{t^6dt}{(1+tt)^4} + \frac{1}{4}\int \frac{t^6dt}{(1+tt)^5} + \text{etc.}$$

But by reduction of integral formulas it is in general

$$\int \frac{t^{2m}}{(1+tt)^{m+1}} = \frac{-t^{2m-1}}{2m(1+tt)^m} + \frac{2m-1}{2m} \int \frac{t^{2m-2}}{(1+tt)^m}.$$

Therefore, because it is

$$\int \frac{dt}{1+tt} = \frac{\pi}{2},$$

it will be

$$\int \frac{ttdt}{(1+tt)^2} = \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \cdot \frac{t}{1+tt},$$

$$\int \frac{t^4dt}{(1+tt)^3} = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2} - \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{t}{1+tt} - \frac{1}{4} \cdot \frac{t^3}{(1+tt)^2},$$

$$\int \frac{t^6dt}{(1+tt)^4} = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{t}{1+tt} - \frac{1 \cdot 5}{4 \cdot 6} \cdot \frac{t^3}{(1+tt)^2} - \frac{1}{6} \cdot \frac{t^5}{(1+tt)^3},$$

$$\int \frac{t^8dt}{(1+tt)^5} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{t}{1+tt} - \frac{1 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8} \cdot \frac{t^3}{(1+tt)^2}$$
$$- \frac{1 \cdot 7}{6 \cdot 8} \cdot \frac{t^5}{(1+tt)^3} - \frac{1}{8} \cdot \frac{t^7}{(1+tt)^4},$$

$$\int \frac{t^{10}dt}{(1+tt)^6} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{\pi}{2} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{t}{1+tt} - \frac{1 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{t^3}{(1+tt)^2}$$
$$- \frac{1 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10} \cdot \frac{t^5}{(1+tt)^3} - \frac{1 \cdot 9}{8 \cdot 10} \cdot \frac{t^7}{(1+tt)^4} - \frac{1}{10} \cdot \frac{t^9}{(1+tt)^5}$$

etc.

Form this substitutions it will arise

$$\int \frac{dt}{1+tt} \log(1+tt)$$
$$= \frac{\pi}{2} \left(\frac{1}{2 \cdot 1} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 5} + \text{etc.} \right)$$

$$- \quad \frac{t}{1+tt} \left(\frac{1}{2 \cdot 1} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 4} + \text{etc.} \right)$$

$$-\frac{t^3}{4(1+tt)^2}\left(\frac{1}{2}+\frac{5}{6\cdot 3}+\frac{5\cdot 7}{6\cdot 8\cdot 4}+\frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5}+\text{etc.}\right)$$
$$-\frac{t^5}{6(1+tt)^3}\left(\frac{1}{3}+\frac{7}{8\cdot 4}+\frac{7\cdot 9}{8\cdot 10\cdot 5}+\frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6}+\text{etc.}\right)$$
$$-\frac{t^7}{8(1+tt)^4}\left(\frac{1}{4}+\frac{9}{10\cdot 5}+\frac{9\cdot 11}{10\cdot 12\cdot 6}+\frac{9\cdot 11\cdot 13}{10\cdot 12\cdot 14\cdot 7}+\text{etc.}\right)$$
$$\text{etc.}$$

§36 Let us at first search for the sum of this series

$$\frac{1}{2\cdot 1}+\frac{1\cdot 3}{2\cdot 4\cdot 2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4}+\text{etc.}$$

and let us put

$$s=\frac{x}{2\cdot 1}+\frac{1\cdot 3x^2}{2\cdot 4\cdot 2}+\frac{1\cdot 3\cdot 5x^3}{2\cdot 4\cdot 6\cdot 3}+\frac{1\cdot 3\cdot 5\cdot 7x^4}{2\cdot 4\cdot 6\cdot 8\cdot 4}+\text{etc.};$$

it will be

$$s=\int\frac{dx}{x\sqrt{1-x}}-\log x,$$

as it will become clear to everyone doing the expansion. But it is

$$\int\frac{dx}{x\sqrt{1-x}}=c-\log(1+\sqrt{1-x})+\log(1-\sqrt{1-x})$$

and hence

$$s=c-\log(1+\sqrt{1-x})+\log(1-\sqrt{1-x})-\log x,$$

where the constant c has to be conditioned in such a way, that for $x=0$ s becomes $=0$. So let x become infintely small; it will be

$$\sqrt{1-x}=1-\frac{x}{2}$$

and

$$\log(1-\sqrt{1-x}) = \log\frac{x}{2} = \log x - \log 2$$

and

$$\log(1+\sqrt{1-x}) = \log 2,$$

whence $c = 2\log 2$. Now put $x = 1$; it will be $s = 2\log 2$ and

$$\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.} = 2\log 2.$$

But from this series the sum of the remaining series are determined in such a way, that it is

$$\frac{1}{2} + \frac{5}{6\cdot 3} + \frac{5\cdot 7}{6\cdot 8\cdot 4} + \frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5} + \text{etc.} = \frac{2\cdot 4}{1\cdot 3}\cdot 2\log 2 - \frac{2\cdot 4}{1\cdot 3\cdot 2},$$

$$\frac{1}{3} + \frac{7}{8\cdot 4} + \frac{7\cdot 9}{8\cdot 10\cdot 5} + \frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6} + \text{etc.} = \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\cdot 2\log 2 - \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 2} - \frac{6}{5\cdot 2},$$

$$\frac{1}{4} + \frac{9}{10\cdot 5} + \frac{9\cdot 11}{10\cdot 12\cdot 6} + \frac{9\cdot 11\cdot 13}{10\cdot 12\cdot 14\cdot 7} + \text{etc.} = \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7}\cdot 2\log 2 - \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7\cdot 2} - \frac{6\cdot 8}{5\cdot 7\cdot 2} - \frac{8}{7\cdot 3}$$

etc.

Having substituted these sums it will arise

$$\int \frac{dt}{1+tt}\log(1+tt)$$

$$= \frac{\pi}{2}\cdot 2\log 2 - \frac{t}{1+tt}\cdot 2\log 2$$

$$- \frac{t^3}{(1+tt)^2}\left(\frac{2}{3}\cdot 2\log 2 - \frac{1}{3\cdot 1}\right)$$

$$- \frac{t^5}{(1+tt)^3}\left(\frac{2\cdot 4}{3\cdot 5}\cdot 2\log 2 - \frac{4}{3\cdot 5\cdot 1} - \frac{1}{5\cdot 2}\right)$$

$$-\frac{t^7}{(1+tt)^4}\left(\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\cdot2\log2-\frac{4\cdot6\cdot8}{3\cdot5\cdot7\cdot1}-\frac{6}{5\cdot7\cdot2}\right)$$

$$-\frac{t^9}{(1+tt)^5}\left(\frac{2\cdot4\cdot6\cdot8}{3\cdot5\cdot7\cdot9}\cdot2\log2-\frac{4\cdot6\cdot8}{3\cdot5\cdot7\cdot9\cdot1}-\frac{6\cdot8}{5\cdot7\cdot9\cdot2}-\frac{1}{9\cdot4}\right)$$

etc.

§37 But because for our purpose after the done integration one has to put $t=\infty$, it will be

$$\int\frac{dt}{1+tt}\log(1+tt)=\pi\log2$$

and

$$s=\frac{1}{2}\pi\int\frac{dt}{1+tt}\log(1+tt)=\frac{\log2}{2},$$

which is that value itself, we foresaw to have to arise (§ 34). For, the remaining terms in the expression, that we found for

$$\int\frac{dt}{1+tt}\log(1+tt),$$

if one puts $t=\infty$, all vanish, because in the single denominators of the single terms t has more dimensions in the numerators and in addition to that the numerical coefficients decrease. If this would happen, we could not surely conclude, that the sum of all terms, which all vanish, is $=0$. For, if for example one takes only the first parts of the numerical coefficients, that this series arises

$$\frac{t}{1+tt}+\frac{2t^3}{3(1+tt)^2}+\frac{2\cdot4t^5}{3\cdot5(1+tt)^5}+\frac{2\cdot4\cdot6t^7}{3\cdot5\cdot7(1+tt)^4}+\text{etc.},$$

the sum of it in the case, in which $t=\infty$, becomes finite and $=\frac{\pi}{2}$, even though the single terms vanish; but if one takes integer coefficients, because of their very convergent series, also the whole series becomes $=0$.

§38 Now let us look for the sum of this series

$$1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\text{etc.}=B\pi^3,$$

which sum by § 32 will be

$$B\pi^3=\frac{\pi^2\log 2}{1\cdot 2\cdot 3}-2\pi^2\left(\frac{P\pi^2}{2\cdot 3\cdot 4\cdot 5}+\frac{Q\pi^4}{4\cdot 5\cdot 6\cdot 7}+\frac{R\pi^6}{6\cdot 7\cdot 8\cdot 9}+\text{etc.}\right).$$

To find the value of this quantity let

$$s=\frac{P\pi^2}{2\cdot 3\cdot 4\cdot 5}+\frac{Q\pi^4}{4\cdot 5\cdot 6\cdot 7}+\frac{R\pi^6}{6\cdot 7\cdot 8\cdot 9}+\text{etc.};$$

it will be

$$\begin{aligned}
\frac{d.\pi^3 s}{d\pi} &= \frac{P\pi^4}{2\cdot 3\cdot 4}+\frac{Q\pi^6}{4\cdot 5\cdot 6}+\frac{R\pi^8}{6\cdot 7\cdot 8}+\text{etc.},\\
\frac{dd.\pi^3 s}{d\pi^2} &= \frac{P\pi^3}{2\cdot 3}+\frac{Q\pi^5}{4\cdot 5}+\frac{R\pi^7}{6\cdot 7}+\text{etc.},\\
\frac{d^3.\pi^3 s}{d\pi^3} &= \frac{P\pi^2}{2}+\frac{Q\pi^4}{4}+\frac{R\pi^6}{6}+\text{etc.},\\
\frac{d^4.\pi^3 s}{d\pi^4} &= P\pi+Q\pi^3+R\pi^5+\text{etc.}=\frac{1}{4}\tan\frac{\pi}{2}.
\end{aligned}$$

By going backwards it will therefore be

$$\begin{aligned}
\frac{d^3.\pi^3 s}{d\pi^3} &= \frac{1}{4}\int d\pi\tan\frac{\pi}{2},\\
\frac{dd.\pi^3 s}{d\pi^2} &= \frac{1}{4}\int d\pi\int d\pi\tan\frac{\pi}{2},\\
\frac{d.\pi^3 s}{d\pi} &= \frac{1}{4}\int d\pi\int d\pi\int d\pi\tan\frac{\pi}{2},\\
\pi^3 s &= \frac{1}{4}\int d\pi\int d\pi\int d\pi\int d\pi\tan\frac{\pi}{2}.
\end{aligned}$$

And hence one will have the sum of the propounded series

$$B\pi^3 = \frac{\pi^2 \log 2}{6} - \frac{1}{2\pi} \int d\pi \int d\pi \int d\pi \int d\pi \tan\frac{\pi}{2},$$

and all integrals have to be taken so, that they vanish for $\pi = 0$.

§39 Put $\frac{\pi}{2} = q$, so that having done the integrations q denotes the fourth part of the circumference of the circle, whose diameter $= 1$, or the arc of 90 degrees. And further let be

$$\sin q = y \quad \text{and} \quad \cos q = x = \sqrt{1 - yy};$$

it will be

$$\tan\frac{\pi}{2} = \frac{y}{x}.$$

Hence because of $\pi = 2q$ the sum of our series will be

$$B\pi^3 = \frac{2qq \log 2}{3} - \frac{4}{q} \int dq \int dq \int dq \int \frac{ydq}{x}.$$

So let us throughout put

$$\int dq \int dq \int dq \int \frac{ydq}{x} = u;$$

it will be

$$B\pi^3 = \frac{2qq \log 2}{3} - \frac{4u}{q},$$

where in the finding of the quantity u all integrations have to be taken so, that the single integrals vanish for $q = 0$ and $y = 0$; but having done the integrals, it will be $y = 1$ and $x = 0$. But it is

$$\int \frac{ydq}{x} = \int \frac{ydy}{1 - yy} = -\log\sqrt{1 - yy} = \log\frac{1}{x}$$

and

$$\log\frac{1}{x} = \frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \frac{y^{10}}{10} + \text{etc.}$$

Because now it is

$$u = \int dq \int dq \int dq \log \frac{1}{x},$$

by a reduction of the integrals it will be

$$u = q \int dq \int dq \log \frac{1}{x} - \int qdq \int dq \log \frac{1}{x}$$

and further

$$\int dq \int dq \log \frac{1}{x} = q \int dq \log \frac{1}{x} - \int qdq \log \frac{1}{x},$$

$$\int qdq \int dq \log \frac{1}{x} = \frac{qq}{2} \int dq \log \frac{1}{x} - \frac{1}{2} \int qqdq \log \frac{1}{x},$$

hence

$$u = \frac{1}{2}qq \int dq \log \frac{1}{x} - q \int qdq \log \frac{1}{x} + \frac{1}{2} \int qqdq \log \frac{1}{x},$$

that we now have three simple integral formulas, we have to integrate.

§40 So let us consider these three single formulas seperately and at first this one $\int dq \log \frac{1}{x}$; even if we already integrated it above [§ 35], let us nevertheless integrate the same without consideration of sines and cosines again, that an easir way is opened to integrate the remaining one. So it is

$$\int dq \log \frac{1}{x} = \int dq \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \frac{y^{10}}{10} + \text{etc.} \right)$$

To find this integral, just consider any of its members - $\int y^{n+2} dq$ - and because it is

$$dq = \frac{dy}{x} = \frac{-dx}{y} \quad \text{and} \quad xx + yy = 1,$$

it will be

$$\int y^{n+2} dq = - \int y^{n+1} dx = -y^{n+1} x + (n+1) \int y^n x dy;$$

but it is

$$\int y^n x dy = \int y^n x^2 dq = \int y^n dq - \int y^{n+2} dq$$

- because of $xx = 1 - yy$; therefore it is

$$\int y^{n+2}dq = -y^{n+1}x + (n+1)\int y^n dq - (n+1)\int y^{n+2}dq$$

and

$$\int y^{n+2}dq = \frac{-y^{n+2}}{n+2} + \frac{n+1}{n+2}\int y^n dq.$$

Hence the integral of a certain member is reduced to the integral of the preceding, and because having done the integration x becomes $= 0$, it will be for this case

$$\int y^{n+2}dq = \frac{n+1}{n+2}\int y^n dq.$$

From this formula therefore all the single parts of the integral will be found, as follows,

$$\int y^2 dq = \frac{1}{2}q,$$

$$\int y^4 dq = \frac{1\cdot 3}{2\cdot 4}q,$$

$$\int y^6 dq = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}q,$$

$$\int y^8 dq = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}q$$

etc.

Therefore one will have

$$\int dq \log\frac{1}{x} = \frac{1}{q}\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.}\right);$$

because the sum of this series was already found above (§ 36) as $= 2\log 2$, it will be

$$\int dq \log\frac{1}{x} = q\log 2.$$

§41 Now let us proceed to the second integral formula $\int qdq \log \frac{1}{x}$, which merges into

$$\int qdq \log \frac{1}{x} = \int qdq \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.} \right),$$

of which we want to consider any part

$$\begin{aligned}\int y^{n+2} qdq &= -\int y^{n+1} qdx = -y^{n+1}qx + \int y^{n+1}xdq + (n+1)\int y^n qxdy \\ &= -y^{n+1}qx + \frac{y^{n+2}}{n+2} + (n+1)\int y^n qdq - (n+1)\int y^{n+2}qdq.\end{aligned}$$

Therefore having put $y = 1$ and $x = 0$ it will be

$$\int y^{n+2}qdq = \frac{1}{(n+2)^2} + \frac{n+1}{n+2}\int y^n qdq.$$

The integrals of the single members so will behave as follows:

$$\begin{aligned}\int y^2 qdq &= \frac{1}{2^2} + \frac{1}{2}\cdot\frac{q^2}{2}, \\ \int y^4 qdq &= \frac{1}{4^2} + \frac{3}{4\cdot 2^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{q^2}{2}, \\ \int y^6 qdq &= \frac{1}{6^2} + \frac{5}{6\cdot 4^2} + \frac{3\cdot 5}{4\cdot 6\cdot 2^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{q^2}{2}, \\ \int y^8 qdq &= \frac{1}{8^2} + \frac{7}{8\cdot 6^2} + \frac{5\cdot 7}{6\cdot 8\cdot 4^2} + \frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 2^2} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{q^2}{2} \\ &\text{etc.}\end{aligned}$$

Hence one will obtain the integral

$$\int qdq \log\frac{1}{x} = +\frac{qq}{4}\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.}\right)$$

$$+\frac{1}{2\cdot 2^2}\left(\frac{1}{1}+\frac{3}{4\cdot 2}+\frac{3\cdot 5}{4\cdot 6\cdot 3}+\frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 4}+\text{etc.}\right)$$
$$+\frac{1}{2\cdot 4^2}\left(\frac{1}{2}+\frac{5}{6\cdot 3}+\frac{5\cdot 7}{6\cdot 8\cdot 4}+\frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5}+\text{etc.}\right)$$
$$+\frac{1}{2\cdot 6^2}\left(\frac{1}{3}+\frac{7}{8\cdot 4}+\frac{7\cdot 9}{8\cdot 10\cdot 5}+\frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6}+\text{etc.}\right)$$
$$\text{etc.}$$

or also in this form

$$\begin{aligned}\int qdq\log\frac{1}{x}=\frac{qq}{4}\Big(&\frac{1}{2\cdot 1}+\frac{1\cdot 3}{2\cdot 4\cdot 2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4}+\text{etc.}\Big)\\
&+\frac{1}{2^2\cdot 2}+\frac{1}{4^2\cdot 4}+\frac{1}{6^2\cdot 6}+\frac{1}{8^2\cdot 8}+\text{etc.}\\
&+\frac{3}{2^2\cdot 4^2}+\frac{5}{4^2\cdot 6^2}+\frac{7}{6^2\cdot 8^2}+\frac{9}{8^2\cdot 10^2}+\text{etc.}\\
&+\frac{3\cdot 5}{2^2\cdot 4\cdot 6^2}+\frac{5\cdot 7}{4^2\cdot 6\cdot 8^2}+\frac{7\cdot 9}{6^2\cdot 8\cdot 10^2}+\frac{9\cdot 11}{8^2\cdot 10\cdot 12^2}+\text{etc.}\\
&+\frac{3\cdot 5\cdot 7}{2^2\cdot 4\cdot 6\cdot 8^2}+\frac{5\cdot 7\cdot 9}{4^2\cdot 6\cdot 8\cdot 10^2}+\frac{7\cdot 9\cdot 11}{6^2\cdot 8\cdot 10\cdot 12^2}+\frac{9\cdot 11\cdot 13}{8^2\cdot 10\cdot 12\cdot 14^2}+\text{etc.}\\
&\text{etc.}\end{aligned}$$

but these series involve that itself, what is in question, of course the summation of the cubes of the terms of the harmonic series.

§42 If we follow the first form, all series become summable (§ 36) and one will have

$$\begin{aligned}\int qdq\log\frac{1}{x}=\frac{qq}{2}\log 2&+\frac{1}{2^2}\left(\frac{2}{1}\log 2\right)\\
&+\frac{1}{4^2}\left(\frac{2\cdot 4}{1\cdot 3}\log 2-\frac{4}{3\cdot 2}\right)\end{aligned}$$

$$+\frac{1}{6^2}\left(\frac{2\cdot4\cdot6}{1\cdot3\cdot5}\log2-\frac{4\cdot6}{3\cdot5\cdot2}-\frac{6}{5\cdot4}\right)$$

$$+\frac{1}{8^2}\left(\frac{2\cdot4\cdot6\cdot8}{1\cdot3\cdot5\cdot7}\log2-\frac{4\cdot6\cdot8}{3\cdot5\cdot7\cdot2}-\frac{6\cdot8}{5\cdot7\cdot4}-\frac{8}{7\cdot6}\right)$$

etc.,

which, if the series are again summed columnwise, give

$$\int qdq\log\frac{1}{x}=\frac{qq}{2}\log2+\log2\left(\frac{1}{2}+\frac{2}{3\cdot4}+\frac{2\cdot4}{3\cdot5\cdot6}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7\cdot9}+\text{etc.}\right)$$

$$-\frac{1}{2\cdot3}\left(\frac{1}{4}+\frac{4}{5\cdot6}+\frac{4\cdot6}{5\cdot7\cdot8}+\frac{4\cdot6\cdot8}{5\cdot7\cdot9\cdot11}+\text{etc.}\right)$$

$$-\frac{1}{4\cdot5}\left(\frac{1}{6}+\frac{6}{7\cdot8}+\frac{6\cdot8}{7\cdot9\cdot10}+\frac{6\cdot8\cdot10}{7\cdot9\cdot11\cdot12}+\text{etc.}\right)$$

$$-\frac{1}{6\cdot7}\left(\frac{1}{8}+\frac{8}{9\cdot10}+\frac{8\cdot10}{9\cdot11\cdot12}+\frac{8\cdot10\cdot12}{9\cdot11\cdot13\cdot14}+\text{etc.}\right)$$

etc.

But it is

$$\frac{1}{2}+\frac{2}{3\cdot4}+\frac{2\cdot4}{3\cdot5\cdot6}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7\cdot8}+\text{etc.}=\frac{qq}{2},$$

whence it will be

$$\frac{1}{4}+\frac{4}{5\cdot6}+\frac{4\cdot6}{5\cdot7\cdot8}+\text{etc.}=\frac{3}{2}\cdot\frac{qq}{2}-\frac{3}{2}\cdot\frac{1}{2},$$

$$\frac{1}{6}+\frac{6}{7\cdot8}+\frac{6\cdot8}{7\cdot9\cdot10}+\text{etc.}=\frac{3\cdot5}{2\cdot4}\cdot\frac{qq}{2}-\frac{3\cdot5}{2\cdot4}\cdot\frac{1}{2}-\frac{5}{4}\cdot\frac{1}{4},$$

$$\frac{1}{8}+\frac{8}{9\cdot10}+\frac{8\cdot10}{9\cdot11\cdot12}+\text{etc.}=\frac{3\cdot5\cdot7}{2\cdot4\cdot6}\cdot\frac{qq}{2}-\frac{3\cdot5\cdot7}{2\cdot4\cdot6}\cdot\frac{1}{2}-\frac{5\cdot7}{4\cdot6}\cdot\frac{1}{4}-\frac{7}{6}\cdot\frac{1}{6}$$

etc.

Therefore one will have

$$\int qdq\log\frac{1}{x} = qq\log 2 - \frac{qq}{2}\left(\frac{1}{2^2}+\frac{3}{2\cdot 4^2}+\frac{3\cdot 5}{2\cdot 4\cdot 6^2}+\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2}+\text{etc.}\right)$$

$$+\frac{1}{2^2}\cdot\frac{1}{2}$$

$$+\frac{3}{2\cdot 4^2}\cdot\frac{1}{2}+\frac{1}{4^2}\cdot\frac{1}{4}$$

$$+\frac{3\cdot 5}{2\cdot 4\cdot 6^2}\cdot\frac{1}{2}+\frac{5}{4\cdot 6^2}+\frac{1}{6^2}\cdot\frac{1}{6}$$

$$+\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2}\cdot\frac{1}{2}+\frac{5\cdot 7}{4\cdot 6\cdot 8^2}\cdot\frac{1}{4}+\frac{7}{6\cdot 8^2}\cdot\frac{1}{6}+\frac{1}{8^2}\cdot\frac{1}{8}$$

etc.

§43 But maybe the difficulty to find a convenient expression is diminished, if we collect those three integral formulas. Therefore let us take the third formula

$$\int qqdq\log\frac{1}{x},$$

which merges into

$$\int qqdq\left(\frac{yy}{2}+\frac{y^4}{4}+\frac{y^6}{6}+\frac{y^8}{8}+\text{etc.}\right).$$

Consider the formula

$$\int y^{n+2}qqdq,$$

which merges into

$$-\int y^{n+1}qqdx = -y^{n+1}qqx + 2\int y^{n+1}qxdq + (n+1)\int y^n qqxdy$$

$$= -y^{n+1}qqx + 2\int y^{n+1}qdy + (n+1)\int y^n qqdq - (n+1)\int y^{n+2}qqdq;$$

hence it will be

$$\int y^{n+2}qqdq = \frac{-y^{n+1}qqx}{n+2} + \frac{2}{n+2}\int y^{n+1}qdy + \frac{n+1}{n+2}\int y^{n}qqdq.$$

But it is

$$\int y^{n+1}qdy = \frac{y^{n+2}q}{n+2} - \frac{1}{n+2}\int y^{n+2}dq = \frac{q}{n+2} - \frac{1}{n+2}\cdot\frac{1\cdot 3\cdot 5\cdot 7\cdots(n+1)}{2\cdot 4\cdot 6\cdot 8\cdots(n+2)}q,$$

having put $y = 1$ (§ 40). Consequently it will be

$$\int y^{n+2}qqdq = \frac{2q}{(n+2)}\left(1 - \frac{1\cdot 3\cdot 5\cdot 7\cdots(n+1)}{2\cdot 4\cdot 6\cdot 8\cdots(n+2)}\right) + \frac{n+1}{n+2}\int y^{n}qqdq$$

and the single members of the integral in question will be

$$\int y^{2}qqdq = \frac{1}{2^2}2q - \frac{1}{2^2\cdot 2}2q + \frac{1}{2}\cdot\frac{q^3}{3},$$

$$\int y^{4}qqdq = \frac{1}{4^4}2q - \frac{1}{4^2}\cdot\frac{1\cdot 3}{2\cdot 4}2q + \frac{1}{2^2}\cdot\frac{3}{4}2q - \frac{1}{2^2}\cdot\frac{1\cdot 3}{2\cdot 4}2q + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{q^3}{3},$$

$$\int y^{6}qqdq = \frac{2q}{6}\left(1 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right) + \frac{5}{6}\cdot\frac{2q}{4^2}\left(1 - \frac{1\cdot 3}{2\cdot 4}\right) + \frac{3\cdot 5}{4\cdot 6}\cdot\frac{2q}{2^2}\left(1 - \frac{1}{2}\right) + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{q^3}{3}$$

etc.

Having finally substituted and arrayed the terms one will finally find

$$\begin{aligned}
\int qqdq\log\frac{1}{x} = \quad & \frac{q^3}{6}\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \text{etc.}\right)\\
+ \frac{1}{2^2}q\left(1 - \frac{1}{2}\right) & \left(1 + \frac{3}{4\cdot 2} + \frac{3\cdot 5}{4\cdot 6\cdot 3} + \frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 4} + \text{etc.}\right)\\
+ \frac{1}{4^2}q\left(1 - \frac{1\cdot 3}{2\cdot 4}\right) & \left(\frac{1}{2} + \frac{5}{6\cdot 3} + \frac{5\cdot 7}{6\cdot 8\cdot 4} + \frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5} + \text{etc.}\right)\\
+ \frac{1}{6^2}q\left(1 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right) & \left(\frac{1}{3} + \frac{7}{8\cdot 4} + \frac{7\cdot 9}{8\cdot 10\cdot 5} + \frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6} + \text{etc.}\right)
\end{aligned}$$

etc.

But because it is

$$= \frac{1}{2}qq\int dq\log\frac{1}{x} - q\int qdq\log\frac{1}{x} + \frac{1}{2}\int qqdq\log\frac{1}{x},$$

having added the integrals, as they were found, it will be

$$\begin{aligned}
u = \quad & \frac{q^3}{12}\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \text{etc.}\right)\\
& - \frac{q}{2\cdot 2^2}\cdot\frac{1}{2}\left(\frac{1}{1} + \frac{3}{4\cdot 2} + \frac{3\cdot 5}{4\cdot 6\cdot 3} + \frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 4} + \text{etc.}\right)\\
& - \frac{q}{2\cdot 4^2}\cdot\frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{2} + \frac{5}{6\cdot 3} + \frac{5\cdot 7}{6\cdot 8\cdot 4} + \frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5} + \text{etc.}\right)\\
& - \frac{q}{2\cdot 6^2}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{3} + \frac{7}{8\cdot 4} + \frac{7\cdot 9}{8\cdot 10\cdot 5} + \frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6} + \text{etc.}\right)
\end{aligned}$$

etc.

But having integrated the series as above [§ 36] it will be

$$\begin{aligned}
u = \frac{q^3}{6}\log 2 & - \frac{q}{2\cdot 2^2}\quad 2\log 2,\\
& - \frac{q}{2\cdot 2^4}\left(2\log 2 - \frac{1}{2\cdot 1}\right),\\
& - \frac{q}{2\cdot 2^6}\left(2\log 2 - \frac{1}{2\cdot 1} - \frac{1\cdot 3}{2\cdot 4\cdot 2}\right),\\
& - \frac{q}{2\cdot 2^8}\left(2\log 2 - \frac{1}{2\cdot 1} - \frac{1\cdot 3}{2\cdot 4\cdot 2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3}\right),\\
& - \frac{q}{2\cdot 2^{10}}\left(2\log 2 - \frac{1}{2\cdot 1} - \frac{1\cdot 3}{2\cdot 4\cdot 2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} - \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4}\right)
\end{aligned}$$

etc.

But it is

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\text{etc.}=\frac{qq}{6},$$

whence

$$\begin{aligned}
u = {} & \frac{q}{2\cdot 2} && \left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\text{etc.}\right)\\
& +\frac{1\cdot 3q}{2\cdot 4\cdot 4} && \left(\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{10^2}+\text{etc.}\right)\\
& +\frac{1\cdot 3\cdot 5q}{2\cdot 4\cdot 6\cdot 6} && \left(\frac{1}{8^2}+\frac{1}{10^2}+\frac{1}{12^2}+\text{etc.}\right)\\
& +\frac{1\cdot 3\cdot 5\cdot 7q}{2\cdot 4\cdot 6\cdot 8\cdot 8} && \left(\frac{1}{10^2}+\frac{1}{12^2}+\frac{1}{14^2}+\text{etc.}\right)\\
& \qquad\text{etc.}
\end{aligned}$$

Therefore it will be

$$\begin{aligned}
u = {} & \frac{q}{2\cdot 2} && \left(\frac{qq}{6}-\frac{1}{2^2}\right)\\
& +\frac{1\cdot 3q}{2\cdot 4\cdot 4} && \left(\frac{qq}{6}-\frac{1}{2^2}-\frac{1}{4^2}\right)\\
& +\frac{1\cdot 3\cdot 5q}{2\cdot 4\cdot 6\cdot 6} && \left(\frac{qq}{6}-\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{6^2}\right)\\
& \qquad\text{etc.}
\end{aligned}$$

or having actually summed the first vertical series

$$\begin{aligned}
u = \frac{q^3}{6}\log 2 & -\frac{q}{2\cdot 2}\cdot\frac{1}{2^2}\\
& -\frac{1\cdot 3q}{2\cdot 4\cdot 4}\left(\frac{1}{2^2}+\frac{1}{4^2}\right)\\
& -\frac{1\cdot 3\cdot 5q}{2\cdot 4\cdot 6\cdot 6}\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}\right)
\end{aligned}$$

etc.

§44 Because now the sum of our propounded series

$$1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\text{etc.}$$

is

$$=B\pi^3=\frac{2qq\log 2}{3}-\frac{4u}{q},$$

the same sum will become

$$\begin{aligned}=&+\frac{1}{2\cdot 2}\cdot 1\\ &+\frac{1\cdot 3}{2\cdot 4\cdot 4}\left(1+\frac{1}{2^2}\right)\\ &+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\left(1+\frac{1}{2^2}+\frac{1}{3^2}\right)\\ &+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 8}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\right)\end{aligned}$$

etc.

Or because it is

$$\frac{1}{2\cdot 2}+\frac{1\cdot 3}{2\cdot 4\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}+\text{etc.}=\log 2,$$

the sum of the propounded series will be

$$\begin{aligned}B\pi^3=\log 2&+\frac{1}{2^2}\left(\log 2-\frac{1}{2\cdot 2}\right)+\frac{1}{3^2}\left(\log 2-\frac{1}{2\cdot 2}-\frac{1\cdot 3}{2\cdot 4\cdot 4}\right)\\ &+\frac{1}{4^2}\left(\log 2-\frac{1}{2\cdot 2}-\frac{1\cdot 3}{2\cdot 4\cdot 4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\right)+\text{etc.},\end{aligned}$$

or this same sum can be expressed so, that it is

$$\begin{aligned}
B\pi^3 = \frac{\pi^2}{6}\log 2 - \frac{1}{2^2}\cdot\frac{1}{2\cdot 2} \\
&- \frac{1}{3^2}\left(\frac{1}{2\cdot 2} + \frac{1\cdot 3}{2\cdot 4\cdot 4}\right) \\
&- \frac{1}{4^2}\left(\frac{1}{2\cdot 2} + \frac{1\cdot 3}{2\cdot 4\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\right) \\
&- \frac{1}{5^2}\left(\frac{1}{2\cdot 2} + \frac{1\cdot 3}{2\cdot 4\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 8}\right) \\
&\text{etc.}
\end{aligned}$$

But because, no matter how we transform this series, are not able to reduce it to a simple series, whose sum is known, we abrupt this task, content with these many expressions, we found the propounded series

$$1 - \frac{1}{2^3} - \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \text{etc.}$$

equal to.