Consideration of certain series having singular properties*

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§1 Many times the consideration of series, which we quasi discover accidentally, yields artifices not to be contemned, which afterwards can be used in the whole doctrine of series with greatest earnings. Therefore, since the doctrine of series is of greatest importance in Analysis, speculations of this kind are to be considered as completely worth one's while that they are evolved with all eagerness. For this purpose, I decided to consider the following series, which both because of the singular properties, which it is detected to have, and on the other hand for the extraordinary uses, which it exhibits for us, seems to be worth one's complete attention. But the series behaves this way

$$\frac{1-x}{1-a}+\frac{(1-x)(a-x)}{a-a^3}+\frac{(1-x)(a-x)(a^2-x)}{a^3-a^6}+\frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}}+\text{etc.}$$

The law of the numerators is manifest from the inspection alone; for; they are formed from the multiplication of the terms of this series

$$1-x, \quad a-x, \quad a^2-x, \quad a^3-x, \quad a^4-x, \quad a^5-x, \quad a^6-x \quad \text{etc.}$$

The denominators on the other hand all consist of two terms, which are powers of a, whose exponents are triangular numbers. Hence the term of order n of the propounded series will be

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$$\frac{(1-x)(a-x)(a^2-x)(a^3-x)\cdot(a^{n-1}-x)}{a^{n(n-1):2}-a^{n(n+1):2}}.$$

§2 At first it is certainly plain, if the quantity x is taken equal to a certain power of a, that then the series will terminate at a certain point, that all following terms go over into zero. Therefore, let us in general put s for the sum of the propounded series that it is

$$s=\frac{1-x}{1-a}+\frac{(1-x)(a-x)}{a-a^3}+\frac{(1-x)(a-x)(a^2-x)}{a^3-a^6}+\frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}}+\text{etc.},$$

and first set $x=1$ or $x=a^0$ and because of all the vanishing terms it will be $s=0$. Further, let $x=a$ that only the first term remains, and it will be $s=1$. Let $x=a^2$ and it will be

$$s=\frac{1-a^2}{1-a}+\frac{(1-a^2)(a-a^2)}{a-a^3}$$

or $s=2$. Put $x=a^3$ and it will arise

$$s=\frac{1-a^3}{1-a}+\frac{(1-a^3)(a-a^3)}{a-a^3}+\frac{(1-a^3)(a-a^3)(a^2-a^3)}{a^3-a^6}.$$

The first of these terms gives $1+a+aa$, the second gives $1-a^3$ and the third $1-a-aa+a^3$, having collected which it will be $s=3$.

§3 In similar manner, if one puts $x=a^4$, having done the calculation one will find $s=4$ and having put $x=a^5$ $s=5$ will arise. Hence it seems that it can be concluded by induction safely enough that, as often as x is set equal to a certain power of a, whose exponent is $=n$, that so often this exponent n itself will yield the value of s. For, if it would hold for each fractional number, then s would be equal to the logarithm of x having taken a for the number, whose logarithm is $=1$. So, if this was true, having put $a=10$ the sum of the series s would always have to express the common logarithm of x and it would be

$$\begin{aligned}s=&-\frac{(1-x)}{9}-\frac{(1-x)(10-x)}{990}-\frac{(1-x)(10-x)(100-x)}{999000}\\&-\frac{(1-x)(10-x)(100-x)(1000-x)}{9999000000}-\text{etc.}=\log x.\end{aligned}$$

But from the following it will become perspicuous that this equation can only hold, if x is a power of a having an positive integer power.

§4 But that having put $x = a^n$ it only is $s = n$, if n is a positive integer number, is easily concluded from the case $x = 0$. For, in this case, if the superior induction would extend to completely all powers, it would have to be $s = -\infty$, since $-\infty$ is always the logarithm of zero. But having put $x = 0$ it will be

$$s = \frac{1}{1-a} + \frac{1}{1-aa} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.};$$

even though this series cannot be summed, it will nevertheless easily become perspicuous that its sum must be finite and therefore cannot express the logarithm of $x = 0$. In similar manner, if having put $a = 10$ x is not put equal to a power of 10, by summation the value will be found to differ rather notably from $\log x$. For, let $x = 9$ having put $a = 10$ and it will be

$$s = \frac{8}{9} + \frac{8 \cdot 1}{990} + \frac{8 \cdot 1 \cdot 91}{999000} + \frac{8 \cdot 1 \cdot 91 \cdot 991}{9999000000} + \frac{8 \cdot 1 \cdot 91 \cdot 991 \cdot 9991}{999990000000000} + \text{etc.};$$

if the terms are expressed in decimal fractions, it will arise

$$\begin{aligned} s = {} & 0.888888888889 \\ & 0.008080808081 \\ & 0.000728728729 \\ & 0.000072152015 \\ & 0.000007208059 \\ & 0.000000720735 \\ & 0.000000072073 \\ & 0.000000007207 \\ & 0.000000000721 \\ & 0.000000000007 \\ \hline s = {} & 0.897778586588 \end{aligned}$$

which value certainly is greater than the logarithm of nine.

§5 Therefore, our series is of such a nature that, if for x rational powers of x are substituted, that the sum of the series becomes equal to the exponent of that power; of course, if it is

$$x = a^0, \quad a^1, \quad a^2, \quad a^3, \quad a^4, \quad a^5, \quad a^6, \quad a^7, \quad a^8 \quad \text{etc.},$$

it will be

$$s = \ 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8 \quad \text{etc.};$$

even though this is the property of logarithms, it does nevertheless only hold, if the exponents of a are integer numbers. Therefore, if one imagines a curved line, whose abscissas are $= s$ and ordinates $= x$, the curve will intersect the logarithmic curve in innumerable points; of course, as often as the abscissa s is expressed by means of an integer number, so often the ordinate will pass through an intersection. Hence it is plain that the logarithmic curve in not even determined by infinitely many curves, what also happens in all other curved lines. Therefore, hence it is understood that any arbitrary series, even though its terms corresponding to integer coefficients are given, can be interpolated in infinitely many different ways, which subject I will treat further on another occasion.

§6 But to get closer to a cognition of our series, it is possible to transform it into this form

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)$$
$$+\frac{1}{1-a^4}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\left(1-\frac{x}{a^3}\right) + \text{etc.},$$

which therefore is simpler than the preceding one, since here the triangular numbers went out. Now let us put ax instead of x and let t denote the sum of the series resulting from this; it will be

$$t = \frac{1}{1-a}(1-ax) + \frac{1}{1-a^2}(1-ax)(1-x) + \frac{1}{1-a^3}(1-ax)(1-x)\left(1-\frac{x}{a}\right)$$
$$+\frac{1}{1-a^4}(1-ax)(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) + \text{etc.}$$

subtract the first series from the second and one will find

$$t - s = x + \frac{x}{a}(1-x) + \frac{x}{aa}(1-x)\left(1-\frac{x}{a}\right) + \frac{x}{a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) + \text{etc.};$$

subtract this series from unity, and since the residue is divisible by $1-x$, it will be

$$1+s-t=(1-x)\left(1-\frac{x}{a}-\frac{x}{aa}\left(1-\frac{x}{a}\right)-\frac{x}{a^3}\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)-\text{etc.}\right).$$

This last factor is further divisible by $1-\frac{x}{a}$, whence it is

$$1+s-t=(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}-\frac{x}{a^3}\left(1-\frac{x}{aa}\right)-\text{etc.}\right).$$

Here, again the factor $1-\frac{x}{aa}$ is detected and having expressed this separately the factor $1-\frac{x}{a^3}$ will appear and so forth, whence it is finally found that it will be

$$1+s-t=(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right)\left(1-\frac{x}{a^5}\right)\quad\text{etc.}$$

§7 Therefore, hence it is plain, as often as x is taken equal to a certain power of a, that because of the one vanishing factor of this expression it will be

$$1+s-t=0\quad\text{and}\quad t=1+s.$$

Hence, if having put $x=a^n$ while n denotes a positive integer number the sum of the propounded series was $s=n$, having put $x=a^{n+1}$ the sum of the series will be

$$t=s+1=n+1.$$

Therefore, having taken $n=0$ or $x=1$ the sum of the series is $s=0$, having put $x=a$ the sum of the series will be $s=1$; and hence it further follows, if one puts $x=a^2$, that it will be $s=2$, and if $x=a^3$, that it will be $s=3$. And now it is plain in general, what we found by induction alone before, if it was $x=a^n$, while n denotes a positive integer number, that it will always be $s=n$. But if n is not a positive integer and s denotes the sum of the series propounded initially having put $x=a^n$, then having put $x=a^{n+1}$ the sum of the series, which shall be $=t$, will not be $=s+1$; for, it will be

$$t=1+s-(1-a^n)(1-a^{n-1})(1-a^{n-2})(1-a^{n-3})(1-a^{n-4})\quad\text{etc.}$$

Therefore, in these cases the value of the series manifestly recedes from the nature of logarithms.

§8 As here by multiplying the values of x by a from the value of s we found the value of t, so vice versa by dividing the values of x by a from the value of t we will obtain the value of s; and hence we will be able to descend to negative values of the exponent n. Of course, in the series propounded at the beginning or reduced to this form

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right) + \text{etc.}$$

for the following cases let us indicate the sum of the series this way:

$$\begin{array}{llll} \text{if} & x = 1, & \text{it is} & s = A = 0, \\ & x = \frac{1}{a}, & & s = B, \\ & x = \frac{1}{a^2}, & & s = C, \\ & x = \frac{1}{a^3}, & & s = D, \\ & x = \frac{1}{a^4}, & & s = E, \\ & & \text{etc.} & \end{array}$$

If one now puts $x = \frac{1}{a}$, it will be $s = B$ and $t = A = 0$, since t arises from s, if instead of x one writes ax; from the preceding it arises

$$1 + B = \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right) \quad \text{etc.}$$

or

$$B = -1 + \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right) \quad \text{etc.};$$

so, if it is $a = 10$, it will be

$$B = -0.109989900001001.$$

§9 Let $x = \frac{1}{a^2}$ and it will be $s = C$ and $t = B$, whence one will have

$$1 + C - B = \left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \quad \text{etc.};$$

to this add the first $1 + B$ and it will be

$$2 + C = \left(2 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \quad \text{etc.}$$

and

$$C = -2 + \left(2 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right) \quad \text{etc.}$$

Or having eliminated the series it will be

$$1 + B = \left(1 - \frac{1}{a}\right)(1 + C + -B);$$

or

$$C - 2B = \frac{1}{a}(1 + C - B).$$

In similar manner, if one puts $x = \frac{1}{a^3}$, it will be $s = D$ and $t = C$, whence it will be

$$1 + D - C = \left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right)\left(1 - \frac{1}{a^6}\right) \quad \text{etc.},$$

the first series added to which will yield

$$3 + D = \left(3 - \frac{1}{a} - \frac{2}{a^2} + \frac{1}{a^3}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right)\left(1 - \frac{1}{a^6}\right) \quad \text{etc.}$$

And since having put $x = \frac{1}{a^4}$ it is

$$1 + E - D = \left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right)\left(1 - \frac{1}{a^6}\right)\left(1 - \frac{1}{a^7}\right) \quad \text{etc.},$$

it will be

$$4 + E = \left(4 - \frac{1}{a} - \frac{2}{a^2} - \frac{2}{a^3} + \frac{1}{a^4} + \frac{2}{a^5} - \frac{1}{a^6}\right)\left(1 - \frac{1}{a^4}\right)\left(1 - \frac{1}{a^5}\right)\left(1 - \frac{1}{a^6}\right) \quad \text{etc.};$$

and so it is possible to proceed arbitrarily far.

§10 But between three values of the sum of the series s one can exhibit a relation for the successive values of x by means of a finite expression. For, while for the value x the sum is still $= s$, if instead of x one puts ax, let the sum of the series be $= t$, and if instead of x one puts aax, let the sum of the series be $= u$. Therefore, since between s and t we found this relation

$$1+s-t=(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right) \quad \text{etc.},$$

if here for x we write ax, a similar relation between u and t will arise

$$1+t-u=(1-ax)(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right) \quad \text{etc.}$$

Therefore, hence it will be

$$1+t-u=(1-ax)(1+s-t)$$

or

$$u=2t-s+ax(1+s-t)$$

or

$$s=\frac{2t-u+ax(1-t)}{1-ax}.$$

And hence for the values A, B, C, D etc. assumed above the following relations will arise.

If it is $x=\frac{1}{a^2}$, it will be

$$A=2B-C+\frac{1}{a}(1+C-B)$$

or

$$C=\frac{1+(2a-1)B-aA}{a-1}=B+\frac{1+a(B-A)}{a-1};$$

if it is $x=\frac{1}{a^3}$, it will be

$$D=C+\frac{1+a^2(C-B)}{a^2-1};$$

if it is $x=\frac{1}{a^4}$, it will be

$$E = D + \frac{1 + a^3(D - C)}{a^3 - 1};$$

if it is $x = \frac{1}{a^5}$, it will be

$$F = E + \frac{1 + a^4(E - D)}{a^4 - 1}$$

etc.

But these relations can be expressed more conveniently in the following way:

$$\begin{aligned}
C &= 2B - A + \frac{1 + B - A}{a - 1},\\
D &= 2C - B + \frac{1 + C - B}{a^2 - 1},\\
E &= 2D - C + \frac{1 + D - C}{a^3 - 1},\\
F &= 2E - D + \frac{1 + E - D}{a^4 - 1}
\end{aligned}$$

etc.

Therefore, since it is $A = 0$, if only the value of the letter B was found

$$B = -1 + \left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{a^2}\right)\left(1 - \frac{1}{a^3}\right)\left(1 - \frac{1}{a^4}\right) \quad \text{etc.},$$

hence the values of all following letters C, D, E, F etc. can be assigned exactly.

§11 But since while n is an positive integer number, if one puts $x = a^n$, it is $s = n$, from our assumed series we will obtain this summable one

$$n = \frac{1 - a^n}{1 - a} + \frac{(1 - a^n)(1 - a^{n-1}}{1 - a^2} + \frac{(1 - a^n)(1 - a^{n-1})(1 - a^{n-2}}{1 - a^3} + \text{etc.}$$

But then in the case, since it is $t = n + 1$, it will be

$$1 = a^n + a^{n-1}(1 - a^n) + a^{n-2}(1 - a^n)(1 - a^{n-1}) + a^{n-3}(1 - a^n)(1 - a^{n-1})(1 - a^{n-2}) + \text{etc.},$$

whose truth having brought all terms to he same side is manifest; for, it will be

$$(1-a^n)(1-a^{n-1})(1-a^{n-2})(1-a^{n-3})(1-a^{n-4}) \text{ etc.} = 0.$$

Hence this provides us with an opportunity to consider forms of this kind more generally. For, let

$$A, \quad B, \quad C, \quad D, \quad E, \quad F \quad \text{etc.}$$

a series of certain quantity and let

$$(1-A)(1-B)(1-C)(1-D)(1-E)\text{etc.} = S.$$

And hence it will be obtained

$$1-A-B(1-A)-C(1-A)(1-B)-D(1-A)(1-B)(1-C)-\text{etc.} = S;$$

for, this formula will most easily be reduced to that one. Therefore, we will have

$$A+B(1-A)+C(1-A)(1-B)+B(1-A)(1-B)(1-C)+\text{etc.} = 1-S.$$

§12 Therefore, if a certain of these quantities A, B, C etc. becomes equal to unity, it will be $S=0$ and a series will arise, whose sum is $=1$. For the sake of an example take this series

$$\begin{array}{ccccccc} A, & B, & C, & D, & E, & F & \text{etc.} \\ \frac{1}{2}, & \frac{2}{3}, & \frac{3}{4}, & \frac{4}{5}, & \frac{5}{6}, & \frac{6}{7} & \text{etc.}; \end{array}$$

since the infinitesimal of these fractions is $=1$, it will be $S=0$ and the following series will arise

$$1 = \frac{1}{2} + \frac{2}{2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{4}{2\cdot 3\cdot 4\cdot 5} + \frac{5}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.},$$

whose truth is certainly easily seen; for, it arises this way; let

$$z = 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{etc.};$$

it will be

$$z - 1 = \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

and hence by subtraction it arises

$$1 = \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}.$$

§13 Let

$$A = \frac{1}{9}, \quad B = \frac{1}{25}, \quad C = \frac{1}{49}, \quad D = \frac{1}{81} \quad \text{etc.};$$

it will be

$$S = \frac{8}{9} \cdot \frac{24}{25} \cdot \frac{48}{49} \cdot \frac{80}{81} \cdot \frac{120}{121} \cdot \text{etc.} = \frac{\pi}{4}$$

while π denotes the circumference of the circle whose diameter is $= 1$. Therefore, hence this series for the quadrature of the circle will arise:

$$-\frac{\pi}{4} + 1 = \frac{1}{9} + \frac{8}{9 \cdot 25} + \frac{8 \cdot 24}{9 \cdot 25 \cdot 49} + \frac{8 \cdot 24 \cdot 48}{9 \cdot 25 \cdot 49 \cdot 81} + \text{etc.}$$

or

$$-\frac{9}{4}\pi + 8 = \frac{2 \cdot 4}{5 \cdot 5} + \frac{2 \cdot 4 \cdot 4 \cdot 6}{5 \cdot 5 \cdot 7 \cdot 7} + \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}{5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9} + \text{etc.}$$

Therefore, because one has innumerable products of this kind, whose value S can be exhibited, from each one this way an infinite series, whose sum can be assigned, will be derived. Therefore, a very broad field to find arbitrarily many summable series is opened.

§14 But I return to the series mentioned initially

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1 - \frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a^2}\right) + \text{etc.},$$

which I want to transform into another form, in which the terms proceed according to the powers of x. This first could certainly be done by actual expansion of the single terms, but since this way the single coefficients would arise in infinite series, most conveniently the formula found above will be used for this aim

$$u = 2t - s + ax(1 - t + s) \quad \text{or} \quad u - 2t + s = ax + ax(s - t),$$

where t arises from s, if instead of x one puts ax, and in equal manner from t u arises, if instead of x it is put ax again. Hence, if for the series in question we assume

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.},$$

it will be

$$t = A + Bax + Ca^2x^2 + Da^3x^3 + Ea^4x^4 + Fa^5x^5 + \text{etc.}$$

and

$$u = A + Ba^2x + Ca^4x^2 + Da^6x^3 + Ea^8x^4 + Fa^{10}x^5 + \text{etc.}$$

From these one will therefore conclude

$$u - 2t + s = B(1-a)^2x + C(1-aa)^2x^2 + D(1-a^3)^2x^3 + E(1-a^4)^2x^4 + \text{etc.},$$

$$ax(1 + s - t) = ax + Ba(1-a)x^2 + Ca(1-aa)x^3 + Da(1-a^3)x^4 + \text{etc.}$$

From the equality of these series it is concluded that it will be:

$$B = \frac{a}{(1-a)^2}, \quad C = \frac{Ba(1-a)}{(1-aa)^2}, \quad D = \frac{Ca(1-aa)}{(1-a^3)^2}, \quad E = \frac{Da(1-a^3)}{(1-a^4)^2} \quad \text{etc.}$$

§15 Therefore, hence the following values of the assumed coefficients will be obtained:

$$\begin{aligned}
B &= \frac{a}{(1-a)^2},\\
C &= \frac{a^2}{(1-a)(1-aa)^2},\\
D &= \frac{a^3}{(1-a)(1-aa)(1-a^3)^2},\\
E &= \frac{a^4}{(1-a)(1-aa)(1-a^3)(1-a^4)^2},\\
F &= \frac{a^5}{(1-a)(1-aa)(1-a^3)(1-a^4)(1-a^5)^2}\\
&\text{etc.}
\end{aligned}$$

But the first term A is hence not defined. And since A yields the value of s, if one puts $x = 0$, it is perspicuous that it will be

$$A = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.}$$

Therefore, having defined these values the series propounded initially

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right) + \text{etc.}$$

will be transformed into this form

$$s = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.}$$
$$+\frac{ax}{(1-a)^2} + \frac{a^2x^2}{(1-a)(1-aa)^2} + \frac{a^3x^3}{(1-a)(1-aa)(1-a^3)^2} + \frac{a^4x^4}{(1-a)(1-aa)(1-a^3)(1-a^4)^2} + \text{etc.}$$

§16 Therefore, since having put $x = a^n$ while n denotes a positive integer number it is $s = n$, one will have this summation

$$n + \frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.}$$
$$= \frac{a^{n+1}}{(a-1)^2} - \frac{a^{2n+2}}{(a-1)(aa-1)^2} + \frac{a^{3n+3}}{(a-1)(aa-1)(a^3-1)^2} - \frac{a^{4n+4}}{(a-1)(aa-1)(a^3-1)(a^4-1)^2} + \text{etc.}$$

Therefore, if it was $n=0$, it will be

$$\frac{1}{a-1}+\frac{1}{a^2-1}+\frac{1}{a^3-1}+\text{etc.}$$
$$=\frac{a}{(a-1)^2}-\frac{a^2}{(a-1)(aa-1)^2}+\frac{a^3}{(a-1)(a^2-1)(a^3-1)^2}-\text{etc.},$$

and if one puts $n=1$, it will be

$$\frac{1}{a-1}+\frac{1}{a^2-1}+\frac{1}{a^3-1}+\text{etc.}$$
$$=\frac{a^2}{(a-1)^2}-\frac{a^4}{(a-1)(aa-1)^2}+\frac{a^6}{(a-1)(a^2-1)(a^3-1)^2}-\text{etc.}-1.$$

Therefore, in general it will be

$$\frac{1}{a-1}+\frac{1}{a^2-1}+\frac{1}{a^3-1}+\frac{1}{a^4-1}+\text{etc.}$$
$$=\frac{a^{n+1}}{(a-1)^2}-\frac{a^{2n+2}}{(a-1)(a^2-1)^2}+\frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)^2}-\text{etc.}-n$$

while n denotes an arbitrary positive integer.

§17 If instead of n one puts $n-1$, one will have

$$\frac{1}{a-1}+\frac{1}{a^2-1}+\frac{1}{a^3-1}+\frac{1}{a^4-1}+\text{etc.}$$
$$=\frac{a^{n}}{(a-1)^2}-\frac{a^{2n}}{(a-1)(a^2-1)^2}+\frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)^2}-\text{etc.}-n+1;$$

if from this series the superior one is subtracted, it will arise

$$1=\frac{a^n}{a-1}-\frac{a^{2n}}{(a-1)(a^2-1)}+\frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)}-\frac{a^{4n}}{(a-1)(a^2-1)(a^3-1)(a^4-1)}+\text{etc.}$$

Therefore, the sum of this series is equal to the unity, whatever value is attributed to a and whatever positive integer number is substituted for n. But in the case, in which it is $n=1$, this summation is easily seen. For, since it is

$$1=\frac{a}{a-1}-\frac{a^2}{(a-1)(a^2-1)}+\frac{a^3}{(a-1)(a^2-1)(a^3-1)}-\text{etc.},$$

it clearly follows from the consideration of this series

$$z = 1 - \frac{1}{a-1} + \frac{1}{(a-1)(a^2-1)} - \frac{1}{(a-1)(a^2-1)(a^3-1)} + \text{etc.},$$

whence it is

$$1 - z = \frac{1}{a-1} - \frac{1}{(a-1)(a^2-1)} + \frac{1}{(a-1)(a^2-1)(a^3-1)}$$
$$- \frac{1}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.},$$

which added to each other will give

$$1 = \frac{a}{a-1} - \frac{aa}{(a-1)(a^2-1)} + \frac{a^3}{(a-1)(a^2-1)(a^3-1)} - \frac{a^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)} + \text{etc.}$$

§18 But further the truth of this series can be shown for the remaining values of n in the following way. If it was

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.},$$

I say that it will also be

$$1 = \frac{a^{n+1}}{a-1} - \frac{a^{2n+2}}{(a-1)(a^2-1)} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.},$$

Since now by assumption it is

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.},$$

it will also be

$$0 = a^n - \frac{a^{2n}}{a-1} + \frac{a^{3n}}{(a-1)(a^2-1)} - \text{etc.},$$

which series added to each other will give

$$1 = \frac{a^{n+1}}{a-1} - \frac{a^{2n+2}}{(a-1)(a^2-1)} + \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}.$$

Since this series

$$1 = \frac{a^n}{a-1} - \frac{a^{2n}}{(a-1)(a^2-1)} + \frac{a^{3n}}{(a-1)(a^2-1)(a^3-1)} - \text{etc.}$$

was shown to be true in the case $n = 1$, it will also be true in the case $n = 2$ and hence further in the cases $n = 3$, $n = 4$ etc., such that, whatever positive integer number is substituted for n, the sum will always be $= 1$.

§19 Since I ordered the series propounded initially $s = \frac{1}{1-a}(1-x) + \text{etc.}$ according to powers of x by means of the property demonstrated above

$$u - 2t + s = ax + ax(s-t),$$

it will not be out of place to derive the same transformation immediately from the series s itself; for, so we will get to the summation of innumerable new series. Therefore, it will be necessary that the terms of the series s are actually expanded by multiplication; that this can be done more easily, I will consider an arbitrary term

$$\frac{1}{(1-a^m)}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right)$$

Therefore, I will put

$$P = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right)$$

and it will be

$$\log P = \log(1-x) + \log\left(1-\frac{x}{a}\right) + \log\left(1-\frac{x}{a^2}\right) + \cdots + \left(1-\frac{x}{a^{m-1}}\right)$$

and by differentiation it will be

$$\frac{dP}{P} = \frac{-dx}{1-x} - \frac{dx}{a-x} - \frac{dx}{aa-x} - \cdots - \frac{dx}{a^{m-1}-x}$$

or

$$\frac{dP}{P} = -dx \left\{ \begin{array}{cccccccccccccc} & 1 & + & x & + & x^2 & + & x^3 & + & x^4 & + & x^5 & + \text{etc.} \\ + & \frac{1}{a} & + & \frac{x}{a^2} & + & \frac{x^2}{a^3} & + & \frac{x^3}{a^4} & + & \frac{x^4}{a^5} & + & \frac{x^5}{a^6} & + \text{etc.} \\ + & \frac{1}{a^2} & + & \frac{x}{a^4} & + & \frac{x^2}{a^6} & + & \frac{x^3}{a^8} & + & \frac{x^4}{a^{10}} & + & \frac{x^5}{a^{12}} & + \text{etc.} \\ & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\ + & \frac{1}{a^{m-1}} & + & \frac{x}{a^{2m-2}} & + & \frac{x^2}{a^{3m-3}} & + & \frac{x^3}{a^{4m-4}} & + & \frac{x^4}{a^{5m-5}} & + & \frac{x^5}{a^{6m-6}} & + \text{etc.} \end{array} \right\}.$$

Now by summing the single vertical series it will arise

$$dP = -Pdx \left(\frac{a^m - 1}{a^m - a^{m-1}} + \frac{a^{2m}-1}{a^{2m} - a^{2m-2}} x + \frac{a^{3m}-1}{a^{3m} - a^{3m-3}} x^2 + \frac{a^{4m}-1}{a^{4m} - a^{4m-4}} x^3 + \text{etc.} \right).$$

§20 Now assume this series for P

$$P = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}$$

and it will

$$\frac{dP}{dx} = \beta + 2\gamma x + 3\delta x^2 + 4\varepsilon x^3 + 5\zeta x^4 + \text{etc.}$$

Now having done the substitution it will be

$$\begin{aligned} \beta + \frac{a^m-1}{a^m - a^{m-1}} \alpha &= 0, \\ 2\gamma + \frac{a^m-1}{a^m - a^{m-1}} \beta + \frac{a^{2m}-1}{a^{2m} - a^{2m-2}} \alpha &= 0, \\ 3\delta + \frac{a^m-1}{a^m - a^{m-1}} \gamma + \frac{a^{2m}-1}{a^{2m} - a^{2m-2}} \beta + \frac{a^{3m}-1}{a^{3m} - a^{3m-3}} \alpha &= 0 \text{etc.}, \end{aligned}$$

and since having put $x = 0$ it is $P = 1$, it is plain that it is $\alpha = 1$. Therefore, it will be

$$\beta = \frac{-a^m + 1}{a^m - a^{m-1}}$$

and

$$2\gamma - \frac{(a^m-1)^2}{(a^m-a^{m-1})^2} + \frac{a^{2m}-1}{a^{2m}-a^{2m-2}} = 0$$

or

$$2\gamma = \frac{a^m-1}{a^m-a^{m-1}}\left(\frac{a^m-1}{a^m-a^{m-1}} - \frac{a^m+1}{a^m+a^{m-1}}\right) = \frac{2a^m(a^{m-1}-1)(a^m-1)}{(a^m-a^{m-1})(a^{2m}-a^{2m-2})}$$

and hence

$$\gamma = \frac{(a^m-1)(a^{m-1}-1)}{(a^m-a^{m-1})(a^m-a^{m-2})}.$$

In similar manner the remaining coefficients, although not with a lot of work, will be found and will finally be detected to be expressed conveniently enough.

§21 Therefore, that this determination of the coefficients can be done more easily, I will apply the method used here already several times. If course, in the series

$$P = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}$$

instead of x I set $\frac{x}{a}$ and the sum of the resulting series shall be $= Q$, namely

$$Q = \alpha + \frac{\beta x}{a} + \frac{\gamma x^2}{a^2} + \frac{\delta x^3}{a^3} + \frac{\varepsilon x^4}{a^4} + \text{etc.}$$

But since it is

$$P = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right),$$

it will be

$$Q = \left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\cdots\left(1-\frac{x}{a^m}\right)$$

and hence

$$P\left(1-\frac{x}{a^m}\right) = Q(1-x) \quad \text{or} \quad a^mP - Px - a^mQ + a^mQx = 0;$$

here, substitute the series assumed for P and Q and it will be

$$\left.\begin{array}{rcccc} \alpha a^m + & \beta a^m x & + \ \gamma a^m x^2 & + \ \delta a^m x^3 & + \text{etc.} \\ - & \alpha x & - \ \beta x^2 & - \ \gamma x^3 & - \text{etc.} \\ -\alpha a^m - & \beta a^{m-1} x & - \gamma a^{m-2} x^2 & + \delta a^{m-3} x^3 & + \text{etc.} \\ + & \alpha a^m x & + \beta a^{m-1} x^2 & + \gamma a^{m-2} x^3 & + \text{etc.} \end{array}\right\} = 0.$$

From the comparison of the homogeneous terms hence it is found

$$\begin{aligned} \beta &= \frac{-\alpha(a^m-1)}{a^{m-1}(a-1)}, \\ \gamma &= \frac{-\beta(a^{m-1}-1)}{a^{m-2}(aa-1)}, \\ \delta &= \frac{-\gamma(a^{m-1}-1)}{a^{m-3}(a^3-1)}, \\ \varepsilon &= \frac{-\delta(a^{m-3}-1)}{a^{m-4}(a^4-1)}, \\ &\text{etc.} \end{aligned}$$

§22 Therefore, since it is $\alpha = 1$, the coefficients will behave this way:

$$\begin{aligned} \alpha &= 1, \\ \beta &= \frac{-(a^m-1)}{a^{m-1}(a-1)}, \\ \gamma &= \frac{+(a^m-1)(a^{m-1}-1)}{a^{2m-3}(a-1)(aa-1)}, \\ \delta &= \frac{-(a^m-1)(a^{m-1}-1)(a^{m-2}-1)}{a^{3m-6}(a-1)(aa-1)(a^3-1)}, \\ \varepsilon &= \frac{+(a^m-1)(a^{m-1}-1)(a^{m-2}-1)(a^{m-3}-1)}{a^{4m-10}(a-1)(a^2-1)(a^3-1)(a^4-1)} \\ &\text{etc.} \end{aligned}$$

Therefore, the general term of the series s,

$$\frac{1}{1-a^m}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right),$$

expanded will give this progression

$$\frac{1}{1-a^m}-\frac{x}{a^{m-1}(1-a)}+\frac{(1-a^{m-1})x^2}{a^{2m-3}(1-a)(1-a^2)}-\frac{(1-a^{m-1})(1-a^{m-2})x^3}{a^{3m-6}(a-1)(1-a^2)(1-a^3)}+\text{etc.}$$

Therefore, if for m the numbers 1, 2, 3, 4 etc. are successively substituted, the following formulas or terms of the series s will arise:

$$\begin{aligned}
&\text{First Term:} && =\frac{1}{1-a}-\frac{x}{1-a},\\
&\text{Second Term:} && =\frac{1}{1-a^2}-\frac{x}{a(1-a)}+\frac{(1-a)x^2}{a(1-a)(1-a^2)},\\
&\text{Third Term:} && =\frac{1}{1-a^3}-\frac{x}{a^2(1-a)}+\frac{(1-a^2)x^2}{a^3(1-a)(1-a^2)}-\frac{(1-a)(1-a^2)x^3}{a^3(1-a)(1-a^2)(1-a^3)},\\
&\text{Fourth Term:} && =\frac{1}{1-a^4}-\frac{x}{a^3(1-a)}+\frac{(1-a^3)xx}{a^5(1-a)(1-a^2)}-\frac{(1-a^2)(1-a^3)x^3}{a^6(1-a)(1-a^2)(1-a^3)}\\
& && \quad+\frac{(1-a)(1-a^2)(1-a^3)x^4}{a^6(1-a)(1-a^2)(1-a^3)(1-a^4)}\\
& && \quad\text{etc.}
\end{aligned}$$

§23 Therefore, if all these terms are collected into one sum, a large amount of infinite series will arise, which taken at the same time will be equal to the series propounded initially. Of course, because it is

$$s=\frac{1}{1-a}(1-x)+\frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right)+\frac{1}{1-a^3}\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)+\text{etc.},$$

it will be

$$\begin{aligned}
s=&\frac{1}{1-a}+\frac{1}{1-a^2}+\frac{1}{1-a^3}+\frac{1}{1-a^4}+\frac{1}{1-a^5}+\text{etc.}\\
&-\frac{x}{1-a}\left(1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\text{etc.}\right)\\
&+\frac{x^2}{a(1-a)(1-a^2)}\left(\frac{1-a}{1}+\frac{1-a^2}{a^2}+\frac{1-a^3}{a^4}+\frac{1-a^4}{a^6}+\text{etc.}\right)
\end{aligned}$$

$$-\frac{x^3}{a^3(1-a)(1-a^2)(1-a^3)}\left(\frac{(1-a)(1-a^2)}{1}+\frac{(1-a^2)(1-a^3)}{a^3}+\frac{(1-a^3)(1-a^4)}{a^6}+\text{etc.}\right)$$
$$+\frac{x^4}{a^6(1-a)(1-a^2)(1-a^3)(1-a^4)}\left(\frac{(1-a)(1-a^2)(1-a^3)}{1}+\frac{(1-a^2)(1-a^3)(1-a^4)}{a^4}+\text{etc.}\right)$$
$$\text{etc.}$$

Therefore, since this series must agree with the one found before, from the agreement of these single series the sums will be found

$$\begin{aligned}
&1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\text{etc.} &&=\frac{-a}{1-a},\\
&\frac{1-a}{1}+\frac{1-a^2}{a^2}+\frac{1-a^3}{a^4}+\frac{1-a^4}{a^6}+\text{etc.} &&=\frac{+a^3}{1-aa},\\
&\frac{(1-a)(1-a^2)}{1}+\frac{(1-a^2)(1-a^3)}{a^3}+\frac{(1-a^3)(1-a^4)}{a^6}+\text{etc.} &&=\frac{-a^6}{1-a^3},\\
&\frac{(1-a)(1-a^2)(1-a^3)}{1}+\frac{(1-a^2)(1-a^3)(1-a^4)}{a^4}+\text{etc.} &&=\frac{+a^{10}}{1-a^4},\\
&\frac{(1-a)(1-a^2)(1-a^3)(1-a^4)}{1}+\frac{(1-a^2)(1-a^3)(1-a^4)(1-a^5)}{a^5}+\text{etc.} &&=\frac{-a^{15}}{1-a^5}\\
& &&\text{etc.}
\end{aligned}$$

§24 These series can be cast in the following forms, from which the law of the progression will be seen more clearly:

$$\begin{aligned}
\frac{a}{a-1} &= 1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\text{etc.}\\
\frac{a^2}{a^2-1} &= \left(1-\frac{1}{a}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)+\frac{1}{a^2}\left(1-\frac{1}{a^3}\right)+\frac{1}{a^3}\left(1-\frac{1}{a^4}\right)+\frac{1}{a^4}\left(1-\frac{1}{a^5}\right)+\text{etc.}\\
\frac{a^3}{a^3-1} &= \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)+\frac{1}{a^2}\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)+\text{etc.},\\
\frac{a^4}{a^4-1} &= \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)+\text{etc.},\\
\frac{a^5}{a^5-1} &= \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right)+\text{etc.}\\
&\text{etc.}
\end{aligned}$$

Hence it is concluded that it will be in general

$$\frac{a^{m+1}}{a^{m+1}-1} = \frac{1}{1-\frac{1}{a^{m+1}}} = \left(1-\frac{1}{a}\right)\left(1-\frac{1}{a}^2\right)\cdots\left(1-\frac{1}{a^m}\right)$$
$$+\frac{1}{a}\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{a^3}\right)\cdots\left(1-\frac{1}{a^{m+1}}\right)+\frac{1}{a^2}\left(1-\frac{1}{a^3}\right)\left(1-\frac{1}{a^4}\right)\cdots\left(1-\frac{1}{a^{m+2}}\right)$$
$$+\frac{1}{a^3}\left(1-\frac{1}{a^4}\right)\left(1-\frac{1}{a^5}\right)\cdots\left(1-\frac{1}{a^{m+3}}\right)+\text{etc.}$$

§25 The sum of this series can also be investigated in this way. For the sake of brevity let $\frac{1}{a}=b$ and put the sum in question

$$z=(1-b)(1-b^2)\cdots(1-b^m)+b(1-b^2)(1-b^3)\cdots(1-b^{m+1})$$
$$+b^2(1-b^3)(1-b^4)\cdots(1-b^{m+2})+b^3(1-b^4)(1-b^5)\cdots(1-b^{m+3})+\text{etc.}$$

Multiply by $1-b^{m+1}$ on both sides and it will arise

$$(1-b^{m+1})z=(1-b)(1-b^2)\cdots(1-b^m)(1-b^{m+1})+(1-b^2)(1-b^3)\cdots(1-b^{m+1})(b-b^{m+2})$$
$$+(1-b^3)(1-b^4)\cdots(1-b^{m+2})(b^2-b^{m+3})+\text{etc.}$$

But it is

$$\begin{aligned}
b\ \ -b^{m+2} &= 1-b^{m+2}-(1-b),\\
b^2-b^{m+3} &= 1-b^{m+3}-(1-bb),\\
b^3-b^{m+4} &= 1-b^{m+4}-(1-b^3)\\
&\text{etc.};
\end{aligned}$$

these values substituted for the last products will give

$$\begin{aligned}
(1-b^{m+1})z=&(1-b)\ (1-b^2)\cdots(1-b^{m+1})+(1-b^2)(1-b^3)\cdots(1-b^{m+2})\\
&-(1-b)\ (1-b^2)\cdots(1-b^{m+1})-(1-b^2)(1-b^3)\cdots(1-b^{m+2})\\
&+(1-b^3)(1-b^4)\cdots(1-b^{m+3})-(1-b^4)(1-b^5)\cdots(1-b^{m+4})\\
&-(1-b^3)(1-b^4)\cdots(1-b^{m+3})-(1-b^4)(1-b^5)\cdots(1-b^{m+4})\\
&\text{etc.}
\end{aligned}$$

Therefore, since all terms cancel each other, only the last one will remain

$$(1-b^{m+1})z = (1-b^{\infty})(1-b^{\infty+1})\cdots(1-b^{m+\infty}),$$

whence it is plain, if it was $b<1$, this means $a>1$, as we assumed, that it will be $(1-b^{m+1})z=1$ and hence

$$z=\frac{1}{1-b^{m+1}}=\frac{a^{m+1}}{a^{m+1}-1},$$

as we had found.

§26 From the things, which were given in § 21, one easily finds a power series in x, which is equal to this product of infinitely many factors

$$P=(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right)\quad\text{etc.}$$

For, having put

$$P=1-\alpha x+\beta x^2-\gamma x^3+\delta x^4-\varepsilon x^5+\text{etc.}$$

write ax instead of x and the resulting value shall be $=Q$; it will be

$$Q=(1-ax)(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\left(1-\frac{x}{a^3}\right)\quad\text{etc.}=P-axP$$

and

$$Q=1-\alpha ax+\beta a^2x^2-\gamma a^3x^3+\delta a^4x^4-\varepsilon a^5x^5+\text{etc.};$$

but it is

$$axP=ax-\alpha ax^2+\beta ax^3-\gamma ax^4+\delta ax^5-\text{etc.},$$

$$-P=-1+\alpha x-\beta x^2+\gamma x^2+\gamma x^3-\delta x^4+\varepsilon x^5-\text{etc.},$$

whence it is

$$\alpha=\frac{a}{a-1},\quad \beta=\frac{\alpha a}{a^2-1},\quad \gamma=\frac{\beta a}{a^3-1},\quad \delta=\frac{\gamma a}{a^4-1}\quad\text{etc.}$$

Therefore, the infinite product

$$P=(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{aa}\right)\quad\text{etc.}$$

is resolved in this infinite series:

$$P = 1 - \frac{ax}{a-1} + \frac{a^2x^2}{(a-1)(a^2-1)} - \frac{a^3x^3}{(a-1)(a^2-1)(a^3-1)}$$
$$+ \frac{a^4x^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)} - \text{etc.}$$

§27 Therefore, if this product P is put equal to zero, this infinite equation

$$0 = 1 - \frac{ax}{a-1} + \frac{a^2x^2}{(a-1)(a^2-1)} - \frac{a^3x^3}{(a-1)(a^2-1)(a^3-1)} + \text{etc.}$$

will have only real roots in x and the values of x will be equal to the terms of this geometric progression

$$1, \quad a, \quad a^2, \quad a^3, \quad a^4, \quad a^5, \quad a^6, \quad a^7 \quad \text{etc.;}$$

hence, if one puts $x = a^n$ while n denotes an arbitrary positive integer number, it will be

$$0 = 1 - \frac{a^{n+1}}{a-1} + \frac{a^{2n+2}}{(a-1)(a^2-1)} - \frac{a^{3n+3}}{(a-1)(a^2-1)(a^3-1)} + \text{etc.},$$

whose truth was already demonstrated above.

§28 But especially that series is remarkable to which innumerable other were found to be equal (§ 16), which is

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.;}$$

its sum, if $a > 1$, though it is finite and can easily be assigned by approximations, can nevertheless not be expressed by rational or irrational numbers. Therefore, it seems especially worth that the Geometers investigate the nature of that transcendental quantity by which its sum is expressed.

§29 But I will show, how the sum of series of this kind can be found approximately quickly, and I will certainly consider this series in a bit broader sense. Let

$$s = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^3-z} + \frac{1}{a^4-z} + \frac{1}{a^5-z} + \text{etc.}$$

Convert the single terms in geometric series and it will be

$$\begin{aligned} s = & \quad \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^5} + \text{etc.} \\ & + z\left(\frac{1}{a^2} + \frac{1}{a^4} + \frac{1}{a^6} + \frac{1}{a^8} + \frac{1}{a^{10}} + \text{etc.}\right) \\ & + z^2\left(\frac{1}{a^3} + \frac{1}{a^6} + \frac{1}{a^9} + \frac{1}{a^{12}} + \frac{1}{a^{15}} + \text{etc.}\right) \\ & \qquad \text{etc.,} \end{aligned}$$

which series summed again will give

$$s = \frac{1}{a-1} + \frac{z}{aa-1} + \frac{zz}{a^3-1} + \frac{z^3}{a^4-1} + \frac{z^4}{a^5-1} + \text{etc.}$$

Therefore, if it was $z = 1$, these two series reduce to the same and this transformation causes no difference.

§30 To sum this series let us put that n terms of the first form have actually already been summed, whose sum shall be $= A$, such that it is

$$A = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^4-z} + \cdots + \frac{1}{a^n-z}.$$

Therefore, the whole sum in question will be

$$s = A + \frac{1}{a^{n+1}-z} + \frac{1}{a^{n+2}-z} + \frac{1}{a^{n+3}-z} + \frac{1}{a^{n+4}-z} + \text{etc.}$$

Now expand these fractions into geometric series and it will be

$$
\begin{aligned}
s = A + {} & \frac{1}{a^{n+1}} + \frac{1}{a^{n+2}} + \frac{1}{a^{n+3}} + \frac{1}{a^{n+4}} + \text{etc.} \\
& + z\left(\frac{1}{a^{2n+2}} + \frac{1}{a^{2n+4}} + \frac{1}{a^{2n+6}} + \frac{1}{a^{2n+8}} + \text{etc.}\right) \\
& + z^2\left(\frac{1}{a^{3n+3}} + \frac{1}{a^{3n+6}} + \frac{1}{a^{3n+9}} + \frac{1}{a^{3n+12}} + \text{etc.}\right) \\
& \qquad\qquad \text{etc.},
\end{aligned}
$$

which series summed again will give

$$s = A + \frac{1}{a^n(a-1)} + \frac{z}{a^{2n}(aa-1)} + \frac{zz}{a^{3n}(a^3-1)} + \frac{z^3}{a^{4n}(a^4-1)} + \text{etc.},$$

which converges the more quickly than the first the greater the number n was.

§31 Let $a = 2$, that it is

$$s = \frac{1}{2-z} + \frac{1}{4-z} + \frac{1}{8-z} + \frac{1}{16-z} + \text{etc.}$$

Therefore, if it was

$$A = \frac{1}{2-z} + \frac{1}{4-z} + \frac{1}{8-z} + \cdots + \frac{1}{2^n - z},$$

it will be

$$s = A + \frac{1}{1 \cdot 2^n} + \frac{z}{3 \cdot 2^{2n}} + \frac{z^2}{7 \cdot 2^{3n}} + \frac{z^3}{15 \cdot 2^{4n}} + \frac{z^4}{31 \cdot 2^{5n}} + \text{etc.}$$

But let us put $z = 1$, such that the sum of this series is in question

$$s = 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{63} + \text{etc.}$$

For the sake of an example let us actually add the four initial terms that it is $n = 4$; it will be

$$\begin{aligned}
1 &= 1.000000000000000 \\
\frac{1}{3} &= 0.333333333333333 \\
\frac{1}{7} &= 0.142857142857142 \\
\frac{1}{15} &= 0.066666666666666 \\
\hline
A &= 1.542857142857141
\end{aligned}$$

Hence it will be

$$s = A + \frac{1}{16 \cdot 1} + \frac{1}{16^2 \cdot 3} + \frac{1}{16^3 \cdot 7} + \frac{1}{16^4 \cdot 15} + \text{etc.}$$

and these terms in decimal fractions will give

$$\begin{aligned}
& 0.063838009558149 \\
A &= 1.542857142857142 \\
\hline
\text{Therefore } s &= 1.606695152415291
\end{aligned}$$

§32 Furthermore, if the single terms of the series

$$s = \frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \text{etc.}$$

are resolved into geometric series and the equal powers of a are collected, one will find this form

$$s = \frac{1}{a} + \frac{2}{a^2} + \frac{2}{a^3} + \frac{3}{a^4} + \frac{2}{a^5} + \frac{4}{a^6} + \frac{2}{a^7} + \frac{4}{a^8} + \frac{3}{a^9} + \text{etc.},$$

which series has this property that the numerator of each fraction indicates, how many divisors the exponent of a in the denominator has. So the numerator of the fraction $\frac{4}{a^6}$ is $= 4$, since the exponent 6 has the four divisors 1, 2, 3, 6. Hence, if the exponent of a in the denominator is a prime number, the numerator will always be $= 2$; but for non-prime numbers it will be greater than two. Hence it easily becomes clear, if $a = 10$, that it will be

$$s = 0.1223242434262445262644283344628.$$