On Maxima and Minima *

Leonhard Euler

§250 If a function of x was of such a nature that while the values of x increase the function itself continuously increases or decreases, then this function will have no maximum or minimum value. For, whatever value of this function is considered, the following will be larger, the preceding on the other hand will be smaller. A function of this kind is $x^3 + x$, whose values for increasing x increase continuously, but for decreasing x decreases continuously; therefore, this function cannot take on another maximum value, if not the maximum point, this means an infinity, is attributed to x; and in like manner, it will take on the minimum value, if one puts $x = -\infty$. But if the function was not of such a nature, that while x increases it either increases or decreases continuously, it will have a maximum or minimum somewhere else, this means a value of such a kind, which is either greater or smaller than the preceding and following values. So, this function $xx - 2x + 3$ takes on a minimum value, if one puts $x = 1$; for, whatever other value is attributed to x, the function will always have a larger value.

§251 But that the nature of maxima and minima is better understood, let us put that y is a function of x of such a kind, which has a maximum value, if for $x = f$, and it is seen, if x is then assumed to be either greater or smaller than f, that the value of y to result from this will be smaller than the latter, which it takes on for $x = f$. In like manner, if the function y has a minimum value for $x = f$, it is necessary, that, no matter whether x is assumed to be

*Original title: " De Maximis et Minimis", first published as part of the book „*Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, 1755", reprinted in in „*Opera Omnia*: Series 1, Volume 10, pp. 446 - 474 ", Eneström-Number E212, translated by: Alexander Aycock for the „Euler-Kreis Mainz"

larger or smaller than f, always a larger value of y results; and this is the definition of absolute maxima and minima. But furthermore, this function y is also said to have a maximum value for the value $x = f$, if this value was larger than the closest ones, either the following or the preceding, which result, if x is assumed to be a little either larger or smaller than f, even though by substituting other values for x the function y might have larger values. Similarly, the function y is said to have a minimum value for $x = f$, if that value was smaller than those, which it obtains, if the closest larger or smaller values than f are substituted for x. And in this last sense we will use the terms maxima and minima.

§252 But before we show the way to find these maxima and minima, it is convenient to note that this investigation extends only to those functions of x, which we called *uniform* above and which are of such a nature, that for the single values of x they in like manner have one single values. But we called functions *biform* and *multiform* which for single values of x take on two or several values at the same time; examples of this kind of functions are the roots of quadratic equations and higher order equations. Therefore, if y was a biform or multiform function of such a kind of x, then it cannot be said to take on a maximum or minimum value for $x = f$; for, since it has either two or more values at the same time for $x = f$ and the same is true for the preceding and the following contiguous values, one cannot decide whether $x = f$ is a maximum or minimum, if not by coincidence all values of the function y, which correspond to the single values of x, are imaginary except for one; in this case functions of this kind are counted to the class of uniform functions. Therefore, we will at first consider the class of uniform functions in this chapter; but then, in the following chapters, we will show, how maxima and minima of multiform functions must be considered.

§253 Therefore, let y be a uniform function, which hence, no matter which value is substituted for x, always has one single real value, and let x denote the value, which induces the maximal or minimal value to the function y. Therefore, in the first case, no matter whether one substitutes $x + \alpha$ or $x - \alpha$ for x, the value of y will be smaller than for $\alpha = 0$, in the second case on the other hand larger. Therefore, because having put $x + \alpha$ instead of x the function y goes over into

$$y + \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} + \frac{\alpha^3 d^3 y}{6dx^3} + \text{etc.},$$

but having put $x - \alpha$ for x it goes over into

$$x - \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} - \frac{\alpha^3 d^3 y}{6dx^3} + \text{etc.},$$

it is necessary that in the case of a maximum it is

$$y > y + \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} + \frac{\alpha^3 d^3 y}{6dx^3} + \text{etc.}$$

and

$$y > y - \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} - \frac{\alpha^3 d^3 y}{6dx^3} + \text{etc.}$$

But in the case, in which the value of y is a minimum value, it will be

$$y < y + \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} + \frac{\alpha^3 d^3 y}{6dx^3} + \text{etc.}$$
$$y < y - \frac{\alpha dy}{dx} + \frac{\alpha^2 ddy}{2dx^2} - \frac{\alpha^3 d^3 y}{6dx^3} + \text{etc.}$$

§254 Since these equations have to hold, if α denotes a very small quantity, let us assume α to be so small that its higher powers can be omitted, and then so for the case of the maximum as the minimum it has to be $\frac{\alpha dy}{dx} = 0$. For, if $\frac{\alpha dy}{dx}$ was not $= 0$, the value of y could be neither be maximum nor minimum value. Therefore, to investigate maxima or minima one has the general rule, that the differential of propounded y is to be put equal to zero, and that value of x, which renders the function either maximal or minimal, will be a root of that equation. But whether the value of y found this way is a maximum or a minimum value, is not clear at this point; it can even happen, that y is neither a maximum nor a minimum value in this case; for, we only found that in both cases it will be $\frac{dy}{dx} = 0$ and we did not vice versa prove, if $\frac{dy}{dx} = 0$, that also a maximum or minimum value of y results.

§255 Nevertheless, to investigate cases, in which the value of y is either a maximum or minimum value, first all the roots of equation $\frac{dy}{dx} = 0$ are to be found. Having found these, it is to be checked, whether for those values the function y has a maximum or minimum value or none of both is the case. For, we will show that it can happen that there is neither a maximum or minimum, even though it is $\frac{dy}{dx} = 0$.
Let f be one of the values of x, which the function obtains from the equation

$$\frac{dy}{dx} = 0,$$

and substitute this value in the expressions $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. and by this substitution let

$$\frac{ddy}{dx^2} = p, \quad \frac{d^3y}{dx^3} = q, \quad \frac{d^4y}{dx^4} = r \quad \text{etc.}$$

But having put f instead of x let the function y go over into F, and if one puts $f + \alpha$ instead of x, this function will go over into

$$F + \frac{1}{2}\alpha^2 p + \frac{1}{6}\alpha^3 q + \frac{1}{24}\alpha^4 r + \text{etc.};$$

but if one puts $f - \alpha$ instead of x, this expression will result

$$F + \frac{1}{2}\alpha^2 p - \frac{1}{6}\alpha^3 q + \frac{1}{24}\alpha^4 r - \text{etc.};$$

hence it is plain, if p was a positive quantity that both values will be larger than F, at least if α denotes a very small quantity, and therefore the value F, which the function y has for $x = f$, will be a minimum value. But if p is a negative quantity, then the value $x = f$ will induce a maximal value to the function y.

§256 But if it was $p = 0$, then the value of q is to be considered; if it was not $= 0$, the value of y will be neither a maximum nor a minimum value; for, having put $x = f + \alpha$ it will be $F + \frac{1}{6}\alpha^3 q > F$ and having put $x = f - \alpha$ it will be $F - \frac{1}{6}\alpha^3 q < F$. But if it also was $q = 0$, the quantity r is to be considered; if it had a positive value, the value of the function F, which is has for $x = f$, will be a minimum value; but if r has a negative value, F will be a maximum value. But if also r vanishes, the one has to consider the value of the following letter s, and has to argue as it was done for the letter q. If s was not $= 0$, then the value F will be neither a maximum nor a minimum value; but if also $s = 0$, then the following letter t, if it has a positive value, will indicate a minimum; but if it has a negative value, it will indicate a maximum. But if also this letter t vanishes, then one has to proceed further to the next letter and argue as it was done for the preceding ones. And so one can decide for any root of the equation $\frac{dy}{dx} = 0$, whether the function y has a maximum or minimum value or none of both; and this way all maxima and minima which the function y can have will be found.

§257 Therefore, if the equation $\frac{dy}{dx} = 0$ has two equal roots, such that it has the quadratic factor $(x - f)^2$, then at the same time $\frac{ddy}{dx^2}$ will vanish for $x = f$ and it will be $p = 0$, but not $q = 0$. In this case the function y will have neither a maximum nor a minimum value. But if the equation $\frac{dy}{dx} = 0$ has three equal roots or $\frac{dy}{dx}$ has the cubic factor $(x - f)^3$, then having put $x = f$ it will be $\frac{ddy}{dx^2} = 0$ and $\frac{d^3y}{dx^3} = 0$, but not $\frac{d^4y}{dx^4}$. Therefore, if the value of this term was positive, it will indicate a minimum value, if negative, a maximum value. Therefore, the rule explained before reduces to this, that, if the fraction $\frac{dy}{dx}$ had a factor $(x - f)^n$, while n is an odd number, the function y, if in it one puts $x = f$, will have either a maximum or minimum value, but if the exponent n was an even number, then the substitution $x = f$ will produce neither a maximum nor a minimum value.

§258 Further, the invention of a maximum or minimum often is tremendously simplified by the following considerations. In cases in which the function y has a maximum or minimum value, each multiple of it, say ay, if a was a positive quantity, will also be a maximum or minimum value, and in the same way y^3, y^5, y^7 etc. and in general y^n, if n was a positive odd number, since formulas of this kind are of such a nature, that for increasing y they also increase and for decreasing y they decrease. But in these cases, in which y has maximum or minimum value, $-y$, $-ay$, $b - ay$ and in general $b - ay^n$, while n is an odd positive integer, in reversed order will have either minimum or maximum values. In the same way in the cases, in which y has maximum or minimum value, these formulas $\frac{a}{y}$, $\frac{a}{y^3}$, $\frac{a}{y^5}$ etc. and in general $\frac{a}{y^n} \pm b$, while a denotes a positive quantity and n a positive odd number, will in reverse order have a minimum or a maximum value; but if a was a negative quantity, then these formulas will have a maximum value, if y was a maximum value, and a minimum value, if y is a minimum value.

§259 But these rules cannot be transferred to even powers the same way; for, since, if y has a negative value, its even powers y^2, y^4 etc. induce positive values, it can happen, that, if y takes on a minimum value, a negative one of course, that its even powers have maximum values then. Therefore, having taken this into account, we will be able to affirm, if y has a maximum or minimum value, while its value is positive, that then its even powers y^2, y^4 etc. will also have maximum or minimum, but if a negative value of y was a maximum value, that then its square yy will be a maximum value, and otherwise, if a negative value of y was a minimum value, that then y^2, y^4 etc. will have a maximum value. But if the even exponents of y were negative, then the opposite will happen. Furthermore, what we mentioned here on the even and odd exponents, will not only hold for integer numbers, but also for fractional ones, whose denominators are odd numbers; for, the fractions $\frac{1}{3}$, $\frac{5}{3}$, $\frac{7}{3}$, $\frac{1}{5}$, $\frac{3}{5}$ etc. are equivalent to odd numbers in this case, but $\frac{2}{3}$, $\frac{4}{3}$, $\frac{2}{5}$, $\frac{4}{5}$, $\frac{6}{7}$ etc. are equivalent to even numbers.

§260 But if the denominators were even numbers, then, because, if y has a negative value, its powers $y^{\frac{1}{2}}$, $y^{\frac{3}{4}}$ etc. become imaginary, here one can say only the following about them: If a positive value of y was a maximum or a minimum value, then also $y^{\frac{1}{2}}$, $y^{\frac{3}{2}}$, $y^{\frac{1}{4}}$ etc. will also have either a maximum or minimum value, but on the other hand $y^{-\frac{1}{2}}$, $y^{-\frac{3}{2}}$, $y^{-\frac{1}{4}}$ etc. can have minimum or maximum values. But if these irrationalities take on two values at the same, one positive, the other negative, for the negative ones the contrary of that, what we said about the positive ones here, holds. But if a negative value of y is a maximum or minimum value, since all powers if this kind become imaginary, one will not be able to count them to maxima or minima. Therefore, by means of these auxiliary remarks, the investigation of maxima and minima is often simplified and otherwise would be extremely difficult.

§261 Because these things extend only to rational functions, which are the only uniform functions, at first let us expand polynomial functions and find their maxima and minima. Therefore, because functions of this kind are reduced to this form

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc.},$$

at first it is plain that their values cannot be greater than if one sets $x = \infty$; then on the other hand, if $x = -\infty$, the value of these formulas become $= \infty^n$, if n is an even number, but $-\infty^n$, if n is an odd number, which value therefore will be the smallest of all. But furthermore often other maxima and minima are given in the sense we understand those terms, what we will illustrate in the following examples.

EXAMPLE 1

To find the values of x for which the function $(x-a)^n$ takes on maximum or a minimum value.

Having put $(x-a)^n = y$ it will be

$$\frac{dy}{dx} = n(x-a)^{n-1};$$

having put it $= 0$ it will be $x = a$. Therefore, because $\frac{dy}{dx}$ contains the factor $(x-a)^{n-1}$, from § 257 it is understood that y cannot have a maximum or a minimum value, if $n-1$ is not an odd number or n is not even. But since then it is

$$\frac{d^y}{dx^n} = n(n-1)(n-2)\cdots 1,$$

this means a positive number, it follows that the value of y for $x = a$ will turn out to be a minimum value. This is obvious, of course; for, having put $x = a$ it is $y = 0$, and if x is put to be either greater or smaller than a, because of the even number n the function y will be positive, this means greater than zero; but if n was an odd number, then the function $y = (x-a)^n$ can have neither a maximum nor a minimum value. But it is perspicuous that the same holds, if n was a fractional number, no matter whether it is odd or even, $(x-a)^{\frac{\mu}{\nu}}$ will have a minimum value for $x = a$, if μ was an even number and ν was an odd number; but if both were odd, neither a maximum nor a minimum value will exist.

EXAMPLE 2

To find the cases in which the value of this formula $xx + 3x + 2$ has a maximum or a minimum value.

Put $xx + 3x + 2 = y$; it will be

$$\frac{dy}{dx} = 2x + 3, \quad \frac{ddy}{2dx^2} = 1.$$

Therefore, set $2x + 3 = 0$; it will be $x = -\frac{3}{2}$. Whether this case produces a maximum or minimum value, will become known from the value $\frac{ddy}{2dx^2} = 1$; since it is affirmative, whatever x is, it indicates a minimum value. But having put $x = -\frac{3}{2}$, it is $y = -\frac{1}{4}$, and if any other values are attributed to x, the value of y to result from this will always be larger than $-\frac{1}{4}$. From the nature of the formula $xx + 3x + 2$ it is also seen that it has to have a minimum value; for, because it grows to infinity, if one puts $x = +\infty$ or $x = -\infty$, it is necessary, that a certain value of x leads to a smallest quantity of y.

EXAMPLE 3

To find the cases in which this expression $x^3 - axx + bx - c$ takes on the maximum or minimum value.

Having put $y = x^3 - axx + bx - c$ it will be

$$\frac{dy}{dx} = 3xx - 2ax + b \quad \text{and} \quad \frac{ddy}{2dx^2} = 3x - a, \quad \frac{d^3y}{6dx^3} = 1.$$

Therefore, set $\frac{dy}{dx} = 3xx - 2ax + b = 0$; it will be

$$x = \frac{a \pm \sqrt{aa - 3b}}{3},$$

from which it it clear, if it is not $aa > 3b$, that the propounded formula will neither have a maximum nor a minimum value. Therefore, this equation results

$$\frac{ddy}{2dx^2} = \pm\sqrt{aa - 3bb},$$

whence it is understood, if it not $aa = 3b$, that the value $x = \frac{a+\sqrt{aa-3b}}{3}$ renders the formula $y = x^3 - axx + bx - c$ minimal, the other value $x = \frac{a-\sqrt{aa-3b}}{3}$ on the other hand renders it maximal. But how large will these value of y be? Because it is $3xx - 2ax + b = 0$ or $x^3 - \frac{2}{3}axx + \frac{1}{3}bx = 0$, it will be

$$y = -\frac{1}{3}axx + \frac{2}{3}bx - c$$

and because of $\frac{1}{3}axx - \frac{2aa}{9}x + \frac{ab}{9} = 0$ it is

$$y = \frac{2}{9}(3b - aa)x + \frac{ab}{9} - c = -\frac{2a(aa-3b)}{27} - \mp \frac{2(aa-3b)\sqrt{aa-3b}}{27} + \frac{ab}{9} - c$$

or

$$y = -\frac{2a^3}{27} + \frac{ab}{3} - c \mp \frac{2}{27}(aa-3b)^{\frac{3}{2}},$$

where the upper sign holds for the minimum value, but the lower signs for the maximum value.

Therefore, the case remains, in which it is $aa = 3b$; because in it it is $\frac{ddy}{dx^2} = 0$, but the following term $\frac{d^3y}{6dx^3} = 1$ is not $= 0$, it follows that in this case the propounded formula has neither a maximum nor a minimum value.

EXAMPLE 4

To find the cases in which this function of x, $x^4 - 8x^3 + 22x^2 - 24x + 12$ has a maximum or minimum value.

Having put $y = x^4 - 8x^3 + 22x^2 - 24x + 12$ it will be

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24, \quad \frac{ddy}{2dx^2} = 6x^2 - 24x + 22.$$

Now set

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24 = 0 \quad \text{or} \quad x^3 - 6x^2 + 11x - 6 = 0;$$

three real values for x are found, namely

$$\text{I. } x = 1, \quad \text{II. } x = 2, \quad \text{III.,} \quad x = 3.$$

From the first value it is $\frac{ddy}{2dx^2} = 4$ and hence having put $x = 1$ the propounded function has a minimum value. From the second value $x = 2$ it is $\frac{ddy}{2dx^2} = -2$ and hence the propounded function has a maximum value. From the third value $x = 2$ it is $\frac{ddy}{2dx^2} = +4$ and hence the propounded function has a minimum value again.

EXAMPLE 5

Let this function be propounded $y = x^5 - 5x^4 + 5x^3 + 1$; *it is in question in which cases it has a maximum or a minimum value.*

Because it is

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 15xx,$$

form the equation $x^4 - 4x^3 + 3xx = 0$, whose roots are

$$\text{I. and} \quad \text{II. } x = 0, \quad \text{III. } x = 1, \quad \text{IV. } x = 3.$$

Since the first and second root are equal, from them neither a maximum nor a minimum value follows; for, it is $\frac{ddy}{dx^2} = 0$, but $\frac{d^3y}{dx^3}$ does not vanish. But the third root $x = 1$ because of $\frac{ddy}{2dx^2} = 10x^3 - 30x^2 + 15x$ yields $\frac{ddy}{2dx^2} = -5$ and in this case the function takes on a maximum value. From the fourth root $x = 3$ it is $\frac{ddy}{2dx^2} = 45$ and hence the propounded function has minimum value.

EXAMPLE 6

To find the cases in which this formula $y = 10x^6 - 12x^5 + 15x^4 - 20x^3 + 20$ *has a maximum or a minimum value.*

Therefore, it will be

$$\frac{dy}{dx} = 60x^5 - 60x^4 + 60x^3 - 60x^2 \quad \text{and} \quad \frac{ddy}{60dx^4} = 5x^4 - 4x^3 + +3x^2 - 2x.$$

Form the equation $x^5 - x^4 + x^3 - xx = 0$; since it, having resolved it into factors, is $x^2(x-1)(xx+1) = 0$, it has two equal roots $x = 0$ and furthermore the root $x = 1$ and additionally two imaginary ones from $xx + 1 = 0$. Therefore, since the two equal roots $x = 0$ exhibit neither a maximum nor a minimum point, only the root $x = 1$ is to be considered, from which it is $\frac{ddy}{60dx^2} = 2$, whose positive value indicates a minimum value.

§262 Therefore, the determination of maxima and minima depends on the roots of the differential equation $\frac{dy}{dx} = 0$; because its highest power is one degree lower than the highest power in the propounded equation, it is obvious, if in general this function is propounded

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} = y,$$

that its maxima and minima are determined by means of the roots of this equation

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} + (n-3)Cx^{n-4} + \text{etc.} = 0.$$

Let us put that the real roots of this equation ordered according to their magnitude are α, β, γ, δ etc. such that α is the largest, $\beta < \alpha$, $\gamma < \beta$ etc. And first, if these roots are all different, every single one will lead to a maximum or minimum value of the propounded formula y and hence the function y will have as many maxima or minima as the equation $\frac{dy}{dx} = 0$ had real different roots. But if two or more roots were equal to each other, it will be as follows: two equal roots will exhibit neither a maximum nor a minimum value, but three on the other hand will be equivalent to a single one; and in general, if the number of equal roots was an even number, hence no maximum nor minimum results; but if the number is odd, one maximum or minimum results from this.

§263 But which roots produce maxima and which produce minima, can be defined without using the rule given before as follows. Since the function y having put $x = \infty$ also becomes infinite and values of x within the boundaries ∞ and α produce neither a maximum nor a minimum value, it is perspicuous that the values of the function y, if successively all values from ∞ up to α are substituted for x, have to decrease continuously; and hence the value $x = \alpha + \omega$ will lead to a larger value of the function y than the value $x = \alpha$; therefore, because $x = \alpha$ produces a maximum or minimum, it is necessary that in this case the function y takes on a minimum value. Therefore, diminishing x or putting $x = \alpha - \omega$ the value of y will increase again, until it finally is $x = \beta$, which is the second root of the equation $\frac{dy}{dx} = 0$ producing a maximum or minimum; therefore, this second root $x = \beta$ will yield a maximum point and the value $x = \beta - \omega$ will cause the function y to be smaller than for $x = \beta$, until one gets to $x = \gamma$, which as a logical consequence will generate a minimum value again. From this reasoning it is understood that the first, third, fifth etc. root of the equation $\frac{dy}{dx} = 0$ will exhibit minima but the first, second, fourth, sixth etc. exhibit maxima. But at the same time it is hence understood that in the case of two equal roots a maximum and minimum coalesce and so none of both actually occurs.

§264 Therefore, if in the propounded function

$$y = x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc.}$$

the greatest exponent n was an even number, the equation

$$\frac{dy}{dx} = x^{n-1} + (n-1)Ax^{n-2} + \text{etc.} = 0$$

will be of odd degree and will hence have one or three or five or any odd number of real roots. If just one root was real, it will give a minimum point; if three were real, the largest will yield a minimum point, the middle one a maximum point and the smallest a minimum point again; and if five roots were real, the function y will have three minimum values and two maximum values; and so forth.

But if the exponent n was an odd number, the equation $\frac{dy}{dx} = 0$ will have an odd degree and will have either no or two or four or six etc. real roots. In the first case the function y will have neither a maximum nor a minimum value; in the other case, in which two roots are given, the greater one will indicate a minimum value, the smaller a maximum value; but the first (which is the largest) and the third of four roots will produce a minimum value, the second and the fourth on the other hand a maximum value. But no matter how many real roots were there, they will give maxima and minima alternately.

§265 Let us proceed to rational functions which constitute the other kind of uniform functions. Therefore, let

$$y = \frac{P}{Q}$$

where P and Q are any polynomial functions of x; and at first it is certainly clear, if a value of such a kind is attributed to x that it is $Q = 0$, if not at the same time P vanishes, that the function y becomes infinite, what appears to be a maximum value. Nevertheless, this case cannot be considered as a maximum value; for, because the inverse fraction $\frac{P}{Q}$ takes on a minimum value in the same cases, in which the propounded $\frac{P}{Q}$ takes on a maximum value, the fraction $\frac{Q}{P}$ would have to take on a minimum value, if Q vanishes; but this does not always happen, since even smaller values, negative ones of course, could occur. Therefore, this at the same time confirms the rule given before, that maxima and minima must be found from the equation $\frac{dy}{dx} = 0$. Therefore, in the propounded cases it will be

$$\frac{dy}{dx} = \frac{QdP - PdQ}{QQdx}$$

and hence the roots of this equation

$$QdP - PdP = 0$$

will give the a maximum or a minimum values of the function y. And if there is any doubt, whether y takes on a maximum or a minimum value, one has to check the value $\frac{ddy}{dx^2}$; if it was positive, it will indicate a minimum value, but if it was negative a minimum value. If also this value $\frac{ddy}{dx^2}$ vanishes, which happens, if the equation $\frac{dy}{dx} = 0$ has two or more equal roots, just recall that an equal number of equal roots produce neither a maximum nor a minimum value.

Example 1

To find the cases in which the function $\frac{x}{1+xx}$ takes on a maximum or a minimum value.

At first it is certainly clear that this function goes over into zero in the three cases $x = \infty$, $x = 0$ and $x = -\infty$, whence it will have at least either two maximum or two minimum values. To find them put $y = \frac{x}{1+xx}$ and it will be

$$\frac{dy}{dx} = \frac{1 - xx}{(1 + xx)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{-6x + 2x^3}{(1 + xx)^3}.$$

Now set $\frac{dy}{dx} = 0$; it will be $1 - xx = 0$ and either $x = +1$ or $x = -1$. In the first case $x = +1$ it is $\frac{ddy}{dx^2} = -\frac{4}{2^3}$ and hence y has a maximum value $= \frac{1}{2}$; in the second case $x = -1$ it is $\frac{ddy}{dx^2} = +\frac{4}{2^3}$ and hence y has a minimum value $= -\frac{1}{2}$.

These are also found more easily, if the propounded fraction $\frac{x}{1+xx}$ is inverted or by putting $y = \frac{1+xx}{x} = x + \frac{1}{x}$, if we recall that then all values, which were found to be maximum values, are to be turned into minimum values and vice versa. But it will be

$$\frac{dy}{dx} = 1 - \frac{1}{xx} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{2}{x^3}.$$

Therefore, having set $\frac{dy}{dx} = 0$ it is $xx - 1 = 0$ and hence either $x = +1$ or $x = -1$ as before. And in the case $x = +1$ it is $\frac{ddy}{dx^2} = 2$ and hence y has a minimum value and the propounded formula $\frac{1}{y}$ a maximum value. But in the case $x = -1$ it is $\frac{ddy}{dx^2} = -2$, whence y has a maximum value and $\frac{1}{y}$ a minimum value.

EXAMPLE 2

To find the cases in which the formula $\frac{2-3x+xx}{2+3x+xx}$ has a maximum or a minimum value.

Having put $y = \frac{xx-3x+2}{xx+3x+2}$ it will be

$$\frac{dy}{dx} = \frac{6x^2 - 12}{(xx+3x+2)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{-12x^3 + 72x + 72}{(xx+3x+2)^3}.$$

Set $\frac{dy}{dx} = 0$; it will be either $x = +\sqrt{2}$ or $x = -\sqrt{2}$. In the first case it will be

$$\frac{ddy}{dx^2} = \frac{48\sqrt{2} + 72}{(4+3\sqrt{2})^3}$$

and hence affirmative because of the affirmative numerator and denominator; Therefore, y will take on a minimum value

$$= \frac{4 - \sqrt{32}}{4 + \sqrt{32}} = 12\sqrt{2} - 17 = -0.02943725.$$

In the second case $x = -\sqrt{2}$ it is

$$\frac{ddy}{dx^2} = \frac{-48\sqrt{2} + 72}{(4-3\sqrt{2})^3} = \frac{24(3-2\sqrt{2})}{(4-3\sqrt{2})^3},$$

whose value because of the affirmative numerator and negative denominator will be negative and hence y will take on a maximum value

$$= \frac{4+3\sqrt{2}}{4-3\sqrt{2}} = -12\sqrt{2} - 17 = -33.97056274.$$

This value, even though it is smaller than the first minimum value, is nevertheless a maximum value, since it is larger than the values in its domain, which result, if a little bit greater or smaller values than $-\sqrt{2}$ are substituted for x. Therefore, because $\sqrt{2}$ is contained within the limits $\frac{4}{3}$ and $\frac{3}{2}$, the crosscheck will easily be done this way:

$$\begin{array}{llllll}
\text{if} & x = & \frac{4}{3}, & \text{it is} & y = & -\frac{2}{70} = -0.0285 & \\
\text{if} & x = & \sqrt{2}, & \text{it is} & y = & +12\sqrt{2} - 17 = 0.0294 & \text{minimum} \\
\text{if} & x = & \frac{3}{2}, & \text{it is} & y = & -\frac{1}{35} = -0.0285 &
\end{array}$$

$$\begin{array}{llllll}
\text{if} & x = & -\frac{4}{3}, & \text{it is} & y = & -35 & \\
\text{if} & x = & -\sqrt{2}, & \text{it is} & y = & -33.970 & \text{maximum} \\
\text{if} & x = & -\frac{3}{2}, & \text{it is} & y = & -35. &
\end{array}$$

EXAMPLE 3

To find the cases in which the formula $\frac{xx-x+1}{xx+x-1}$ *has a maximum or a minimum value.*

Put $y = \frac{xx-x+1}{xx+x-1}$ and it will be

$$\frac{dy}{dx} = \frac{2xx - 4x}{(xx + x - 1)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{-4x^3 + 12xx + 4}{(xx + x - 1)^3}.$$

Set $\frac{dy}{dx} = 0$; it will be either $x = 0$ or $x = 1$; in the first case it is $\frac{ddy}{dx^2} = \frac{4}{-1}$ and hence y will have a maximum value $= -1$. In the second case it is $\frac{ddy}{dx^2} = \frac{20}{5^2}$ and hence y has the minimum value $= \frac{3}{5}$, even though that maximum value is smaller than this minimum value. That this is indeed the case is seen from the following calculations

$$\begin{array}{llll} \text{if} \quad x = -\frac{1}{3}, & \text{it will be} \quad y = -\frac{13}{11} & \\ \text{if} \quad x = +0, & \text{it will be} \quad y = -1 & \text{maximum} \\ \text{if} \quad x = +\frac{1}{3}, & \text{it will be} \quad y = -\frac{7}{5} & \end{array}$$

$$\begin{array}{llll} \text{if} \quad x = 2-\frac{1}{3}, & \text{it will be} \quad y = -\frac{19}{13} & \\ \text{if} \quad x = 2, & \text{it will be} \quad y = +\frac{3}{5} & \text{minimum} \\ \text{if} \quad x = 2+\frac{1}{3}, & \text{it will be} \quad y = -\frac{37}{61}. & \end{array}$$

But that, if one puts $x = 1$, it is $y = 1$ and hence > -1, is the reason why between the values 0 and 1 there is one value of x, for which it is $y = \infty$.

EXAMPLE 4

To find the the cases in which this fraction $\frac{x^3+x}{x^4-xx+1}$ has a maximum or a minimum value.

Having put $y = \frac{x^3+x}{x^4-xx+1}$ it will be

$$\frac{dy}{dx} = \frac{-x^6-4x^4+4xx+1}{(x^4-xx+1)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{2x^9+18x^7-30x^5-16x^3+12x}{(x^4-xx+1)^3}.$$

Therefore, we will have this equation

$$x^6+4x^4-4xx-1=0,$$

which is resolved into these two

$$xx-1=0 \quad \text{and} \quad x^4+5x^2+1=0;$$

the roots of the first of these equations are $x = +1$ and $x = -1$, the other gives $x = -\frac{5\pm\sqrt{21}}{2}$, from which no real root emerges. Therefore, the first of the two roots, $x = +1$, gives $\frac{ddy}{dx^2} = -14$ and therefore y has a maximum value $= 2$; the other root $x = -1$ gives $\frac{ddy}{dx^2} = +14$ and therefore y has a minimum value $= -2$.

EXAMPLE 5

To find the cases in which this fraction $\frac{x^3-x}{x^4-xx+1}$ has a maximum or a minimum value.

Having put $y = \frac{x^3-x}{x^4-xx+1}$ it will be

$$\frac{dy}{dx} = \frac{-x^6+2x^4+2x^2-1}{(x^4-x^2+1)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{2x^9-6x^7-18x^5+10x^3}{(x^4-x^2+1)^3}.$$

But having put $\frac{dy}{dx} = 0$ it will be

$$x^6 - 2x^4 - 2x^2 + 1 = 0,$$

which divided by $xx + 1$ gives

$$x^4 - 3x^2 + 1 = 0,$$

and this is further resolved into

$$xx - x - 1 = 0 \quad \text{and} \quad xx + x - 1 = 0,$$

whence the following four real roots result

$$\begin{array}{llll} \text{I.} & x = \dfrac{1+\sqrt{5}}{2} & \text{II.} & x = \dfrac{1-\sqrt{5}}{2}, \\ \text{III.} & x = -\dfrac{1+\sqrt{5}}{2} & \text{IV.} & x = -\dfrac{1-\sqrt{5}}{2}. \end{array}$$

Since all are contained in the equation $x^4 - 3xx + 1 = 0$, having put $x^4 = 3xx - 1$ the following equations will hold for all of them

$$\frac{ddy}{dx^2} = \frac{2x(10-20xx)}{8x^6} = \frac{5(1-2xx)}{2x^5} = \frac{5(1-2xx)}{2x(3xx-1)} \quad \text{and} \quad y = \frac{x^3-x}{2xx} = \frac{xx-1}{2x}.$$

But for the first two resulting from the equation $xx = x + 1$ it will be

$$\frac{ddy}{dx^2} = -\frac{5(2x+1)}{2x(3x+2)} = -\frac{5(2x+1)}{2(5x+3)} \quad \text{and} \quad y = \frac{1}{2}.$$

Therefore, the first root $x = \frac{1+\sqrt{5}}{2}$ gives

$$\frac{ddy}{dx^2} = -\frac{5(2+\sqrt{5})}{11+5\sqrt{5}}$$

and hence y has a maximum value. The second root $x = \frac{1-\sqrt{5}}{2}$ gives

$$\frac{ddy}{dx^2} = -\frac{5(2-\sqrt{5})}{11-5\sqrt{5}} = -\frac{5(\sqrt{5}-2)}{5\sqrt{5}-11}$$

and hence $y = \frac{1}{2}$ will also have a maximum value. The two remaining roots give $y = -\frac{1}{2}$, a minimum value.

§266 Therefore, in these examples the exploration, whether a certain found value produces a maximum or a minimum, can be simplified; for, because it is $\frac{dy}{dx} = 0$, the value of the term $\frac{ddy}{dx^2}$ having taken into account its equation can be expressed in an easier way. For, let the fraction $y = \frac{P}{Q}$ be propounded; because it is

$$dy = \frac{QdP - PdQ}{QQ} \quad \text{and} \quad QdP - PdQ = 0,$$

it will be

$$ddy = \frac{d(QdP - PdQ)}{Q^2} - \frac{2dQ(QdP - PdQ)}{Q^3}.$$

But because of $QdP - PdQ = 0$ this last term vanishes and it will be

$$ddy = \frac{d(QdP - PdQ)}{QQ} = \frac{QddP - PddQ}{Q^2}.$$

But because the decision is to be made from the either affirmative or negative value of this term, and the denominator Q^2 is always affirmative, this can be done considering only the numerator in such a way, that, if $QddP - PddQ$ or $\frac{d(QdP-PdQ)}{dx^2}$ was positive, a minimum will be indicated, if it is negative, a maximum. Or after $\frac{dy}{dx}$ was found, whose form will be of this kind $\frac{R}{QQ}$, only find $\frac{dR}{dx}$, and from the root, which causes this expression to be to positive, a minimum value will result and otherwise a maximum value.

§267 If the denominator of the propounded fraction was a square or any higher power such that it is $y = \frac{P}{Q^n}$, it will be

$$dy = \frac{QdP - nPdQ}{Q^{n+1}}$$

and having put $\frac{QdP - nPdQ}{dx} = R$ it will be

$$\frac{dy}{dx} = \frac{R}{Q^{n+1}}$$

and the maximum and minimum values will be determined from the roots of the equation $R = 0$. Further, because it is

$$\frac{ddy}{dx} = \frac{QdR - (n+1)RdQ}{Q^{n+2}},$$

because of $R = 0$ it will be

$$\frac{ddy}{dx} = \frac{dR}{Q^{n+1}};$$

its positive value will indicate a minimum value, a negative a maximum value. But it is perspicuous, if n was an odd number, that because of the always positive Q^{n+1} the decision can be made considering only $\frac{dR}{dx}$; but if n is an even number, use the formula $\frac{QdR}{dx}$.

But let us further put that a fraction of this kind is propounded $\frac{P^m}{Q^n} = y$; it will be

$$dy = \frac{(mQdP - nPdQ)P^{m-1}}{Q^{n+1}};$$

therefore, if one puts $\frac{mQdP - nPdQ}{dx} = R$, the roots of the equation $R = 0$ will indicate the cases, in which the function y has a maximum or a minimum value. Therefore, because it is

$$\frac{dy}{dx} = \frac{P^{m-1}R}{Q^{n+1}},$$

it will be

$$\frac{ddy}{dx} = \frac{P^{m-2}R((m-1)QdP - (n+1)PdQ)}{Q^{n+2}} + \frac{P^{m-1}dR}{Q^{n+1}},$$

and because of $R = 0$ it will be

$$\frac{ddy}{dx^2} = \frac{P^{m-1}dR}{Q^{n+1}dx};$$

this can additionally be divided by any square $\frac{P^{2\mu}}{Q^{2\nu}}$ to make the decision. Furthermore, also the equation $P = 0$ will give a maximum or a minimum point, if m was an even number; and in like manner considering the inverse formula $\frac{Q^n}{P^m}$ a maximum or minimum point will result by putting $Q = 0$, if n was an even number, as we showed above (§ 257); but here we do not consider the maximum or minimum values to result from this, but only, in order to explain the use of the method, investigate those, which result from the equation $R = 0$.

EXAMPLE 1

Let the fraction $\frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$ *be propounded; in which case it takes on a minimum or a maximum value, is in question.*

Having put $y = \frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$ at first it is certainly clear that it will be $y = 0$, if $x = -\frac{\alpha}{\beta}$, and $y = \infty$, if $x = -\frac{\gamma}{\delta}$; the latter of these cases will give a minimum value, the first a maximum value, if m and n were even numbers. Furthermore, it will be

$$\frac{dy}{dx} = \frac{(\alpha+\beta x)^{m-1}}{(\gamma+\delta x)^{n+1}}((m-n)\beta\delta x + m\beta\gamma - n\alpha\delta)$$

and hence

$$R = (m-n)\beta\delta x + m\beta\gamma - n\alpha\delta.$$

Therefore, having put $R = 0$ it will be

$$x = \frac{n\alpha\delta - m\beta\gamma}{(m-n)\beta\delta}.$$

Further, because of $\frac{dR}{dx} = (m-n)\beta\delta$ it is perspicuous, whether

$$\frac{P^{m-1}dR}{Q^{n+1}dx} = \frac{m^{m-1}\beta^{n+1}}{n^{n+1}\delta^{m-1}}\left(\frac{\alpha\delta-\beta\gamma}{m-n}\right)^{m-n-2}\frac{dR}{dx}$$

is a positive or negative quantity. In the first case, the propounded formula will have minimum value, in the second a maximum value.

So, if it was $y = \frac{(x+3)^3}{(x+2)^2}$, it will be $\frac{P^{m+1}dR}{Q^{n+1}dx} = \frac{9}{8}$ and hence the formula $\frac{(x+3)^3}{(x+2)^2}$ will have a minimum value for $x = 0$.

But if it is $y = \frac{(x-1)^m}{(x+1)^m}$, it will be

$$\frac{P^{m-1}dR}{Q^{n+1}dx} = \frac{m^{m-1}}{n^{n+1}}\left(\frac{n-m}{2}\right)^{n-m+2}(m-n)$$

and $x = \frac{n+m}{n-m}$. But because m and n are put to be positive numbers, the decision will be to made from the formula $(n-m)^{n-m+2}(m-n)$ or $(n-m)^{n-m}(m-n)$. Therefore, if it was $n > m$, the found value $x = \frac{n+m}{n-m}$ will always give a maximum value; but if $n < m$, the number $m - n$ will give a minimum value, but an odd a maximum value; so $\frac{(x-1)^3}{(x+1)^2}$ will have a maximum value for $x = -5$; for, it is $y = -\frac{6^3}{4^2} = -\frac{27}{2}$.

Example 2

Let the formula $y = \frac{(1+x)^3}{(1+xx)^2}$ *be propounded.*

It will be

$$\frac{dy}{dx} = \frac{(1+x)^2}{(1+xx)^3}(3-4x-xx) \quad \text{and} \quad \frac{P^{m-1}}{Q^{n+1}} \cdots \frac{dR}{dx} = -\frac{(1+x)^2}{(1+xx)^3}(2x+4);$$

because here $(1+x)^2$ and $(1+xx)^3$ always have a positive value, the decision is to be made from the formula $-x-2$; if it was positive, it indicates a minimum value, if negative, a maximum value. But from the equation $3 - 4x - xx = 0$ it follows either

$$x = -2 + \sqrt{7} \quad \text{or} \quad x = -2 - \sqrt{7}.$$

In the first case it is $-x - 2 = -\sqrt{7}$ and hence the propounded fraction will have a maximum value, in the other case a minimum value because of $-x - 2 = +\sqrt{7}$. But having put $x = -2 + \sqrt{7}$ it will be $1 + x = -1 + \sqrt{7}$ and $1 + xx = 12 - 4\sqrt{7}$, whence

$$y = \left(\frac{-1+\sqrt{7}}{12-4\sqrt{7}}\right)^2(\sqrt{7}-1) = \frac{(2+\sqrt{7})^2(\sqrt{7}-1)}{16} = \frac{17+7\sqrt{7}}{16} = 2.220.$$

But having put $x = -2 - \sqrt{7}$ it will be

$$y = \frac{17 - 7\sqrt{7}}{16} = -0.0950.$$

§268 Also irrational and transcendental functions exist which have the property of uniform functions, and therefore their maxima and minima can be found the same way. For, the cube and all odd roots are indeed uniform, since they exhibit only one single real value; but even though square roots and roots of all even powers, if they are real, denote actually two values, the one positive, the other negative, each of them can nevertheless be considered separately and in this sense one can even investigate the maxima and minima. So, if y was any function of x, even though $\sqrt{y}$ takes on two values, one can nevertheless can treat each single one separately. $+\sqrt{y}$ will have a maximum or minimum value, if y had such a one, if it was affirmative, since otherwise $\sqrt{y}$ would become imaginary. But vice versa $-\sqrt{y}$ will have a maximum or minimum value in the same cases, in which $+\sqrt{y}$ has a maximum or minimum value. But any power $y^{\frac{m}{n}}$ takes on a maximum or minimum value in the same cases, if n was an odd number; but if n was an even number, only those cases remain, in which y has a positive value, and in these cases because of the two values two maximum or minimum values will result.

§269 Since the differential equation, which results from the power of the function y^m, is $\frac{y^{m-1}dy}{dx} = 0$, whose roots at the same time indicate the cases, in which a surdic power $y^{\frac{m}{n}}$ has a maximum or minimum value, to investigate this value one has two equations, the one $y^{m-1} = 0$, the other $\frac{dy}{dx} = 0$, the latter of which goes over into $y = 0$ and exhibits maxima and minima only, if $m - 1$ was an odd number or if m was an even number, for the reasons mentioned in § 257. Therefore, because n is an odd number, if m was an even number, if we denote the even numbers by 2μ and the odd numbers by $2\nu - 1$, the function $y^{2\mu:(2\nu-1)}$ will have a maximum or minimum value, if those values are attributed to x which are found so from this equation $y = 0$ as from this $\frac{dy}{dx} = 0$. But if m is an odd number, the function $y^{(2\mu-1):2\nu}$ or $y^{(2\mu-1):(2\nu)}$ has a maximum or minimum value only, if a value resulting from this equation $\frac{dy}{dx} = 0$ is substituted for x. And in the second case $y^{(2\mu-1):2\nu}$ maxima and minima only result, if y has affirmative values for the values found from the equation $\frac{dy}{dx} = 0$.

§270 So, this formula $x^{\frac{2}{3}}$ takes on a minimum value for $x = 0$, because in this case x^2 has minimum value. But if we do not reduce the formula $x^{\frac{2}{3}}$ to the form x^2, the method given before would not indicate this at all, since in the case $x = 0$ the terms of the series

$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \text{etc.},$$

whence the decision is to be made, except for the first all become infinite. For, having put $y = x^{\frac{2}{3}}$ it will be

$$\frac{dy}{dx} = \frac{2}{3x^{\frac{1}{3}}}, \quad \frac{ddy}{dx^2} = \frac{-2}{9x^{\frac{4}{3}}}, \quad \frac{d^3y}{dx^3} = \frac{2 \cdot 4}{27x^{\frac{7}{3}}} \quad \text{etc.}$$

And hence neither the equation $\frac{dy}{dx} = \frac{2}{3x^{\frac{1}{3}}} = 0$ shows the value $x = 0$ nor the following terms indicate whether it is a maximum or a minimum value. Therefore, since we assumed that the series

$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \text{etc.}$$

becomes convergent, if ω is assumed to be a very small quantity, in those cases, in which this series becomes divergent, the general method is not applicable, what happens in the example $y = x^{\frac{2}{3}}$ mentioned here, if one puts $x = 0$. Therefore, in these cases the same reduction we used before will be necessary to reduce the propounded expression to another form, which is not subjected to this inconvenience. But this only happens in very few cases which are contained in the formula $y^{\frac{2\mu}{2\nu-1}}$ or are easily reduced to it. So, if the maxima and minima of the formula $y^{\frac{2\mu}{2\nu-1}}z$ are in question, where z is any function of x, investigate this form $y^{2\mu}z^{2\nu-1}$, which has a maximum or minimum value in the same cases as the propounded one.

§271 Having excluded this case, which is now easily handled, functions, which contain irrational quantities, can be treated the same way as rational functions and their maxima and minima can be determined, what we will illustrate in the following examples.

EXAMPLE 1

Let the formula $\sqrt{aa+xx}-x$ *be propounded; in which cases it has maximum or minimum values is in question.*

Having put $y=\sqrt{aa+xx}-x$ it will be

$$\frac{dy}{dx}=\frac{x}{\sqrt{aa+xx}}-1 \quad \text{and} \quad \frac{ddy}{dx^2}=\frac{aa}{(aa+xx)^{3:2}}.$$

Having put $\frac{dy}{dx}=0$ it will be $x=\sqrt{aa+xx}$ and hence $x=\infty$ and it is $\frac{ddy}{dx^2}=0$. In a like manner the following terms $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc. all become $=0$; hence one cannot decide, whether it is a maximum or a minimum. The reason for this is that it actually is so $x=-\infty$ as $x=+\infty$. Putting $x=\infty$ because of

$$\sqrt{aa+xx}=x+\frac{aa}{2x}$$

it is $y=0$, which value is the smallest of all.

EXAMPLE 2

Let the cases be in question in which this form $\sqrt{aa+2bx+mxx}-nx$ *takes on a maximum or a minimum value.*

Having put $y=\sqrt{aa+2bx+mxx}-nx$ it will be

$$\frac{dy}{dx}=\frac{b+mx}{\sqrt{aa+2bx+mxx}}-n;$$

having put this $=0$ it will be

$$bb+2mbx+mmxx=nnaa+2nnbx+mnnxx$$

or

$$xx=\frac{2bx(nn-m)+nnaa-bb}{mm-mnn}$$

and hence

$$x=\frac{(nn-m)b\pm\sqrt{mnn(m-nn)aa-nn(m-nn)bb}}{m(m-nn)}$$

or

$$x = -\frac{b}{m} \pm \frac{n}{m}\sqrt{\frac{maa - bb}{m - nn}};$$

hence it is

$$\sqrt{aa + 2bx + mxx} = \frac{b + mx}{n} = \pm\sqrt{\frac{maa - bb}{m - nn}}.$$

Therefore, because it is

$$\frac{ddy}{dx^2} = \frac{maa - bb}{(aa + 2bx + mxx)^{\frac{3}{2}}},$$

it will be

$$\frac{ddy}{dx^2} = \frac{maa - bb}{\pm\left(\frac{maa-bb}{m-nn}\right)^{\frac{3}{2}}} = \frac{\pm(m - nn)\sqrt{m - nn}}{\sqrt{maa - bb}}.$$

Therefore, a maximum or minimum exists only, if $\frac{m-nn}{maa-bb}$ was a positive quantity. But if it is a positive quantity, the upper sign will give a minimum, if $m > nn$, a maximum on the other hand, if $m < nn$; the contrary happens, if the lower sign holds. Therefore, if it is $m = 2$, $n = 1$ and $b = 0$, the formula $\sqrt{aa + 2xx} - x$ has a minimum value for $x = +\frac{1}{2}\sqrt{2aa} = \frac{a}{\sqrt{2}}$, but a maximum value for $x = -\frac{a}{\sqrt{2}}$. Therefore, the minimum value will be $= a\sqrt{2} - \frac{a}{\sqrt{2}} = \frac{a}{\sqrt{2}}$ and the maximum value $= a\sqrt{2} + \frac{a}{\sqrt{2}} = \frac{3a}{\sqrt{2}}$.

EXAMPLE 3

To find the cases in which this expression $\sqrt[4]{1 + mx^4} + \sqrt[4]{1 - nx^4}$ has a maximum or a minimum value.

Because it is $\frac{dy}{dx} = \frac{mx^3}{(1+mx^4)^{\frac{3}{4}}} - \frac{nx^3}{(1-nx^4)^{\frac{3}{4}}}$, it will be

$$mx^3(1 - nx^4)^{\frac{3}{4}} = nx^3(1 + mx^4)^{\frac{3}{4}} \quad \text{and hence} \quad m^4(1 - nx^4)^3 = n^4(1 + mx^4)^3$$

or

$$n^4 - m^4 + 3mn(n^3 + m^3)x^4 + 3m^2n^2(n^2 - m^2)x^8 + m^3n^3(n + m)x^{12} = 0.$$

Therefore, only if this equation has a positive root for x^4, a maximum or minimum exists. For, this equation in general cannot be solved in a convenient manner, since it will be

$$x^4 = \frac{m^{\frac{4}{3}} - n^{\frac{4}{3}}}{mn(\sqrt[3]{m} + \sqrt[3]{n})} \quad \text{or} \quad x^4 = \frac{m - \sqrt[3]{m^2 n} + \sqrt[3]{mn^2} - n}{mn}$$

let us consider a special case and put $m = 8n$, and then it will be

$$-4095 + 24 \cdot 513x^4 - 3 \cdot 63 \cdot 64n^2x^8 + 9 \cdot 512x^{12} = 0$$

or

$$512n^3x^{12} - 1344n^2x^8 + 1368nx^4 - 455 = 0;$$

put $8nx^4 = z$; it will be

$$z^3 - 21z^2 + 171z - 455 = 0,$$

which has the divisor $z - 5$, and the other factor will be $zz - 16zz + 91 = 0$ containing imaginary roots. Therefore, it will only be $z = 8nx^4 = 5$ and hence $x = \sqrt[4]{\frac{5}{8n}}$, which value is a maximum or minimum point of the expression $\sqrt[4]{1 + 8nx^4} + \sqrt[4]{1 - nx^4}$. To find out which of both is the case, consider

$$\frac{ddy}{dx^2} = \frac{3mxx}{(1 + mx^4)^{\frac{7}{4}}} - \frac{3nxx}{(1 - nx^4)^{\frac{7}{4}}}.$$

But because of $m = 8n$ having put $x^4 = \frac{5}{8n}$ it will be

$$\frac{ddy}{dx^2} = \left(\frac{24n}{6^{\frac{7}{4}}} - \frac{3n}{(3:8)^{\frac{7}{4}}} \right) xx = -\frac{360nxx}{6^{\frac{7}{4}}}$$

and hence negative; therefore, $\sqrt[4]{1 + 8nx^4} + \sqrt[4]{1 - nx^4}$ will have a maximum value for $x = \sqrt[4]{\frac{5}{8n}}$. This maximum value will be $= \sqrt[4]{6} + \sqrt[4]{\frac{3}{8}} = \frac{3\sqrt[4]{6}}{2}$. If we put u instead of nx^4, it is plain that this expression $\sqrt[4]{1 + 8u} + \sqrt[4]{1 - u}$ has a maximum value for $u = \frac{5}{8}$ and that this maximum value will be $= \frac{3\sqrt[4]{6}}{2} = 2.347627$. Therefore, whatever value except for $\frac{5}{8}$ is written for u, the expression will have a smaller value.

§272 Maxima and minima will be determined in like manner, if also transcendental quantities are contained in the propounded expression. For, if the propounded function was not multiform and one has not to consider several values of it at the same time, the roots of the differential equation will show maxima or minima, if they were not equal roots, whose number is even. Therefore, we will demonstrate this investigation in several examples.

EXAMPLE 1

To find the number which has the smallest ratio to its logarithm.

That a smallest ratio $\frac{x}{\log x}$ exist is obvious, because this ratio becomes infinite so for $x = 1$ as for $x = \infty$. Therefore, vice versa the fraction $\frac{\ln x}{x}$ will take on a maximum value somewhere; of course in the same case, in which $\frac{x}{\ln x}$ has a minimum value. To find this case put $y = \frac{\log x}{x}$ and it will be

$$\frac{dy}{dx} = \frac{1}{xx} - \frac{\log x}{xx}.$$

Having put this equal to zero it will be $\log x = 1$, and since we assume the hyperbolic logarithm here, if e is put for the number whose hyperbolic logarithm is $= 1$, it will be $x = e$. Therefore, because all logarithms have a certain ratio to the hyperbolic ones, $\frac{e}{\log e}$ will also be a minimum point for the common logarithm or $\frac{\log e}{e}$ will be a maximum value. Since assuming tabulated logarithms it is $\log e = 0.4342944819$, the fraction $\frac{\log x}{x}$ will always be smaller than $\frac{4342944819}{27182818284}$ or approximately $\frac{47}{305}$ and no other number exists, which has a smaller ratio than 305 to 47 to its logarithm. That in this case $\frac{\log x}{x}$ has a maximum value is obvious, since because of $\frac{dy}{dx} = \frac{1-\log x}{xx}$ it is

$$\frac{ddy}{dx^2} = -\frac{1}{x^3} - \frac{2(1-\log x)}{x^3} = -\frac{1}{x^3}$$

since $1 - \log x = 0$ and hence the differential is negative.

EXAMPLE 2

To find the number x that this power $x^{1:x}$ takes on a maximum value.

That a maximal value of this formula exists is obvious, because by substituting numbers for x it is

$$1^{1:1} = 1.000000$$
$$2^{1:2} = 1.414213$$
$$3^{1:3} = 1.442250$$
$$1^{1:4} = 1.414213.$$

Therefore, put $x^{1:x} = y$ and it will be

$$\frac{dy}{dx} = x^{1:x}\left(\frac{1}{xx} - \frac{\log x}{xx}\right).$$

Having put this value equal to zero it will be $\log x = 1$ and $x = e$ where $e = 2.718281828$. And because it is $\frac{dy}{dx} = (1 - \log x)\frac{x^{1:x}}{xx}$, it will be

$$\frac{ddy}{dx^2} = -\frac{x^{1:x}}{x^3} + (1 - \log x)\frac{d}{dx} \cdot \frac{x^{1:x}}{xx} = -\frac{x^{1:x}}{x^3}$$

because $1 - \log x = 0$. Therefore, because $\frac{ddy}{dx^2}$ is a negative quantity, $x^{1:x}$ has a maximum value in the case $x = e$. But because it is $e = 2.718281828$, one finds that it will $e^{\frac{1}{e}} = 1.444667861009764$, which value is easily obtained from the series

$$e^{\frac{1}{e}} = 1 + \frac{1}{e} + \frac{1}{2e^2} + \frac{1}{6e^3} + \frac{1}{24e^4} + \text{etc.}$$

This example is also solved using the result of the preceding example; for, if $x^{1:x}$ takes on a maximum value, also its logarithm, which is $\frac{\log x}{x}$, will have to take on a maximum value; for this to happen, it has to be $x = e$, as we found.

Example 3

To find the arc x that its sine has a maximum or minimum value.

Having put $\sin x = y$ it will be $\frac{dy}{dx} = \cos x$ and hence $\cos x = 0$, whence the following values for x result: $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$ etc. But it is $\frac{ddy}{dx^2} = -\sin x$. Therefore, because these values, having substituted them for x, give either $+1$ or -1 for $\sin x$, the latter will be maximum point, the first a minimum point, as it is known.

EXAMPLE 4

To find the arc x such that the rectangle $x \sin x$ takes on a maximum value.

That a maximum value exists is obvious, because having put either $x = 0^\circ$ or 180° the propounded rectangle vanishes. Therefore, let $y = x \sin x$; it will be

$$\frac{dy}{dx} = \sin x + x \cos x$$

and hence

$$\tan x = -x.$$

Let $x = 90^\circ + u$; it will be $\tan x = -\cot u$, therefore $\cot u = 90^\circ + u$. To resolve the equation in the way explained above [§ 234] put $z = 90^\circ + u - \cot u$ and let f be the value in question of the arc u. Because it is $dz = du + \frac{du}{\sin^2 u}$, it will be

$$p = \frac{du}{dz} = \frac{\sin^2 u}{1 + \sin^2 u} \quad dp = \frac{2du \sin u \cos u}{(1 + \sin u)^2}$$

and hence

$$\frac{dp}{dz} = q = \frac{2 \sin^3 u \cos u}{(1 + \sin^2 u)^3}, \quad dq = \frac{6du \sin^2 u \cos^2 u - 2du \sin^4 u}{(1 + \sin^2 u)^3} - \frac{12du \sin^4 u \cos^2 u}{(1 + \sin^2 u)^4}.$$

Therefore,

$$\frac{dq}{dz} = r = \frac{6 \sin^4 u \cos^2 u - 2 \sin^6 u}{(1 + \sin^2 u)^4} - \frac{12 \sin^2 u^6 \cos^2 u}{(1 + \sin^2 u)^5} = \frac{6 \sin^4 - 14 \sin u^6 + 4 \sin^8 u}{(1 + \sin^2 u)^5}.$$

From these it will be

$$f = u - pz + \frac{1}{2} qzz - \frac{1}{6} rz^3 + \text{etc.}$$

After by trying several values an approximate value of f was detected, put $u = 26^\circ 15'$; it will be $90^\circ + u = 116^\circ 15'$ and the arc equal to the cotangent u is defined this way. From

$$\begin{array}{llrl}
 & \log\cot u & = & 10.3070250 \\
\text{subtract} & & & 4.6855749 \\
\hline
 & & & 5.6214501 \\
\hline
\end{array}$$

$$\begin{array}{llrl}
\text{Therefore} & \cot u & = & 418263.7'' \\
\text{or} & \cot u & = & 116^\circ 11' 3\tfrac{7}{10}'' \\
\text{hence} & z & = & 3' 56\tfrac{3}{10}'' = 256.3''.
\end{array}$$

Now to find the value of the term pz do this calculation:

$$\begin{array}{llrl}
 & \log\sin u & = & \underline{9.6457058} \\
 & \log\sin^2 u & = & \underline{9.2914116} \\
 & 1+\sin^2 u & = & 1.19561 \\
\hline
 & \log(1+\sin^2 u) & = & 0.0775895 \\
\hline
 & \log p & = & 9.2138221 \\
 & \log z & = & 2.2724637 \\
\hline
 & \log pz & = & 1.5872858 \\
\text{Therefore} & pz & = & 38.6621 \text{ seconds} \\
\text{or} & pz & = & 38'' 39''' 43'''' \\
\text{from} & u & = & 26^\circ 15' \\
\hline
\text{it will be} & f & = & 26^\circ 14' 21'' 20''' 17'''' \\
\text{and the arc in question} & x & = & 116^\circ 14' 21'' 20''' 17''''
\end{array}$$

But the third term $\frac{1}{2}qzz = \frac{\sin^3 u \cos u}{(1+\sin^2 u)^3}zz$ has to added additionally. To find its value, z must be expressed in parts of the radius this way:

$$
\begin{array}{llcl}
 & \log \sin z & = & 2.3734637 \\
\text{add} & & & 4.6855749 \\
\hline
 & & & 7.0590386 \\
\text{add} & \log \dfrac{\sin^2 u}{1+\sin^2 u} z & = & 1.5872858 \\
\hline
 & & & 8.6463244 \\
\text{add} & \log \sin u & = & 9.6457058 \\
 & \log \cos u & = & 9.9527308 \\
\hline
 & & & 8.2447600 \\
\text{subtract} & \log (1+\sin^2 u)^2 & = & 0.1551790 \\
\hline
 & \log \frac{1}{2} qzz & = & 8.0895810
\end{array}
$$

Therefore

$$\frac{1}{2}qzz = 0.012291$$

or

$$\frac{1}{2}qzz = 44''''15'''''.$$

Hence, having also used this term the arc in question will be

$$x = 116^\circ 14' 21'' 21''' 0'''';$$

but taking into account even more terms one finds

$$x = 116^\circ 14' 21'' 20''' 35'''' 47'''''.$$