On transcendental Progressions or those whose general terms can not be given algebraically *

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§1 After I recently on the occasion of those things Goldbach had communicated on series to the Society, I sought after a certain general expression, which would give all terms of this progression

$$1 + 1 \cdot 2 + 1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 3 \cdot 4 + \text{etc.},$$

considering that it continued to infinity is finally confounded with the geometric series, I stumbled upon the following expression

$$\frac{1 \cdot 2^2}{1+n} \cdot \frac{2^{1-n} \cdot 3^n}{2+n} \cdot \frac{3^{1-n} \cdot 4^n}{3+n} \cdot \frac{4^{1-n} \cdot 5^n}{4+n} \cdot \text{etc.},$$

which expresses the term of order n in the mentioned progression. It certainly does not terminate in any case, neither if n is an integer number nor if it is a fraction, but to find a certain term it only gives an approximation, if the cases $n = 0$ and $n = 1$ are excluded, in which it actually goes over into 1. Put $n = 2$; one will have

$$\frac{2 \cdot 2}{1 \cdot 2} \cdot \frac{3 \cdot 3}{3 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{5 \cdot 5}{4 \cdot 6} = \text{ the second term } 2$$

If it is $n = 3$, one will have

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$$\frac{2\cdot 2\cdot 2}{1\cdot 1\cdot 4}\cdot\frac{3\cdot 3\cdot 3}{2\cdot 2\cdot 5}\cdot\frac{4\cdot 4\cdot 4}{3\cdot 3\cdot 6}\cdot\frac{5\cdot 5\cdot 5}{4\cdot 4\cdot 4}\cdot\text{etc} = \text{to the third term } 6$$

§2 But although this expression seems to have no use for finding the terms, it can nevertheless be accommodated to the interpolation of the series or the terms, whose indices are fractional numbers, in extraordinary manner. But I decided to explain nothing about this here, since below more suitable ways occur to achieve the same. I just want to mention about the general term, what leads to the things which follow. I asked for the term, whose index is $n = \frac{1}{2}$ or which falls in the middle between the first, 1, and the preceding one, which is also 1. But having put $n = \frac{1}{2}$ I obtain the series

$$\sqrt{\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot\text{etc.}},$$

which expresses the term in question. But this series to me seemed to be similar to the one, which I remembered to have seen in Wallis's works one the circular area. For, Wallis found that the circle behaves to the square of the diameter as

$$2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot\text{etc.}\quad\text{to}\quad 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot\text{etc.}$$

Therefore, if the diameter was $= 1$, the area of the circle will be

$$= \frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\text{etc.}$$

Therefore, from the agreement of this series with mine it is possible to conclude that the term of the index $\frac{1}{2}$ is equal to the square root of the circle, whose diameter is $= 1$.

§3 I had believed before that the general term of the series 1, 2, 6, 24 etc., if not algebraically, but nevertheless is given exponentially. But after I had understood that certain intermediate terms depend on the quadrature of the circle, I realized that neither algebraic nor exponential quantities are suitable to express it. For, the general term of the progression must be of such a nature that the one time it comprehends algebraic quantities the other time quantities depending on the quadrature of the circle and even on other quadratures; and no algebraic nor exponential formula meets this condition.

§4 But after I had considered that among differential quantities formulas of such a kind are given, which in certain cases admit an integration and then yields algebraic quantities, in other cases on the other hand do not admit an integration and then exhibit quantities depending on the quadratures of curves, it came to mind that maybe formulas of this kind are apt to express the general terms of the mentioned progression and others similar to it. But it call progressions, which require such general terms, which cannot be given algebraically, *transcendental*; as Geometers use to call all that what supersedes the power of common Algebra, transcendental.

§5 Therefore, I thought about, how differential formulas should be accommodated to express general terms of progressions in the best way possible. But the general term is a formula, into which so constant as certain other non constant quantities as n, which gives the order of the terms or the index, enter, that, if the third term is desired, instead of n one has to put 3. But in a differential formula a certain variable quantity must be contained. For this it is not advisable to use n, since its variability does not extend to the integration, but just after the formula had been integrated or is put to had been integrated serves for forming the progression. Therefore, in the differential formula a certain variable quantity x must be contained, which after the integration is to be equal to another concerning the progression; and hence the term arises, whose index is n.

§6 That this is understood more clearly, I say that $\int pdx$ is the general term of the progression to be found in the following from it; but let p denote an arbitrary function of x and constants, one of which must be n. Now imagine pdx as integrated and augmented by such a constant that having put $x = 0$ the whole integral vanishes; then put x equal to a certain known quantity. Having done this in the found integral only quantities extending to the progression will remain, and it will express the term, whose index $= n$. Or the integral determined in this way will be the general term. If this integration is actually possible, the differential formula in not necessary, but the progression formed from this will have a general algebraic term; but matters change, if the integration only succeeds for certain numbers substituted for n.

§7 Therefore, I assumed many differential formulas of this kind only admitting an integration, if instead of n one puts an positive integer, that the principal terms become algebraic, and hence formed progressions. Therefore, their general terms were immediately clear, and it will be possible to define on which certain quadrature each of its intermediate terms depend. I will certainly not go through many formulas of this kind, but will only treat one a bit more general one, which extends very far and can be accommodated to all progressions, whose arbitrary terms are products consisting of a number of terms depending on the index; the factors are fractions, whose numerators and denominators proceed in an arbitrary arithmetic progression, as

$$\frac{2}{3}+\frac{2\cdot 4}{3\cdot 5}+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}+\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}+\text{ etc.}$$

§8 Let this formula be propounded

$$\int x^e dx(1-x)^n$$

taking the place of the general term, which integrated in such a way that it becomes $=0$, if it is $x=0$, and then having put $x=1$ shall give the term of order n of the progression arising from it. Therefore, let us see, which progression it gives. It is

$$(1-x)^n=1-\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\text{ etc.}$$

and hence

$$dx(1-x)^n=x^e dx-\frac{n}{1}x^{e+1}dx+\frac{n(n-1)}{1\cdot 2}x^{e+2}dx-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{e+3}dx+\text{ etc.}$$

Hence

$$\int x^e dx(1-x)^n=\frac{x^{e+1}}{e+1}-\frac{nx^{e+2}}{1\cdot(e+2)}+\frac{n(n-1)x^{e+3}}{1\cdot 2\cdot(e+3)}-\frac{n(n-1)(n-2)x^{e+4}}{1\cdot 2\cdot 3\cdot(e+4)}+\text{ etc.}$$

Put $x=1$, since the addition of the constant it not necessary, and one will have

$$\frac{1}{e+1}-\frac{n}{1\cdot(e+2)}+\frac{n(n-1)}{1\cdot 2\cdot(e+3)}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot(e+4)}+\text{ etc.}$$

as the general term of the series to be found. It will be such a one that, if it is $n=0$, the term arises as $=\frac{1}{(e+1)(e+2)}$; if $n=1$, the term $\frac{1\cdot 2}{(e+1)(e+2)(e+3)}$; if $n=3$, the term arises as $=\frac{1\cdot 2\cdot 3\cdot 4}{(e+1)(e+2)(e+3)(e+4)}$; the law, according to which these terms proceed, is manifest.

§9 Therefore, I obtained this progression

$$\frac{1}{(e+1)(e+2)}+\frac{1\cdot 2}{(e+1)(e+2)(e+3)}+\frac{1\cdot 2\cdot 3}{(e+1)(e+2)(e+3)(e+4)}+\text{ etc.},$$

whose general term is

$$\int x^e dx(1-x)^n.$$

On the other hand the term of order n will be this form

$$\frac{1\cdot 2\cdot 3\cdot 4\cdots n}{(e+1)(e+2)\cdots(e+n+1)}.$$

This form certainly suffices to find the terms of integer indices, but if the indices were no integers, from it the terms cannot be found. This series will serve for finding them approximately

$$\frac{1}{e+1}-\frac{n}{1\cdot(e+2)}+\frac{n(n-1)}{1\cdot 2\cdot(e+3)}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot(e+4)}+\text{ etc.}$$

If $\int x^e dx(1-x)^n$ is multiplied by $e+n+1$, one will have a progression, whose term of order n has this form

$$\frac{1\cdot 2\cdot 3\cdots n}{(e+1)(e+2)\cdots(e+n)},$$

whose general term will therefore be

$$(e+n+1)\int x^e dx(1-x)^n.$$

Here it is to be noted that the progression always becomes algebraic, whenever instead of e a positive number is assumed. Put, e.g., $e=2$, the n-th term of the progression will be

$$\frac{1\cdot 2\cdot 3\cdots n}{3\cdot 4\cdot 5\cdots(n+2)} \quad \text{or} \quad \frac{1\cdot 2}{(n+1)(n+2)}$$

The general term itself also indicates this, which term will be

$$(n+3)\int xxdx(1-x)^n.$$

For, its integral is

$$\left(C-\frac{(1-x)^{n+1}}{n+1}+\frac{2(1-x)^{n+2}}{n+2}-\frac{(1-x)^{n+3}}{n+3}\right)(n+3);$$

that this becomes $= 0$, if it is $x = 0$, it will be

$$C = \frac{1}{n+1} - \frac{2}{n-2} + \frac{1}{n+3}.$$

Put $x = 1$; the general term will be

$$\frac{n+3}{n+1} - \frac{2(n+3)}{n+2} + 1 = \frac{2}{(n+1)(n+2)}.$$

§10 Therefore, that we obtain transcendental progressions, put e equal to the fraction $\frac{f}{g}$. The term of order n of the progression will be

$$= \frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} g^n$$

or

$$\frac{g \cdot 2g \cdot 3g \cdots ng}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}.$$

The general term on the other hand will be

$$= \frac{f+(n+1)g}{g} \int x^{\frac{f}{g}} dx (1-x)^n.$$

If this is divided by g^n, it will be for the progression

$$\frac{1}{f+g} + \frac{1 \cdot 2}{(f+g)(f+2g)} + \frac{1 \cdot 2 \cdot 3}{(f+g)(f+2g)(f+3g)} + \text{ etc.},$$

whose term of n-th order is

$$= \frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)\cdots(f+ng)}.$$

Therefore, the general term of the progression will be

$$= \frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx (1-x)^n$$

If here the fraction $\frac{f}{g}$ is not equal to an integer number, or if f does not have a multiple ratio to g, the progression will be transcendental and the intermediate terms will depend on quadratures.

§11 I want to mention a certain example, that the general term is shown more clearly. In the first progression of the preceding paragraph let $f = 1$, $g = 2$; the term of order n will be

$$= \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n+1)},$$

the progression itself on the other hand will be this one

$$\frac{2}{3} + \frac{2 \cdot 4}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} + \text{ etc.},$$

whose general term will therefore be

$$\frac{2n+3}{2} \int dx (1-x)^n \sqrt{x}.$$

Let the term be in question, whose index is $\frac{1}{2}$; therefore, it will be $n = \frac{1}{2}$ and one will have the general term in question

$$= 2 \int dx \sqrt{x - xx}$$

Since this denotes the element of the circular area, it is perspicuous that the term in question is the area of the circle, whose diameter is $= 1$.
Further, let this series be propounded

$$1 + \frac{r}{1} + \frac{r(r-1)}{1 \cdot 2} + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3} + \text{ etc.},$$

which is the coefficient of the binomial raised to the power r. Therefore, the term of order n is

$$\frac{r(r-1)(r-2) \cdots (r-n+2)}{1 \cdot 2 \cdot 3 \cdots (n-1)}.$$

In the preceding paragraph one had this one

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g) \cdots (f+ng)}.$$

This, that it can be compared to that one, is to be inverted, that one has

$$\frac{(f+g)(f+2g) \cdots (f+ng)}{1 \cdot 2 \cdots n};$$

multiply this one by $\frac{n}{f+ng}$ and it will be

$$= \frac{(f+g)(f+2g)\cdots(f+(n-1)g)}{1\cdot 2\cdots(n-1)};$$

therefore, it must be $f+g=r$ and $f+2g=r-1$, whence it will be $g=-1$ and $f=r+1$. Treat the following general term in the same way

$$\frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx(1-x)^n.$$

For the propounded progression

$$1+\frac{r}{1}+\frac{r(r-1)}{1\cdot 2}+\text{ etc.}$$

this general term will arise

$$\frac{n(-1)^{n+1}}{(r-n)(r-n+1)\int x^{-r-1}dx(1-x)^n}.$$

Let $r=2$; the general term of this progression

$$1, \quad 2, \quad 1, \quad 0, \quad 0, \quad 0\text{etc.}$$

will be

$$\frac{n(-1)^{n+1}}{(2-n)(3-n)\int x^{-3}dx(1-x)^n}.$$

Here it must be noted that this cases and other, in which $e+1$ become a negative number, cannot be deduced from the general one, since then the integral does not become $=0$, if it is $x=0$. For these on the other hand in is convenient to integrate

$$\int x^e dx(1-x)^n$$

in a peculiar way; for, after the integration and infinite constant is to be added. But whenever $+1$ is a positive number, as I put in § 8, the addition of the constant is not necessary. But having considered the progression, whose term of order n was the following

$$\frac{r(r-1)(r-2)\cdots(r-n+2)}{1\cdot 2\cdot 3\cdots(n-1)},$$

that for of the exponent n can be transformed into this one

$$\frac{r(r-1)\cdots 1}{(1\cdot 2\cdot 3\cdots(n-1))(1\cdot 2\cdots(r-n+1))}.$$

But by means of § 14 it is

$$r(r-1)\cdots 1 = \int dx(-\ln(x))^r$$

and

$$1\cdot 2\cdot 3\cdots(n-1) = \int dx(-\ln(x))^{n-1}$$

and

$$1\cdot 2\cdots(r-n+1) = \int dx(-\ln(x))^{r-n+1}.$$

Therefore, the progression treated there

$$1+\frac{r}{1}+\frac{r(r-1)}{1\cdot 2}+\frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}+\text{ etc.}$$

has this general term

$$\frac{\int dx(-\ln(x))^r}{\int dx(-\ln(x))^{n-1}\int dx(-\ln(x))^{r-n+1}}-$$

If it was $r=2$, the general term will be

$$\frac{2}{\int dx(-\ln(x))^{n-1}\int dx(-\ln(x))^{3-n}},$$

to which this progression corresponds

$$1, \quad 2, \quad 1, \quad 0, \quad 0, \quad 0 \text{ etc.};$$

as if the term of the index $\frac{3}{2}$ is in question, it will be

$$\frac{2}{\int dx(-\ln(x))^{\frac{1}{2}}\int dx(-\ln(x))^{\frac{3}{2}}}.$$

Therefore, having called the area of the circle, whose diameter is $=1$, A, since it is

$$\int dx(-\ln(x))^{\frac{1}{2}} = \sqrt{A} \quad \text{and} \quad \int dx(-\ln(x))^{\frac{3}{2}} = \frac{3}{2}\sqrt{A},$$

the term falling into the middle between the first two terms of the progression 1, 2, 1, 0, 0, 0 etc. will be of this form $\frac{4}{3A}$, this means approximately $\frac{5}{3}$.

§12 Now, I proceed to the progression, I talked about at the beginning,

$$1+1\cdot 2+1\cdot 2\cdot 3+\text{ etc.}$$

and in which the term of order n is $1\cdot 2\cdot 3\cdot 4\cdots n$. This progression is contained in our general one, but the general term must be derived from it in a peculiar way. Until now I had the general term, if the term of order n is

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)\cdots(f+ng)},$$

which, if one puts $f=1$ and $g=0$, goes over into $1\cdot 2\cdot 3\cdots n$, whose general term is in question; therefore, in the general term

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n$$

substitute these values for f and g; the general term in question will be

$$\int\frac{x^{\frac{1}{0}}dx(1-x)^n}{0^{n+1}}.$$

What is the value of this expression, I investigate in the following way.

§13 From the condition, according to which general terms of this kind must be accommodated to use, it is understood that instead of x other functions can be assumed, as long as they were of such a kind that they are $=0$, if it is $x=0$, and $=1$, if $x=1$. For, if a function of this kind are substituted for x, the general term will therefore satisfy the same condition as before. Therefore, put $x^{\frac{p}{f+g}}$ instead of x and as a logical consequence $\frac{g}{f+g}x^{\frac{-f}{g+f}}dx$ instead of x, having done which one will have

$$\frac{f+(n+1)g}{g^{n+1}}\int\frac{g}{f+g}dx(1-x^{\frac{g}{f+g}})^n.$$

Now put $f=1$ and $g=0$ here; one will have

$$\int\frac{dx(1-x^n)}{0^n}.$$

But because it is $x^0=1$, here we have the case, in which the numerator and the denominator vanish, $(1-x^0)^n$ and 0^n. Therefore, by means of the known rule let us find the value of the fraction $\frac{1-x^0}{0}$. This will by done by asking for the value of the fraction $\frac{1-x^z}{z}$ then, when z vanishes; therefore, differentiate the numerator and the denominator with respect to variable z only; one will have $\frac{-x^z dt \ln(x)}{dz}$ or $-x^z \ln(x)$; if now one puts $z = 0$, $-\ln(x)$ will arise. Therefore, it is

$$\frac{1-x^0}{0} = -\ln(x).$$

§14 Therefore, because it is

$$\frac{1-x^0}{0} = -\ln(x),$$

it will be

$$\frac{(1-x^0)^n}{0^n} = (-\ln(x))^n$$

and therefore the general term in question $\int \frac{dx(1-x^0)^n}{0^n}$ was transformed into $\int dx(-\ln(x))^n$. Its value can be found by means of quadratures. Therefore, the general term of this progression

$$1, \quad 2, \quad 6, \quad 24, \quad 120, \quad 720 \quad \text{etc.}$$

is

$$\int dx(-\ln(x))^n,$$

to be used in the same way as it was prescribed above. That this is the general term of the propounded progression is also seen from this, that it indeed yields the terms, whose indices are positive integers. Let, for the sake of an example, be $n = 3$; it will be

$$\int dx(-\ln(x))^3 = \int -dx(\ln(x))^3 = -x(\ln(x))^3 + 3x(\ln(x))^2 - 6x\ln(x) + 6x;$$

the addition of a constant is not necessary, since for $x = 0$ everything vanishes; therefore, put $x = 1$; since $\ln(1) = 0$, all terms affected which logarithms will vanish and 6 will remain, which is the third term.

§15 It is clear that this method to find the terms of this series is too laborious, even for those, whose indices are integer numbers, which are certainly obtained easier by continuing the progression. But nevertheless to find the terms of fractional indices it is more than suitable, which terms could not even be by defined by the most laborious method. If one puts $x = \frac{1}{2}$, one will have the corresponding term $= \int dx\sqrt{-\ln(x)}$, whose value is given by quadratures. But at the beginning [§ 11] I showed that this term is equal to the square root of the area the circle, whose diameter is 1. Hence it is certainly not possible to conclude the same because of the missing Analysis; but below a method will follow to reduce the same intermediate terms to quadratures of algebraic curves. From their comparison to this one maybe a lot for the amplification of Analysis can be derived.

§16 The general term of the progression, whose term of order n is indicated by

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)},$$

by means of § 10 is

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n.$$

But if the term of order n was

$$1\cdot 2\cdot 3\cdots n,$$

then the general term is

$$\int dx(-\ln(x))^n.$$

If this formula is substituted for $1\cdot 2\cdot 3\cdots n$, one will have

$$\frac{\int dx(-\ln(x))^n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} = \frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n.$$

Hence it is caused

$$(f+g)(f+2g)(f+3g)\cdots(f+ng) = \frac{g^{n+1}\int dx(-\ln(x))^n}{(f+(n+1)g)\int x^{\frac{f}{g}}dx(1-x)^n}.$$

Therefore, this expression is the general term of this general progression

$$(f+g) \quad ,(f+g)(f+2g) \quad ,(f+g)(f+2g)(f+3g) \quad \text{etc.}$$

Therefore, by means of the general term all terms of any arbitrary index of all progressions of this kind are defined. What will follow below about the reduction of $\int dx(-\ln(x))^n$ to more known quadratures or quadratures of algebraic curves, will also have its use here.

§17 Let $f+g=1$ and $f+2g=3$; it will be $g=2$ and $f=-1$. Hence this particular progression will arise

$$1, \quad 1\cdot 3, \quad 1\cdot 3\cdot 5, \quad 1\cdot 3\cdot 5\cdot 7\text{etc.}$$

Therefore, its general term is

$$\frac{2^{n+1}\int dx(-\ln(x))^n}{(2n+1)\int x^{-\frac{1}{2}}dx(1-x)^n}.$$

Although this exponent of x is negative, nevertheless the inconvenience, about which was talked above, does not occur here, since it is smaller than unity. Put $n=\frac{1}{2}$ that the term of order $\frac{1}{2}$ is found; it will be

$$=\frac{2^{\frac{3}{2}}\int dx\sqrt{-\ln(x)}}{2\int x^{-\frac{1}{2}}dx\sqrt{1-x}}=\frac{\sqrt{2}\int dx\sqrt{-\ln(x)}}{\int\frac{dx-xdx}{\sqrt{x-xx}}}.$$

But by means of § 15 it is known that $\int dx\sqrt{-\ln(x)}$ gives the square root of the circle, whose diameter is $=1$; let the circumference of that circle be p; the area will be $=\frac{1}{4}p$ and hence $\int dx\sqrt{-\ln(x)}$ gives $\frac{1}{2}\sqrt{p}$. Further,

$$\int\frac{dx-xdx}{2\sqrt{x-xx}}=\int\frac{dx}{2\sqrt{x-xx}}+\sqrt{x-xx},$$

but $\int\frac{dx}{2\sqrt{x-xx}}$ gives the arc of the circle, whose sinus versus is x. Therefore, having put $x=1$ $\frac{1}{2}p$ will arise. Therefore, the term in question will be

$$=\sqrt{\frac{2}{p}}.$$

§18 Since the general term of the progression, whose term of order n is indicated by

$$(f+g)(f+2g)\cdots(f+ng),$$

by means of § 16 is

$$\frac{g^{n+1}\int dx(-\ln(x))^n}{(f+(n+1)g)\int x^{\frac{f}{g}}dx(1-x)^n},$$

similarly, if the term of order n was

$$(h+k)(h+2k)\cdots(h+nk),$$

the general term will be

$$\frac{k^{n+1}\int dx(-\ln(x))^n}{(h+(n+1)k)\int x^{\frac{h}{k}}dx(1-x)^n}.$$

Divide that progression by this one, namely the first term by the first, the second by the second and so forth; one will get to a new progression, whose term of order n will be

$$\frac{(f+g)(f+2g)\cdots(f+ng)}{(h+k)(h+2k)\cdots(h+nk)}.$$

And the general term of this progression composited of these two will be

$$\frac{g^{n+1}(h+(n+1)k)\int x^{\frac{h}{k}}dx(1-x)^n}{k^{n+1}(f+(n+1)g)\int x^{\frac{f}{g}}dx(1-x)^n}.$$

This term is free from the logarithm integral $\int dx(-\ln(x))^n$

§19 In all general terms of this kind it is especially to be noted that not even for f, g, h, k one has to put constant numbers, but they can be assumed to depend on n in arbitrary manner. For, in the integration these letters are treated in the same way as n, all as constants. Therefore, let the term of order n be this one

$$(f+g)(f+2g)\cdots(g+ng);$$

put $g=1$, but $f=\frac{nnn}{2}$. Since the progression itself is

$$f+g, \quad (f+g)(f+2g), \quad (f+g)(f+2g)(f+3g) \quad \text{etc.}$$

instead of g put 1 everywhere; the progression will be

$$f+1, \quad (f+1)(f+2), \quad (f+1)(f+2)(f+3) \quad \text{etc.}$$

But instead of f one has to write 0 in the first term, 1 in the second, 3 in the third, 6 in the fourth and so forth; this progression will arise

$$1, \quad 1 \cdot 2, \quad 4 \cdot 5 \cdot 6, \quad 7 \cdot 8 \cdot 9 \cdot 10 \quad \text{etc.},$$

whose general term is

$$\frac{2\int dx(-\ln(x))^n}{(nn+n+2)\int x^{\frac{nn-n}{2}}dx(1-x)^n} = \frac{2\int dx(-\ln(x))^n}{(nn+n+2)\int dx(x^{\frac{n-1}{2}}-x^{\frac{n+1}{2}})^n}$$

§20 Now I proceed to the progression, whence I found this shortcut in defining the intermediate terms of this progression

$$1, \quad 2, \quad 6, \quad 24, \quad 120 \quad \text{etc.}$$

Hence, this shortcut extends further than only to this progression, since its general term

$$\int dx(-\ln(x))^n$$

also enters the general terms of infinitely many other progressions. I assume this general term

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n,$$

to which this term of order n corresponds

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}.$$

Here I put $f=n$, $g=1$; the general term will be

$$(2n+1)\int x^n dx(1-x)^n \quad \text{or} \quad (2n+1)\int dx(x-xx)^n$$

and its form of order n

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots 2n}.$$

The progression itself on the other hand is this one

$$\frac{1}{2}, \quad \frac{1\cdot 2}{3\cdot 4}, \quad \frac{1\cdot 2\cdot 3}{4\cdot 5\cdot 6} \quad \text{etc.}$$

or this one

$$\frac{1\cdot 1}{1\cdot 1}, \quad \frac{1\cdot 2\cdot 1\cdot 2}{1\cdot 2\cdot 3\cdot 4}, \quad \frac{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}.$$

In this the numerators are the squares of the progression 1, 2, 6, 24 etc., among two contiguous denominators on the other hand the equidistant one is easily found. In the progression 1, 2, 6, 24 etc. let the term, whose index is $\frac{1}{2}$, be A; the term of order $\frac{1}{2}$ of that progression will be $= \frac{AA}{1}$.

§21 In the general term

$$(2n+1)\int x^n dx(1-x)^n$$

put $n = \frac{1}{2}$; the term of this exponent will be

$$2\int dx\sqrt{x-xx} = \frac{AA}{1},$$

whence

$$A = \sqrt{1\cdot 2\int dx\sqrt{x-xx}}$$

$=$ to the term of the progression 1, 2, 6, 24 etc., whose index is $\frac{1}{2}$, which therefore, as it is clear from it, is the square root of the circle with diameter 1. Now call the term of order $\frac{3}{2}$ of this progression A; the corresponding one in the assumed progression will be

$$\frac{A\cdot A}{1\cdot 2\cdot 3} = 4\int dx(x-xx)^{\frac{3}{2}},$$

therefore

$$A = \sqrt{1\cdot 2\cdot 3\cdot 4\int dx(x-xx)^{\frac{3}{2}}}.$$

In similar manner the term of order $\frac{5}{2}$ will be found

$$= \sqrt{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \int dx(x - xx)^{\frac{5}{2}}}.$$

From these I conclude in general that the term of order $\frac{p}{2}$ will be

$$= \sqrt{1 \cdot 2 \cdot 3 \cdot 4 \cdots (p+1) \int dx(x - xx)^{\frac{p}{2}}}.$$

Therefore, this way one finds all terms of the progression 1, 2, 6, 24 etc., whose indices are fractions, while the denominator is 2.

§22 Further, in the general term

$$\frac{f + (n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx(1-x)^n$$

I put $f = 2n$, while g remains $= 1$; it will arise

$$(3n+1) \int dx(xx - x^3)^n$$

as general term of the progression

$$\frac{1}{3}, \quad \frac{1 \cdot 2}{5 \cdot 6}, \quad \frac{1 \cdot 2 \cdot 3}{7 \cdot 8 \cdot 9} \quad \text{etc.}$$

Multiply that one by the preceding $(2n+1) \int dx(x - xx)^n$; it will arise

$$(2n+1)(3n+1) \int dx(x - xx)^n \int dx(xx - x^3)^n.$$

This will give this progression

$$\frac{1 \cdot 1 \cdot 1}{1 \cdot 2 \cdot 3}, \frac{1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \quad \text{etc.},$$

where the numerators are the cubes of the corresponding terms of the progression 1, 2, 6, 24 etc. Let the term of order $\frac{1}{3}$ of this progression be A; the corresponding term of that progression will be

$$\frac{A^3}{1} = 2\left(\frac{2}{3} + 1\right) \int dx(x - xx)^{\frac{1}{3}} \int dx(xx - x^3)^{\frac{1}{3}},$$

therefore, the term of order $\frac{1}{3}$ is

$$\sqrt[3]{1 \cdot 2 \cdot \frac{5}{3} \int dx(x - xx)^{\frac{1}{3}} \int dx(xx - x^3)^{\frac{1}{3}}} :$$

similarly the term of order $\frac{2}{3}$ is

$$\sqrt[3]{1 \cdot 2 \cdot 3 \cdot \frac{7}{3} \int dx(x - xx)^{\frac{2}{3}} \int dx(xx - x^3)^{\frac{2}{3}}}.$$

And the term of order $\frac{4}{3}$ is

$$\sqrt[3]{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \frac{11}{3} \int dx(x - xx)^{\frac{4}{3}} \int dx(xx - x^3)^{\frac{4}{3}}}$$

and in general the term of $\frac{p}{3}$ is

$$\sqrt[3]{1 \cdot 2 \cdots p \cdot \frac{2p+3}{3} \cdot (p+1) \int dx(x - xx)^{\frac{p}{3}} \int dx(xx - x^3)^{\frac{p}{3}}}.$$

§23 If we want to proceed further by putting $f = 3n$, it will be necessary to multiply the general term

$$(4n+1) \int dx(x^3 - x^4)^n$$

by the preceding ones, whence one has

$$(2n+1)(3n+1)(4n+1) \int dx(x - xx)^n \int dx(x^2 - x^3)^n,$$

which is the general term for this series

$$\frac{1 \cdot 1 \cdot 1 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}, \quad \frac{1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \quad \text{etc.}$$

From this the terms of the progression 1, 2, 6, 24 etc. will be defined, whose indices are fractions having the denominator 4. For, the term, whose index is $\frac{p}{4}$, will be found

$$= sqrt[4] 1 \cdot 2 \cdot 3 \cdots p \left(\frac{2p}{4} + 1\right) \left(\frac{3p}{4} + 1\right) (p+1)$$
$$\times \int dx(x - xx)^{\frac{p}{4}} \int dx(xx - x^3)^{\frac{p}{4}} \int dx(x^3 - x^4)^{\frac{p}{4}}.$$

Hence it is possible to conclude in general that the term of order $\frac{p}{q}$ is

$$= \sqrt[q]{1 \cdot 2 \cdot 3 \cdots p \left(\frac{2p}{q}+1\right)\left(\frac{3p}{q}+1\right)\left(\frac{4p}{q}+1\right) \cdots (p+1)}$$
$$\times \int dx(x-xx)^{\frac{p}{q}} \int dx(x^2-x^3)^{\frac{p}{q}} \int dx(x^3-x^4)^{\frac{p}{q}} \cdots \int dx(x^{q-1}-x^q)^{\frac{p}{q}}.$$

Therefore, from this formula the terms of any arbitrary fractional index are found by means of the quadarture of algebraic curves; but for this $1 \cdot 2 \cdot 3 \cdots p$ is required, the term, whose index is the numerator of the propounded fraction.

§24 In the same way it is possible to proceed further to higher composited progressions by assuming higher composited numbers, but I will not persecute this any further here. Integral sings can also be multiplied, that the general term is

$$\int qdx \int pdx;$$

for, the integral of pdx must be multiplied by the qdx and what results must be integrated again, which just then having put $x = 1$ will give the term of the series. But in each of the two integrations, that it is determined, one has to cause by a constant to be added that having put $x = 0$ the integral likewise becomes $= 0$.

In similar manner general terms are to be treated, which are contained in several integral signs, as

$$\int rdx \int qdx \int pdx$$

But nevertheless instead of p, q, r always functions of such a kind are to be taken, that, as often as n was a positive integer, at least algebraic terms arise.

§25 Let the general term be

$$\int \frac{dx}{x} \int x^e dx(1-x)^n,$$

this converted into a series gives

$$\frac{x^{e+1}}{(e+1)^2} - \frac{nx^{e+2}}{1 \cdot (e+2)^2} + \frac{n(n-1)x^{e+3}}{1 \cdot 2 \cdot (e+3)^2} - \text{etc.}$$

Having put $x = 1$ one will have the term of order n expressed by this series

$$\frac{1}{(e+1)^2} - \frac{n}{1 \cdot (e+2)^2} + \frac{n(n-1)}{1 \cdot 2 \cdot (e+3)^2} - \text{etc.}$$

The progression itself will be this beginning from the term, whose index is 0,

$$\frac{1}{(e+1)^2}, \quad \frac{(e+2)^2 - (e+1)^2}{(e+2)^2(e+1)^2}, \quad \frac{(e+3)^2(e+2)^2 - 2(e+3)^2(e+1)^2 + (e+2)^2(e+1)^2}{(e+3)^2(e+2)^2(e+1)^2}$$

$$\frac{(e+4)^2(e+3)^2(e+2)^2 - 3(e+4)^2(e+3)^2(e+1)^2 + 3(e+4)^2(e+2)^2(e+1)^2 - (e+3)^2(e+2)^2(e+1)^2}{(e+4)^2(e+3)^2(e+2)^2(e+1)^2}$$

etc.

The law of this progression is manifest and does not require any explanation. Let $e = 0$; it will be

$$\int dx(1-x)^n = \frac{1-(1-x)^{n+1}}{n+1};$$

the general term therefore is

$$\int \frac{dx - dx(1-x)^{n+1}}{(n+1)x},$$

the progression on the other hand will be this one

$$\frac{1}{1}, \quad \frac{4-1}{4 \cdot 1}, \quad \frac{9 \cdot 4 - 2 \cdot 9 \cdot 1 + 4 \cdot 1}{9 \cdot 4 \cdot 1}, \quad \frac{16 \cdot 9 \cdot 4 - 3 \cdot 16 \cdot 9 \cdot 1 + 3 \cdot 16 \cdot 4 \cdot 1 - 9 \cdot 4 \cdot 1}{16 \cdot 9 \cdot 4 \cdot 1}.$$

The difference will constitute this progression

$$\frac{-1}{4 \cdot 1}, \quad \frac{-9+4}{9 \cdot 4 \cdot 1}, \quad \frac{-16 \cdot 9 + 2 \cdot 16 \cdot 4 - 9 \cdot 4}{16 \cdot 9 \cdot 4 \cdot 1} \text{etc.}$$

§26 Therefore, in this dissertation I achieved this, what I mainly intended, namely, that I find the general terms of all progressions, whose single terms are products of factors proceeding in an arithmetic progression, and in which the number of factors depends on the index in an arbitrary manner. But although here the number of factors is always put equal to the index, if it is desired to depend on it in another way, this will not cause any difficulties. The index is denoted by the letter n; if now anyone would require that the number of factors is $\frac{nn+n}{2}$, no other operation is necessary, besides that everywhere for n one substitutes $\frac{nn+n}{2}$.

§27 Instead of ending the dissertation here I want to add something more curious than useful. It is known that by $d^n x$ the differential of order n of x is understood and $d^n p$, if p denotes a certain function of x and dx is put constant, is homogeneous to dx^n; but always, whenever n is an positive integer number, the ratio, which $d^n p$ has to $d^n x$, can be expressed algebraically; as if $n = 2$ and $p = x^3$, $d^2(x^3)$ will be to dx^2 as $6x$ to 1. Now it is in question, if n is a fractional number, of what kind the ratio will be then. The difficulty is easily understood in these cases; for, if n is an positive integer number, d^n is found by continued differentiation; but there is no such way, if n is a fractional number. But nevertheless by means of interpolations of the progressions, about which I talked in this dissertation, it will be possible to handle this.

§28 Let the ratio between $d^n(z^e)$ and dz^n to be found for constant dz, or the value of the fraction $\frac{d^n(z^e)}{dz^n}$ is required. First, let us see, what are its values, if n is an integer number, that afterwards the transfer can be done in general. If $n = 1$, its value will be

$$ez^{e-1} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-1)} z^{e-1};$$

I express e this way, that later the things, which were given, can be transferred more easily to this.
If $n = 2$, the value will be

$$e(e-1)z^{e-2} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-2)} z^{e-2}.$$

If $n = 3$ one will have

$$e(e-1)(e-2)z^{e-3} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-3)} z^{e-3}.$$

Hence, I generally infer, whatever n is, that it will always be

$$\frac{d^n(z^e)}{dz^n} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-n)} z^{e-n}.$$

But by means of § 14 it is

$$1 \cdot 2 \cdot 3 \cdots e = \int dx(-\ln(x))^e \text{ und } 1 \cdot 2 \cdot 3 \cdots (e-n) = \int dx(-\ln(x))^{e-n}.$$

Hence one has

$$\frac{d^n(z^e)}{dz^n} = z^{e-n} \frac{\int dx(-\ln(x))^e}{\int dx(-\ln(x))^{e-n}}$$

or

$$d^n(z^e) = z^{e-n} dz^n \frac{\int dx(-\ln(x))^e}{\int dx(-\ln(x))^{e-n}}$$

Here dz is put constant and $\int dx(-\ln(x))^s$ and $\int dx(-\ln(x))^{s-n}$ must be integrated in such a way as it was prescribed above, and then one has to put $x = 1$.

§28 It is not necessary to show, how the true value is found; this will become clear by putting an arbitrary integer number instead of n. But let it be in question, what $d^{\frac{1}{2}}z$ is, whence dz is constant. Therefore, it will be $e = 1$ and $n = \frac{1}{2}$. Therefore, one will have

$$d^{\frac{1}{2}}z = \frac{\int dx(-\ln(x))}{\int dx\sqrt{-\ln(x)}}\sqrt{zdz}.$$

But it is

$$\int dx(-\ln(x)) = 1$$

and having called the area of the circle, whose diameter is 1, A, it will be

$$\int dx\sqrt{-\ln(x)} = \sqrt{A},$$

whence it is

$$d^{\frac{1}{2}}z = \sqrt{\frac{zdz}{A}}.$$

Therefore, let this equation for a certain curve be propounded

$$yd^{\frac{1}{2}}z = z\sqrt{dy},$$

where dz is put constant, and it shall be in question, what kind of curve it is. Since it is $d^{\frac{1}{2}}z = \sqrt{\frac{zdz}{A}}$, the equation will go over into this one

$$y\sqrt{\frac{zdz}{A}} = z\sqrt{dz},$$

which squared gives

$$\frac{yydz}{A} = zdy,$$

whence one finds

$$\frac{1}{A}\ln(z) = C - \frac{1}{y}$$

or

$$y\ln(z) = cAy - A,$$

which is the equation for the curve in question.