On the Determination of Series or a new Method to find the general Terms of Series*

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§1 Since the law of progression, which the terms of a series follow, can vary to infinity, it seems that not only all different species of series, but not even all classes, no matter how far they extend, can actually be enumerated. Hence two or more series are given, which, even though they have as many common terms as one wants, nevertheless differ and are contained in very different laws. Who has inspected the very broad field of series even only for a short time, will easily understand that the nature of a series is not determined, no matter how many of its terms are exhibited. So if it is in question, what the series is, which starts with these terms

$$1, \quad 3, \quad 5, \quad 7, \quad 9, \quad 11, \quad 13, \quad 15,$$

the question is most undetermined; and except for the series of the odd natural numbers proceeding in natural order innumerable other series can be assigned, which start with the same terms; and this defect of determination is not restricted to a certain number of given terms, but, no matter how large that number was, can be common to all infinite series.

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§2 But this will be seen more clearly, if we transfer the nature of series to Geometry. For, each arbitrary series can be represented by means of a curved line, whose ordinate is expressed by suitable terms of the series itself, while the abscissas denote their indices or numbers, which represent the order of each term. This way any arbitrary term of the series defines a point on the curved line, which corresponds to the given abscissa. Hence, if a series is required, which has arbitrary many given terms, the question reduces to this, that a curved line is found, which goes through as many given points. But it is perspicuous that always innumerable curved lines can be assigned which go through the single points at the same time. Although Newton showed this only for parabolic curves, if not only all algebraic curves but also transcendental ones are admitted, there is no doubt that the number of satisfying curves additionally becomes infinitely larger.

§3 It will seem more remarkable, if I say that the series is not even determined, even though innumerable of its terms are given. So if I define this series

$$1 + 2 + 3 + 4 + 5 + 6 + \text{etc.}$$

in such a way that I say that in it all integer numbers in natural order are contained, who will then not believe that this series is completely determined, since for each place in the series its term was assigned? For, in the place, which is x units away from the beginning, the term will be $=$ the number x itself, or the term, whose index is $= x$, will also be $= x$. But how that series, as it was done, is described, there is not more known than that to the index x, if x was an integer number, corresponds the term x; but if for the index x a fractional number is assumed, there is now reason, by which it would be clear that the term corresponding to that index x is $= x$. But I will show, if for this series the term corresponding to the index x is put $= y$, that it can happen in infinitely many ways that, as often as x is an integer number, so often it always is $y = x$, even though by taking fractional numbers for x the value of y differs from x. Hence, even though all terms of the series, which correspond to integer indices, are determined, the intermediate ones, which have fractional indices, can nevertheless be determined in infinitely many ways, such that the interpolation of this series remains undetermined.

§4 That this is seen more clearly, one has to recur to circular arcs; for, since having put the semicircumference of the circle, whose radius is $= 1$, $= \pi$ the sine of the arc $n\pi$ is $= 0$, as often as n is an integer, it is manifest, if one puts $y = x + P \sin \pi x$ while P denotes either a constant quantity or an arbitrary function of x and for x one successively puts the integer numbers 1, 2, 3, 4, 5 etc., that then the values of y will be $= 1, 2, 3, 4, 5$, etc., as if it was $P = 0$. And nevertheless the intermediate values, which correspond to fractional indices, will not be equal to these indices. For, for the sake of an example let $P = xx$ and put $x = \frac{1}{2}$; because of $\sin \frac{1}{2}\pi = 1$ the term corresponding to the index $\frac{1}{2}$ will become

$$\frac{1}{2} + \frac{1}{4} \cdot 1 = \frac{3}{4}.$$

But one can think of infinitely many other expressions of this kind, which satisfy equally, of which kind are

$$y = x + P \sin \pi x + Q \sin 2\pi x + R \sin 3\pi x + S \sin 4\pi x + \text{etc.},$$

by which the interpolation is rendered much more undetermined.

§5 I already exhibited a similar example of a series, which could seem determined, some time ago; for, I had found an expression or a function of x, which, if for x any power of 10 is substituted, becomes equal to this power, if this exponent is a positive integer, of course. That function of x, which I will indicate by the letter y, was of such a nature that having put $x = 1$ it becomes $y = 0$ and, if one puts $x = 10^n$ while n is a positive integer number, always becomes $y = n$; hence it seemed to follow that the function y will always be the common logarithm of x. Nevertheless, I showed, if for x not a certain power of ten is substituted, that the value of y often differs a lot from the logarithm of ten. Therefore, having set up the series, for which we have

$$\begin{array}{lcccccccc} \text{Indices} & 1, & 10^1, & 10^2, & 10^3, & 10^4, & 10^5, & 10^6 & \text{etc.} \\ \text{and} & & & & & & & & \\ \text{Terms} & 0, & 1, & 2, & 3, & 4, & 5, & 6 & \text{etc.} \end{array}$$

for the description of logarithms it does not suffice, if someone says that the logarithms are the middle terms, which correspond to the indices assumed in the superior series, of the inferior series.

§6 Therefore, since the nature of a series is not determined from some of its terms, even though their number is infinite, since the interpolation nevertheless remains undetermined and can be done in infinitely many ways, it is easily seen, how uncertain all these interpolation methods are, which teach to complete the task from the terms having integer indices alone. For, the interpolation can only be considered as certain, if the nature of the series is taken into account in the operation. But the nature of a series is seen perfectly, if its general term or a formula, which for each index x, whether integer or fractional or even surdic, exhibits the corresponding terms, was known. For, this way not only all terms of the series, which correspond to integer indices, are determined, but also the terms, which correspond to arbitrary non-integer indices, are defined without any ambiguity; and so the task of interpolation is no longer impeded by any uncertainty.

§7 But except for the general term one has innumerable other ways to form series; nevertheless all these ways can be conveniently reduced to three classes. To the first class I count these ways of forming series, in which each term of the series is only determined by the corresponding index; since this is caused by certain operations to be done for this aim, the formula containing these operations in general will be the general term of the series itself, that by which the series is perfectly and absolutely determined I already noted. To the second class all these ways of forming series extend, in which the general term of the series is determined by some of the preceding terms according to a certain rule, which way is usually especially applied in recurring series. But whenever to find a certain term of a series not only the preceding terms are to be taken into account, but also the index itself must be used, I hence constitute the third class of determination.

§8 If an arbitrary term of the series is determined from the index alone, then, whether an integer or a fractional number is assumed for the index, the corresponding term is equally defined and so the interpolation of the series will have neither any difficulty nor uncertainty. But if, as we put in the second class, an arbitrary term is determined from the preceding one or several preceding ones, then having assumed the first or some first terms ad libitum the single terms, which correspond to integer indices, will be found, but it is not possible to define the intermediate terms corresponding to fractional terms from this, which is also be said about the third class. But although this way in the second and third class not only all terms, which correspond to integer indices, are assigned, but also the law between the terms and its preceding ones is prescribed, which equally extends to terms of fractional indices, nevertheless not even this way the series is completely determined, but for any arbitrary series of this class infinitely many general terms can be exhibited, which, while they yield the same terms for integer indices, nevertheless deviate for the fractional ones.

§9 Since this justly seems to be paradoxical, it will be worth one's while to consider this defect of determination in the series, in which each term is determined from the preceding ones, more diligently. Therefore, let us take the simplest case and assume that the series is defined in such a way, that each term is equal to the preceding one itself. If now the first term of the series is set $= 1$, the second will also be $= 1$ and all following ones, which correspond to integer indices, will become equal to the unity and this series will arise:

$$\begin{array}{rcccccccccccc} \text{Indices:} & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10 & \text{etc.} \\ \text{Terms:} & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1 & \text{etc.} \end{array}$$

and it is manifest that to each integer index x corresponds the term $= 1$. But how the terms corresponding to fractional indices will behave, is hence not defined; only this is known, if the term corresponding to the index $\frac{1}{2}$ was $= a$, that also all terms, which correspond to the indices

$$\frac{3}{2}, \quad \frac{5}{2}, \quad \frac{7}{2}, \quad \frac{9}{2} \quad \text{etc.}$$

will also be $= a$. For, all terms, whose indices differ by one unit or several units, must be equal by the prescribed law, since each preceding term is understood as the one, whose index is the one smaller by one unit.

§10 Therefore, this series is defined in such a way that, if the term corresponding to the index x is put $= y$, the following term corresponding to the index $x+1$ is $= y'$, one has $y' = y$; but then furthermore it is assumed, if it was $x = 1$, that it will also be $y = 1$. Hence, if for this series the general term is desired, it must be a function of x of such a kind, which shall be $= y$, that, if instead of x one puts $x+1$, the resulting value y' of the function y will be equal to y itself, and that having put $x = 1$ it also is $y = 1$. But it is manifest, if in general one puts $y = 1$, that this condition is satisfied and in this case not only the terms, which correspond to integer indices, but also those, which correspond to fractional ones, will be smaller than unity. But on the other hand these conditions can also be satisfied in infinitely many other ways; for, if one puts

$$y = 1 + \alpha \sin 2\pi x,$$

while π denotes the half of the circumference of the circle, whose radius is $= 1$, it will be

$$y' = 1 + \alpha \sin 2\pi x;$$

but it is

$$\sin 2\pi(x+1) = \sin 2\pi x$$

and hence $y' = y$, but then for $x = 1$ it will also be $y = 1$. In this case the intermediate terms or the ones, corresponding to fractional indices, are not longer equal to the unity; for, having put $x = \frac{1}{4}$ it will be $y = 1 + \alpha$.

§11 Since here not only α can be assumed ad libitum, but one can also think of innumerable other formulas of this kind, which fulfill the prescribed conditions, of which kind these are

$$y = 1 + \alpha \sin 2\pi x + \beta \sin 4\pi x + \gamma \sin 6\pi x + \text{etc.},$$

it is perspicuous that the interpolation even of this most simple series $1 + 1 + 1 +$ etc., if it is only defined in such a way that each term is equal to the preceding one, but the first is said to be expressed by the unity, is highly undetermined, since the intermediate terms having fractional indices can be equal to any numbers. Nevertheless, even though innumerable general terms can be exhibited for this series, they are all contained in the same general law and without any deviation can be found by means of Analysis. Of course, a very far extending method can be given, by means of which it is possible to define the general terms of all series, whose terms are determined by the preceding ones, whether without the index or with the index, in most universal manner; this method since it does not only lead to a more complete understanding of series, but also contains augmentations not to be contemned for whole Analysis, I constituted to develop here more diligently; for this aim I will consider the following problems.

PROBLEM 1

§12 *To find the general term of the series, any arbitrary term of which is equal to the preceding one, but whose first term* $= 1$.

SOLUTION

Let the general term or the one which corresponds to the index x be $= y$ and put the following term (whose index is $= x+1) = y'$ and it must be $y' = y$; and having put $x = 1$ is must be $y = 1$. Since now y is a certain function of x, by the nature of differential calculus, if instead of x one puts $x + 1$, it will be

$$y' = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.},$$

having assumed the differential dx to be constant. Therefore, it must be

$$0 = \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.},$$

And this equation contains completely all satisfying values of y, as long as the integration is tempered in such a way that having put $x = 1$ it is $y = 1$ or, what reduces to the same, that having put $x = 0$ it is $y = 1$. Therefore, the question was reduced to the resolution of this differential equation, which not only consists of an infinite number of terms, but also contains all orders of differentials. But since the variable y everywhere does not have more than one dimension and of the other variable x only the differential dx, which is assumed to be constant, occurs, this equation can be treated in the same way which I explained in MISCELLANEA BEROLOLIN. Volume 7. Therefore, by putting z instead of $\frac{dy}{dx}$, z^2 instead of $\frac{ddy}{dx^2}$ and in general z^n instead of $\frac{d^ny}{dx^n}$ form the algebraic equation

$$0 = \frac{z}{1} + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.},$$

which having taken e for the number, whose hyperbolic logarithm is $= 1$, goes over into this finite form $0 = e^z - 1$. Now one has to investigate all roots of this equation, whose number is infinite, or one has to assign all factors of the formula $e^z - 1$. But it is

$$e^z = \left(1 + \frac{z}{n}\right)^n,$$

having put n to be an infinite number; if this value is substituted, one will have to resolve this equation

$$\left(1+\frac{z}{n}\right)^n - 1,$$

of which certainly one simple factor is $= \frac{z}{n}$ or z, which the infinite equation immediately reveals. To find the remaining ones one has to recall the theorem, in which it is demonstrated that a factor of the binomial form $a^n - b^n$ is

$$aa - 2ab\cos\frac{2k\pi}{n} + bb,$$

while k denotes an integer number. Therefore, in the present case it is

$$a = 1+\frac{z}{n} \quad \text{and} \quad b = 1,$$

whence all factors of the propounded formula $e^z - 1$ are contained in this general form

$$1+\frac{2z}{n}+\frac{zz}{nn}-2\left(1+\frac{z}{n}\right)\cos\frac{2k\pi}{n}+1$$

or

$$2\left(1+\frac{z}{n}\right)\text{versin}\,\frac{2k\pi}{n}+\frac{zz}{nn};$$

hence by dividing this factor by the constant quantity $2\,\text{versin}\,\frac{2k\pi}{n}$ the general factor will be

$$= 1+\frac{z}{n}+\frac{zz}{2nn\ \text{versin}\frac{2k\pi}{n}}.$$

Since now n is an infinite number, it will be

$$\cos\frac{2k\pi}{n} = 1-\frac{2kk\pi\pi}{nn} \quad \text{und} \quad \text{versin}\frac{2k\pi}{n} = \frac{2kk\pi\pi}{nn};$$

having substituted this value the general factor of the formula $e^z - 1$ will be

$$= 1+\frac{z}{n}+\frac{zz}{4kknn},$$

and by successively putting the integer numbers 1, 2, 3, 4 etc. instead of k completely all factors of the formula $e^z - 1$ will arise. But the first factor z

gives the constant part of the integral, which shall be $= C$; but if the remaining factors, which are reduced to this form

$$4kk\pi\pi + \frac{4kk\pi\pi}{n}z + zz,$$

are compared to the form of the factors, which I expanded in the dissertation mentioned before,

$$ff - 2fz\cos\varphi + zz,$$

it will be

$$f = 2k\pi \quad \text{and} \quad \cos\varphi = -\frac{k\pi}{n}$$

and $\sin\varphi = 1$ because of the infinite number n, in which case it is $\cos\varphi = 0$. Therefore, the part of the integral to arise from this will be

$$\alpha e^{\frac{-2kk\pi\pi}{n}x}\sin 2k\pi x + \mathfrak{A}e^{\frac{-2kk\pi\pi}{n}x}\cos 2k\pi x$$

or because of $n = \infty$

$$\alpha\sin 2k\pi x + \mathfrak{A}\cos 2k\pi x,$$

Therefore, having successively substituted all integer numbers 1, 2, 3, 4 etc. for k the integral of the found equation will arise

$$0 = \frac{dy}{1\cdot dx} + \frac{ddy}{1\cdot 2\cdot dx^2} + \frac{d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \frac{d^4y}{1\cdot 2\cdot 3\cdot 4\cdot x^4} + \text{etc.}$$

will arise expressed in the following form

$$\begin{aligned} y = C + \alpha\sin 2\pi x &+ \mathfrak{A}\cos 2\pi x \\ + \beta\sin 4\pi x &+ \mathfrak{B}\cos 4\pi x \\ + \gamma\sin 6\pi x &+ \mathfrak{C}\cos 6\pi x + \text{etc.} \end{aligned}$$

Now define the constant C in such a way that having put $x = 0$ it is $y = 1$, and one will find the general term of the propounded series

$$\begin{aligned} y = 1 + \alpha\sin 2\pi x &+ \mathfrak{A}(\cos 2\pi x - 1) \\ + \beta\sin 4\pi x &+ \mathfrak{B}(\cos 4\pi x - 1) \\ + \gamma\sin 6\pi x &+ \mathfrak{C}(\cos 6\pi x - 1) + \text{etc.} \end{aligned}$$

Therefore, whatever values are substituted for α, β, γ, δ etc., $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., always a formula will arise, which exhibits the general term of the propounded series. Q. E. I.

COROLLARY 1

§13 If the first term, to which all remaining ones having integer exponents are equal, must not be the unity but an arbitrary quantity, the general term of the series y or the term, which corresponds to the index x, is found as

$$\begin{aligned} y = a + \alpha\ \sin 2\pi x\ + \beta\ \ \sin 4\pi x\ + \gamma \sin 6\pi x\ + \delta\ \ \sin 8\pi x + \text{etc.} \\ + \mathfrak{A} \cos 2\pi x + \mathfrak{B} \cos 4\pi x + \mathfrak{C} \cos 6\pi x + \mathfrak{D} \cos 8\pi x + \text{etc.} \end{aligned}$$

and an arbitrary term having in integer number index will be

$$= a + \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \text{etc.}$$

COROLLARY 2

§14 Since sines and cosines of the arcs $4\pi x$, $6\pi x$, $8\pi x$ etc. can be expressed by means of powers of $\sin 2\pi x$ and $\cos 2\pi x$ and vice versa all rational functions, or which do not have the ambiguity of the sign, can be exhibited by series of this kind we found for y, we will be able to define the general term y in such a way that we say that y is an arbitrary function of $\sin 2\pi x$ and $\cos 2\pi x$, as long as no formulas of this kind

$$\sqrt{1 \pm \cos 2\pi x}$$

and other similar ones occur, which involve the sines and cosines of submultiple angles of $2\pi x$.

COROLLARY 3

§15 Therefore, having excluded these cases, if we put $\sin 2\ pix = p$ and $\cos 2\pi x = q$, y will be equal to an arbitrary function of p and q; hence this differential infinite equation

$$0 = \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

will be integrated in general in such a way that y is an arbitrary function of p and q.

COROLLARY 4

§16 But if we call $\sin \pi x = r$ and $\cos \pi x = s$, it will be $p = 2s$ and $q = ss - rr$ and functions of p and q will be functions of even dimensions of r and s. Hence from the infinite differential equation the value of y will in general become equal to an arbitrary function of even dimensions of r and s, where it is to be noted that because of the whole sine $= 1$ it is $rr + ss = 1$.

COROLLARY 5

§17 Put $\frac{x}{a}$ instead of x that one has this equation

$$0 = \frac{ady}{1 \cdot dx} + \frac{a^2ddy}{1 \cdot 2 \cdot dx^2} + \frac{a^3d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{a^4d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

If we now put

$$\sin \frac{\pi x}{a} = r \quad \text{and} \quad \cos \frac{\pi x}{a} = s.$$

the integral of this equation will be described in such a way that $y =$ an arbitrary function of even dimensions of r and s.

COROLLARY 6

§18 Therefore, two formulas for the value of this integral can be exhibited, the one of which is

$$y = \frac{A + Br^2 + Crs + Ds^2 + Er^4 + Fr^3s + Gr^2s^2 + Hrs^3 + Is^4 + \text{etc.}}{\alpha + \beta r^2 + \gamma rs + \delta s^2 + \varepsilon r^4 + \zeta r^3 s + \eta r^2 s^2 + \theta rs^3 + \iota s^4 + \text{etc.}}$$

The other form will be

$$y = \frac{Ar + Bs + Cr^3 + Dr^2s + Ers^2 + Fs^3 + Gr^5 + \text{etc.}}{\alpha r + \beta s + \gamma r^3 + \delta r^2 s + \varepsilon rs^2 + \zeta s^3 + \eta r^5 + \text{etc.}}$$

COROLLARY 7

§19 Therefore, whatever value of this kind is substituted for y in the equation

$$0 = \frac{ady}{1 \cdot dx} + \frac{a^2ddy}{1 \cdot 2 \cdot dx^2} + \frac{a^3d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{a^4d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.,}$$

the identical equation will arise or an infinite series will result, whose sum will be $= 0$. But for the continued differentiations it is to be noted that it is

$$\frac{dr}{dx} = \frac{\pi s}{a} \quad \text{and} \quad \frac{ds}{dx} = -\frac{\pi r}{a}$$

and hence by means of the substitution the differentials dx will cancel each other everywhere.

SCHOLIUM 1

§20 But the factors deserve it to be mentioned, into which this infinite algebraic expression

$$\frac{z}{1} + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{z^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{etc.},$$

was resolved above. For, since the first simple factor is $= z$ and the remaining trinomial ones are contained in this general form

$$1 + \frac{z}{n} + \frac{zz}{4kk\pi\pi},$$

if we successively the numbers 1, 2, 3, 4 etc. are substituted for k, let us for the sake of brevity put

$$Z = \frac{z}{1} + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.},$$

and by means of infinitely many factors it will be

$$Z = z\left(1 + \frac{z}{n} + \frac{zz}{4\pi\pi}\right)\left(1 + \frac{z}{n} + \frac{zz}{16\pi\pi}\right)\left(1 + \frac{z}{n} + \frac{zz}{36\pi\pi}\right)\left(1 + \frac{z}{n} + \frac{zz}{64\pi\pi}\right)\text{etc.},$$

the number of which factors having excluded the first is infinite and $= \frac{1}{2}n$. Therefore, let $\frac{1}{2}n = m$ or $n = 2m$ and put $z = 2v$;

$$\frac{2v}{1} + \frac{2^2v^2}{1 \cdot 2} + \frac{2^3v^3}{1 \cdot 2 \cdot 3} + \frac{2^4v^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

$$= 2v\left(1 + \frac{v}{m} + \frac{vv}{\pi\pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{4\pi\pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{9\pi\pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{16\pi\pi}\right)\text{etc.}$$

and hence the following product of infinitely many factors, whose number is $= m$, will be

$$\left(1+\frac{v}{m}+\frac{vv}{\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{4\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{9\pi\pi}\right)\left(1+\frac{v}{m}+\frac{vv}{16\pi\pi}\right)\text{etc.}$$

$$= 1+\frac{2}{1\cdot 2}v+\frac{4}{1\cdot 2\cdot 3}v^2+\frac{8}{1\cdot 2\cdot 3\cdot 4}v^3+\frac{16}{1\cdot 2\cdot 3\cdot 4\cdot 5}v^5+\text{etc.}$$

If now this product is actually expanded, since the number of factors is $= m$ while m is an infinite number, it will arise

$$\begin{aligned} &1+v+\frac{m(m-1)}{1\cdot 2}\cdot\frac{vv}{mm}+\frac{vv}{\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}\right)\\ +&\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cdot\frac{v^3}{m^3}+\frac{(m-1)v^3}{m\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}\right)\\ &\text{etc.,}\end{aligned}$$

which terms compared to the series already found will give

$$\begin{aligned} 1=\frac{2}{1\cdot 2},\quad \frac{1}{1\cdot 2}\quad &+\frac{1}{\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}\right)=\frac{4}{1\cdot 2\cdot 3},\\ \frac{1}{1\cdot 2\cdot 3}&+\frac{1}{\pi\pi}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}\right)=\frac{8}{1\cdot 2\cdot 3\cdot 4}\end{aligned}$$

Hence in each of both one has

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\text{etc.}=\frac{\pi\pi}{6},$$

which is the same summation, I first had found already many years before and have confirmed it with several proofs. Furthermore, hence it is perspicuous, even though in these factors the number m is infinite, that it is nevertheless not possible to omit the other term $\frac{v}{m}$, since in the expansion because of the infinite repetition from the infinitely small terms $\frac{v}{m}$ finite terms arise. But whenever an arbitrary term is considered separately, as we did it in the formation of the integral, then it is possible to omit these infinitely small terms without error.

SCHOLIUM 2

But it is also possible to sum higher powers of the terms of the series

$$1+\frac{1}{4}+\frac{1}{9}+\text{etc.}$$

from this source and the same progressions will arise, which I had found once. But that this calculation does not become too long, it can be done easily in the following manner. Put

$$V=1+\frac{2v}{1\cdot 2}+\frac{2^2v^2}{1\cdot 2\cdot 3}+\frac{2^3v^3}{1\cdot 2\cdot 3\cdot 4}+\frac{2^4v^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.};$$

it will be

$$V=\frac{e^{2v}-1}{2v}$$

and

$$\frac{dV}{Vdv}=\frac{2e^{2v}}{e^{2v}-1}-\frac{1}{v},$$

which is reduced to this more convenient form

$$\frac{dV}{Vdv}=\frac{2e^{v}}{e^{v}-e^{-v}}-\frac{1}{v}=\frac{1+\frac{v}{1}+\frac{v^2}{1\cdot 2}+\frac{v^3}{1\cdot 2\cdot 3}+\frac{v^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}}{\frac{v}{1}+\frac{v^3}{1\cdot 2\cdot 3}+\frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{v^7}{1\cdot 2\cdots 7}+\text{etc.}}-\frac{1}{v},$$

such that it is

$$\frac{dV}{Vdv}-1=\frac{1+\frac{v^2}{1\cdot 2}+\frac{v^4}{1\cdot 2\cdot 3\cdot 4}+\frac{v^6}{1\cdot 2\cdots 6}+\text{etc.}}{\frac{v}{1}+\frac{v^3}{1\cdot 2\cdot 3}+\frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{v^7}{1\cdot 2\cdots 7}\text{etc.}}-\frac{1}{v}$$

or

$$\frac{dV}{Vdv}-1=\frac{\frac{2v}{1\cdot 2\cdot 3}+\frac{4v^3}{1\cdot 2\cdots 5}+\frac{4v^5}{1\cdot 2\cdots 7}+\frac{8v^7}{1\cdot 2\cdots 9}+\text{etc.}}{1+\frac{v^2}{1\cdot 2\cdot 3}+\frac{v^4}{1\cdot 2\cdots 5}+\frac{v^6}{1\cdot 2\cdots 7}++\frac{v^8}{1\cdot 2\cdots 9}+\text{etc.}}-\frac{1}{v}.$$

Put

$$\frac{dV}{Vdv}=1+\mathfrak{A}v-\mathfrak{B}v^3+\mathfrak{C}v^5-\mathfrak{D}v^7+\mathfrak{E}v^9-\text{etc.};$$

it will be

$$\begin{aligned}
\mathfrak{A} &= \frac{2}{1\cdot 2\cdot 3},\\
\mathfrak{B} &= \frac{\mathfrak{A}}{1\cdot 2\cdot 3} - \frac{4}{1\cdot 2\cdots 5},\\
\mathfrak{C} &= \frac{\mathfrak{B}}{1\cdot 2\cdot 3} - \frac{\mathfrak{A}}{1\cdot 2\cdots 5} + \frac{6}{1\cdot 2\cdots 7},\\
\mathfrak{D} &= \frac{\mathfrak{C}}{1\cdot 2\cdot 3} - \frac{\mathfrak{B}}{1\cdot 2\cdots 5} + \frac{\mathfrak{A}}{1\cdot 2\cdots 7} - \frac{8}{1\cdot 2\cdots 9}
\end{aligned}$$

etc.

Having found these values consider this other form of the quantity V expressed by means of factors

$$V = \left(1 + \frac{v}{m} + \frac{vv}{1\pi\pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{4\pi\pi}\right)\left(1 + \frac{v}{m} + \frac{vv}{9\pi\pi}\right)\text{etc.},$$

from which one finds by means of differentiation

$$\frac{dV}{Vdv} = \frac{\frac{1}{m} + \frac{2v}{1\pi\pi}}{1 + \frac{v}{m} + \frac{vv}{1\pi\pi}} + \frac{\frac{1}{m} + \frac{2v}{4\pi\pi}}{1 + \frac{v}{m} + \frac{vv}{4\pi\pi}} + \frac{\frac{1}{m} + \frac{2v}{9\pi\pi}}{1 + \frac{v}{m} + \frac{vv}{9\pi\pi}} + \text{etc.}$$

But in general it is

$$\begin{aligned}
\frac{\frac{1}{m} + \frac{2v}{\lambda\pi\pi}}{1 + \frac{v}{m} + \frac{vv}{\lambda\pi\pi}} = \frac{1}{m} &+ \frac{2}{\lambda\pi\pi}v - \frac{3}{m\lambda\pi\pi}v^2 + \frac{4}{m^2\lambda\pi\pi}v^3 - \text{etc.}\\
&- \frac{1}{mm} \qquad + \frac{1}{m^3} \qquad - \frac{1}{m^4}\\
&\qquad\qquad\qquad\qquad\qquad - \frac{2}{\lambda\lambda\pi^4}.
\end{aligned}$$

But since m is an infinite number and is equal to the number of factors, having excluded the first terms one will be able to omit the remaining ones divided by m without any error

$$\frac{\frac{1}{m} + \frac{2v}{\lambda\pi\pi}}{1 + \frac{v}{m} + \frac{vv}{\lambda\pi\pi}} = \frac{1}{m} + \frac{2v}{\lambda\pi\pi} - \frac{2v^3}{\lambda^2\pi^4} + \frac{2v^5}{\lambda^3\pi^6} - \frac{2v^7}{\lambda^4\pi^8} + \text{etc.};$$

therefore, having successively substituted the square numbers 1, 4, 9, 16 etc. and having combined these series, whose number is m, into one single sum one will find

$$\begin{aligned}\frac{dV}{Vdv} = 1 + \frac{2v}{\pi\pi}&\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}\right)\\
-\frac{2v^3}{\pi^4}&\left(1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}+\text{etc.}\right)\\
+\frac{2v^5}{\pi^6}&\left(1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\frac{1}{25^3}+\text{etc.}\right)\\
-\frac{2v^7}{\pi^8}&\left(1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\frac{1}{25^4}+\text{etc.}\right)\\
&\text{etc.}\end{aligned}$$

If now this series is compared to the one found first, one will have

$$\begin{aligned}1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.} &= \frac{1}{2}\mathfrak{A}\pi^2 = \frac{1}{6}\pi^2,\\
1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\text{etc.} &= \frac{1}{2}\mathfrak{B}\pi^4 = \frac{1}{90}\pi^4,\\
1+\frac{1}{4^3}+\frac{1}{9^3}+\frac{1}{16^3}+\text{etc.} &= \frac{1}{2}\mathfrak{C}\pi^6 = \frac{1}{945}\pi^6,\\
1+\frac{1}{4^4}+\frac{1}{9^4}+\frac{1}{16^4}+\text{etc.} &= \frac{1}{2}\mathfrak{D}\pi^8 = \frac{1}{9450}\pi^8\\
&\text{etc.}\end{aligned}$$

And this way all summations exhibited already once by me will be confirmed more, since the principle, which I had used then, could seem erroneous to some people.

PROBLEM 2

§22 *To find the general term of the series, whose arbitrary term exceeds the preceding one by a given quantity and whose first term is given.*

SOLUTION

Let the first term be $= a$ and the excess of each term over the preceding shall be $= g$; the terms corresponding to the integer indices will of course be these

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a, & a+g, & a+2g, & a+3g, & a+4g, & a+5g, & a+6g \quad \text{etc.}, \end{array}$$

such that to the integer index x corresponds the term $y = a + (x-1)g$. But while x is an arbitrary number infinitely many other formulas take the place of y. For let y' be the term corresponding to the index $x+1$; it will be

$$y' = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

Since now by assumption it must be $y' = y + g$, it will be

$$g = \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

Although I gave the resolution of equations of this kind, where except for the terms, which contain the differentials of y, a either a constant term or an arbitrary function is present, some time ago, it will nevertheless be helpful to get rid of this term g by means of the substitution $y = gx + u$; for, it will be

$$dy = gdx + du, \quad ddy = ddu, \quad d^3y = d^3u \quad \text{etc.},$$

because of the constant dx. Therefore, it will be

$$0 = \frac{du}{1 \cdot dx} + \frac{ddu}{1 \cdot 2 \cdot dx^2} + \frac{d^3u}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4u}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

Since this equation agrees with the one which we found in the preceding problem, if we put $\sin \pi x = r$ and $\cos \pi x = s$, u will be an arbitrary function of even dimensions of r and s, of which kind we exhibited one in § 18; and having found this the general term in question will be $y = A + gx + u$, as long as the constant A is defined in such a way, that having put $x = 1$ it is $y = a$. Q.E.I.

PROBLEM 3

§23 *To find the general term of the series, whose arbitrary term arises, if the preceding is multiplied by a given number m, and whose first term shall be $= a$.*

Therefore, the terms of this series, which have integer indices, will constitute the following geometric progression

$$\begin{array}{cccccc c} 1 & 2 & 3 & 4 & 5 & 6 & \\ a, & ma, & m^2s, & m^3a, & m^4a, & m^5a & \text{etc.}, \end{array}$$

such that to the integer index x the term am^{x-1} corresponds. Therefore, let in general be y the term corresponding to the index x and y' the term corresponding to the index $x+1$ and it will be $y' = my$. But it is

$$y' = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.} = my.$$

To resolve this equation according to the prescriptions put 1 for y, z for $\frac{dy}{dx}$, z^2 for $\frac{ddy}{dx^2}$ etc. that the following algebraic equation arises

$$m = 1 + \frac{z}{1} + \frac{zz}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.},$$

whose single roots must be investigated. But it will be $m = e^z$; but let the hyperbolic logarithm of m be $= \lambda$, that it is $m = e^\lambda$ and hence $e^\lambda - e^m = 0$. But since having taken an infinite number for n it is

$$e^\lambda = \left(1 + \frac{\lambda}{n}\right)^n \quad \text{and} \quad e^z = \left(1 + \frac{z}{n}\right)^n,$$

one will have this equation, whose roots are to be investigated

$$\left(1 + \frac{\lambda}{n}\right)^n - \left(1 + \frac{z}{n}\right)^n = 0,$$

of which the one root $z - \lambda = 0$ is certainly known immediately, whence the part $y = \alpha e^{\lambda x} = \alpha m^x$ of the integral because of $e^\lambda = m$ is obtained. The remaining roots are imaginary and are contained in this trinomial factor

$$\left(1 + \frac{\lambda}{n}\right)^2 - 2\left(1 + \frac{\lambda}{n}\right)\left(1 + \frac{z}{n}\right)\cos\frac{2k\pi}{n} + \left(1 + \frac{z}{n}\right)^2,$$

while k is an arbitrary integer number; this form goes over into this one

$$2 + \frac{2\lambda}{n} - 2\left(1 + \frac{\lambda}{n}\right)\cos\frac{2k\pi}{n} + \frac{\lambda\lambda}{nn} - \frac{2z}{n} - \frac{2z}{n}\left(1 + \frac{\lambda}{n}\right)\cos\frac{2k\pi}{n} + \frac{zz}{nn}.$$

But because of the infinite number n it is

$$\cos\frac{2k\pi}{n} = 1 - \frac{2kk\pi\pi}{nn}.$$

Therefore, having multiplied that form by nn the general factor will be

$$= 2n(n+\lambda)\left(1-\cos\frac{2k\pi}{n}\right) + \lambda\lambda + 2nz\left(1-\cos\frac{2k\pi}{n}\right) - 2\lambda z\cos\frac{2k\pi}{n} + zz$$

$$= \lambda\lambda + 4kk\pi\pi + \frac{4kk\pi\pi z}{n} - 2\lambda z + zz,$$

having neglected the vanishing terms; with respect to this even the term $\frac{4kk\pi\pi z}{n}$ can be omitted, such that the general factor is

$$\lambda\lambda + 4kk\pi\pi - 2\lambda z + zz,$$

and the number of these factors, if for k successively the numbers 1, 2, 3, 4 etc. are substituted, will be $= \frac{n}{2}$. But this compared to the general form given in my dissertation printed in Volume 7 of the Miscellanea Berolin.

$$ff - 2fz\cos\varphi + zz$$

will give

$$f = \sqrt{\lambda\lambda + 4kk\pi\pi} \quad \text{and} \quad \cos\varphi = \frac{\lambda}{\sqrt{\lambda\lambda + 4kk\pi\pi}}$$

and hence

$$\sin\varphi = \frac{2k\pi}{\sqrt{\lambda\lambda + 4kk\pi\pi}}.$$

Hence this part of the integral y arises

$$y = e^{\lambda x}(\alpha\sin 2k\pi x + \mathfrak{A}\cos 2k\pi x).$$

Therefore, having successively substituted the values for k because of e^{λ} one will find

$$y = m^x \begin{Bmatrix} C + \alpha\sin 2\pi x + \beta\sin 4\pi x + \gamma\sin 6\pi x + \text{etc.} \\ +\mathfrak{A}\cos 2\pi x + \mathfrak{B}\cos 4\pi x + \mathfrak{C}\cos 6\pi x + \text{etc.} \end{Bmatrix}.$$

Therefore, since having put $x = 1$ it must be $y = a$, it will be

$$a = m(C + \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \text{etc.}),$$

whence the constant C is defined. Or if having put $\sin\pi x = r$ and $\cos\pi x = s$ Q was an arbitrary function of even dimension of r and s, the general term in question will be $y = m^x$. Q.E.I.

COROLLARY 1

§24 Therefore, in the geometric progression, insofar it is only described in such a way, that each term is said to have a constant ratio to the preceding one, the interpolation is not determined, since the intermediate terms can be expressed in infinitely many different ways, they can even receive any value.

COROLLARY 2

§25 Therefore, the complete integral of this infinite differential equation can be expressed in general

$$(m-1)y = \frac{dy}{1\cdot dx} + \frac{ddy}{1\cdot 2\cdot dx^2} + \frac{d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \frac{d^4y}{1\cdot 2\cdot 3\cdot 4\cdot dx^4} + \text{etc.}$$

For, having put $\sin \pi x = r$ and $\cos \pi x = s$, if Q denotes an even function of r and s, it will be $y = m^x Q$ and hence $m^{-x}y$ becomes equal to an arbitrary function of even dimension of r and s.

COROLLARY 3

§26 If for x one writes $\frac{x}{a}$, this equation will arise

$$(m-1)y = \frac{ady}{1\cdot dx} + \frac{aaddy}{1\cdot 2\cdot dx^2} + \frac{a^3d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \frac{a^4d^4y}{1\cdot 2\cdot 3\cdot 4\cdot dx^4} + \text{etc.}$$

To integrate this put

$$\sin\frac{\pi x}{a} = r \quad \text{and} \quad \cos\frac{\pi x}{a} = s$$

and let Q denote an arbitrary function of even dimensions of r and s, such that Q retains the same value, even though for r and s one writes $-r$ and $-s$. Having done this it will be $y = m^{x:a}Q$.

COROLLARY 4

§27 And the solution of this problem could even be reduced to the solution of the first problem. For, since it must be $y' = my$, it will be $\log y' = \log y + \log m$.

Put $\log y = v$, that it is $\log y' = v'$, and let it be $\log m = \lambda$; it must be $v' = v + \lambda$, whence because of

$$v' = v + \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.}$$

it is

$$\lambda = \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4v}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

and having put $v = u + \lambda x$ one will have

$$0 = \frac{du}{1 \cdot dx} + \frac{ddu}{1 \cdot 2 \cdot dx^2} + \frac{d^3u}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4u}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.},$$

which is the equation, to which we got in the first problem. Therefore, if one puts $\sin \pi x = r$ and $\cos \pi x = s$ and Q denotes a function of even dimensions of r and s, it will be $u = Q$ and hence

$$v = \lambda x + Q = \log y = x \log m + Q.$$

Therefore, by taking numbers one has $y = m^x e^Q$; since there e^Q also is a function of even dimensions of r and s, if for it one writes Q, it will be, as we found before, be $y = m^x Q$.

Scholium

§28 Since we found all roots of the algebraic equation

$$m = 1 + \frac{z}{1} + \frac{zz}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.},$$

we will hence be able to exhibit all factors of this infinite expression

$$Z = 1 + \frac{z}{1(1-m)} + \frac{zz}{1 \cdot 2(1-m)} + \frac{z^3}{1 \cdot 2 \cdot 3(1-m)} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4(1-m)} + \text{etc.}$$

For, having put $\log m = \lambda$ the first simple factor will be $1 - \frac{z}{\lambda}$ and the remaining trinomial factors will be contained in this general form

$$1 + \frac{4kk\pi\pi}{n(\lambda\lambda + 4kk\pi\pi)} - \frac{2\lambda z - zz}{\lambda\lambda + 4kk\pi\pi},$$

which is transformed into this one

$$1+\frac{z}{n}-\frac{\lambda\lambda z}{n(\lambda\lambda+4kk\pi\pi)}-\frac{2\lambda z-zz}{\lambda\lambda+4kk\pi\pi},$$

if instead of k successively the numbers 1, 2, 3, 4 etc. are substituted and n is an infinitely large number, whose half $\frac{n}{2}$ exhibits the number of factors itself. For the sake of brevity let it be

$$\lambda\lambda+4kk\pi\pi=\Phi$$

and it will be

$$Z=\left(1-\frac{z}{\lambda}\right)\left(1+\frac{z}{n}-\frac{\lambda\lambda z}{n\Phi}-\frac{2\lambda z}{\Phi}+\frac{zz}{\Phi}\right),$$

where the second factors holds the place of all infinitely many factors, which arise from the variation of the quantity Φ. Therefore, having taken logarithms and differentiated them one will obtain

$$\frac{dZ}{Zdz}=\frac{-1}{\lambda-z}+\frac{\frac{1}{n}-\frac{\lambda\lambda}{n\Phi}-\frac{2\lambda}{\Phi}+\frac{2z}{\Phi}}{1+\frac{z}{n}-\frac{\lambda\lambda z}{n\Phi}-\frac{2\lambda z}{\Phi}+\frac{zz}{\Phi}}.$$

And having resolved these terms into infinite series

$$\begin{aligned}\frac{dZ}{Zdz}=&-\frac{1}{\lambda}-\frac{z}{\lambda^2}-\frac{zz}{\lambda^3}-\frac{z^3}{\lambda^4}-\frac{z^4}{\lambda^5}-\frac{z^5}{\lambda^6}-\text{etc.}\\&-\frac{1}{n}-\frac{4\lambda^2 z}{\Phi^2}-\frac{8\lambda^3 zz}{\Phi^3}-\frac{16\lambda^4 z^3}{\Phi^4}-\frac{32\lambda^5 z^4}{\Phi^5}-\frac{64\lambda^6 z^5}{\Phi^6}\\&-\frac{\lambda\lambda}{n\Phi}+\frac{2z}{\Phi}+\frac{6\lambda zz}{\Phi\Phi}+\frac{16\lambda^2 z^3}{\Phi^3}+\frac{40\lambda^3 z^4}{\Phi^4}+\frac{96\lambda^4 z^5}{\Phi^5}\\&-\frac{2\lambda}{\Phi}\qquad\qquad\qquad\;-\frac{-2z^3}{\Phi^3}-\frac{10\lambda z^4}{\Phi^3}-\frac{36\lambda^2 z^5}{\Phi^4},\end{aligned}$$

put

$$\frac{dZ}{Zdz}=A+Bz+Cz^2+Dz^3+Ez^4+Fz^5+\text{etc.},$$

and because it is $\Phi=\lambda\lambda+4kk\pi\pi$, where it is to be understood that successively for k all numbers 1, 2, 3, 4 etc. up to $\frac{1}{2}n$ are substituted, it will be

$$A=\frac{1}{2}-\frac{1}{\lambda}-2\lambda\left(\frac{1}{\lambda\lambda+4\pi\pi}+\frac{1}{\lambda\lambda+16\pi\pi}+\frac{1}{\lambda\lambda+36\pi\pi}+\text{etc.}\right)$$

If for the sake of brevity one sets

$$\frac{1}{(\lambda\lambda+4\pi\pi)}+\frac{1}{(\lambda\lambda+16\pi\pi)}+\frac{1}{(\lambda\lambda+36\pi\pi)}+\text{etc.}=\mathfrak{A},$$
$$\frac{1}{(\lambda\lambda+4\pi\pi)^2}+\frac{1}{(\lambda\lambda+16\pi\pi)^2}+\frac{1}{(\lambda\lambda+36\pi\pi)^2}+\text{etc.}=\mathfrak{B},$$
$$\frac{1}{(\lambda\lambda+4\pi\pi)^3}+\frac{1}{(\lambda\lambda+16\pi\pi)^3}+\frac{1}{(\lambda\lambda+36\pi\pi)^3}+\text{etc.}=\mathfrak{C},$$
$$\frac{1}{(\lambda\lambda+4\pi\pi)^4}+\frac{1}{(\lambda\lambda+16\pi\pi)^4}+\frac{1}{(\lambda\lambda+36\pi\pi)^4}+\text{etc.}=\mathfrak{D}$$

etc.,

it will be

$$\begin{aligned}
A &= \frac{1}{2}-\frac{1}{\lambda}-2\lambda\mathfrak{A},\\
B &= -\frac{1}{\lambda\lambda}+2\mathfrak{A}-4\lambda^2\mathfrak{B}\\
C &= -\frac{1}{\lambda^3}+6\lambda\mathfrak{B}-8\lambda^3\mathfrak{C},\\
D &= -\frac{1}{\lambda^4}-2\mathfrak{B}+16\lambda^2\mathfrak{C}-16\lambda^4\mathfrak{D},\\
E &= -\frac{1}{\lambda^5}-10\lambda\mathfrak{C}+40\lambda^3\mathfrak{D}-32\lambda^5\mathfrak{E},\\
F &= -\frac{1}{\lambda^6}+2\mathfrak{C}-36\lambda^2\mathfrak{D}+96\lambda^4\mathfrak{E}-64\lambda\mathfrak{F}
\end{aligned}$$

etc.

Since now it is

$$Z=1+\frac{z}{1(1-m)}+\frac{zz}{1\cdot 2(1-m)}+\frac{z^3}{1\cdot 2\cdot 3(1-m)}+\text{etc.},$$

it will be

$$Z=\frac{e^z-m}{1-m}=\frac{e^z-e^\lambda}{1-e^\lambda}\quad\text{und}\quad\frac{dZ}{dz}=\frac{e^z}{1-e^z};$$

hence

$$\frac{dZ}{Zdz}=\frac{e^z}{e^z-e^\lambda}=\frac{1}{1-e^\lambda e^{-z}}=\frac{1}{1-me^{-m}}$$

and from this

$$\frac{dZ}{Zdz} = \frac{1}{1 - m + mz - \frac{mzz}{1\cdot 2} + \frac{mz^3}{1\cdot 2\cdot 3} - \frac{mz^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}}.$$

Now because of

$$\frac{dZ}{Zdz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$$

it will become

$$\begin{array}{rcccccccccc}
1 = (1-m)A & + & (1-m)Bz & + & (1-m)Cz^2 & + & (1-m)Dz^3 & + & (1-m)Ez^4 & + & \text{etc.},\\
 & + & mA & + & mB & + & mC & + & mD & & \\
 & & & - & \frac{1}{2}mA & - & \frac{1}{2}mB & - & \frac{1}{2}mC & & \\
 & & & & & + & \frac{1}{6}mA & + & \frac{1}{6}mB & & \\
 & & & & & & & - & \frac{1}{24}mA & &
\end{array}$$

whence the following determinations will arise

$$\begin{aligned}
A &= \frac{1}{1-m},\\
B &= \frac{-mA}{1-m} = \frac{-m}{(1-m)^2},\\
C &= \frac{-mB + \frac{1}{2}mA}{1-m} = \frac{mm}{(1-m)^3} + \frac{m}{2(1-m)^2},\\
D &= \frac{-mC + \frac{1}{2}mB - \frac{1}{6}mA}{1-m} = \frac{-m^3}{(1-m)^4} - \frac{mm}{(1-m)^3} - \frac{m}{6(1-m)^2},\\
E &= \frac{-mD + \frac{1}{2}mC - \frac{1}{6}mB + \frac{1}{24}mA}{1-m} = \frac{m^4}{(1-m)^5} + \frac{3m^3}{2(1-m)^4} + \frac{7mm}{12(1-m)^3} + \frac{m}{24(1-m)^2}
\end{aligned}$$

etc.

Therefore, the following summations of the series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{B}$, $\mathfrak{D}$ etc. will arise:

$$\text{I.} \quad \frac{1}{1-m} = \frac{1}{2} - \frac{1}{\lambda} - 2\lambda\mathfrak{A}$$

or

$$\mathfrak{A} = \frac{1}{4\lambda} - \frac{1}{2\lambda\lambda} - \frac{1}{2\lambda(1-m)};$$

$$\text{II.} \quad \frac{-m}{(1-m)^2} = -\frac{1}{\lambda\lambda} + 2\mathfrak{A} - 4\lambda\lambda\mathfrak{B} = \frac{1}{2\lambda} - \frac{2}{\lambda\lambda} - \frac{1}{\lambda(1-m)} - 4\lambda\lambda\mathfrak{B},$$

whence it is

$$\mathfrak{B} = \frac{1}{8\lambda^3} - \frac{1}{2\lambda^4} - \frac{1}{4\lambda^3(1-m)} + \frac{m}{4\lambda\lambda(1-m)^2};$$

$$\begin{aligned}\text{III.} \quad \frac{mm}{(1-m)^3} + \frac{m}{2(1-m)^2} = -\frac{1}{\lambda^3} + 6\lambda\mathfrak{B} - 8\lambda^3\mathfrak{C} = \frac{3}{4\lambda\lambda} - \frac{4}{\lambda^3} - \frac{3}{2\lambda^2(1-m)} \\ + \frac{3m}{2\lambda(1-m)^2} - 8\lambda\mathfrak{C},\end{aligned}$$

therefore

$$\mathfrak{C} = \frac{3}{32\lambda^5} - \frac{1}{2\lambda^6} - \frac{3}{16\lambda^5(1-m)} + \frac{3m}{16\lambda^4(1-m)^2} - \frac{m}{16\lambda^3(1-m)^2} - \frac{mm}{8\lambda^3(1-m)^3}.$$

And so the following sums of the propounded series $\mathfrak{D}$, $\mathfrak{E}$ etc. will be found.

COROLLARY 1

§29 Therefore, because it is $m = e^\lambda$, it will be

$$\frac{1}{\lambda\lambda + 4\pi\pi} + \frac{1}{\lambda\lambda + 16\pi\pi} + \frac{1}{\lambda\lambda + 36\pi\pi} + \text{etc.} = \frac{1}{4\lambda} - \frac{1}{2\lambda\lambda} - \frac{1}{2\lambda(1-e^\lambda)};$$

let $\lambda = \frac{2\pi a}{b}$; it will be

$$\begin{aligned}\frac{bb}{4(aa+bb)\pi^2} + \frac{bb}{4(aa+4bb)\pi^2} + \frac{bb}{4(aa+9bb)\pi^2} + \text{etc.} \\ = \frac{b}{8\pi a} - \frac{bb}{8\pi\pi aa} - \frac{b}{4\pi a(1-e^{2\pi a:b})}\end{aligned}$$

and hence by multiplying by $\frac{4\pi\pi}{bb}$ one will have

$$\frac{1}{aa+bb} + \frac{1}{aa+4bb} + \frac{1}{aa+9bb} + \text{etc.} = \frac{\pi}{2ab} - \frac{1}{2aa} + \frac{\pi}{ab(e^{2\pi a:b}-1)},$$

which sum I exhibited already elsewhere deduced from another source.

COROLLARY 2

§30 Therefore, if one sets $b = 1$, one has this sum

$$\frac{1}{aa+1} + \frac{1}{aa+4} + \frac{1}{aa+9} + \text{etc.} = \frac{\pi}{2a} - \frac{1}{2aa} + \frac{\pi}{a(e^{2\pi a}-1)},$$

and if furthermore one sets $a = 0$ that this series arises

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.},$$

the sum of this series because of the terms growing to infinity will be derived this way from the formula: Assume a to be infinitely small; it will be

$$e^{2\pi a} = 1 + 2\pi a + 2\pi\pi aa + \frac{4}{3}\pi^3 a^3$$

and hence the sum will be

$$= \frac{\pi}{2a} - \frac{1}{2aa} + \frac{1}{2aa + 2\pi a^3 + \frac{4}{3}\pi^2 a^4}$$

$$= \frac{\pi a + \pi\pi aa + \frac{2}{3}\pi^3 a^3 - 1 - \pi a - \frac{2}{3}\pi^2 aa + 1}{2aa\left(1 + \pi a + \frac{2}{3}\pi\pi a^2\right)} = \frac{1}{6}\pi^2,$$

which, as it is known, is the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$

COROLLARY 3

§31 If in the series found before

$$\frac{1}{aa+bb} + \frac{1}{aa+4bb} + \frac{1}{aa+9bb} + \text{etc.} = \frac{\pi}{2ab} - \frac{1}{2aa} + \frac{\pi}{ab(e^{2\pi a:b}-1)}$$

the quantity a is considered as a variable and a differentiation is done, the sum of the series $\mathfrak{B}$ will arise; and so forth by means of continued differentiation from the series $\mathfrak{A}$ one will find the sums of the following series $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc.

Corollary 4

§32 The sum of this series can be expressed more conveniently this way

$$\frac{1}{aa+bb}+\frac{1}{aa+4bb}+\frac{1}{aa+9bb}+\text{etc.}=\frac{-1}{2aa}+\frac{\pi(e^{2\pi a:b}+1)}{2ab(e^{2\pi a:b}-1)}$$

$$=\frac{-1}{2aa}+\frac{\pi(e^{\pi a:b}+e^{-\pi a:b})}{2ab(e^{\pi a:b}-e^{-\pi a:b})}.$$

From this form the value of the series, if b is imaginary, is easily calculated; for, let $b=\frac{c}{\sqrt{-1}}$; it will be

$$\frac{1}{aa-cc}+\frac{1}{aa-4cc}+\frac{1}{aa-9cc}+\text{etc.}=\frac{-1}{2aa}+\frac{\pi\left(e^{\frac{\pi a\sqrt{-1}}{c}}+e^{\frac{-\pi a\sqrt{-1}}{c}}\right)\sqrt{-1}}{2ac\left(e^{\frac{\pi a\sqrt{-1}}{c}}-e^{\frac{-\pi a\sqrt{-1}}{c}}\right)}.$$

But it is

$$e^{\frac{\pi a\sqrt{-1}}{c}}+e^{\frac{-\pi a\sqrt{-1}}{c}}=2\cos\frac{\pi a}{c}$$

and

$$e^{\frac{\pi a\sqrt{-1}}{c}}-e^{\frac{-\pi a\sqrt{-1}}{c}}=2\sqrt{-1}\cdot\sin\frac{\pi a}{c},$$

whence it is

$$\frac{1}{aa-cc}+\frac{1}{aa-4cc}+\frac{1}{aa-9cc}+\text{etc.}=\frac{-1}{2aa}+\frac{\pi\cos\pi a:c}{2ac\sin\pi a:c}.$$

Corollary 5

§33 Since it is $\cos\frac{(2k+1)\pi}{2}=0$, in the cases, in which it is $a=2k+1$ and $c=2$, the sum of the series is

$$=\frac{-1}{2aa}=-\frac{1}{2(2k+1)^2},$$

while k is an arbitrary integer number. Hence it will be

$$\frac{1}{(2k+1)^2-4}+\frac{1}{(2k+1)^2-16}+\frac{1}{(2k+1)^2-36}+\frac{1}{(2k+1)^2-64}+\text{etc.}=\frac{-1}{2(2k+1)^2},$$

which summation I demonstrated elsewhere. For, if the single fractions are resolved into partial fractions, it arises

$$\frac{-1}{2k+1} = \frac{1}{2k-1} + \frac{1}{2k-3} + \frac{1}{2k-5} + \frac{1}{2k-7} + \frac{1}{2k-9} + \text{etc.}$$
$$+ \frac{1}{2k+3} + \frac{1}{2k+5} + \frac{1}{2k+7} + \frac{1}{2k+9} + \text{etc.}$$

COROLLARY 6

§34 Having brought the term $\frac{-1}{2k+1}$ to the other side and having collected each two terms a new series will arise, whose sum is $= 0$. Of course, having divided the single terms by $4k$ it will be

$$0 = \frac{1}{4kk-1} + \frac{1}{4kk-9} + \frac{1}{4kk-25} + \frac{1}{4kk-49} + \frac{1}{4kk-81} + \text{etc.},$$

whose truth will easily reveal itself in the single cases.

PROBLEM 4

§35 *To find the general term of the series, whose arbitrary term arises, if the preceding is multiplied by a given number m and to that product a given number c is added, and the first term of which series is equally given as $= a$.*

SOLUTION

Therefore, the terms, which corresponds to integer indices, will behave this way

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ a, & ma+c, & m^2a+mc+c, & m^3a+m^2c+mc+c \quad \text{etc.}; \end{array}$$

hence, if the index x is an integer number, the corresponding term will be

$$= m^{x-1}a + \frac{m^{x-1}-1}{m-1}c.$$

But if x is not an integer, infinitely many other formulas except for this one will equally satisfy; to find them let y be the term corresponding to the index x and y' the following or the one corresponding to $x+1$; it will be

$$y' = my + c,$$

whence it will be

$$my + c = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.}$$

Put

$$y = v - \frac{c}{m-1}$$

and it will be

$$mv = v + \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.};$$

since this equation agrees with the one, which we found in the preceding problem, if one puts $\sin \pi x = r$ and $\cos \pi x = s$ and Q is taken for an arbitrary function of even dimensions of r and s, it will be

$$v = m^x Q$$

and hence

$$y = m^x Q - \frac{c}{m-1}.$$

Put $x = 1$, in which case it is $r = 0$ and $s = -1$, and let Q go over into C; it must be

$$a = mC - \frac{c}{m-1}$$

and hence it will be

$$C = \frac{a}{m} + \frac{c}{m(m-1)}.$$

Hence, if for Q the constant quantity C itself is taken, it will be

$$y = m^{x-1}a + \frac{(m^{x-1} - 1)c}{m-1}$$

for the simplest case. And if P is such a function of even dimensions of r and s, which vanishes for $x = 1$, one will be able to put $Q = C + P$ and the form of the general term in question will be this one extending very far

$$y = m^{x-1}a + \frac{(m^{x-1} - 1)c}{m-1} + m^x P.$$

Q.E.I.

PROBLEM 5

§36 *To find the general term of recurring series of second order, in which each term becomes equal to the aggregate of the two preceding terms multiplied by any arbitrary numbers.*

SOLUTION

Let

the term to which the index	x	corresponds	$=y$,
the term to which the index	$x-1$	corresponds	$={}'y$,
V	$x-2$	corresponds	$={}''y$,

and let this law of the recurring series be propounded that it is

$$y = \alpha' y + \beta'' y.$$

Therefore, since it is

$$\begin{aligned} {}'y &= y - \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} - \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.}, \\ {}''y &= y - \frac{2dy}{1 \cdot dx} + \frac{4ddy}{1 \cdot 2 \cdot dx^2} - \frac{8d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{16d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.}, \end{aligned}$$

having substituted these formulas it will be

$$\begin{aligned} y = &+ \alpha \left(y - \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} - \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.} \right) \\ &+ \beta \left(y - \frac{2dy}{1 \cdot dx} + \frac{4ddy}{1 \cdot 2 \cdot dx^2} - \frac{8d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{16d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.} \right) \end{aligned}$$

To resolve this equation according to the general prescription put 1 for y, z for $\frac{dy}{dx}$, z^2 for $\frac{ddy}{dx^2}$ etc. and it will be

$$1 = +\alpha\left(1 - \frac{z}{1} + \frac{zz}{1\cdot 2} - \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}\right)$$
$$1 = +\alpha\left(1 - \frac{2z}{1} + \frac{4zz}{1\cdot 2} - \frac{8z^3}{1\cdot 2\cdot 3} + \frac{16z^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}\right),$$

which equation is reduced to this finite form

$$1 = \alpha e^{-z} + \beta e^{-2z},$$

whose factors must be investigated. Therefore, having put $e^{+z} = u$ resolve this equation

$$uu = \alpha u + \beta,$$

of which either both roots are real or both are imaginary or finally both are equal to each other. These three cases must be expanded separately.

I. Therefore, first let the two roots be real and different to each other, or let

$$uu - \alpha u - \beta = (u - A)(u - B)$$

and hence by putting e^z for u we will have the two general factors $e^z - A$ and $e^z - B$. But we saw above that the formula $e^z - m$ gave this integral

$$y = \ + A^x \left\{\begin{matrix} C + \ \alpha\sin 2\pi x \ + \ \beta\sin 4\pi x \ + \ \gamma\sin 6\pi x \ + \text{etc.} \\ + \ \mathfrak{A}\cos 2\pi x \ + \ \mathfrak{B}\cos 4\pi x \ + \ \mathfrak{C}\cos 6\pi x \ + \text{etc.} \end{matrix}\right\}.$$

Therefore, both factors $e^z - A$ and $e^z - B$ taken together will give this value for the general term y

$$y = \ + A^x \left\{\begin{matrix} C + \ \alpha\sin 2\pi x \ + \ \beta\sin 4\pi x \ + \ \gamma\sin 6\pi x \ + \text{etc.} \\ + \ \mathfrak{A}\cos 2\pi x \ + \ \mathfrak{B}\cos 4\pi x \ + \ \mathfrak{C}\cos 6\pi x \ + \text{etc.} \end{matrix}\right\}$$

$$+ B^x \left\{\begin{matrix} C + \ \alpha'\sin 2\pi x \ + \ \beta'\sin 4\pi x \ + \ \gamma'\sin 6\pi x \ + \text{etc.} \\ + \ \mathfrak{A}'\cos 2\pi x \ + \ \mathfrak{B}'\cos 4\pi x \ + \ \mathfrak{C}'\cos 6\pi x \ + \text{etc.} \end{matrix}\right\}.$$

Or put $\sin\pi x = r$ and $\cos\pi x = s$ and let P and Q be arbitrary functions of even dimensions of r and s and, if it was

$$uu - \alpha u - \beta = (u - A)(u - B)$$

or if A and B are the roots of the equation $uu - \alpha u - \beta = 0$, in this case it will be

$$y = A^x P + B^x Q.$$

II. If both roots were imaginary, then certainly the same formula already found can be used, since in each case the imaginary quantities cancel each other; nevertheless, one can exhibit a formula for y free from all imaginary quantities. For, in this case the equation $uu - \alpha u - \beta = 0$ will obtain a form of such a kind

$$uu - 2fu\cos\omega + ff = 0,$$

whose roots are

$$u = f\cos\omega \pm f\sqrt{-1} \cdot \sin\omega,$$

such that it is

$$A = f\cos\omega + f\sqrt{-1} \cdot \sin\omega \quad \text{und} \quad B = f\cos\omega - f\sqrt{-1} \cdot \sin\omega.$$

But hence it will be

$$A^x = f^x \cos\omega x + f^x\sqrt{-1} \cdot \sin\omega x$$

and

$$B^x = f^x \cos\omega x - f^x\sqrt{-1} \cdot \sin\omega x.$$

Therefore, if these values are substituted for A^x and B^x, it will be

$$y = (P + Q)f^x \cos\omega x + (P - Q)\sqrt{-1} \cdot f^x \sin\omega x.$$

Since now P and Q are arbitrary functions of r and s, as long as they have even dimensions, instead of $P + Q$ write P and instead of $(P - Q)\sqrt{-1}$ put Q and from the equation

$$uu - \alpha u - \beta = uu - 2fu\cos\omega + ff = 0$$

the general term in question will be

$$y = f^x P \cos \omega x + f^x Q \sin \omega x.$$

III. If both roots A and B of the equation $uu - \alpha u - \beta = 0$ were equal, say $A = B = m$, one will have the equation

$$(e^z - m)^2 = 0.$$

As in § 23 put $m = e^\lambda$; the first factor of the formula $(e^z - e^\lambda)^2$ will be the square $= (z - \lambda)^2$, whence the this part of the integral arises

$$(\mathfrak{A} + \mathfrak{B}x)e^{\lambda x} = (\mathfrak{A} + \mathfrak{B}x)m^x = (\mathfrak{A} + \mathfrak{B}x)A^x.$$

All remaining ones will equally be quadratic and will be contained in this general form

$$(\lambda\lambda + 4kk\pi\pi - 2\lambda z + zz)^2,$$

from which according to the prescriptions given by me once this part of the integral arises

$$A^x(\mathfrak{A} + \mathfrak{B}x) \sin 2k\pi x + A^x(\mathfrak{C} + \mathfrak{D}x) \cos 2k\pi x.$$

By collecting all these if follows, if it was

$$uu - \alpha u - \beta = (u - A)^2 = uu - 2Au + AA,$$

that the general term in question will be

$$y = A^x \left\{ \begin{array}{r} \mathfrak{A} + \mathfrak{B}x(\mathfrak{C} + \mathfrak{D}x) \sin 2\pi x + (\mathfrak{G} + \mathfrak{H}x) \sin 4\pi x + \text{etc.} \\ (\mathfrak{E} + \mathfrak{F}x) \cos 2\pi x + (\mathfrak{I} + \mathfrak{K}x) \cos 4\pi x + \text{etc.} \end{array} \right\}.$$

Put $\sin \pi x = r$ and $\cos \pi x = s$ again and let P and Q be arbitrary even functions of r and s and one will be able to express the general term in such a way, that it is

$$y = A^x(P + Qx).$$

Q.E.I.

COROLLARY 1

§37 Therefore, if in a recurring series the arbitrary term y is determined by the two preceding ones $'y$ and $''y$ in such a way that it is $y = \alpha'y + \beta''y$, or if according to de Moivre $+\alpha$, $+\beta$ was the scale of relation, and if x was the index of the term y, y will be a highly undetermined function of x, since innumerable formulas can be exhibited, which yield satisfying values for y.

COROLLARY 2

§38 But to find all expressions for y from the scale of relation $+\alpha$, $+\beta$ form this equation $uu - \alpha u - \beta = 0$, from whose resolution the form of the general term y will be found in the following manner.

COROLLARY 3

§39 Let the roots of the equation

$$uu - \alpha u - \beta = 0$$

be A and B such that it is

$$A = \frac{1}{2}\alpha + \sqrt{\frac{1}{4}\alpha\alpha + \beta} \quad \text{and} \quad B = \frac{1}{2}\alpha - \sqrt{\frac{1}{4}\alpha\alpha + \beta},$$

and having put $\sin \pi x = r$ and $\cos \pi x = s$ take any arbitrary functions of r and s, which shall be P and Q; it will be

$$y = A^x P + B^x Q = \left(\frac{1}{2}\alpha + \sqrt{\frac{1}{4}\alpha\alpha + \beta}\right)^x + \left(\frac{1}{2}\alpha - \sqrt{\frac{1}{4}\alpha\alpha + \beta}\right)^x Q.$$

COROLLARY 4

§40 But if both roots of the equation $uu = \alpha u + \beta$ were equal, that formula because of $\beta + \frac{1}{4}\alpha\alpha = 0$ is useless. But in this case, since both roots are $\frac{1}{2}\alpha$, if one puts $\frac{1}{2}\alpha = A$, the general term will be

$$y = A^x(P + Qx).$$

COROLLARY 5

§41 But if $\frac{1}{4}\alpha\alpha + \beta$ is a negative quantity, the parts found before will be imaginary. Therefore, to find the imaginary form compare the equation

$$uu - \alpha u - \beta = 0$$

to this one

$$uu - 2fu\cos\omega + ff = 0;$$

it will be

$$f = \sqrt{-\beta} \quad \text{und} \quad \alpha = 2\sqrt{-\beta}\cdot\cos\omega$$

or

$$\cos\omega = \frac{\alpha}{2\sqrt{-\beta}} \quad \text{and} \quad \sin\omega = \frac{\sqrt{-4\beta - \alpha\alpha}}{2\sqrt{-\beta}} = \sqrt{1 + \frac{\alpha\alpha}{4\beta}},$$

whence the angle ω will be found, from which it will be

$$y = f^x(P\cos\omega x + Q\sin\omega x).$$

COROLLARY 6

§42 If for P and Q constant quantities are assumed, the same form of the general term arises, which is usually exhibited and is considered as the only one, which satisfies the condition. But having propounded an arbitrary determined series one has to define these two constant quantities from the first two terms, which are assumed to be given. But in general, since the two arbitrary functions P and Q enter, which, as often as x is an integer number, will obtain the same constant values, it is plain that two terms of the series corresponding to integer indices can be assumed ad libitum.

SCHOLIUM

§43 This method to find the general terms of recurring series is mainly remarkable for that reason that it not only exhibits all possible forms but also proceeds a priori and completes the task from analytical principles alone, whereas others, which treated those series, all got to the special form of the general term on an indirect way. For, it is the principal property and quasi a criterion of a direct method, that it not only from the principles of the subject themselves finds its nature, but also contains all ways of determinations at the same time. But an indirect method, even though they often yield short and elegant solutions of problems, nevertheless very rarely exhaust the nature of the question in consideration. An extraordinary example of this difference is seen in the preceding problem, but will even occur more clearly in the following problem, where in general the general terms of all recurring series will be investigated.

PROBLEM 6

§44 *To find the general term of recurring series of arbitrary order, whose arbitrary term becomes equal to an aggregate of several of the preceding terms multiplied by arbitrary numbers.*

SOLUTION

Let the term corresponding to the index x be $= y$, but denote the preceding terms, which correspond to the indices $x-1$, $x-2$, $x-3$, $x-4$ etc., by $'y$, $''y$, $'''y$, ${}^{\mathrm{IV}}y$ and let this law of the series be propounded that one everywhere has

$$y = \alpha\,'y + \beta\,''y + \gamma\,'''y + \delta\,{}^{\mathrm{IV}}y + \text{etc.}$$

Since now from the nature of differentials it is

$$\begin{aligned}
'y &= y - \frac{dy}{1\cdot dx} + \frac{ddy}{1\cdot 2\cdot dx^2} - \frac{d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \text{etc.},\\
''y &= y - \frac{2dy}{1\cdot dx} + \frac{2^2ddy}{1\cdot 2\cdot dx^2} - \frac{2^3d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \text{etc.},\\
'''y &= y - \frac{3dy}{1\cdot dx} + \frac{3^2ddy}{1\cdot 2\cdot dx^2} - \frac{3^3d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \text{etc.}\\
&\text{etc.,}
\end{aligned}$$

if these values are substituted there, an equation will arise, in whose single terms one dimension of the variable y occurs, but of the other variable x only the differential dx, which is assumed to be constant, enters. Hence, if everywhere 1 is set instead of y, z instead of $\frac{dy}{dx}$ and generally z^m instead of $\frac{d^my}{dx^m}$, having done the reduction this equation will emerge

$$1 = \alpha e^{-z} + \beta e^{-2z} + \gamma e^{-3z} + \delta e^{-4z} + \text{etc.}$$

Now let $e^z = u$, and having got rid of the fractions an algebraic equation of this kind will arise

$$u^n = \alpha u^{n-1} + \beta u^{n-2} + \gamma u^{n-3} + \delta u^{n-4} + \text{etc.},$$

which will be of so many dimensions as preceding terms are required for the determination of the term y or of what order the recurring series itself was. Now the form of the general term y will be concluded from the roots of this equation or from the factors of this formula

$$u^n - \alpha u^{n-1} - \beta u^{n-2} - \gamma u^{n-3} - \delta u^{n-4} - \text{etc.} = U$$

in the same way we did it in the solutions of the problems propounded until now: Of course, if it is $\sin \pi x = r$ and $\cos \pi x = s$ and P, Q, R, S, T etc. denote arbitrary functions of even dimensions of r and s, further, investigate all real so simple as trinomial factors of the formula U and, if some of them were equal, express them combined by means of powers. But these single factors will yield as many parts of the general terms y, which parts will be formed by means of the following rules:

I. If one factor is $u - A$, the part of the integral will be

$$y = A^x P.$$

II. If the factor is $(u - A)^2$, the part of the integral will be

$$y = A^x (P + Qx).$$

III. If the factor is $(u - A)^3$, the part of the integral will be

$$y = A^x (P + Qx + Rx^2).$$

IV. If the factor is $(u - A)^4$, the part of the integral will be

$$y = A^x (P + Qx + Rx^2 + Sx^3).$$

etc.

1. If the factors is $u - 2Au\cos\omega + AA$, it will be

$$y = A^x(P\cos\omega x + Q\sin\omega x).$$

2. If the factor is $(u - 2Au\cos\omega + AA)^2$, it will be

$$y = A^x(P + Qx)\cos\omega x + A^x(R + Sx)\sin\omega x$$

3. If the factor is $(u - 2Au\cos\omega + AA)^3$, the part will be

$$\begin{aligned} y = &+ A^x(P+Qx+Rxx)\cos\omega x \\ &+ A^x(S+Tx+Vxx)\sin\omega x \end{aligned}$$

etc.

Therefore, if for the single factors of the formula U hence the parts of the integral are found and the are combined into one sum, one will have the complete value for the general term in question. Q.E.I.

COROLLARY 1

§45 Therefore, this way one obtains the complete integral of the following infinite differential equation

$$
\begin{array}{rcl}
y = & y & (\alpha + \quad \beta + \quad \gamma + \quad \delta + \text{etc.}) \\
- & \dfrac{dy}{1 \cdot dx} & (\alpha + 2\ \beta + 3\ \gamma + 4\ \delta + \text{etc.}) \\
+ & \dfrac{ddy}{1 \cdot 2 \cdot dx^2} & (\alpha + 2^2\beta + 3^2\gamma + 4^2\delta + \text{etc.}) \\
- & \dfrac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} & (\alpha + 2^3\beta + 3^3\gamma + 4^3\delta + \text{etc.}) \\
+ & \dfrac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} & (\alpha + 2^4\beta + 3^4\gamma + 4^4\delta + \text{etc.}) \\
 & & \text{etc.}
\end{array}
$$

or the value of y will be expressed by means of a function of x.

Corollary 2

§46 Therefore, the complete difficulty is reduced to the resolution of the algebraic equation

$$u^n = \alpha u^{n-1} + \beta u^{n-2} + \gamma u^{n-3} + \delta u^{n-4} + \text{etc.}$$

For having found its roots or factors it is easy to determine the value of y by means of the rules given before.

Corollary 3

§47 Since by integration so many arbitrary quantities P, Q, R, S etc. are introduced as the exponent n contains unities or as preceding terms enter into the determination of the following, it is manifest that as many terms can be taken ad libitum, from which all remaining ones, whose indices are integer numbers, are determined. This is nevertheless no obstruction that the terms of the non integer indices stay most undetermined, as it was already noted in the preceding problems.

Problem 7

§48 *If any arbitrary term of the series becomes equal to a certain constant quantity c together with an aggregate of several preceding terms multiplied by given numbers (as in the preceding problem), to find the general term of this series.*

Solution

Having as before put the term corresponding to the undetermined index $x = y$ let the preceding ones corresponding to the indices $y - 1$, $x - 2$, $x - 3$ etc. be $'y$, $''y$, $'''y$ etc. and let this law of progression be propounded

$$y = c + \alpha\, 'y + \beta\, ''y + +\gamma\, '''y + \delta\, ^{\mathrm{IV}}y + \text{etc.};$$

therefore, having propounded the values exhibited above for $'y$, $''y$, $'''y$, $^{\mathrm{IV}}y$ etc. it will be

$$\begin{aligned} y = c + & \quad y\,(\alpha + \beta + \gamma + \delta + \text{etc.}) \\ & - \frac{dy}{1 \cdot dx}(\alpha + 2\,\beta + 3\,\gamma + 4\,\delta + \text{etc.}) \\ & + \frac{ddy}{1 \cdot 2 \cdot dx^2}(\alpha + 2^2\beta + 3^2\gamma + 4^2\delta + \text{etc.}) \\ & - \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3}(\alpha + 2^3\beta + 3^3\gamma + 4^3\delta + \text{etc.}) \\ & \qquad \text{etc.} \end{aligned}$$

Now, to get rid of the constant term c from the equation put $y = v + g$ and let

$$g = c + g(\alpha + \beta + \gamma + \delta + \text{etc.})$$

and hence

$$g = \frac{c}{1 - \alpha - \beta - \gamma - \delta - \text{etc.}}.$$

Having done this because of $dy = dv$, $ddy = ddv$ etc. one will have this equation:

$$\begin{aligned} v = & \quad v\,(\alpha + \beta + \gamma + \delta + \text{etc.}) \\ & - \frac{dv}{1 \cdot dx}(\alpha + 2\,\beta + 3\,\gamma + 4\,\delta + \text{etc.}) \\ & + \frac{ddv}{1 \cdot 2 \cdot dx^2}(\alpha + 2^2\beta + 3^2\gamma + 4^2\delta + \text{etc.}) \\ & \qquad \text{etc.} \end{aligned}$$

Since this equation is similar to the one we resolved in the preceding problem, the value of v will be found by means of the rules given there. Having found this one will have the general term in question

$$y = v + \frac{c}{1 - \alpha - \beta - \gamma - \delta - \text{etc.}},$$

whence the nature of the propounded series will become known. Q.E.I.

COROLLARY 1

§49 Therefore, the constant quantity c, which is added to the formula

$$\alpha\,{}'y + \beta\,{}''y + \gamma\,{}'''y + \text{etc.},$$

does only affect the general term y in that regard that it adds a constant to it. Therefore, find the general term for the pure recurring series, whose relation scale is $+\alpha$, $+\beta$, $+\gamma$, $+\delta$ etc., and to it add the number $\frac{c}{1-\alpha-\beta-\gamma-\text{etc.}}$.

COROLLARY 2

§50 But this constant quantity to be added $\frac{c}{1-\alpha-\beta-\gamma-\text{etc.}}$ becomes infinite and hence uncertain, if the denominator vanishes or if it is

$$1-\alpha-\beta-\gamma-\delta-\text{etc.}=0.$$

But in this case the equation

$$u^n-\alpha u^{n-1}-\beta u^{n-2}-\gamma u^{n-3}-\text{etc.}=0$$

will have the root $u-1=0$, whence the part $y=P$ of the integral arises; this quantity P, that not all terms become infinite, must be infinite in such a way that it together with that infinite constant yields a finite quantity, which will be $=P+Qx$.

SCHOLIUM 1

That this becomes more clear, it is to be observed that series of this kind, as we considered here, can always be reduced to pure recurring series higher by one degree. For, if it is

$$y = c + \alpha\,{}'y + \beta\,{}''y + \gamma\,{}'''y + \delta\,{}^{\mathrm{IV}}y,$$

it will be

$${}'y = c + \alpha\,{}''y + \beta\,{}'''y + \gamma\,{}^{\mathrm{IV}}y + \delta\,{}^{\mathrm{V}}y,$$

whose difference gives

$$y = (\alpha+1)\,{}'y + (\beta-\alpha)\,{}''y + (\gamma-\beta)\,{}'''y + (\delta-\gamma)\,{}^{\mathrm{IV}}y - \delta\,{}^{\mathrm{V}}y,$$

which is the law for a pure recurring series, whose general term will be formed from the resolution of this equation:

$$u^{n+1} - (\alpha+1)u^n - (\beta-\alpha)u^{n-1} - (\gamma-\beta)u^{n-2} - \text{etc.} = 0.$$

But one factor of this is already known, namely $u-1$, because it is

$$(u-1)(u^n - \alpha u^{n-1} - \beta u^{n-2} - \gamma u^{n-3} - \text{etc.}) = 0.$$

But the factor $u-1$ gives the part $1^x P$ of the integral only then, whenever not at the same time it is a factors of the other form $u^n - \alpha u^{n-1} - \text{etc.}$; but if this also has the factor $u-1$ or its power as a factor, the exponent of this must be augmented by the unity and hence the corresponding part of the integral must be investigated. But having found the general term y this way, because in it the quantity c is not contained, it will be too general; therefore, it must be restricted to the propounded case. Of course, from the value of y find the values of the preceding terms $'y$, $''y$, $'''y$ etc. by putting $x-1$, $x-2$, $x-3$ etc. instead of x etc., where it is to be noted that the functions P, Q, R etc. retain the same values and hence experience no change. Further, substitute these values in the equation

$$y = c + \alpha\, 'y + \beta\, ''y + \gamma\, '''y + \delta\, ^{\mathrm{IV}}y + \text{etc.}$$

and this way one of the those functions P, Q, R etc. will be determined. So, if this law of the series is propounded

$$y = c + 3'y - 2''y,$$

hence this equation will arise

$$(u-1)(u^2 - 3u - 2)) = 0,$$

whose factors are

$$(u-1)^2(u-2) = 0,$$

from which this general term is concluded

$$y = P + Qx + 2^x R;$$

therefore, it will be

$$'y = P + Qx - Q + 2^{x-1}R$$

and

$$''y = P + Qx - 2Q + 2^{x-2}R,$$

which substituted will give this equality

$$P + Qx + 4 \cdot 2^{x-2}R = c + Qx + Q + 4 \cdot 2^{x-2}R,$$

whence one finds $Q = -c$; and so the general term corresponding to the propounded law will be

$$y = P - cx + 2^x R,$$

where for P and R one can assume arbitrary functions of even dimensions of r and s.

SCHOLIUM 2

§52 Therefore, since we gave a universal method to find the general term of series, each term of which is determined by means of the preceding ones, if no powers of the preceding terms occur, let us accommodate this same method to series, each term of which is not only determined from the preceding ones but also the index itself, in which we constituted the third class of the formation of series. But if squares or higher powers enter into the determination of the following term, as if it was

$$y' = yy + ay,$$

that the infinite differential equation, by which the general term is found, is certainly easily exhibited, which in this case will be

$$yy + ay = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.};$$

but since until now no artifice is known to resolve equations of this kind, we are forced to omit the treatment of this class of series here.

PROBLEM 8

§53 *To find the general term of the series, whose arbitrary term corresponding to the index x becomes equal to an multiple of the preceding together with a multiple of the index and a certain constant quantity.*

SOLUTION

Let y be the term corresponding to the index x and let y' denote the following terms and let this law of the series be propounded

$$y' = my + a + bx,$$

from which the value of y must be defined. Therefore, if for y' we substitute its value, we will have this equation:

$$a + bx + my = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.}$$

Even though I taught to resolve equations of this kind in general, it will nevertheless be helpful to resolve this equation into another one, in which all terms contain only one dimension of y. Therefore, put

$$y = A + Bx + v;$$

it will be

$$dy = Bdx + dv, \qquad ddy = ddv \quad \text{etc.}$$

and it will be

$$\begin{array}{ll} a + bx = & A + Bx + v + \dfrac{dv}{1 \cdot dx} + \dfrac{ddv}{1 \cdot 2 \cdot dx^2} + \dfrac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.} \\ +mA + mBx & +B \end{array}$$

Now let

$$A + B = a + mA \quad \text{and} \quad B = b + mB$$

and one will find

$$B = \frac{-b}{m-1} \quad \text{and} \quad A = \frac{-b}{(m-1)^2} - \frac{a}{m-1}.$$

Therefore, this equation will remain

$$mv = v + \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.};$$

since this is reduced to $e^z - m = u - m = 0$, it will be

$$v = m^x P$$

and hence the general term in question

$$y = \frac{-b}{(m-1)^2} - \frac{a+bx}{m-1} + m^x P.$$

Here, the one case is excluded, in which it is $m = 1$, because of the vanishing denominator $m-1$. For, since in this case one will have

$$a + bx = v + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.},$$

to get rid of the term bx one has to take a value of this kind for y

$$y = A + Bx + Cxx + v,$$

whence it is

$$\frac{dy}{dx} = B + 2Cx + \frac{dv}{dx} \quad \text{und} \quad \frac{ddy}{dx^2} = 2C + \frac{ddv}{dx^2};$$

and so one will have

$$\begin{aligned} a + bx = \quad & B + 2Cx + \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \text{etc.} \\ +C & \end{aligned}$$

Therefore, let it be

$$C = \frac{1}{2}b \quad \text{and} \quad B = a - \frac{1}{2}b$$

and it will be $v = P$ and the general term

$$y = A + \left(a - \frac{1}{2}b\right)x + \frac{1}{2}bxx + P,$$

or since A can be contained in the function P, it will be

$$y = \left(a - \frac{1}{2}\right)x + \frac{1}{2}bxx + P.$$

Q.E.I.

SCHOLIUM

§54 But series of this kind can be reduced to the law of simple recurring series. For, because it is

$$y' = a + bx + my,$$

it will be

$$y'' = a + b(x+1) + my',$$

whence by subtracting it will be

$$y'' - y' = b + my' - my;$$

in similar manner it will be

$$y''' - y'' = b + my'' - my'$$

and by subtracting again

$$y''' - 2y'' + y' = my'' - 2my' + my$$

or

$$y''' = (m+2)y'' - (2m+1)y' + my$$

or for the preceding terms

$$y = (m+2)\ {}'y - (2m+1)\ {}''y + m\ {}'''y.$$

Therefore, hence according to § 51 this equation will be formed

$$u^3 - (m+2)u^2 + (2m+1)u - m = 0,$$

which has these factors

$$(u-1)^2(u-m) = 0,$$

from which this general term arises

$$y = P + Qx + m^x R.$$

Now to accommodate this too far extending formula to the propounded case $y' = a + bx + my$, because of

$$y' = P + Qx + Q + m \cdot m^x R$$

it will be

$$P + Q + Qx + m \cdot m^x R = a + bx + mP + mQx + m \cdot m^x R$$

and hence

$$P + Q = a + mP \quad \text{and} \quad Q = b + mQ,$$

whence it is found

$$Q = \frac{-b}{m-1} \quad \text{und} \quad P = \frac{-b}{(m-1)^2} - \frac{a}{m-1},$$

such that the general term, as it was found before, is

$$y = \frac{-b}{(m-1)^2} - \frac{a+bx}{m-1} + m^x R.$$

But if it is $m = 1$, it is immediately plain that the three factors of the equation $(u-1)^2(u-m) = 0$ will be equal and that it is $(u-1)^3 = 0$, whence the general term is

$$y = P + Qx + Rxx$$

and hence

$$\begin{aligned} y' = P + {} & Qx + Rxx = P + Qx + Rxx \\ {} + Q + {} & 2Rx \qquad\quad a + b\,x \\ {} + R & \end{aligned}$$

Therefore, it arises

$$R = \frac{1}{2}b \quad \text{und} \quad Q = a - \frac{1}{2}b,$$

such that the general term is

$$y = P + \left(a - \frac{1}{2}b\right)x + \frac{1}{2}bxx,$$

as before. In similar manner it is clear, if the law of progression in general is

$$y = X + \alpha\,'y + \beta\,''y + \gamma\,'''y + +\delta\,^{\mathrm{IV}}y + \text{etc.}$$

and X is a polynomial function of x, as

$$X = a + bx + cxx + dx^3 + \text{etc.},$$

that by continued subtraction one finally gets to a law by which the single terms are determined by the preceding ones alone, and so the series will always be recurring, whose general term can be defined by the prescriptions given before. But this term will extend too far and therefore by finding the values of the terms $'y$, $''y$, $'''y$ etc. must be accommodated to the propounded law, having done which always so many functions P, Q, R etc. will be determined as letters a, b, c, d etc. were eliminated by subtraction. Therefore, because series of this kind do no longer cause any difficulties, let us consider other ones, in which X is neither a rational nor a polynomial function of x.

Problem 9

§55 *To find the general term of the series, whose arbitrary term becomes equal to the preceding one together with an arbitrary function of the index itself.*

Solution

Let the term corresponding to the index x be $= y$ and its preceding one

$$'y = y - \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} - \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.}$$

But let the law of progression be

$$y =' y + X,$$

whence it will be

$$X = \frac{dy}{1 \cdot dx} - \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} - \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.},$$

which equation is resolved by means of the rules I gave some time ago. Of course, by putting z^n for $\frac{d^ny}{dx^n}$ form this expression

$$Z = z - \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} - \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.} = 1 - e^{-z},$$

all factors of which must be found, the first of which will be z; the remaining one are contained in this general form $zz+4kk\pi\pi$. But from the factor $z-0$ this part of the integral will arise

$$y=\int Xdx+\text{etc.}$$

But from the factor $zz+4kk\pi\pi$, if it is compared to the formula $zz-2kz\cos\varphi+kk$, it will be $k=2k\pi$ and $\cos\varphi=0$, whence it is $\varphi=90^\circ$, and hence the letters $\mathfrak{M}$ and $\mathfrak{N}$ because of

$$A=0,\quad B=1,\quad C=\frac{-1}{1\cdot 2},\quad D=\frac{1}{1\cdot 2\cdot 3}\quad \text{etc.}$$

will be determined in such a way

$$\begin{aligned}\mathfrak{M}&=1-\frac{4k^2\pi^2}{1\cdot 2}+\frac{16k^4\pi^4}{1\cdot 2\cdot 3\cdot 4}-\frac{64k^6\pi^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.},\\ \mathfrak{N}&=-\frac{2k\pi}{1}+\frac{8k^3\pi^3}{1\cdot 2\cdot 3}-\frac{32k^5\pi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.},\end{aligned}$$

that it is

$$\mathfrak{M}=\cos 2k\pi\quad\text{und}\quad\mathfrak{N}=-\sin 2k\pi.$$

Having found these values the part of the integral to arise from the factor $zz+4kk\pi\pi$ will be

$$y=2\left\{\begin{matrix}(\cos 2k\pi\cos 2k\pi x-\sin 2k\pi\sin 2k\pi x)\int Xdx\cos 2k\pi x\\ (\cos 2k\pi\sin 2k\pi x+\sin 2k\pi\cos 2k\pi x)\int Xdx\sin 2k\pi x\end{matrix}\right\};$$

but it is $\sin 2k\pi=0$, $\cos 2k\pi=1$, whence it will be

$$v=2\cos 2k\pi x\int Xdx\cos 2k\pi x+2\sin 2k\pi x\int Xdx\sin 2k\pi x.$$

If now all these values to arise from the variability of the number k are collected into one sum, the general term in question will arise:

$$y = \int Xdx + 2\cos 2\pi x \int Xdx \cos 2\pi x + 2\cos 4\pi x \int Xdx \cos 4\pi x$$
$$+ 2\cos 6\pi x \int Xdx \cos 6\pi x + \text{etc.}$$
$$+ 2\sin 2\pi x \int Xdx \sin 2\pi x + 2\sin 4\pi x \int Xdx \sin 4\pi x$$
$$+ 2\sin 6\pi x \int Xdx \sin 6\pi x + \text{etc.}$$

Q.E.I.

COROLLARY 1

§56 Since it is $y =' y + X$, it is manifest that y expresses the summatory term of the series, whose general term is $= X$. For, if the sum of all terms from the first to this one X, whose index is $= x$, is put $= y$, it will be the sum of all except for the last $=' y$ and hence $y =' y + X$.

COROLLARY 2

§57 Therefore, the found expression y or the general term of the propounded series at the same time is the summatory term of the series, whose general term is $= X$; and so we obtained a new expression for the sum of a series, whose general term is given; but because of the infinite amount of integrals it will very rarely be of any use.

SCHOLIUM

§58 If except for the arbitrary function of the index x not only the closet preceding term but also more of the preceding terms are used for the formation of the following term of the series, in similar manner one will get to the resolution of an infinite differential equation, which can be treated by means of the method propounded by me. Therefore, not only the series, whose law of formation extends to the second class, by the method explained here can be reduced to a calculation and their general terms can be found, but it also equally extends to the third class and shows the true nature of those series even more clearly.

Problem 10

§59 *To find the general term of the series, whose arbitrary term is equal to the preceding term multiplied by its index.*

Solution

If the first term is put equal to unity, Wallis's hypergeometric series will arise

Indices:	1,	2,	3,	4,	5,	6,	7,	8,	9	etc.
Terms:	1,	1,	2,	6,	24,	120,	720,	5040,	40320	etc.

Put the term corresponding to the index $x = y$ and the one following it $= y'$; it will be

$$y' = yx,$$

whence this equation arises

$$yx = y + \frac{dy}{1 \cdot dx} + \frac{ddy}{1 \cdot 2 \cdot dx^2} + \frac{d^3y}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \text{etc.};$$

but for solving such an equation no general rule is known. But in a simple task this equation is transformed into another one, which can be resolved. Of course, out $y = e^v$; it will be $y' = e^{v'}$ and hence it will be $e^{v'} = e^v x$ and having taken logarithms

$$v' = v + \log x,$$

whence one will have

$$\log x = \frac{dv}{1 \cdot dx} + \frac{ddv}{1 \cdot 2 \cdot dx^2} + \frac{d^3v}{1 \cdot 2 \cdot 3 \cdot dx^3} + \frac{d^4v}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + + \frac{d^4v}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot dx^5} + \text{etc.},$$

which equation is contained in the preceding one by taking $X = \log x$; therefore, the integral will be

$$\begin{aligned} v = \int dx \log x + 2\cos 2\pi x \int dx \log x \cos 2\pi x + 2\cos 4\pi x \int dx \log x \cos 4\pi x + \text{etc.} \\ + 2\sin 2\pi x \int dx \log x \sin 2\pi x + 2\sin 4\pi x \int dx \log x \sin 4\pi x + \text{etc.} \end{aligned}$$

But having found the value of v the general term in question will be e^v while e denotes a number, whose hyperbolic logarithm is $= 1$. Q.E.I.

SCHOLIUM

§60 The first term of this expression is $\int dx \log x = x \log x - x$, the remaining single terms can be integrated by means of infinite series. For, it is

$$\begin{aligned}\int dx \log x \cos mx = &+\frac{1}{m}\sin mx\left(\log x + \frac{1}{m^2x^2} - \frac{1\cdot 2\cdot 3}{m^4x^4} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{m^6x^6} - \text{etc.}\right)\\ &+\frac{1}{m}\cos mx\left(\frac{1}{mx} - \frac{1\cdot 2}{m^3x^3} + \frac{1\cdot 2\cdot 3\cdot 4}{m^5x^5} - \text{etc.}\right)\\ \int dx \log x \sin mx = &-\frac{1}{m}\cos mx\left(\log x + \frac{1}{m^2x^2} - \frac{1\cdot 2\cdot 3}{m^4x^4} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{m^6x^6} - \text{etc.}\right)\\ &+\frac{1}{m}\sin mx\left(\frac{1}{mx} - \frac{1\cdot 2}{m^3x^3} + \frac{1\cdot 2\cdot 3\cdot 4}{m^5x^5} - \text{etc.}\right)\end{aligned}$$

Hence it is concluded that it will be

$$2\cos mx \int dx \log x \cos mx + 2\sin mx \int dx \log x \sin mx$$

$$= \frac{2}{mmx}\left(1 - \frac{1\cdot 2}{m^2x^2} + \frac{1\cdot 2\cdot 3\cdot 4}{m^4x^4} - \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{m^6x^6} + \text{etc.}\right) + \alpha\cos mx + \mathfrak{A}\sin mx.$$

Now having successively substituted the values 2π, 4π, 6π etc. for m and having collected all these expressions one will find

$$\begin{aligned} v = C + x\log x - x + &\frac{1}{2\pi^2 x}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}\right) + \alpha\cos 2\pi x + \mathfrak{A}\sin 2\pi x\\ - &\frac{1\cdot 2}{8\pi^4x^3}\left(1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \text{etc.}\right) + \beta\cos 4\pi x + \mathfrak{B}\sin 4\pi x\\ - &\frac{1\cdot 2\cdot 3\cdot 4}{32\pi^6x^5}\left(1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \text{etc.}\right) + \gamma\cos 6\pi x + \mathfrak{C}\sin 6\pi x\end{aligned}$$

etc.

If now for the these series of powers the sums found be me a long time ago are substituted, one will have

$$v = C + x\log x - x + \alpha\ \cos 2\pi x + \beta\ \cos 4\pi x + \gamma\cos 6\pi x + \text{etc.}$$
$$+\mathfrak{A}\sin 2\pi x + \mathfrak{B}\sin 4\pi x + \mathfrak{C}\sin 6\pi x + \text{etc.}$$

$$+\frac{1}{1\cdot 2\cdot 3}\cdot\frac{1}{2x} - \frac{1}{3\cdot 4\cdot 5}\cdot\frac{1}{6x^3} + \frac{1}{5\cdot 6\cdot 7}\cdot\frac{1}{6x^5} - \frac{1}{7\cdot 8\cdot 9}\cdot\frac{3}{10x^7} + \frac{1}{9\cdot 10\cdot 11}\cdot\frac{5}{6x^9} - \text{etc.},$$

or if P is a function of even dimensions of $r = \sin\pi x$ and $s = \cos\pi x$, it will be

$$v = P + x\log x - x + \frac{1}{1\cdot 2\cdot 3}\cdot\frac{1}{2x} - \frac{1}{3\cdot 4\cdot 5}\cdot\frac{1}{6x^3} + \frac{1}{5\cdot 6\cdot 7}\cdot\frac{1}{6x^5} - \text{etc.}$$

Since now having put $x = 1$ it is $y = 1$ and $v = 0$, in this case it must be

$$P = 1 - \frac{1}{1\cdot 2\cdot 3\cdot 2} + \frac{1}{3\cdot 4\cdot 5\cdot 6} - \frac{1}{5\cdot 6\cdot 7\cdot 6} + \frac{3}{7\cdot 8\cdot 9\cdot 10} - \text{etc.},$$

whose value I showed elsewhere to be

$$P = \frac{1}{2}\log 2\pi.$$

and it will have this value, as often as x is an arbitrary integer. Hence by going back to numbers, one will find the general terms in question

$$y = \frac{x^x}{e^x}\cdot e^{\frac{1}{1\cdot 2\cdot 3}\cdot\frac{1}{2x} - \frac{1}{3\cdot 4\cdot 5}\cdot\frac{1}{6x^3} + \frac{1}{5\cdot 6\cdot 6}\cdot\frac{1}{x^5} - \text{etc.}}\sqrt{2\pi}$$

or

$$y = \frac{x^x}{e^x}\cdot e^{\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}.\text{etc.}}\sqrt{2\pi}.$$

Hence, if x is a very large number, it will approximately be

$$y = \frac{x^x}{e^x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} + \text{etc.}\right)\sqrt{2\pi}$$

and so the magnitude of each terms moved from the beginning very far is easily approximately assigned.