On Maxima and Minima of multiform functions and such containing several variables *

Leonhard Euler

§273 If y was a multiform function of x such that for any value of x it has several real values, then having varied x those several values of y are connected to each other in such a way that they represent several series of successive values. For, if we consider y as ordinate of a curve, while x is the abscissa, so many different branches of the same curve line will correspond to the same abscissa x as many real different values y had; and hence those successive values of y which constitute the same branch are to be considered to be connected; but the values related to the different branches will be distinct from each other. Therefore, we will have as many series of connected values of y as many different real values it will have for each value of x; and in each arbitrary series the values of y, while x is assumed to increase, either grow or decrease or, after they had increased, decrease again or vice versa. From this it is perspicuous that in each series of connected values maxima and minima exist as in uniform functions.

§274 To determine these maxima and minima also the same method which we treated in the preceding chapter for uniform functions can be used. For,

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$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \text{etc.},$$

it is necessary, that in the case of a maximum or minimum the term $\frac{\omega dy}{dx}$ vanishes and it is $\frac{dy}{dx} = 0$. Therefore, the roots of this equation $\frac{dy}{dx} = 0$ will indicate those values of x, the maxima or minima in the single series of the connected values of y correspond to. And it will indeed not be ambiguous, in which series of connected values a maximum or minimum is given. For, because in the equation $\frac{dy}{dx} = 0$ both variables x and y are contained, the values of x can only be defined, if by means of the equation containing the relation of the function y to x the variable y is eliminated; but before this happens, one gets to an equation expressing the value of y by means of a rational or uniform function of x. Hence, having found the values of x the corresponding value of y will be found, which will be the maximum or minimum values in the series of connected successive values one get to.

§275 But the decision, whether these values of y are maxima or minima, will be made the same way as we explained it before: Find the value of $\frac{ddy}{dx^2}$ expressed in finite terms and in it successively substitute a value of x found before for x; but at the same time substitute the value for y which corresponds to it for any arbitrary value of x; having done this see, whether the expression $\frac{ddy}{dx^2}$ will have a positive or a negative value, and in the first case a minimum will be indicated, in the second a maximum. But if also $\frac{ddy}{dx^2}$ vanishes, then one will have to proceed to the formula $\frac{d^3y}{dx^3}$; but if also $\frac{d^3y}{dx^3}$ vanishes, the decision is to be made from the formula $\frac{d^4y}{dx^4}$ the same way as we described it for the formula $\frac{ddy}{dx^2}$. And if also $\frac{d^4y}{dx^4}$ vanishes in a certain case, one will have to proceed to the fifth differential of y; but, no matter how far one had to proceed, the decisions to be made from the differentials of odd order are always those which we explained are to be made from the formula $\frac{d^3y}{dx^3}$. In these cases in the formulas $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc. one will have to proceed so far until one gets to such a one, which does not vanish in a certain case; if this was a differential of odd order, neither a maximum nor a minimum will be indicated; but if it was of even order, its positive value will imply a minimum, its negative value a maximum.

§276 Let us put that the function y is determined by x by means of an arbitrary equation; if this equation is differentiated, it will have a form of this kind $Pdx + Qdy = 0$. Therefore, having put $\frac{dy}{dx} = 0$ it will be $\frac{P}{Q} = 0$ and hence either $P = 0$ or $Q = \infty$. The second equation, if the relation among x and y is expressed by means of a polynomial equation, cannot hold, because either x or y or both would have to become infinite. Therefore, the decision is to be made from the equation $P = 0$, whose roots or values of x, which it obtains, after by means of the propounded equation the variable y was completely eliminated, will indicate the cases, in which the values of y are maximum or minimum values. But to make the decision, whether a maximum or a minimum value results, examine the formula $\frac{ddy}{dx^2}$. But having differentiated the differential equation $Pdx + Qdy = 0$ again, if we put

$$dP = Rdx + Sdy \quad \text{and} \quad dQ = Tdx + Vdy$$

and assume dx to be constant, will give

$$Rdx^2 + Sdxdy + Tdxdy + Vdy^2 + Qddy = 0.$$

But because now it is $\frac{dy}{dx} = 0$, having divided by dx^2 it will be

$$R + \frac{Qddy}{dx^2} = 0 \quad \text{and hence} \quad \frac{ddy}{dx^2} = -\frac{R}{Q}.$$

Therefore, in the differential equation $Pdx + Qdy = 0$ differentiate only the quantity P, while assuming y to be constant, and Rdx will result; then investigate the value of the fraction $\frac{R}{Q}$, which, if it was affirmative, will indicate a maximum, if negative, a minimum value.

§277 Let y be a biform function of x, which is determined by this equation $yy + py + q = 0$ while p and q denote any uniform functions of x. Therefore, by differentiating it will be $2ydy + pdy + ydp + sq = 0$ and hence $Pdx = ydp + dq$. Therefore, having put $P = 0$ it will be $ydp + dq = 0$ and $y = -\frac{dq}{dp}$ will result and so y is expressed by means of an uniform function of x such that, whatever value was found for x, from it and y it acquires one single determined value. But now the elimination of y will be easy; for, if in the propounded equation $yy + py + q = 0$ the value $-\frac{dq}{dp}$ is substituted for y, one will have $dq^2 - pdpdq + qdp^2 = 0$, which equation, having divided it by dx^2 and resolved it, will yield all values of x corresponding to maxima or minima; this will become more clear in the following examples.

EXAMPLE 1

Having propounded the equation $yy + mxy + aa + bx + nxx = 0$ to define the maxima or minima of the function y.

Having differentiated the equation we will have

$$2ydy + mxdy + mydx + bdx + 2nxdx = 0,$$

whence it is

$$P = my + b + 2nx \quad \text{and} \quad Q = 2y + mx.$$

Therefore, having put $P = 0$ it will be $y = -\frac{b+2nx}{m}$; this value, having substituted it in the equation, gives

$$\frac{4nn}{mm}xx + \frac{4nb}{mm}x + \frac{bb}{mm} - 2nxx - bx + aa + nxx + bx = 0$$

or

$$xx = \frac{4nbx + bb + mmaa}{mmn - 4nn},$$

whence it is

$$x = \frac{2nb \pm \sqrt{mmnbb + mmn(mm - 4n)aa}}{mmn - 4nn}$$

or

$$x = \frac{2nb \pm m\sqrt{nbb + n(mm - 4n)aa}}{n(mm - 4n)} \quad \text{and} \quad y = \frac{-mb \mp \sqrt{nbb + n(mm - 4n)aa}}{mm - 4n}.$$

Then having put only x to be variable it is $dP = 2ndx$ and hence $R = 2n$. But it is

$$Q = 2y + mx = \pm\frac{\sqrt{nbb + n(mm - 4n)aa}}{n},$$

whence

$$\frac{R}{Q} = \frac{\pm 2nn}{\sqrt{nbb + n(mm - 4n)aa}};$$

because its numerator $2nn$ is always positive, if the upper sign holds, a maximum value of y will result, if the lower holds, a minimal value will result. Here, the following things have to be mentioned.

I. If $m = 0$, from the equation $P = 0$ it immediately follows $x = -\frac{b}{2n}$ so that no elimination is necessary. And a double value of y corresponds to this value, because it is $y = \pm\frac{1}{2n}\sqrt{nbb - 4nnaa}$, of which the positive one is a maximum value, the other negative one a minimum value.

II. If it is $n = 0$, it is $y = -\frac{b}{m}$ and x grows to infinity and y always has the same value such that it is neither a maximum nor a minimum value.

III. If it is $mm = 4n$, it will be $4nbx + bb + mmaa = 0$ or $x = \frac{bb+mmaa}{-mmb}$ and it will be

$$y = -\frac{b + 2nx}{m} = -\frac{2b + mmx}{2m} = -\frac{2b}{2m} + \frac{bb + mmaa}{2mb} = \frac{mmaa - bb}{2mb}.$$

Therefore, the other value of y, $\frac{mmaa-bb}{2mb}$, which will be a maximum or a minimum value, corresponds to this value of $x = \frac{-mmaa+bb}{mmb}$. But because for this value of y to result in the expression

$$y = \frac{-mb \mp 2\sqrt{nbb + n(mm - 4n)aa}}{mm - 4n}$$

the lower sign has to hold, the value of y will be a minimum value.

EXAMPLE 2

Having propounded the equation $yy - xxy + x - x^3 = 0$ to define the maximum or minimum values of y.

Having differentiated the equation this equation results

$$2ydy - xxdy - 2xydx + dx - 3xxdx = 0.$$

And it is

$$P = 1 - 3xx - 2xy \quad \text{and} \quad Q = 2y - xx.$$

Therefore, having put $P = 0$ it will be $y = \frac{1-3xx}{2x}$ and hence having substituted this value it will be

$$\frac{1}{4xx}-\frac{3}{2}+\frac{9xx}{4}-\frac{x}{2}+\frac{3}{2}x^3+x-x^3=0$$

or

$$1-6xx+2x^3+9x^4+2x^5=0.$$

One of its roots is $x=-1$ and $y=1$ corresponds to it. But having assumed y to be constant it is $R=-6x-2y$, therefore

$$\frac{ddy}{dx^2}=\frac{2y+6x}{2y-xx};$$

in the case $x=-1$ and $y=1$ it goes over into -4 so that the value of $y=1$ is a maximum value. But two values of y from the equation $yy-y=0$ correspond to $x=-1$; therefore, the other is $y=0$, which is neither a maximum nor a minimum value. If this equation of degree five is divided by $x+1$, an equation results whose roots cannot be exhibited in a simple manner.

EXAMPLE 3

Let this equation be propounded $yy+2xxy+4x-3=0$; the maximum and minimum values of y are in question.

By means of differentiation this equation will result

$$2ydy+2xxdy+4xydx+4dx=0.$$

Having put $\frac{dy}{dx}=0$ it will be $xy+1=0$ and hence $y=-\frac{1}{x}$, which value substituted in the the propounded equation gives

$$\frac{1}{xx}-2x+4x-3=0=2x^3-3xx+1,$$

whose roots are $x=1$, $x=1$ and $x=-\frac{1}{2}$. Since now it is

$$\frac{dy}{dx}=-\frac{4xy+4}{2y+2xx}=-\frac{2xy+2}{y+xx},$$

by differentiating this equation it will be $\frac{ddy}{dx^2}=-\frac{2y}{y+xx}$, having put y to be constant and so $dy=0$ and having put $xy+1=0$. Therefore, these values will be as follows:

$$
\begin{array}{r|r|c}
x & y & \dfrac{ddy}{dx^2} \\
1 & -1 & \infty \\
1 & -1 & \infty \\
-\dfrac{1}{2} & 2 & \dfrac{-16}{9}
\end{array}
\quad \text{for a maximum.}
$$

Since for the equal roots it is $\frac{ddy}{dx^2} = \infty$, it is not determined, whether in this case a maximum or a minimum value results. But because at the same time it is $y + xx = 0$, it will not even be $\frac{dy}{dx} = 0$ in this case, since $P = 0$ and $Q = 0$ in the fraction $\frac{dy}{dx} = -\frac{P}{Q}$; therefore, because the primary property is missing, neither a maximum nor a minimum can exist in this case. But it is indicated that in this case $x = 1$ both values of y become equal to each other. We will explain this nature in more detail below, when we will get to the application of the differential calculus in the doctrine of curved lines. For, even though this subject also extend to this, we nevertheless discuss it completely just in the following treatise, that it is not necessary to do it twice.

§278 Additionally, another kind of maxima and minima is exists in multiform functions; those are maxima and minima, whose nature can be explained most easily considering biform functions, are not found by the method given up to now. To see this, let y be any biform function of x, such that, whatever value is attributed to x, for y two values result, either both real or both imaginary. Let us put that these values of y become imaginary, if one puts $x > f$, but are real, if one sets $x < f$; and having put $x = f$ both values will coalesce into one, which value we want to be $y = g$. Therefore, because, if one takes $x > f$, the function y has no real value, if it happens that for $x < f$ both values of y become either greater than g or smaller than g, in the first case the value $y = g$ will be a minimum value, in the second a maximum value, since in that case it is smaller than both preceding ones, but larger in the other case. And this maximum or minimum value cannot be found by means of the method treated up to now, because here it is not $\frac{dy}{dx} = 0$. But these maxima or minima are also of a different kind, since such are not maxima or minima with respect to the preceding or following values connected by a series, but with respect to only two distinct either preceding or following values.

§279 This happens, if the propounded equation was of this kind

$$y = p \pm (f-x)\sqrt{f-x}q$$

while p and q are functions of x not divisible by $f-x$; let q obtain a positive value, if one puts either $x=f$ or assumes x to be little larger or a little smaller. Let $p=g$ for $x=f$ and it is obvious that in the case $x=f$ the two values of y coalesce into the single one $y=g$; but for $x>f$ both values of y will become imaginary. Therefore, if we put x to be a little bit smaller than f, say $x=f-\omega$, the function p will go over into

$$g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2dx^2} - \text{etc.}$$

and q into

$$q - \frac{\omega dq}{dx} + \frac{\omega^2 ddq}{2dx^2} - \text{etc.},$$

whence in this case it will be

$$y = g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2dx^2} - \text{etc.} \pm \omega\sqrt{\omega}\left(q - \frac{\omega dq}{dx} + \frac{\omega^2 ddq}{2dx^2} - \text{etc.}\right).$$

Let us put ω to be very small that with respect to ω its higher powers vanish, and it will be $y = g - \frac{\omega dp}{dx} \pm \omega\sqrt{\omega}q$; these two values of y will both be smaller than g, if $\frac{dp}{dx}$ was positive, but larger, if negative. Therefore, the double value of $y=g$ in that case will be a maximum value, in the other a minimum value.

§280 Therefore, these maxima and minima result, first since both values of y become equal for $x=f$, but imaginary for $x>f$ and finally real for $x<f$; further, since having put $x=f-\omega$ the other irrational term yields higher powers of ω than the rational term. Therefore, this also happens, if it was $y = p \pm (f-x)^n\sqrt{f-x}q$, as long as n is an integer >0. But because not only the square root, but also any other root of even power introduces the same ambiguity of the sign, the same will happen, if it was $y = p \pm (f-x)^{\frac{2n+1}{2m}}q$, as long as it is $2n+1 > 2m$; therefore, it will be $(y-p)^{2m} = (f-x)^{2n+1}q^{2m}$ or $(y-p)^{2m} = (f-x)^{2n+1}Q$. Therefore, if the function y is expressed by means of such an equation such that it is $2n+1>2m$, having put $x=f$ the value of y will be a maximum or minimum; the first, if $\frac{dp}{dx}$ was a positive quantity, the latter, if $\frac{dp}{dx}$ is negative for $x=f$. But if in this case it is $\frac{dp}{dx}=0$, then it will be

$$y = g + \frac{\omega^2 ddp}{2dx^2} \pm \omega^{\frac{2n+1}{2m}} q.$$

Therefore, only if $\frac{2n+1}{2m} > 2$, a maximum or a minimum can exist; but if it is $\frac{2n+1}{2m} > 2$, then $y = g$ will have a maximum value, if $\frac{ddp}{dx^2}$ had a negative value, a minimum value, if it was positive; and like this the decision will have to be made, if also $\frac{ddp}{dx^2}$ vanishes.

§281 Therefore, if y was a function of x of such a kind, it can happen, that except for the maxima and minima, which the first method exhibits, also maxima and minima of the other species are given which can be explored by the method explained here. We will show this in the following examples.

EXAMPLE 1

To determine the maxima and minima of the function y which is defined by this equation

$$yy - 2xy - 2xx - 1 + 3x + x^3 = 0.$$

To investigate the maxima and minima of the first kind differentiate the equation and it will be

$$2ydy - 2xdy - 2ydx - 4x + 3dx + 3xxdx = 0$$

and having put $\frac{dy}{dx} = 0$ it will be

$$y = \frac{3}{2} - 2x + \frac{3}{2}xx,$$

which value substituted in the first equation gives

$$9x^4 - 32x^3 + 42xx - 24x + 5 = 0,$$

which is resolved into

$$9xx - 14x + 5 = 0 \quad \text{and} \quad xx - 2x + 1 = 0.$$

The first gives $x = 1$ twice and the corresponding value is $y = 1$, whence in this case in the fraction

$$\frac{dy}{dx} = \frac{2y - 3 + 4x - 3xx}{2y - 2x}$$

the denominator also vanishes and so a maximum or minimum of the first kind is can not exist on this case; the first equation $9xx - 14x + 5 = 0$ on the other hand will give $x = 1$ and $x = \frac{5}{9}$, the first of which values leads to the same inconvenience as the first. But having put $x = \frac{5}{9}$ it is $y = \frac{3}{2} - \frac{10}{9} + \frac{25}{54} = \frac{23}{27}$. And because it is $\frac{dy}{dx} = \frac{2y-3+4x-3xx}{2y-2x}$, it will be

$$\frac{ddy}{dx^2} = \frac{4 - 6x}{2y - 2x} = \frac{-3x + 2}{y - x}$$

because of $dy = 0$ and the numerator $= 0$. Therefore, it will be $\frac{ddy}{dx^2} = \frac{9}{8}$, whence this value $x = \frac{5}{9}$ is a minimum point of the first kind. Further, because it is $(y - x)^2 = (1 - x)^3$, it will be

$$y = x \pm (1 - x)\sqrt{1 - x}$$

and hence having put $x = 1$ a maximum value of the second kind results; for, having put $x = 1 - \omega$ it will be $y = 1 - \omega \pm \omega\sqrt{\omega}$, both of which values are smaller than 1, if a very small ω is taken.

EXAMPLE 2

To find the maxima and minima of the function $y = 2x - xx \pm (1 - x)^2\sqrt{1 - x}$.

To find the maxima and minima of the first kind differentiate the equation and it will be

$$\frac{dy}{dx} = 2 - 2x \mp \frac{5}{2}(1 - x)\sqrt{1 - x},$$

which fraction put $= 0$ at first yields $x = 1$, and because it is

$$\frac{ddy}{dx^2} = -2 \pm \frac{15}{4}\sqrt{1 - x},$$

y in this case will be a maximum value of the first kind and it is $y = 1$. Having divided the equation $\frac{dy}{dx} = 0$ by $1 - x$ it will be

$$4 \mp 5\sqrt{1 - x} = 0 \quad \text{or} \quad 16 = 25 - 25x,$$

whence it is $x = \frac{9}{25}$ and $\frac{ddy}{dx^2} = -2 \pm 3$. Hence, if the upper sign holds, $y = \frac{2869}{3125}$ will be a minimum value; but if the lower sign holds, it will be $y = \frac{821}{3125}$, which might seem to be a maximum value; but indeed only the upper sign can hold, since $4 \mp 5\sqrt{1-x}$ can only be $= 0$, if it is $\sqrt{1-x} = +\frac{4}{5}$. Therefore, we find a maximum value of the first kind in the case $x = 1$ and $y = 1$ and a minimum in the case $x = \frac{9}{25}$ and $y = \frac{2689}{3125}$. A maximum of the other kind also results, if $x = 1$, in which case it is $y = 1$. For, having put $x = 1 - \omega$ it will be $y = 1 - \omega\omega \pm \omega^2\sqrt{\omega}$, this is < 1 in both cases . Therefore, here, if $x = 1$, the a maximum value of the first and the second kind coalesce and constitute a mixed maximum value.

§282 From these examples not only the nature of this other kind of maxima and minima is understood, but also one can also form functions of this kind arbitrarily, which admit maxima or minima of the second kind. But how, if any function was propounded, it can be decided, whether it has a maximum or minimum value of such a kind or not, will be shown in the following section, since the nature of curved lines is illustrated in the best way by this investigation. Furthermore, it is easily understood, if y was a function of x of such a kind which has a maximum or minimum value of the second kind, that then also vice versa x will be a function of such a kind of y. For, since from this equation $(y-x)^2 = (1-x)^3$ for $x = 1$ the function y has a maximum value of the second kind, if the variables x and y are interchanged, this equation $(x-y)^2 = (1-y)^3$ for y will exhibit a function of such a kind of x, which has a maximum value of the second kind. For, having put $x = 1 + \omega$ it will be $(1 + \omega - y)^2 = (1-y)^3$; hence, if we set $y = 1 + \varphi$, it will be $(\omega - \varphi)^2 = (-\varphi)^3 = -\varphi^3$ and hence φ must be negative. Therefore, let $y = 1 - \varphi$; it will be $(\omega + \varphi)^2 = \varphi^3$, because having taken a very small φ φ^3 vanishes with respect to φ^2, ω will have to be negative; therefore, no real values of y correspond to the value $x = 1 + \omega$. But having put $x = 1 - \omega$ and $y = 1 - \varphi$ because of $(\varphi - \omega)^2 = \varphi^3$ it will be $\varphi = \omega \pm \omega\sqrt{\omega}$ and hence $y = 1 - \omega \mp \omega\sqrt{\omega}$, whence both values of y corresponding to $x = 1 - \omega$ are smaller than the value $y = 1$ which corresponds to the value $x = 1$; and as a logical consequence this value of y will be a maximum value of the second kind.

§283 Up to now we only considered biform functions, whose maxima or minima, since both values can easily be expressed by means of the resolution of the quadratic equation, can be checked for their correctness. But if the function y is expressed by means of an equation of higher order, the method explained before we used to investigate maxima and minima of the first kind can be applied with the same success. But let us reserve the discussion of maxima and minima of the second kind for the following treatise. Therefore, let us show how to treat triform and multiform functions in several examples.

EXAMPLE 1

Define the function y whose maxima and minima are in question by means of this equation

$$y^3 + x^3 = 3axy.$$

Having differentiated this equation it is

$$3y^2dy + 3xxdy = 3axdy + 3aydx$$

and hence

$$\frac{dy}{dx} = \frac{ay - xx}{yy - ax}.$$

Therefore, a maximum or minimum will exist, if it was $ay = xx$ or $y = \frac{xx}{a}$, which value, having substituted it in the propounded equation gives

$$\frac{x^6}{a^3} + x^3 = 3x^3 \quad \text{and} \quad x^6 = 2a^3x^3.$$

Therefore, it will be $x = 0$ trice, in which case because of $y = \frac{xx}{a} = 0$ also the denominator $yy - ax = 0$. Whether in this case a maximum or minimum results, will be seen, if we attribute a value differing hardly from 0 to x. Therefore, let $x = \omega$ and $y = \varphi$; because of $\varphi^3 + \omega^3 = 3a\omega\varphi$ it will be either $\varphi = \alpha\sqrt{\omega}$ or $\varphi = \beta\omega^2$. In the first case it will be $\alpha^3\omega\sqrt{\omega} = 3\alpha a\omega\sqrt{\omega}$ and hence $\alpha = \sqrt{3}a$. Therefore, having put $x = \omega$ it will be $y = \pm\sqrt{3}\alpha\omega$. Hence, even though ω cannot be taken negatively, nevertheless one of the two values of y will be greater than 0, the other will be smaller and hence $y = 0$ will be neither a maximum nor a minimum value. But if one sets $\varphi = \beta\omega^2$, it will be $\omega^3 = 3aa\beta\omega^3$ and hence $\beta = \frac{1}{3a}$ and $\varphi = \frac{\omega^2}{3a}$. Therefore, in this case, no matter whether x is taken $= +\omega$ or $= -\omega$, the value of $y = \varphi$ will be greater than zero and hence in this case $y = 0$ will be a minimum value. Therefore, finally the third case to be examined follows from the equation $x^3 = 2a^3$, which gives $x = a\sqrt[3]{2}$ and $y = a\sqrt[3]{4}$. Whether this is a maximum or a minimum, is to be investigated using the second differential of the equation $\frac{dy}{dx} = \frac{ay-xx}{yy-ax}$, which differential because of $dy = 0$ and $ay - xx = 0$ will be $\frac{ddy}{dx^2} = \frac{-2x}{yy-ax}$, whose value in the present case is $-\frac{2a\sqrt[3]{2}}{2a^2\sqrt[3]{2}-aa\sqrt[3]{2}} = -\frac{2}{a}$, which indicates that the value of y is a maximum value.

Example 2

If the function y is defined by means of this equation $y^4 + x^4 + ay^3 + ax^3 = b^3x + b^3y$, *to find its maximum and minimum values.*

Because by differentiation this equation results

$$4y^3dy + 3ayydy - b^3dy = b^3 - 3axxdx - 4x^3dx,$$

it will be

$$\frac{dy}{dx} = \frac{b^3 - 3axx - 4x^3}{4y^3 + 3ayy - b^3}$$

and one has to put $b^3 = 3axx + 4x^3$. Therefore, the question is reduced to this, that the maxima and minima of the uniform function $b^3 - ax^3 - x^4$ are investigated, which at the same time will be the maxima and the minima of the function y. Let $a = 2$ and $b = 3$ or let this equation be propounded $y^4 + x^4 + 2y^3 + 2x^3 = 27x + 27y$; it will be $\frac{dy}{dx} = \frac{27-6xx-4x^3}{4y^3+6yy-27}$ and $4x^3 + 6xx - 27 = 0$, which divided by $2x - 3 = 0$ gives $2xx + 6x + 9 = 0$; because the roots of the second equation are imaginary, it will be $x = \frac{3}{2}$ and $y^4 + 2y^3 - 27y = \frac{459}{16}$, the single roots of which will be either maxima or minima. But because it is $\frac{dy}{dx} = \frac{27-6xx-4x^3}{4y^3+6yy-27}$, it will be $\frac{ddy}{dx^2} = \frac{-12x-12xx}{4y^3+6yy-27}$, which for $x = \frac{3}{2}$, if it was positive, will indicate a minimum, otherwise a maximum.

Example 3

If it was $y^m + ax^n = by^px^q$, *to define the maximum and minimum values of y.*

By means of differentiation it is

$$\frac{dy}{dx} = \frac{qby^px^{q-1} - nax^{n-1}}{my^{m-1} - pby^{p-1}x^q},$$

having put which fraction $= 0$ it will at first be $x = 0$, if n and q are greater than 1, and at the same time it was $y = 0$. To decide, whether in this case a maximum or a minimum value is given, the closest values are to be investigated, since also the denominator becomes $= 0$; this investigation will depend mainly on the exponents. Furthermore, the equation $\frac{dy}{dx} = 0$ will give $y^p = \frac{na}{qb} x^{n-q}$, which value, having substituted it in the propounded equation, by putting $\frac{na}{qp} = g$ will give

$$g^{\frac{m}{p}} x^{\frac{mn-mq}{p}} + ax^n = \frac{na}{q} x^n \quad \text{or} \quad g^{\frac{m}{p}} x^{\frac{mn-mq-np}{p}} = \frac{(n-q)a}{q},$$

whence it is

$$x = \left(\frac{(n-q)a}{q} \right)^{p:(mn-mq-np)} : g^{m:(mn-mq-np)}$$

and at the same time the value of y becomes known. Further, it is to be considered, whether the second differential

$$\frac{ddy}{dx^2} = \frac{q(q-q)by^p x^{q-2} - n(n-1)ax^{n-2}}{my^{m-1} - pby^{p-1}x^q}$$

obtains a positive or a negative value, since the first the first case a minimum value will be indicated, in the second a maximum value.

EXAMPLE 4

If it was $y^4 + x^4 = 4xy - 2$, to assign the maxima and minima of the function y.

By differentiation it is

$$\frac{dx}{dy} = \frac{y - x^3}{y^3 - x}$$

and hence $y = x^3$ results; therefore, it will be $x^{12} = 3x^4 - 2$ or $x^{12} - 3x^4 + 2 = 0$, which equation is resolved into these $x^4 - 1 = 0$ and $x^8 + x^4 - 2 = 0$ and the latter into $x^4 - 1 = 0$ and $x^4 + 2 = 0$. Therefore, it will be either $x = \pm 1$ or $x = -1$ twice; in both cases also the denominator of the fraction $\frac{dy}{dx}$ vanishes. Therefore, to investigate, whether in these cases a maximum or a minimum exists, let us put $x = 1 - \omega$ and $y = 1 - \varphi$; it will be

$$1 - 4\varphi + 6\varphi^2 - 4\varphi^3 + \varphi^4 + 1 - 4\omega + 6\omega^2 - 4\omega^3 + \omega^4$$

$$= 4 - 4\omega - 4\varphi + 4\omega\varphi - 2$$

and hence

$$4\omega\varphi = 6\varphi^2 + 6\omega^2 - 4\varphi^3 - 4\omega^3 + \varphi^4 + \omega^4$$

and because of the very small ω and φ it is $4\omega\varphi = 6\varphi^2 + 6\omega^2$. Therefore, the value of φ will be imaginary, no matter whether ω is taken positively or negatively. Or if y and x denote the coordinates of a curve, it in the case $x = 1$ and $y = 1$ will have a conjugated point. Therefore, also this value can be neither a maximum nor a minimum, since the preceding and following, to which it would have been compared, become imaginary.

§284 If the equation expressing the relation among x and y was of such a nature that the function of y becomes equal to a function of x, say $Y = X$ to find the maxima or minima one will have to put $dX = 0$; therefore, y will have a maximum or a minimum value in the same cases, in which X has a maximum or a minimum value. In like manner, if x is considered as a function of y, x will have a maximum or a minimum value, if $dY = 0$, this means, if Y had a maximum or a minimum value. And from this it does nevertheless not follow that y at the same time has a maximum or a minimum value. For, if it was $2ay - yy = 2bx - xx$, y will have a maximum or a minimum value, if $x = b$, and it will be $y = a \pm \sqrt{aa - bb}$. On the other hand, x has a maximum or a minimum value, if $y = a$, and it is $x = b \pm \sqrt{bb - aa}$ and therefore y is neither a maximum nor a minimum, if $x = b \pm \sqrt{bb - aa}$, in which case x nevertheless has a maximum or a minimum value. Furthermore, in this case, if y has maximum or minimum values, x will have none at all; for, y cannot have a maximum or a minimum value, if it was not $a > b$, in which case the maximum or minimum of x becomes imaginary.

§285 Then it can also happen that not all roots of the equation $dX = 0$ yield maximum or minimum values of y; for, if that equation had two equal roots, from this neither a maximum nor a minimum follows; the same happens, if an even number or roots were equal to each other. So, if the equation $b(y-a)^2 = (x-b)^3 + c^3$ is propounded, since having taken the differentials it is $2bdy(y-a) = 3dx(x-b)^2$, the function y will have neither a maximum nor a minimum value for $x = b$, since here two equal roots occur. But if x is considered as a function of y, it will have a maximum or minimum value, if one sets $y = a$, and $x = b - c$ will be a minimum. Finally, since in equations of this kind $Y = X$ the variables x and y are not mixed, if a value is attributed to x, which is a root of the equation $dX = 0$, all values of y, no matter how many were real, will be maxima or minima; this does not happen, if in the equation the two variables were mixed.

§286 We will reserve the things, which are still to be explained about the nature of maxima and minima, for the following book, since they can be represented and explained in a more convenient way by means of figures. Therefore, let us proceed to functions which are composited of several variables, and let us investigate the values, which have to be attributed to the single variables so that the function has a maximum or a minimum value. And at first it is certainly clear, if the variables were not mixed, such that a function of this kind is propounded $X + Y$, where X is a function only of x and Y is a function only of y, that then the propounded function $X + Y$ will have a maximum, if at the same time X and Y have maximum value, and a minimum, if X and Y have a minimum value at the same time. Therefore, to find the maxima and minima, find the values of x, for which X has a maximum value, and in like manner the values of y, for which Y has a maximum value; analogously for minima, of course. Therefore, one must be careful not to combine two values of x and y of different nature, of which one renders X maximal, the other Y minimal, or vice versa. For, if this would happen, the function $X + Y$ would have neither a maximum nor a minimum value in this case. But a function of this kind $X - Y$ will have maximum value, if X has a maximum value and at the same time Y has a minimum value; on the other hand $X - Y$ will have a minimum value, if X had a minimum value and Y a maximum value. But if both of the functions X and Y would have either a maximum or minimum value, their difference $X - Y$ would have neither a maximum nor a minimum value; all these things are clear and perspicuous from the nature of maxima and minima explained before.

§287 Therefore, if the maximum or minimum values of a function of two variables are in question, the question is a lot more intricate than the same question for only one variable. For, not only for each variable the cases, in which a maximum or a minimum is produced, are to be distinguished carefully, but also from these two of such a kind are to be combined that the propounded function has a maximum or a minimum value; this will become more clear from the consideration of some examples.

EXAMPLE 1

Let this function of the two variables x and y be propounded $y^4 - 8y^3 + 18y^2 - 8y + x^3 - 3xx - 3x$ and let the values for y and x be in question which are to be substituted that this function has a maximum or minimum value.

Since this expression is resolved into two parts of this kind $Y + X$, of which one is a function only of y, the other a function only of x, investigate the cases, in which each of has a maximum or minimum value. Therefore, because it is

$$Y = y^4 - 8y^3 + 18y^2 - 8y,$$

it will be

$$\frac{dY}{dy} = 4y^3 - 24y^2 + 36y - 8;$$

having put this expression equal to zero and having divided by 4 it will be

$$y^3 - 6y^2 + 9y - 2 = 0,$$

whose roots are $y = 2$ and $y = 2 \pm \sqrt{3}$. Therefore, because it is $\frac{ddY}{4dy^2} = 3yy - 12y + 9$, in the case $y = 2$ a maximum will result. For the remaining two roots $y = 2 \pm \sqrt{3}$, which result from the equation $yy - 4y + 1 = 0$, it will be $\frac{ddY}{12dy^2} = yy - 4y + 3 = 2$, whence both of them give a minimum. But in these cases it will be as follows:

$$\begin{array}{l|ll} y = 2 & Y = +8 & \text{maximum} \\ y = 2 - \sqrt{3} & Y = -1 & \text{minimum} \\ y = 2 + \sqrt{3} & Y = -1 & \text{minimum} \end{array}$$

In like manner, because it is

$$X = x^3 - 3xx - 3x,$$

it will be

$$\frac{dX}{dx} = 3xx - 6x - 3,$$

whence this equation results

$$xx = 2x + 1$$

and $x = 1 \pm \sqrt{2}$. But it is $\frac{ddX}{6dx^2} = x - 1 = \pm\sqrt{2}$. Therefore, the root $x = 1 + \sqrt{2}$ gives a minimum, namely $X = -5 - 4\sqrt{2}$, and $x = 1 - \sqrt{2}$ gives a maximum, namely $X = -5 - 4\sqrt{2}$. Therefore, the propounded formula

$$X + Y = y^4 - 8y^3 + 18yy - 8y + x^3 - 3xx - 3x$$

will have a maximum value, if one puts $y = 2$ and $x = 1 - \sqrt{2}$, and $X + Y = 3 + 4\sqrt{2}$ will result. But the same formula $X + Y$ will have a minimum, if one takes either $y = 2 - \sqrt{3}$ or $y = 2 + \sqrt{3}$ and $x = 1 + \sqrt{2}$; in both cases it will be $X + Y = -6 - 4\sqrt{2}$.

Example 2

If this function of two variables is propounded $y^4 - 8y^3 + 18y^2 - 8y - x^3 + 3xx + 3x$, to investigate in which which cases it has either a maximum or minimum value.

Having put, as we did in the preceding example,

$$Y = y^4 - 8y^3 + 18y^2 - 8y \quad \text{and} \quad X = x^3 - 3xx - 3x$$

the propounded formula will be $Y - X$ and will hence have a maximum, if Y had a maximum value and X a minimum value. Therefore, because we found these cases before already, it is plain that $Y - X$ has a maximum value, if one puts $y = 2$ and $x = 1 + \sqrt{2}$; and it will be $Y - X = 13 + 4\sqrt{2}$. The value of $Y - X$ will be a minimum, if Y has a minimum value and X a maximum value, which happens by putting $y = 2 \pm \sqrt{3}$ and $x = 1 - \sqrt{2}$; it will be $Y - X = 4 - 4\sqrt{2}$. Furthermore, in each of the two examples it is plain that these values we found are neither the largest nor the smallest of all; for, if, for the sake of an example, one would put $y = 100$ and $x = 0$, without any doubt a value greater than the one we found would result; and in like manner by putting $y = 0$ and either $x = -100$ or $x = +100$ a smaller value than those we found for the case of a minimum would result. Therefore, recall the definition of maxima and minima we gave above is, of course that we called a value a maximum, which is larger than the closest preceding and the following values, but this value is a minimum, if it was smaller than both the closest preceding and following values. So, in this example the value of $Y - X$, which results by putting $y = 2$ and $x = 1 + \sqrt{2}$, is larger than those which result, if one puts $y = 2 \pm \omega$ and $x = 1 + \sqrt{2} \pm \varphi$ having taken sufficiently small quantities for ω and φ.

§288 Having treated these examples the way to find the general solution will be easier. Let V denote any function of the two variables x and y and let the values to be found for x and y which render the function maximal or minimal. Therefore, because to achieve this a determined value is to be attributed to both variables x and y, let us put that the one y already has the value, which is required to render the function V either maximal or minimal, and having put this it will only be necessary that for the other variable x also an appropriate value is found, which will happen, if the function V is differentiated considering only x as variable and the differential is set equal to zero. In a like manner, if we assume the variable x to already have the value, which is apt to render the function V either maximal or minimal, the value of y will be found by differentiating with respect to y only and putting this differential equal to zero. Hence, if the differential of the function V was $= Pdx + Qdy$, it will be necessary that $P = 0$ and $Q = 0$, from which two equations the value of both variables x and y can be found.

§289 Since this way without any difference the values for x and y are found, by which the function V is rendered either maximal or minimal, the cases, in which either a maximum or a minimum value results, are to be distinguished carefully from each other. For, that a function V has a maximum value, it is necessary, that both variables conspire for this; for, if the one would exhibit a maximum, the other a minimum, the function itself would become neither a maximum nor a minimum. Therefore, having found the values of x and y from the equations P and Q it is to be investigated, whether both a the same time render the function V either maximal or minimal; and just then, after it is certain that the value of both variables found from this lead to a maximum value, we will be able to affirm that the function in this case has a maximum value. The same is to be understood for a minimum, such that the function V can only have a minimum value, if at the same time both variables x and y produce a minimum. Therefore, all those cases are to be rejected, in which the one variable is detected to indicate a maximum, and the other a minimum. Sometimes it even happens that the values resulting from the equations $P = 0$ and $Q = 0$ of one of the variables or of both exhibit neither a maximum nor a minimum, which cases are to be rejected as inept in exactly the same way.

§290 But whether the values found for x and y lead to a maximum or a minimum, will be investigated for each of them separately in the same way as above, as if only one variable was there. To make the decision for the variable x consider the other y as constant, and because it is $dV = Pdx$ or $\frac{dV}{dx} = P$, differentiate P again having put y to be constant that $\frac{ddV}{dx^2} = \frac{dP}{dx}$ results, and consider, whether the value of $\frac{dP}{dx}$, after the values found before were substituted for x and y, becomes positive or negative; for, in the first case a minimum will be indicated, in the second a maximum. Because in like manner for constant x it is $dV = Qdy$ or $\frac{dV}{dy} = Q$, differentiate Q again having put only y to be variable and examine the value $\frac{dQ}{dy}$ after having substituted the values for x and y, which were found from the equations $P = 0$ and $Q = 0$; if it was affirmative, it will indicate a minimum, otherwise a maximum. Therefore, it is concluded, if from the values found for x and y the formulas $\frac{dP}{dx}$ and $\frac{dQ}{dy}$ have values affected with different signs, the one positive, the other negative, then the function V will have neither a maximum nor minimum value; but if both of the formulas $\frac{dP}{dx}$ and $\frac{dQ}{dy}$ become affirmative, a minimum value will result, and otherwise, if both of them become negative, a maximum value.

§291 If one of the formulas $\frac{dP}{dx}$ and $\frac{dQ}{dx}$ or even both vanish, if the found values are substituted for x and y, then one will have to proceed to the following differentials $\frac{ddP}{dx^2}$ and $\frac{ddQ}{dy^2}$; if they not equally vanish, neither a maximum nor a minimum will exist; but if they vanish, then the decision is to be made from the following differentials $\frac{d^3P}{dx^3}$ and $\frac{d^3Q}{dy^3}$ in the same way it was made for the formulas $\frac{dP}{dx}$ and $\frac{dQ}{dy}$. To be explain the cases, in which this happens, in a more clear way, let the value $x = \alpha$ have been resulted; if it renders the formula $\frac{dP}{dx}$ vanishing, it is necessary that $\frac{dp}{dx}$ has the factor $x - \alpha$; if this factor was the only one of this kind, neither maximum nor a minimum will be indicated; the same happens, if $\frac{dP}{dx}$ had the factor $(x - \alpha)^3$ or $(x - \alpha)^5$ etc. But if the factor was either $(x - a)^2$ or $(x - \alpha)^4$ etc., then a maximum or minimum will be indicated; but furthermore, one will have to see, whether it is in agreement with the case indicated by y.

§292 But the work to proceed to higher differentials in these cases can be reduced tremendously; for, if, in order to consider the subject in more generality, we assume that $\alpha x + \beta = 0$ was found and the formula $\frac{dP}{dx}$ has the factor $(\alpha x + \beta)^2$ such that it is $\frac{dP}{dx} = (\alpha x + \beta)^2 T$, since it is $\alpha x + \beta = 0$, it will be $\frac{d^3P}{dx^3} = 2\alpha^2 T$ and hence because of the positive $2\alpha^2$ the decision can be made from the quantity T itself; if it has a positive value it will indicate a minimum, otherwise a maximum. And the same auxiliary theorem can be applied in the investigation of maxima and minima, if one single variable is contained, such that it never necessary to ascend to higher differentials. It is not even necessary to ascend to the second differential; for, if from the equation $P = 0$ it is $\alpha x + \beta = 0$, it is necessary, that P has a factor $\alpha x + \beta$; let $P = (\alpha x + \beta)T$, and because it is

$$\frac{dP}{dx} = \alpha T + (\alpha x + \beta)\frac{dT}{dx},$$

because of $\alpha x + \beta = 0$ it will be $\frac{dP}{dx} = \alpha T$ and hence already the factor T itself, depending on whether the value of αT was positive or negative, will immediately indicate a minimum or a maximum.

§293 Therefore, having given these prescriptions, it will not be difficult, if any function involving two variables was propounded, to investigate the cases, in which this function has maximum or minimum value. If more is to be taken into account, the expansion of some examples will suggest this, which is why it will be convenient to illustrate the given rules by some examples.

EXAMPLE 1

Let this function of two variables be propounded $V = xx + xy + yy - ax - by$; to investigate, in which cases it has maximum or minimum values.

Because it is $dV = 2xdx + ydx + xdy + 2ydy - adx - bdy$, if this expression is compared to the general formula $dV = Pdx + Qdy$, it will be

$$P = 2x + y - a \quad \text{and} \quad Q = 2y + x - b,$$

whence these equations will be formed

$$2x + y - a = 0 \quad \text{and} \quad 2y + x - b = 0,$$

having combined which and thus eliminating y it will be $x - b = 4x - 2a$ and hence

$$x = \frac{2a-b}{3} \quad \text{and} \quad y = a - 2x = \frac{2b-a}{3}.$$

Therefore, because it is

$$\frac{dP}{dx} = 2 \quad \text{and} \quad \frac{dQ}{dy} = 2,$$

both of them indicate a minimum; from this we conclude that the formula

$$xx + xy + yy - ax - by$$

has a minimum value, if one puts $x = \frac{2a-b}{3}$ and $y = \frac{2b-a}{3}$, and this way it will be

$$V = \frac{-3aa + 3ab - 3bb}{9} = \frac{-aa + ab - bb}{3};$$

since this is the only one, it will be the smallest of all. Therefore, it can only in one way be

$$xx + xy + yy - ax - by = \frac{-aa + ab - bb}{3},$$

and since it cannot be smaller, this equation will be impossible

$$xx + xy + yy - ax - by = \frac{-aa + ab - bb}{3} - cc.$$

EXAMPLE 2

If the formula $V = x^3 + y^3 - 3axy$ is propounded, let the cases be in question in which V has a maximum or minimum value.

Because of $dV = 3xxdx + 3yydy - 3aydx - 3axdy$ it will be

$$P = 3xx - 3ay \quad \text{and} \quad Q = 3yy - 3ax,$$

whence it is

$$ay = xx \quad \text{and} \quad ax = yy.$$

Therefore, because if it is $yy = x^4 : aa = ax$, it will be $x^4 - a^3x = 0$ and hence either $x = 0$ or $x = a$. In the first case it is $y = 0$, in the second $y = a$. Therefore, because it is

$$\frac{dP}{dx} = 6x, \quad \frac{ddP}{dx^2} = 6 \quad \text{and} \quad \frac{dQ}{dy} = 6y \quad \text{and} \quad \frac{ddQ}{dy^2} = 6,$$

in the first case, in which it is $x = 0$ and $y = 0$, neither a maximum nor a minimum results. But in the second case, in which both $x = a$ and $y = a$, a minimum results, if a was a positive quantity, and it will be $V = -a^3$, which value is only smaller than the closest preceding and following ones; for, without any doubt V can have much smaller values, if negative values are attributed to both variables x and y.

EXAMPLE 3

Let this function be propounded $V = x^3 + ayy - bxy + cx$, whose maximal or minimal values are to be investigated.

Since it is $dV = 3xxdx + 2aydy - bydx - bxdy + cdx$, it will be

$$P = 3xx - by + c \quad \text{and} \quad Q = 2ay - bx,$$

having put which values equal to zero it will be $y = \frac{bx}{2a}$ and hence

$$3xx - \frac{bbx}{2a} + c = 0 \quad \text{or} \quad xx = \frac{2bbx - 4ac}{12a},$$

whence it is

$$x = \frac{bb \pm \sqrt{b^4 - 48aac}}{12ac}.$$

Therefore, only if it is $b^4 - 48aac > 0$, a maximum or a minimum can exist. Therefore, let us put that it is $b^4 - 48aac = bbff$ that $c = \frac{bb(bb-ff)}{48aa}$; it will be

$$x = \frac{bb \pm bf}{12a} \quad \text{and} \quad y = \frac{bb(b \pm f)}{24aa}.$$

Since further it is

$$\frac{dP}{dx} = 6x \quad \text{and} \quad \frac{dQ}{dy} = 2a,$$

it will be

$$\frac{dP}{dx} = \frac{b(b \pm f)}{2a}.$$

Therefore, only if $2a$ and $\frac{b(b\pm f)}{2a}$ are quantities of the same sign, either a maximum or a minimum can exist. And if they are indeed both either positive or negative, what happens, if their product $b(b \pm f)$ was positive, then the function V has a minimum value, if a was a positive quantity, otherwise a maximum value, if a is a negative quantity. Hence, if it was $f = 0$ or $c = \frac{b^4}{48aa}$, because of the positive quantity bb the function V will have a minimum value, if a was a positive quantity and one sets $x = \frac{bb}{12a}$ and $y = \frac{b^3}{24aa}$; otherwise, if a is negative, these substitutions produce a maximum. If $f < b$, in the two cases either a maximum or a minimum results; but if $f > b$, then only the case $x = \frac{b(b+f)}{12a}$ and $y = \frac{bb(b+f)}{24aa}$ will yield a maximum or a minimum, depending on whether a was negative or positive. Let $a = 1$, $b = 3$ and $f = 1$, that one has this formula $V = x^3 + yy - 3xy + \frac{3}{2}x$; this will have a minimum value because of the positive a, if one puts either $x = 1$ and $y = \frac{3}{2}$ or $x = \frac{1}{2}$ and $y = \frac{3}{4}$. In the first case it is $V = \frac{1}{4}$, in the second $V = \frac{5}{16}$. It is nevertheless plain that by putting negative numbers instead of x a lot smaller values for V can result. Therefore, the value of $V = \frac{1}{4}$ must be understood to be smaller than if one puts $x = 1 + \omega$ and $y = \frac{3}{2} + \varphi$, if ω and φ are small numbers, either positive or negative; but ω must not be larger than $-\frac{15}{4}$; for, if $\omega < -\frac{15}{4}$, it can happen that V becomes smaller than $\frac{1}{4}$.

Example 4

To find the maxima or minima of this function

$$V = x^4 + y^4 - axxy - axyy + ccxx + ccyy.$$

Having taken the differential it will be

$$P = 4x^3 - 2axy - ayy + 2ccx \quad \text{and} \quad Q = 4y^3 - axx - 2axy + 2ccy,$$

having put which equal to zero and having subtracted them from each other it will be

$$4x^3 - 4y^3 + axx - ayy + 2ccx - 2ccy = 0;$$

since it is divisible by $x - y$, it will be $y = x$ at first and then $4x^3 - 3axx + 2ccx = 0$ which gives

$$x = 0 \quad \text{and} \quad 4xx = 3ax - 2cc \quad \text{and} \quad x = \frac{3a \pm \sqrt{9aa - 32cc}}{8}.$$

If we take $x = 0$, it will also be $y = 0$ and because of

$$\frac{dP}{dx} = 12xx - 2ay + 2cc \quad \text{and} \quad \frac{dQ}{dy} = 12yy - 2ax + 2cc$$

the function V has a minimum value $= 0$. If we set $x = y = \frac{3a \pm \sqrt{9aa-32cc}}{8}$, if it was $9aa > 32cc$, because of $4xx = 3ax - 2cc$ it will be

$$\frac{dP}{dx} = \frac{dQ}{dy} = 12xx - 2ax + 2cc = 7ax - 4cc = \frac{21aa - 32cc \pm 7a\sqrt{9aa - 32cc}}{8};$$

because this value is always positive because of $32cc < 9aa$, the value V will also in this case be a minimum and it will be

$$V = -\frac{27}{256}a^4 + \frac{9}{16}aacc - \frac{1}{2}c^4 \mp \frac{a}{256}(9aa - 32cc)^{\frac{3}{2}}.$$

But let us divide the equation $4x^3 - 4y^3 + axx - ayy + 2ccx - 2ccy = 0$ by $x - y$ and it will be $4xx + 4xy + 4yy + ax + ay + 2cc = 0$. But from the equation $P = 0$ it will be $yy = -2xy + \frac{4}{a}x^3 + \frac{2ccx}{a}$, having substituted which value in that equation it is

$$y = \frac{16x^3 + 4axx + aax + 8ccx + 2acc}{4ax - aa}.$$

But this gives

$$y = -x \pm \sqrt{\frac{4x^3 + axx + 2ccx}{a}},$$

whence it is

$$16x^3 + 8axx + 8ccx + 2acc = (4x - a)\sqrt{4ax^3 + aaxx + 2accx},$$

which equation, having reduced it to a rational one, gives

$$\begin{array}{lllllllll} 256x^6 + 192ax^5 & +80aa & x^4 & +4a^3 & x^3 & -a^2 & x^2 & -2a^3ccx & +4a^2c^4 = 0; \\ & +256cc & & +160acc & & +48aacc & & +32ac^4 & \\ & & & & & +64c^4 & & & \end{array}$$

whose real roots, if it has such, will indicate the maxima or minima of the function V, if $\frac{dP}{dx}$ and $\frac{dQ}{dy}$ become rational quantities affected with the same sign.

EXAMPLE 5

To find the maxima and minima of this expression

$$x^4 + mxxyy + y^4 + aaxx + naaxy + aayy = V.$$

After the differentiation it will be

$$\begin{aligned} P &= 4x^3 + 2mxyy + 2aax + naay = 0, \\ Q &= 4y^3 + 2mxxy + 2aay + naax = 0, \end{aligned}$$

which equation either subtracted from or added to each other give

$$\begin{aligned} &(4xx + 4xy + 4yy - 2mxy + 2aa - naa)(x - y) = 0, \\ &(4xx - 4xy + 4yy + 2mxy + 2aa + naa)(x + y) = 0, \end{aligned}$$

which divided by $x - y$ and $x + y$ and either added or subtracted again give

$$4xx + 4yy + 2aa = 0 \quad \text{and} \quad 4xy - 2mxy - naa = 0.$$

From the latter it is $y = \frac{naa}{2(2-m)x}$; but the first does not admit real values. Therefore, we have three cases.

I. Let $y = x$ and it will be $4x^3 + 2mx^3 + 2aax + naax = 0$, whence it is either $x = 0$ or $2(2+m)xx + (2+n)aa = 0$. Let $x = 0$; it will be $y = 0$ and because of

$$\frac{dP}{dx} = 12xx + 2myy + 2aa \quad \text{and} \quad \frac{dQ}{dy} = 12yy + 2mxx + 2aa$$

in this case $V = 0$ will become a minimum, if the coefficient aa was positive. The other case gives $xx = -\frac{(n+2)aa}{2(m+2)}$, which can only be real, if $\frac{n+2}{m+2}$ is a negative number. Let $\frac{n+2}{m+2} = -2kk$ or $n = -2kkm - 4kk - 2$; it will be $x = \pm ka$ and $y = \pm ka$. But

$$\frac{dP}{dx} = 12kkaa + 2mkkaa + 2aa \quad \text{and} \quad \frac{dQ}{dy} = 12kkaa + 2mkkaa + 2aa;$$

since they are equal, V will have a maximum or a minimum value, depending on whether these quantities were positive or negative.

II. Let $y = -x$ and it will be $2(m+2)x^3 = (n-2)aax$, therefore either $x = 0$ or $xx = \frac{(n-2)aa}{2(m+2)}$. The first root $x = 0$ reduces to the preceding. The second on the other hand will be real, if $\frac{(n-2)aa}{2(m+2)}$ was a positive quantity, and because it is $\frac{dP}{dx} = \frac{dQ}{dy}$, either maximum or a minimum will result.

III. Let $y = \frac{naa}{2(2-m)x}$; it will be

$$4x^3 + \frac{mn^2a^4}{2(2-m)^2x} + 2aax + \frac{nna^4}{2(2-m)x} = 0 \quad \text{or} \quad 4x^4 + 2aaxx + \frac{nna^4}{(2-m)^2} = 0,$$

no root of which equation is real, if aa is not a negative quantity.

EXAMPLE 6

Let this determined function be propounded $V = x^4 + y^4 - xx + xy - yy$, whose maximal or minimal values are to be investigated.

Since therefore it is $P = 4x^3 - 2x + y = 0$ and $Q = 4y^3 - 2y + x = 0$, from the first it will be $y = 2x - 4x^3$ which, having substituted in the other, gives

$$256x^9 - 384x^7 + 192x^5 - 40x^3 + 3x = 0.$$

One of its roots is $x = 0$, whence it also is $y = 0$. Therefore, in this case because of

$$\frac{dP}{dx} = 12xx - 2 \quad \text{and} \quad \frac{dQ}{dy} = 12yy - 2$$

a maximum value $V = 0$ results. But having divided the found equation by x it will be

$$256x^8 - 384x^6 + 192x^4 - 40xx + 3 = 0,$$

which has the factor $4xx-1$, whence it is $4xx=1$ and $x=\pm\frac{1}{2}$ and $y=\pm\frac{1}{2}$; then it will be $\frac{dP}{dx}=\frac{dQ}{dy}=1$; therefore, in each of the two cases a maximum value $V=-\frac{1}{8}$ results. Divide that equation by $4xx-1$ and one will obtain

$$64x^6-80x^4+28xx-3=0,$$

which again contains $4xx-1=0$ twice such that the preceding case results. Furthermore, from this it is $4xx-3=0$ and $x=\frac{\pm\sqrt{3}}{2}$; $y=\frac{\mp\sqrt{3}}{2}$ corresponds to this. Therefore, it will also be $\frac{dP}{dx}=\frac{dQ}{dy}=7$ and therefore V will have a minimum value $=-\frac{9}{8}$; this is the smallest value of all, which the function V can have, and therefore, this equation $V=-\frac{9}{8}-cc$ is always impossible. But hence it is obvious, how to determine maxima and minima of functions containing three or more variables is .