On the investigation of summable series *

Leonhard Euler

§19 If the sum of the series whose terms contain the variable quantity x was known and which will therefore be a function of x, then, whatever value is attributed to x, one will always be able to assign the sum of the series. Therefore, if one puts $x + dx$ instead of x, the sum of the resulting series will be equal to the sum of the first and the differential: Therefore, it follows that the differential of the sum will be $=$ the differential of the series. Because this way so the sum as the single terms will be multiplied by dx, if one divides by dx everywhere, one will have a new series, whose sum will be known. In like manner, if this series is differentiated again and it is divided by dx, a new series will result together with its sum and this way new equally summable series will be found from one summable series involving the undetermined quantity x, if that series is differentiated several times.

§20 In order to understand all this better, let the undetermined geometric progression be propounded, whose sum is known, it is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.}$$

If this equation is now differentiated with respect to x, it will be

*Original title: " De Investigatione serierum summabilium", first published as part of the book „*Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, 1755", reprinted in in „*Opera Omnia*: Series 1, Volume 10, pp. 235 - 255 ", Eneström-Number E212, translated by: Alexander Aycock for the „Euler-Kreis Mainz"

$$\frac{dx}{(1-x)^2} = dx + 2xdx + 3x^2dx + 4x^3dx + 5x^4dx + \text{etc.}$$

and having divided by dx one will have

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \text{etc.}$$

If one differentiates again and divides by dx, this equation will result

$$\frac{2}{(1-x)^3} = 2 + 2 \cdot 3x + 3 \cdot 4x^2 + 4 \cdot 5x^3 + 5 \cdot 6x^4 + \text{etc.}$$

or

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \text{etc.}$$

where the coefficients are the triangular numbers. If one differentiates further and divides by $3dx$, one will obtain

$$\frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \text{etc.},$$

whose coefficients are the first pyramidal numbers. And, by proceeding further this way, the same series result, which are known to result from the expansion of the fraction $\frac{1}{(1-x)^n}$.

§21 This investigation will extend even further, if, before the differentiation is done, the series and the sum are multiplied by a certain power of x or even a function of x. So, because it is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \text{etc.},$$

multiply by x^m everywhere and it will be

$$\frac{x^m}{1-x} = x^m + x^{m+1} + x^{m+2} + x^{m+3} + x^{m+4} + \text{etc.}$$

Now differentiate this series and having divided the result by dx it will be

$$\frac{mx^{m-1} - (m-1)x^m}{(1-x)^2} = mx^{m-1} + (m+1)x^m + (m+2)x^{m+1} + (m+3)x^{m+2} + \text{etc.}$$

Now divide by x^{m-1}; one will have

$$\frac{m-(m-1)x}{(1-x)^2} = \frac{m}{1-x} + \frac{x}{(1-x)^2} = m + (m+1)x + (m+2)x^2 + \text{etc.}$$

Before another differentiation is done multiply this equation by x^n that it is

$$\frac{mx^n}{1-x} + \frac{x^{n+1}}{(1-x)^2} = mx^n + (m+1)x^{n+1} + (m+2)x^{n+2} + \text{etc.}$$

Now, do the differentiation and having divided by dx it will be

$$\frac{mnx^{n-1}}{1-x} + \frac{(m+n+1)x^n}{(1-x)^2} + \frac{2x^{n+1}}{(1-x)^3}$$
$$= mnx^{n-1} + (m+1)(n+1)x^n + (m+2)(n+2)x^{n+1} + \text{etc.}$$

But having divided by x^{n-1} it will be

$$\frac{mn}{1-x} + \frac{(m+n+1)x}{(1-x)^2} + \frac{2xx}{(1-x)^3}$$
$$mn + (m+1)(n+1)x + (m+2)(n+2)x^2 + \text{etc.}$$

and it will be possible to proceed further this way; but one will always find the same series which result from the expansions of the fractions constituting the sum.

§22 Since the sum of the geometric progression assumed at first can be assigned up to any given term, this way also series consisting of finite a number of terms will be summed. Because it is

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n,$$

after the differentiation and having divided by dx it will be

$$\frac{1}{(1-x)^2} - \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}.$$

Hence, the sum of the powers of natural numbers up to a certain term can be found. For, multiply this series by x that it is

$$\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}=x+2x^2+3x^3+\cdots+nx^n,$$

which, having differentiated it again and divided it by dx, will give

$$\frac{1+x-(n+1)x^n+(2nn+2n-1)x^{n+1}-nnx^{n+2}}{(1-x)^3}=1+4x+9x^2+\cdots+n^2x^{n-1};$$

this equation multiplied by x will give

$$\frac{x+x^2-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}=x+4x^2+9x^3+\cdots+n^2x^n,$$

which equality, if it is differentiated, divided by dx and multiplied by x, will produce this series

$$x+8x^2+27x^3+\cdots+n^2x^n,$$

whose sum can therefore be found. And from this in like manner it is possible to find the indefinite sum of the fourth powers and higher powers.

§23 Therefore, this method can be applied to all series which contain an undetermined quantity and whose sum is known, of course. Because except for geometric series all recurring series enjoy the same prerogatives that they can be summed not only up to infinity but also to any given term, one will be able to also find innumerable other summable series from these by the same method. Because a lot of work would be necessary, if we wanted to study this in more detail, let us consider only one single case.

Let this series be propounded

$$\frac{x}{1-x-xx}=x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+\text{etc.},$$

which equation, if it is differentiated and divided by dx, will give

$$\frac{1+xx}{(1-x-xx)^2}=1+2x+6x^2+12x^3+25x^4+48x^5+91x^6+\text{etc.}$$

But it easily becomes clear that all series resulting this way will also be recurring whose sums can even be found from their nature itself.

§24 Therefore, in general, if the sum of a certain series contained in this form

$$ax + bx^2 + cx^3 + dx^4 + \text{etc.}$$

was known, which sum we want to put $= S$, one will be able to find the sum of the same series, if the single terms are each multiplied by terms of an arithmetic progression. For, let

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.};$$

multiply by x^m; it will be

$$Sx^m = ax^{m+1} + bx^{m+2} + cx^{m+3} + dx^{m+4} + \text{etc.};$$

differentiate this equation and divide by dx

$$mSx^{m-1} + x^m \frac{dS}{dx} = (m+1)ax^m + (m+2)bx^{m+1} + (m+3)cx^{m+2} + \text{etc.};$$

divide by x^{m-1} and it will be

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \text{etc.}$$

Therefore, if one wants to find the sum of the following series

$$\alpha ax + (\alpha + \beta)bx^2 + (\alpha + 2\beta)cx^3 + (\alpha + 3\beta)dx^4 + \text{etc.},$$

multiply the above series by β and put $m\beta + \beta = \alpha$ that it is $M = \frac{\alpha - \beta}{\beta}$ and the sum of this series will be

$$= (\alpha - \beta)S + \frac{\beta xdS}{dx}.$$

§25 One will also be able to find the sum of this propounded series, if its single terms are each multiplied by terms of series of second order, whose second differences are just constant, of course. For, because we already found

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \text{etc.},$$

multiply this equation by x^n that it is

$$mSx^n + \frac{x^{n+1}dS}{dx} = (m+1)ax^{n+1} + (m+2)bx^{n+2} + \text{etc.};$$

differentiate this equation having put dx to be constant and divide by dx

$$mnSx^{n-1} + \frac{(m+n+1)x^nS}{dx} + \frac{x^{n+1}ddS}{dx^2}$$
$$= (m+1)(n+1)ax^n + (m+2)(n+2)bx^{n+1} + \text{etc.}$$

Divide by x^{n-1} and multiply by k that it is

$$mnkS + \frac{(m+n+1)kxdS}{dx} + \frac{kx^2ddD}{dx^2}$$
$$= (m+1)(n+1)kax + (m+2)(n+2)kbx^2 + (m+3)(n+3)kcx^3 + \text{etc.}$$

Now, compare this series to that one; it will be

	Diff. I	Diff. II
$kmn + 1km + 1kn + 1k = \alpha$		
	$km + kn + 3k = \beta$	
$knm + 2km + 2kn + 4k = \alpha + 1\beta$		$2k = \gamma$
	$km + kn + 5k = \beta + \gamma$	
$lnm + 3km + 3kn + 9k = \alpha + 2\beta + \gamma$		

Therefore, $k = \frac{1}{2}\gamma$ and $m + n = \frac{2\beta}{\gamma} - 3$ and

$$mn = \frac{\alpha}{k} - m - n - 1 = \frac{2\alpha}{\gamma} - \frac{2\beta}{\gamma} + 2 = \frac{2(\alpha - \beta + \gamma)}{\gamma}.$$

Hence, the sum of the series in question will be

$$(\alpha - \beta + \gamma)S + \frac{(\beta - \gamma)xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2}.$$

§26 In like manner, one will be able to find the sum of this series

$$Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.},$$

if the sum S of this series was known, of course,

$$S = a + bx + cx^2 + dx^3 + dx^3 + ex^4 + fx^5 + \text{etc.}$$

and A, B, C, D etc. constitute a series which is reduced to constant differences. For, since it its form is concluded from the preceding, assume this sum

$$\alpha S + \frac{\beta x dS}{dx} + \frac{\gamma x^2 ddS}{2dx^2} + \frac{\Delta x^3 d^3 S}{6dx^3} + \frac{\varepsilon x^4 d^4 S}{24dx^4} + \text{etc.}$$

Now, to find the letters α, β, γ, δ etc., expand the single series and it will be

$$\begin{aligned}
\alpha S &= \alpha a + \alpha bx + \alpha cx^2 + \alpha dx^3 + \alpha ex^4 + \text{etc.} \\
\frac{\beta x dS}{dx} &= \quad + \beta bx + 2\beta cx^2 + 3\beta dx^3 + 4\beta ex^4 + \text{etc.} \\
\frac{\gamma x^2 ddS}{2dx^2} &= \quad + \gamma cx^2 + 3\gamma dx^3 + 6\gamma ex^4 + \text{etc.} \\
\frac{\delta x^3 d^3 S}{6dx^3} &= \quad + \delta dx^3 + 4\delta ex^4 + \text{etc.} \\
\frac{\varepsilon x^4 d^4 S}{24dx^4} &= \quad + \varepsilon ex^4 + \text{etc.} \\
&\text{etc.}
\end{aligned}$$

compare this series, having arranged it according to the powers of x, to the propounded one

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.}$$

and having made the comparison of the single terms we find

$$\begin{aligned}
\alpha &= A \\
\beta &= B - \alpha = B - A \\
\gamma &= C - 2\beta - \alpha = C - 2B + A \\
\Delta &= D - 3\gamma - 3\beta - \alpha = D - 3C + 3B - A \\
&\text{etc.}
\end{aligned}$$

Having found these values the sum in question will therefore be

$$Z = AS + \frac{(B-A)xdS}{1dx} + \frac{(C-2B+A)x^2ddS}{1\cdot 2dx^2} + \frac{(D-3C+3B-A)x^3d^3S}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

or if the differences of the series A, B, C, D, E etc. are indicated in the customary manner, it will be

$$Z = AS + \frac{\Delta A\cdot xdS}{1dx} + \frac{\Delta^2 A\cdot x^2d^2S}{1\cdot 2dx^2} + \frac{\Delta^3 A\cdot x^3d^3S}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

if it was, as we assumed,

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \text{etc.}$$

Therefore, if the series A, B, C, D etc. has finally constant differences, one will be able to express the sum of the series Z in finite terms.

§27 Since having taken e for the number whose hyperbolic logarithm is $= 1$ it is

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.};$$

assume this series for the first, and because it is $S = e^x$, it will be $\frac{dS}{dx} = e^x$, $\frac{ddS}{dx^2} = e^x$ etc. Therefore, the sum of this series which is composed of that one and this one A, B, C, D etc.

$$A + \frac{Bx}{1} + \frac{Cx^2}{1\cdot 2} + \frac{Dx^3}{1\cdot 2\cdot 3} + \frac{Ex^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

will be expressed this way

$$e^x\left(A + \frac{x\Delta A}{1} + \frac{xx\Delta^2 A}{1\cdot 2} + \frac{x^3\Delta^3 A}{1\cdot 2\cdot 3} + \frac{x^4\Delta^4 A}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\right).$$

So, if this series is propounded

$$2 + \frac{5x}{1} + \frac{10x^2}{1\cdot 2} + \frac{17x^2}{1\cdot 2\cdot 3} + \frac{26x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{37x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.},$$

because of the series

$$A, \quad B, \quad C, \quad D, \quad E \quad \text{etc.}$$

$$\begin{array}{llllllll} A & = 2, & 5, & 10, & 17, & 26 & \text{etc.} \\ \Delta A & = & 3, & 5, & 7, & 9 & \text{etc.} \\ \Delta\Delta A & = & & 2, & 2, & 2 & \text{etc.} \end{array}$$

the sum of this series

$$2 + 5x + \frac{10x^2}{2} + \frac{17x^3}{6} + \frac{26x^4}{24} + \text{etc.}$$

will be

$$= e^x(2 + 3x + xx) = e^x(1 + x)(2 + x),$$

which is immediately clear. For, it is

$$\begin{aligned} 2e^x &= 2 + \frac{2x}{1} + \frac{2x^2}{2} + \frac{2x^3}{6} + \frac{2x^4}{24} + \text{etc.} \\ 3xe^x &= \quad + 3x + \frac{3x^2}{1} + \frac{3x^2}{2} + \frac{3x^4}{6} + \text{etc.} \\ xxe^x &= \qquad + xx + \frac{x^3}{1} + \frac{x^4}{2} + \text{etc.} \end{aligned}$$

and in total

$$e^x(1 + 3x + xx) = 2 + 5x + \frac{10xx}{2} + \frac{17x^3}{6} + \frac{24x^4}{24} + \text{etc.}$$

§28 The things treated up to now not only concern infinite series, but also sums of a finite number of terms; for, the coefficients a, b, c, d etc. can either proceed to infinity or can be terminated at any arbitrary point. But because this does not require any further explanation, let us consider in more detail what follows from the things mentioned up to now. Therefore, having propounded any arbitrary series, whose single terms consist of two factors, the one group of which terms constitutes a series leading to constant differences, one will be able to assign the sum of this series, as long as having omitted these factors the sum was summable. Of course, if this series is propounded

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.}$$

in which the quantities A, B, C, D, E etc. constitute a series of such a kind which is finally reduced to constant differences, then one will be able to exhibit the sum of this series, if the sum S of the following series is known

$$S = a + bx + cx^2 + dx^3 + ex^4 + \text{etc.}$$

For, having calculated the continued differences of the progression A, B, C, D, E etc., as we showed at the beginning of this book,

$$\begin{array}{ccccccccccccc}
A, & & B, & & C, & & D, & & E, & & F, & & \text{etc.} \\
& \Delta A & & \Delta B, & & \Delta C, & & \Delta D, & & \Delta E & & \text{etc.} & \\
& & \Delta^2 A & & \Delta^2 B, & & \Delta^2 C, & & \Delta^2 D & & \text{etc.} & & \\
& & & \Delta^3 A & & \Delta^3 B, & & \Delta^3 C, & & \text{etc.} & & & \\
& & & & \Delta^4 A & & \Delta^4 B, & & \text{etc.} & & & & \\
& & & & & \Delta^5 A & & \text{etc.} & & & & & \\
& & & & & & \text{etc.} & & & & & &
\end{array}$$

the sum of the propounded series will be

$$Z = SA + \frac{xdS}{1dx}\Delta A + \frac{x^2 ddS}{1 \cdot 2dx^2}\Delta^2 A + \frac{x^3 d^3 S}{1 \cdot 2 \cdot 3dx^3}\Delta^3 A + \text{etc.}$$

after having put dx to be constant in the higher powers of S.

§29 Therefore, if the series A, B, C, D etc. never leads to constant differences, the sum of the series Z will be expressed by means of a new infinite series which will converge more than the propounded one, and so this series will be transformed into another one equal to it. To illustrate this let this series be propounded

$$Y = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \frac{y^5}{5} + \frac{y^6}{6} + \text{etc.},$$

which is known to express $\ln \frac{1}{1-y}$ such that it is $Y = -\ln(1-y)$. Divide the series by y and put $y = x$ and $Y = yZ$ that it is

$$Z = -\frac{1}{y}\ln(1-y) = -\frac{1}{x}\ln(1-x);$$

it will be

$$Z = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \text{etc.},$$

which compared to this one

$$S = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.} = \frac{1}{1-x}$$

will give these values for the series A, B, C, D, E etc.

$$\begin{array}{ccccccccccc}
1, & & \frac{1}{2}, & & \frac{1}{3}, & & \frac{1}{4}, & & \frac{1}{5} & & \text{etc.} \\
& -\frac{1}{1\cdot 2}, & & -\frac{1}{2\cdot 3}, & & -\frac{1}{3\cdot 4}, & & -\frac{1}{4\cdot 5} & & \text{etc.} & \\
& & \frac{1\cdot 2}{1\cdot 2\cdot 3}, & & \frac{1\cdot 2}{2\cdot 3\cdot 4}, & & \frac{1\cdot 2}{3\cdot 4\cdot 5} & & \text{etc.} & & \\
& & & -\frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4}, & & -\frac{1\cdot 2\cdot 3}{2\cdot 3\cdot 4\cdot 5} & & \text{etc.} & & & \\
& & & & \text{etc.} & & & & & &
\end{array}$$

Therefore, it will be

$$A = 1, \quad \Delta A = -\frac{1}{2}, \quad \Delta^2 A = \frac{1}{3}, \quad \Delta^3 A = -\frac{1}{4} \quad \text{etc.}$$

Further, because it is $S = \frac{1}{1-x}$, it will be

$$\frac{dS}{dx} = \frac{1}{(1-x)^2}, \quad \frac{ddS}{1\cdot 2dx^2} = \frac{1}{(1-x)^3}, \quad \frac{d^3S}{1\cdot 2\cdot 3dx^3} = \frac{1}{(1-x)^4} \quad \text{etc.}$$

Having substituted these values this sum will result

$$Z = \frac{1}{1-x} - \frac{x}{2(1-x)^2} + \frac{x^2}{3(1-x)^3} - \frac{x^3}{4(1-x)^4} + \frac{x^4}{5(1-x)^5} - \text{etc.}$$

Therefore, because it is $x = y$ and $Y = -\ln(1-y) = yZ$, it will be

$$-\ln(1-y) = \frac{y}{1-y} - \frac{y^2}{2(1-y)^2} + \frac{y^3}{3(1-y)^3} - \frac{y^4}{4(1-y)^4} + \text{etc.},$$

which series obviously expresses $\ln\left(1 + \frac{y}{1-y}\right) = \ln\frac{1}{1-y} = -\ln(1-y)$, the validity of which is even clear considering the results demonstrated before.

§30 In order to also see the use, if only odd powers occur and the signs alternate, let this series be propounded

$$Y = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \frac{y^9}{9} - \frac{y^{11}}{11} + \text{etc.},$$

from which it is known that it is $Y = \arctan y$.

Divide this series by y and put $\frac{Y}{y} = Z$ and $yy = x$; it will be

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \text{etc.}$$

If it is compared to this one

$$S = 1 - x + xx - x^3 + x^4 - x^5 + \text{etc.},$$

it will be $S = \frac{1}{1+x}$ and the series of coefficients A, B, C, D etc. will become

$$\begin{array}{llllllllll}
A & = & 1, & & \frac{1}{3}, & & \frac{1}{5}, & & \frac{1}{7}, & \frac{1}{9}, \quad \text{etc.} \\
\Delta A & = & & -\frac{2}{3}, & & -\frac{2}{3\cdot 5}, & & -\frac{2}{5\cdot 7}, & & -\frac{2}{7\cdot 9} \quad \text{etc.} \\
\Delta^2 A & = & & & \frac{2\cdot 4}{3\cdot 5}, & & \frac{2\cdot 4}{3\cdot 5\cdot 7}, & & \frac{2\cdot 4}{5\cdot 7\cdot 9} & \text{etc.} \\
\Delta^3 A & = & & & & -\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}, & & -\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 9} & \text{etc.} & \\
\Delta^4 A & = & & & & & \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9} & \text{etc.} & & \\
& & & & & & \text{etc.} & & &
\end{array}$$

But because it is $S = \frac{1}{1+x}$, it will be

$$\frac{dS}{1dx} = -\frac{1}{(1+x)^2}, \quad \frac{ddS}{1\cdot 2dx^2} = \frac{1}{(1+x)^3}, \quad \frac{d^3S}{1\cdot 2\cdot 3dx^3} = -\frac{1}{(1+x)^4} \quad \text{etc.}$$

Hence, having substituted these values, the form will become

$$Z = \frac{1}{1+x} + \frac{2x}{3(1+x)^2} + \frac{2\cdot 4x^2}{3\cdot 5(1+x)^3} + \frac{2\cdot 4\cdot 6x^3}{3\cdot 5\cdot 7(1+x)^4} + \text{etc.}$$

having substituted $x = yy$ again and multiplied by y it will be

$$Y = \arctan y = \frac{y}{1+yy} + \frac{2y^3}{3(1+yy)^2} + \frac{2 \cdot 4y^5}{3 \cdot 5(1+yy)^3} + \frac{2 \cdot 4 \cdot 6y^7}{3 \cdot 5 \cdot 7(1+yy)^4} + \text{etc.}$$

§31 One can also transform the above series expressing the arc of a circle in another way by comparing it to the logarithmic series.

For, let us consider the series

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \text{etc.},$$

which we want to compare to this one

$$S = \frac{1}{0} - \frac{x}{2} + \frac{xx}{4} - \frac{x^3}{6} + \frac{x^4}{8} - \text{etc.} = \frac{1}{0} - \frac{1}{2}\ln(1+x),$$

and the values of the letters A, B, C, D etc. will be

$$\begin{array}{lllllllll}
A & = \frac{0}{1}, & & \frac{2}{3}, & & \frac{4}{5}, & & \frac{6}{7}, & \frac{8}{9} \quad \text{etc.} \\
\Delta A & = & \frac{2}{3}, & & \frac{+2}{3 \cdot 5}, & & \frac{+2}{5 \cdot 7}, & \frac{+2}{7 \cdot 9} & \text{etc.} \\
\Delta^2 A & = & & \frac{-2 \cdot 4}{3 \cdot 5}, & & \frac{-2 \cdot 4}{3 \cdot 5 \cdot 7}, & \frac{-2 \cdot 4}{5 \cdot 7 \cdot 9} & & \text{etc.} \\
\Delta^3 A & = & & & \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} & \text{etc.} & & & \\
 & & & & \text{etc.} & & & &
\end{array}$$

Further, because it is $S = \frac{1}{0} - \frac{1}{2}\ln(1+x)$, it will be

$$\frac{dS}{1dx} = -\frac{1}{2(1+x)}, \quad \frac{ddS}{1 \cdot 2dx^2} = \frac{1}{4(1+x)^2},$$
$$\frac{d^3S}{1 \cdot 2 \cdot 3dx^3} = -\frac{1}{6(1+x)^3}, \quad \frac{d^4S}{1 \cdot 2 \cdot 3 \cdot 4dx^4} = \frac{1}{8(1+x)^4} \quad \text{etc.}$$

Therefore, it will be $SA = D\frac{0}{1} = 1$ and from the remaining terms it will be

$$Z = 1 - \frac{x}{3(1+x)} - \frac{2xx}{3 \cdot 5(1+x)^2} - \frac{2 \cdot 4x^3}{3 \cdot 5 \cdot 7(1+x)^3} - \text{etc.}$$

Now, let us put $x = yy$ and multiply by y; it will be

$$Y = \arctan y = y - \frac{y^3}{3(1+yy)} - \frac{2y^2}{3\cdot 5(1+yy)^2} - \frac{2\cdot 4y^7}{3\cdot 5\cdot 7(1+yy)^3} - \text{etc.}$$

This transformation will therefore not be obstructed by the infinite term $\frac{1}{0}$ which entered the series S. But if there remains any doubt, just expand the single terms except for the first into power series in y and one will discover that indeed the series first propounded results.

§32 Up to this point we considered only series of such a kind in which all powers of the variable occurred. Now, we want therefore proceed to other series which in the single terms contain the same power of the variable of which kind this series is

$$S = \frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} + \text{etc.}$$

For, if the sum S of this series was known and is expressed by a certain function of x, by differentiating and by dividing by $-dx$ it will be

$$\frac{-dS}{dx} = \frac{1}{(a+x)^2} + \frac{1}{(b+x)^2} + \frac{1}{(c+x)^2} + \frac{1}{(d+x)^2} + \text{etc.}$$

If this series is differentiated again and divided by $-2dx$, one will recognize the series of the cubes

$$\frac{ddS}{2dx^2} = \frac{1}{(a+x)^3} + \frac{1}{(b+x)^3} + \frac{1}{(c+x)^3} + \frac{1}{(d+x)^3} + \text{etc.}$$

and this series differentiated again and divided by $-3dx$ will give

$$\frac{-d^3S}{dx^3} = \frac{1}{(a+x)^4} + \frac{1}{(b+x)^4} + \frac{1}{(c+x)^4} + \frac{1}{(d+x)^4} + \text{etc.}$$

And in the same way, the sum of all following powers will be found, if the sum of the first series was known.

§33 But we found series of fractions of this kind involving an undetermined quantity above in the *Introductio*, where we showed, if the half of the circumference of the circle, whose radius is $= 1$, is set $= \pi$, that it will be

$$\frac{\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.}$$
$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.}$$

Therefore, because it is possible to assume any arbitrary numbers for m and n, let us set $n = 1$ and $m = x$ that we obtain a series similar to that one we had propounded in the preceding paragraph; having done this it will be

$$\frac{\pi}{\sin\pi x} = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \text{etc.}$$
$$\frac{\pi\cos\pi}{\sin\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

Therefore, one will be able to exhibit the sums of any powers of fractions resulting from these fractions by means of differentiations.

§34 Let us consider the first series and for the sake of brevity put $\frac{\pi}{\sin\pi x} = S$, whose higher differentials shall be taken having put dx constant, and it will be

$$\begin{aligned}
S &= \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \text{etc.}\\
\frac{-dS}{dx} &= \frac{1}{xx} - \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(3+x)^2} - \frac{1}{(3-x)^2} - \text{etc.}\\
\frac{ddS}{2d^2x} &= \frac{1}{x^3} + \frac{1}{(1-x)^3} - \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(3+x)^3} + \frac{1}{(3-x)^3} - \text{etc.}\\
\frac{-d^3S}{6d^3x} &= \frac{1}{x^4} - \frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(3+x)^4} - \frac{1}{(3-x)^4} - \text{etc.}\\
\frac{d^4S}{24d^4x} &= \frac{1}{x^5} + \frac{1}{(1-x)^5} - \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(3+x)^5} + \frac{1}{(3-x)^5} - \text{etc.}\\
\frac{-d^5S}{120d^3x} &= \frac{1}{x^6} - \frac{1}{(1-x)^6} - \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(3+x)^6} - \frac{1}{(3-x)^6} - \text{etc.}
\end{aligned}$$

etc.

where it is to be noted that in the even powers the signs follow the same law and in like manner in the odd the same structure of the signs is observed. Therefore, the sums of all these series are found from the differentials of the expression $S = \frac{\pi}{\sin \pi x}$.

§35 To express this differentials in a simpler way let us put

$$\sin \pi = p \quad \text{and} \quad \cos \pi = q;$$

it will be

$$dp = \pi dx \cos \pi x = \pi q dx \quad \text{and} \quad dq = -\pi p dx.$$

Therefore, because it is $S = \frac{\pi}{p}$, it will be

$$\begin{aligned}
\frac{-dS}{dx} &= \frac{\pi^2 q}{pp} \\
\frac{ddS}{dx^2} &= \frac{\pi^3(pp+2qq)}{p^3} = \frac{\pi^3(qq+1)}{p^3} \quad \text{because it is} \quad pp+qq=1 \\
\frac{-d^3S}{dx^3} &= \pi^4\left(\frac{5q}{pp} + \frac{6q^3}{p^4}\right) = \frac{\pi^4(q^3+5q)}{p^4} \\
\frac{d^4S}{dx^4} &= \pi^5\left(\frac{24q^4}{p^5} + \frac{28q^2}{p^3} + \frac{5}{p}\right) = \frac{\pi^5(q^4+18q^2+5)}{p^5} \\
\frac{-d^5S}{dx^5} &= \pi^6\left(\frac{120q^5}{p^6} + \frac{180q^3}{p^4} + \frac{61q}{pp}\right) = \frac{\pi^6(q^5+58q^3+61q)}{p^6} \\
\frac{d^6S}{dx^6} &= \pi^7\left(\frac{720q^6}{p^7} + \frac{1320q^4}{p^5} + \frac{662q^2}{p^3} + \frac{61}{p}\right) = \frac{\pi^7(q^6+179q^4+479q^2+61)}{p^7} \\
\frac{-d^7S}{dx^7} &= \pi^8\left(\frac{5040q^7}{p^8} + \frac{10920q^5}{p^6} + \frac{7266q^3}{p^4} + \frac{1385q}{p^2}\right)
\end{aligned}$$

or

$$\begin{aligned}
&= \frac{\pi^8}{p^8}(q^7+543q^5+3111q^3+1385q) \\
\frac{d^8S}{dx^8} &= \pi^9\left(\frac{40320q^8}{p^9} + \frac{100800q^6}{p^7} + \frac{83664q^4}{p^5} + \frac{24568q^2}{p^3} + \frac{1385}{p}\right)
\end{aligned}$$

or

$$= \frac{\pi^9}{p^9}(q^8 + 1636q^6 + 18270q^4 + 19028q^2 + 1385)$$

etc.

These expressions are easily continued arbitrarily far; for, if it was

$$\pm\frac{d^nS}{dx^n} = \pi^{n+1}\left(\frac{\alpha q^n}{p^{n+1}} + \frac{\beta q^{n-2}}{p^{n-1}} + \frac{\gamma q^{n-4}}{p^{n-3}} + \frac{\delta q^{n-6}}{p^{n-5}} + \text{etc.}\right),$$

then its differential, having changed the signs, will be

$$\mp\frac{d^{n+1}S}{dx^{n+1}}\begin{Bmatrix} (n+1)\alpha\frac{q^{n+1}}{p^{n+2}} + (n\alpha + (n-1)\beta)\frac{q^{n-1}}{p^n} + ((n-2)\beta + (n-3)\gamma)\frac{q^{n-3}}{p^{n-2}} \\ + ((n-4)\gamma + (n-5)\delta)\frac{q^{n-5}}{p^{n-4}} + \text{etc.} \end{Bmatrix}$$

§36 Therefore, from these series one will obtain the following sums of the series exhibited in § 34

$$\begin{aligned} S &= \pi\cdot\frac{1}{p} \\ \frac{-dS}{dx} &= \frac{\pi^2}{1}\cdot\frac{q}{p^2} \\ \frac{ddS}{24dx^2} &= \frac{\pi^3}{2}\left(\frac{2q^2}{p^3} + \frac{1}{p}\right) \\ \frac{-d^3S}{6dx^3} &= \frac{\pi^4}{6}\left(\frac{6q^3}{p^4} + \frac{5q}{p^2}\right) \\ \frac{d^4S}{24dx^4} &= \frac{\pi^5}{24}\left(\frac{24q^4}{p^5} + \frac{28q^2}{p^3} + \frac{5}{p}\right) \\ \frac{-d^5S}{120dx^5} &= \frac{\pi^6}{120}\left(\frac{120q^5}{p^6} + \frac{180q^3}{p^4} + \frac{61q}{p^2}\right) \end{aligned}$$

$$\frac{d^6S}{720dx^6} = \frac{\pi^7}{720}\left(\frac{720q^6}{p^7} + \frac{1320q^4}{p^5} + \frac{662q^2}{p^3} + \frac{61}{p}\right)$$
$$\frac{-d^7S}{720dx^6} = \frac{\pi^8}{5040}\left(\frac{5040q^7}{p^8} + \frac{10920q^5}{p^6} + \frac{7266q^3}{p^4} + \frac{1385q}{p^2}\right)$$
$$\frac{d^8S}{40320dx^8} = \frac{\pi^9}{40320}\left(\frac{40320q^8}{p^9} + \frac{100800q^6}{p^7} + \frac{83664q^4}{p^5} + \frac{24568q^2}{p^3} + \frac{1385}{p}\right)$$
etc.

§37 Let us treat the other series found above [§ 33] in the same way

$$\frac{\pi\cos\pi x}{\sin\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

and having put $\frac{\pi\cos\pi x}{\sin\pi x} = T$ for the sake of brevity the following summations will result

$$T = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \text{etc.}$$
$$\frac{-dT}{dx} = \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} + \text{etc.}$$
$$\frac{ddT}{2dx^2} = \frac{1}{x^3} - \frac{1}{(1-x)^3} + \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^2} - \text{etc.}$$
$$\frac{-d^3T}{6d^3x} = \frac{1}{x^4} + \frac{1}{(1-x)^4} + \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} + \text{etc.}$$
$$\frac{d^4T}{24dx^4} = \frac{1}{x^5} - \frac{1}{(1-x)^5} + \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} - \text{etc.}$$
$$\frac{-d^5T}{120d^5x} = \frac{1}{x^6} + \frac{1}{(1-x)^6} + \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} + \text{etc.}$$
etc.,

where in the even powers all terms are positive, but in the odd powers the signs $+$ and $-$ alternate.

§38 To find the values of these differentials let us, as before, put

$$\sin\pi x = p \quad \text{and} \quad dq = -\pi p dx$$

that it is $pp + qq = 1$; it will be

$$dp = \pi q dx \quad \text{and} \quad dq = -\pi p dx.$$

Having added these values it will be

$$\begin{aligned}
T &= \pi \cdot \frac{q}{p} \\
\frac{-dT}{dx} &= \pi^2 \left(\frac{qq}{pp} + 1 \right) = \frac{\pi^2}{pp} \\
\frac{ddT}{dx^2} &= \pi^3 \left(\frac{2q^3}{p^3} + \frac{2q}{p} \right) = \frac{2\pi^3 q}{p^3} \\
\frac{-d^3T}{dx^3} &= \pi^4 \left(\frac{6q^4}{p^4} + \frac{8qq}{pp} + 2 \right) = \pi^4 \left(\frac{6qq}{p^4} + \frac{2}{pp} \right) \\
\frac{d^4T}{dx^4} &= \pi^5 \left(\frac{24q^3}{p^5} + \frac{16q}{p^3} \right) \\
\frac{-d^5T}{dx^5} &= \pi^6 \left(\frac{120q^4}{p^6} + \frac{120qq}{p^4} + \frac{16}{pp} \right) \\
\frac{d^6T}{dx^6} &= \pi^7 \left(\frac{720q^5}{p^7} + \frac{960q^3}{p^5} + \frac{272q}{p^3} \right) \\
\frac{-d^7T}{dx^7} &= \pi^8 \left(\frac{5040q^6}{p^8} + \frac{8400q^4}{p^6} + \frac{3696q^2}{q^4} + \frac{272}{p^2} \right) \\
\frac{d^8T}{dx^8} &= \pi^9 \left(\frac{40320q^7}{p^9} + \frac{80640q^5}{p^7} + \frac{48384q^3}{p^5} + \frac{7936q}{p^3} \right) \\
&\text{etc.}
\end{aligned}$$

These formulas can easily be continued arbitrarily far. For, if it is

$$\pm \frac{d^nT}{dx^n} = \pi^{n+1} \left(\frac{\alpha q^{n-1}}{p^{n+1}} + \frac{\beta q^{n-3}}{p^{n-1}} + \frac{\gamma q^{n-5}}{p^{n-3}} + \frac{\delta q^{n-7}}{p^{n-5}} + \text{etc.} \right),$$

the expression for the following differential will be

$$\mp \frac{d^{n+1}T}{dx^{n+1}} = \pi^{n+2} \left(\frac{(n+1)\alpha q^n}{p^{n+2}} + \frac{(n-1)(\alpha+\beta)q^{n-2}}{p^n} + \frac{(n-3)(\beta+\gamma)q^{n-4}}{p^{n-2}} + \text{etc.} \right)$$

§39 Therefore, having put $\sin \pi x = p$ and $\cos \pi x = q$, the series of powers given in § 37 will have the following sums

$$\begin{aligned}
T &= \pi \cdot \frac{q}{p} \\
\frac{-dT}{dx} &= \pi^2 \frac{1}{pp} \\
\frac{ddT}{2dx^2} &= \pi^3 \frac{q}{p^3} \\
\frac{-d^3T}{6dx^3} &= \pi^4 \left(\frac{qq}{\pi^4} + \frac{1}{3pp} \right) \\
\frac{-d^4T}{24dx^4} &= \pi^5 \left(\frac{q^3}{p^5} + \frac{2q}{3p^3} \right) \\
\frac{-d^5T}{120dx^5} &= \pi^6 \left(\frac{q^4}{p^6} + \frac{3qq}{p^4} + \frac{2}{15pp} \right) \\
\frac{d^6T}{720dx^6} &= \pi^7 \left(\frac{q^5}{p^7} + \frac{4q^3}{3p^5} + \frac{17q}{45p^3} \right) \\
\frac{-d^7T}{5040dx^7} &= \pi^8 \left(\frac{q^6}{p^8} + \frac{5q^4}{3p^6} + \frac{11q^2}{15p^4} + \frac{17}{315pp} \right) \\
\frac{d^8T}{40320dx^8} &= \pi^9 \left(\frac{q^7}{p^9} + \frac{6q^5}{3p^7} + \frac{6q^3}{5p^5} + \frac{62q}{315p^3} \right) \\
&\text{etc.}
\end{aligned}$$

§40 Except for these series we found several others in the *Introductio* from which in like manner others can be derived by means of differentiation.

For, we showed that it is

$$\frac{1}{2x} - \frac{\pi\sqrt{x}}{2x \tan \pi\sqrt{x}} = \frac{1}{1-x} + \frac{1}{4-x} + \frac{1}{9-x} + \frac{1}{16-x} + \frac{1}{25-x} + \text{etc.}$$

Let us put that the sum of this series is $= S$ that it is

$$S = \frac{1}{2x} - \frac{\pi}{2\sqrt{x}} \cdot \frac{\cos \pi\sqrt{x}}{\sin \pi\sqrt{x}};$$

it will be

$$\frac{dS}{dx} = -\frac{1}{2xx} + \frac{\pi}{4x\sqrt{x}} \cdot \frac{\cos \pi\sqrt{x}}{\sin \pi\sqrt{x}} + \frac{\pi\pi}{4x(\sin \pi\sqrt{x})^2},$$

which expression therefore yields the sum of this series

$$\frac{1}{(1-x)^2}+\frac{1}{(4-x)^2}+\frac{1}{(9-x)^2}+\frac{1}{(16-x)^2}+\frac{1}{(25-x)^2}+\text{etc.}$$

Further, we also showed that it is

$$\frac{\pi}{2\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}-\frac{1}{2x}=\frac{1}{1+x}+\frac{1}{4+x}+\frac{1}{9+x}+\frac{1}{16+x}+\text{etc.}$$

Therefore, if this sum is put $=S$, it will be

$$\frac{-dS}{dx}=\frac{1}{(1+x)^2}+\frac{1}{(4+x)^2}+\frac{1}{(9+x)^2}+\frac{1}{(16+x)^2}+\text{etc.}$$

But it is

$$\frac{dS}{dx}=\frac{-\pi}{4x\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}-\frac{\pi\pi}{x}\cdot\frac{e^{2\pi\sqrt{x}}}{(e^{2\pi\sqrt{x}}-1)^2}+\frac{1}{2xx}.$$

Therefore, the sum of this series will be

$$\frac{-dS}{dx}=\frac{\pi}{4x\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}+\frac{\pi\pi}{x}\cdot\frac{e^{2\pi\sqrt{x}}}{(e^{2\pi\sqrt{x}}-1)^2}-\frac{1}{2xx}.$$

And in like manner the sums of the following powers will be found by means of further differentiation.

§41 If the value of a certain product composed of factors involving the undetermined letter x was known, one will be able to find innumerable summable series from it by means of the same method. For, let the value of this product

$$(1+\alpha x)(1+\beta x)(1+\gamma x)(1+\delta x)(1+\varepsilon x)\text{etc.}$$

be $=S$, a function of x, of course; by taking logarithms it will be

$$\ln S=\ln(1+\alpha x)+\ln(1+\beta x)+\ln(1+\gamma x)+\ln(1+\delta x)+\text{etc.}$$

Now, take the differentials; after division by dx it will be

$$\frac{dS}{Sdx}=\frac{\alpha}{1+\alpha x}+\frac{\beta}{1+\beta x}+\frac{\gamma}{1+\gamma x}+\frac{\delta}{1+\delta x}+\text{etc.},$$

from the further differentiation of which the sums of any powers of these fractions will be found, precisely as we explained it in more detail in the preceding examples.

§42 But, in the *Introductio* we exhibited several expressions of such a kind we want to apply this method to. If π is the arc of 180° of the circle whose radius is $=1$, we showed that it is

$$\sin\frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{4nn-mm}{4nn}\cdot\frac{16nn-mm}{16nn}\cdot\frac{26nn-mm}{36nn}\cdot\text{etc.}$$
$$\cos\frac{m\pi}{2n} = \frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\frac{49nn-mm}{49nn}\cdot\text{etc.}$$

Let us put $n=1$ and $m=2x$ that it is

$$\sin\pi x = \pi x\cdot\frac{1-xx}{1}\cdot\frac{4-xx}{4}\cdot\frac{9-xx}{9}\cdot\frac{16-xx}{16}\cdot\text{etc.}$$

or

$$\sin\pi x = \pi x\cdot\frac{1-x}{1}\cdot\frac{1+x}{1}\cdot\frac{2-x}{2}\cdot\frac{2+x}{2}\cdot\frac{3-x}{3}\cdot\frac{3+x}{3}\cdot\frac{4-x}{4}\cdot\text{etc.}$$

and

$$\cos\pi x = \frac{1-4xx}{1}\cdot\frac{9-4xx}{9}\cdot\frac{25-4xx}{25}\cdot\frac{49-4xx}{49}\cdot\text{etc.}$$

or

$$\cos\pi x = \frac{1-2x}{1}\cdot\frac{1+2x}{1}\cdot\frac{3-2x}{3}\cdot\frac{3+2x}{3}\cdot\frac{5-2x}{5}\cdot\frac{5+2x}{5}\cdot\text{etc.}$$

Therefore, from these expressions, if one takes logarithms, it will be

$$\ln\sin\pi x = \ln\pi x + \ln\frac{1-x}{1} + \ln\frac{1+x}{1} + \ln\frac{2-x}{2} + \ln\frac{2+x}{2} + \ln\frac{3-x}{3} + \text{etc.}$$
$$\ln\cos\pi x = \ln\frac{1-2x}{1} + \ln\frac{1+2x}{1} + \ln\frac{3-2x}{3} + \ln\frac{3+2x}{3} + \ln\frac{5-2x}{5} + \text{etc.}$$

§43 Now, let us take the differentials of these series of logarithms and having divided by dx everywhere the first series will give

$$\frac{\pi\cos\pi x}{\sin\pi x}=\frac{1}{x}-\frac{1}{1-x}+\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}-\frac{1}{3-x}+\text{etc.}$$

which is the series we treated in § 37. The other series on the other hand will give

$$\frac{-\pi\sin\pi x}{\cos\pi x}=-\frac{2}{1-2x}+\frac{2}{1+2x}-\frac{2}{3-2x}+\frac{2}{3+2x}-\frac{2}{5-2x}+\text{etc.}$$

Let us put $2x=z$ that it is $x=\frac{z}{2}$ and divide by -2; it will be

$$\frac{\pi\sin\frac{1}{2}\pi z}{2\cos\frac{1}{2}\pi z}=\frac{1}{1-z}-\frac{1}{1+z}+\frac{1}{3-z}-\frac{1}{3+z}+\frac{1}{5-z}-\text{etc.}$$

But because it is

$$\sin\frac{1}{2}\pi z=\sqrt{\frac{1-\cos\pi z}{2}}\quad\text{and}\quad\cos\frac{1}{2}\pi z=\sqrt{\frac{1+\cos\pi z}{2}},$$

it will be

$$\frac{\pi\sqrt{1-\cos\pi z}}{\sqrt{1+\cos\pi z}}=\frac{2}{1-z}-\frac{2}{1+z}+\frac{2}{3-z}+\frac{2}{5-z}-\text{etc.}$$

or by writing x instead of z it will be

$$\frac{\pi\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}}=\frac{2}{1-x}-\frac{2}{1+x}+\frac{2}{3-x}-\frac{2}{3+x}+\frac{2}{5-x}-\text{etc.}$$

Add this series to the one found first

$$\frac{\pi\cos\pi x}{\sin\pi x}=\frac{1}{x}-\frac{1}{1-x}+\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}-\frac{1}{3-x}+\text{etc.}$$

and one will find the sum of this series

$$\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}+\frac{1}{3-x}-\frac{1}{3+x}-\text{etc.}$$

to be $=\frac{\pi\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi}}+\frac{\pi\cos\pi x}{\sin\pi x}$. But this fraction $\frac{\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}}$, if the numerator and denominator are multiplied by $\sqrt{1-\cos\pi x}$, goes over into $\frac{1-\cos\pi x}{\sin\pi}$. Therefore, the sum of the series will be $=\frac{\pi}{\sin\pi x}$, which is the series we had in § 34; therefore, we will not prosecute this any further.