On Maxima and Minima of multiform functions and such containing several variables *

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§273 If y was a multiform function of x such that for any value of x it obtains several real values, then having varied x those several values of y are connected to each other in such a way that they represent several series of successive values. For, if we consider y as ordinate of a curve, while x is the abscissa, as many real different values y had, as many different branches of the same curve line will correspond to the same abscissa x; and hence those successive values of y, which constitute the same branch, are to be considered to be connected; but the values related to the different branches will be distinct from each other. Therefore, as many series of connected values of y we will have, as many different real values will it receive for each value of x; and in each arbitrary series the values of y, while x is assumed to increase, either grow or decrease or, after they had increased, decrease again or vice versa. From this it is perspicuous that in each series of connected values equally maxima and minima are given as in uniform functions.

§274 To determine these maxima and minima also the same method will be valid, which we treated in the preceding chapter for uniform functions. For,

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$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \text{etc.},$$

it is necessary, that in the case of a maximum or minimum the term $\frac{\omega dy}{dx}$ vanishes and it is $\frac{dy}{dx} = 0$. Therefore, the roots of this equation $\frac{dy}{dx} = 0$ will indicate those values of x, to which in the single series of the connected values of y the maxima or minima correspond. And it will indeed not be ambiguous, in which series of connected values a maximum or minimum is given. For, because in the equation $\frac{dy}{dx} = 0$ both variables x and y are contained, the values of x can only be defined, if by means of the equation, in which the relation of the function y to x is contained, the variable y is eliminated; but before this happens, one gets to an equation, by which the value of y is expressed by means of a rational or uniform function of x. Hence, having found the values of x the corresponding value of y will be found, which will be maximal or minimal in the series of connected successive values, to which one gets.

§275 But the decision, whether these values of y are maxima or minima, will be made the same way as we indicated before: Find the value of $\frac{ddy}{dx^2}$ expressed in finite terms and in it for x successively substitute a value of x found before; but at the same time for y put the value, which corresponds to it for any arbitrary value of x; having done this see, whether the expression $\frac{ddy}{dx^2}$ will obtain an affirmative or an negative value, and in the first cases a minimum will be indicated, in the second a maximum. But if also $\frac{ddy}{dx^2}$ vanishes, then one will have to proceed to the formula $\frac{d^3y}{dx^3}$; but if also $\frac{d^3y}{dx^3}$ vanishes, the decision is to be made from the formula $\frac{d^4y}{dx^4}$ the same way as we described it for the formula $\frac{ddy}{dx^2}$. And if also $\frac{d^4y}{dx^4}$ vanishes in a certain case, one will have to proceed to the fifth differential of y; but, no matter how far one had to proceed, the decisions from the differentials of odd order are always similar to those, which we gave from the formula $\frac{d^3y}{dx^3}$. In these cases in the formulas $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc. one will have to proceed so far until one gets to such a one, which does not vanish in a certain case; if this was a differential of odd order, neither a maximum nor a minimum will be indicated; but if it was of even order, its positive value will imply a minimum, its negative value a maximum.

§276 Let us put that the function y is determined from x by means of an arbitrary equation; if this equation is differentiated, it will obtain an form of this kind $Pdx + Qdy = 0$. Therefore, having put $\frac{dy}{dx} = 0$ it will be $\frac{P}{Q} = 0$ and hence either $P = 0$ or $Q = \infty$. The second equation, if the relation between x and y is expressed by means of a polynomial equation, cannot hold, because either x or y or both would have to become infinite. Hence, the decision is to be made from the equation $P = 0$, whose roots or values of x, which it obtains, after by means of the propounded equation the variable y was completely eliminated, will indicate the cases, in which the values of y become maximal or minimal. But to make the decision, whether a maximum or a minimum arises, examine the formula $\frac{ddy}{dx^2}$. But the differential equation $Pdx + Qdy = 0$ differentiated again, if we put

$$dP = Rdx + Sdy \quad \text{and} \quad dQ = Tdx + Vdy,$$

for constant dx will give

$$Rdx^2 + Sdxdy + Tdxdy + Vdy^2 + Qddy = 0.$$

But because now it is $\frac{dy}{dx} = 0$, having divided by dx^2 it will be

$$R + \frac{Qddy}{dx^2} = 0 \quad \text{and hence} \quad \frac{ddy}{dx^2} = -\frac{R}{Q}.$$

Hence in the differential equation $Pdx + Qdy = 0$ differentiate only the quantity P by putting y constant and Rdx will arise; then investigate the value of the fraction $\frac{R}{Q}$, which, if it was affirmative, will indicate a maximum, if negative, a minimum.

§277 Let y be a biform function of x, which is determined by this equation $yy + py + q = 0$ while p and q denote any uniform of x. Therefore, by differentiating it will be $2ydy + pdy + ydp + sq = 0$ and hence $Pdx = ydp + dq$. Therefore, having put $P = 0$ it will be $ydp + dq = 0$ and $y = -\frac{dq}{dp}$ will arise and so y is expressed by means of an uniform function of x, such that, whatever value for x was found, from it and y it acquires one single determined value. But now the elimination of y will be easy; for, if in the propounded equation $yy + py + q = 0$ for y the value $-\frac{dq}{dp}$ is substituted, one will have $dq^2 - pdpdq + qdp^2 = 0$, which equation divided by dx^2 and resolved will yield all values of x, to which the maxima or minima correspond; this will become more clear in the following examples.

EXAMPLE 7

Having propounded the equation $yy + mxy + aa + bx + nxx = 0$ to define the maxima or minima of the function y.

Having differentiated the equation we will have

$$2ydy + mxdy + mydx + bdx + 2nxdx = 0,$$

whence it is

$$P = my + b + 2nx \quad \text{and} \quad Q = 2y + mx.$$

Therefore, having put $P = 0$ it will be $y = -\frac{b+2nx}{m}$; this value substituted in the equation gives

$$\frac{4nn}{mm}xx + \frac{4nb}{mm}x + \frac{bb}{mm} - 2nxx - bx + aa + nxx + bx = 0$$

or

$$xx = \frac{4nbx + bb + mmaa}{mmn - 4nn},$$

whence it is

$$x = \frac{2nb \pm \sqrt{mmnbb + mmn(mm - 4n)aa}}{mmn - 4nn}$$

or

$$x = \frac{2nb \pm m\sqrt{nbb + n(mm - 4n)aa}}{n(mm - 4n)} \quad \text{and} \quad y = \frac{-mb \mp \sqrt{nbb + n(mm - 4n)aa}}{mm - 4n}.$$

Then having put only x to be variable it is $dP = 2ndx$ and hence $R = 2n$. But it is

$$Q = 2y + mx = \pm\frac{\sqrt{nbb + n(mm - 4n)aa}}{n},$$

whence

$$\frac{R}{Q} = \frac{\pm 2nn}{\sqrt{nbb + n(mm - 4n)aa}};$$

because its numerator $2nn$ is always positive, if the superior sign holds, a maximal value for y will arise, if the inferior holds, a minimal value will arise. Here, the following things have to be mentioned.

I. If $m = 0$, from the equation $P = 0$ it immediately follows $x = -\frac{b}{2n}$, that no elimination is necessary. And to this value a double value of y corresponds because of $y = \pm\frac{1}{2n}\sqrt{nbb - 4nnaa}$, of which the positive one is a maximum, the other negative one a minimum.

II. If it is $n = 0$, it is $y = -\frac{b}{m}$ and x grows to infinity and y through any infinite space retains the same value, such that it is neither a maximum nor a minimum.

III. If it is $mm = 4n$, it will be $4nbx + bb + mmaa = 0$ or $x = \frac{bb+mmaa}{-mmb}$ and it will be

$$y = -\frac{b+2nx}{m} = -\frac{2b+mmx}{2m} = -\frac{2b}{2m} + \frac{bb+mmaa}{2mb} = \frac{mmaa-bb}{2mb}.$$

Therefore, to this value of $x = \frac{-mmaa+bb}{mmb}$ the other value of y corresponds, $\frac{mmaa-bb}{2mb}$, which will be a maximum or a minimum. But because, that this value of y arises, in the expression

$$y = \frac{-mb \mp 2\sqrt{nbb + n(mm-4n)aa}}{mm-4n}$$

the inferior sign has to hold, the value of y will be a minimum.

EXAMPLE 2

Having propounded the equation $yy - xxy + x - x^3 = 0$ to define the maximal or minimal values of y.

Having differentiated the equation it arises

$$2ydy - xxdy - 2xydx + dx - 3xxdx = 0.$$

And it is

$$P = 1 - 3xx - 2xy \quad \text{and} \quad Q = 2y - xx.$$

Hence, having put $P = 0$ it will be $y = \frac{1-3xx}{2x}$ and hence having substituted this value

$$\frac{1}{4xx} - \frac{3}{2} + \frac{9xx}{4} - \frac{x}{2} + \frac{3}{2}x^3 + x - x^3 = 0$$

or

$$1 - 6xx + 2x^3 + 9x^4 + 2x^5 = 0.$$

One of its roots is $x = -1$, to which $y = 1$ corresponds. But having put y constant it is $R = -6x - 2y$, therefore

$$\frac{ddy}{dx^2} = \frac{2y + 6x}{2y - xx};$$

in the case $x = -1$ and $y = 1$ it goes over into -4, such that the value of $y = 1$ is a maximum. But to $x = -1$ corresponds a double value of y from the equation $yy - y = 0$; therefore, the other is $y = 0$, which is neither a maximum nor a minimum. If this equation of degree five is divided by $x + 1$, an equation arises, whose roots cannot be exhibited in a simple manner.

EXAMPLE 3

Let this equation be propounded $yy + 2xxy + 4x - 3 = 0$, of the maximal and minimal values of y are required.

By means of differentiation this equation will arise

$$2ydy + 2xxdy + 4xydx + 4dx = 0.$$

Having put $\frac{dy}{dx} = 0$ it will be $xy + 1 = 0$ and hence $y = -\frac{1}{x}$, which value substituted in the the propounded equation arises as

$$\frac{1}{xx} - 2x + 4x - 3 = 0 = 2x^3 - 3xx + 1,$$

whose roots are $x = 1$, $x = 1$ and $x = -\frac{1}{2}$. Since now it is

$$\frac{dy}{dx} = -\frac{4xy + 4}{2y + 2xx} = -\frac{2xy + 2}{y + xx},$$

by differentiating it will be $\frac{ddy}{dx^2} = -\frac{2y}{y+xx}$ having put y constant because of $dy = 0$ and having put $xy + 1 = 0$. Hence, these values will behave this way:

x	y	$\frac{ddy}{dx^2}$	
1	-1	∞	
1	-1	∞	
$-\frac{1}{2}$	2	$\frac{-16}{9}$	for a maximum.

Since for the equal roots it is $\frac{ddy}{dx^2} = \infty$, whether in this case a maximum or a minimum arises, is not determined. But because at the same time it is $y + xx = 0$, it will not even in this case be $\frac{dy}{dx} = 0$ because of $P = 0$ and $Q = 0$ in the fraction $\frac{dy}{dx} = -\frac{P}{Q}$; hence, because the primary property is missing, neither a maximum nor a minimum can occur. But it is indicated that in this case $x = 1$ both values of y become equal to each other. We will explain this nature in more detail below, when we will get to the use of the differential calculus in the doctrine of curved lines. For, even though this subject also extend to this, we nevertheless, that it is not necessary to do it twice, keep it in total for the following treatise.

§278 Additionally in multiform functions another kind of maxima and minima is given, which are not found by the method given up to now, whose nature can be explained most easily from biform functions. For, let y be any biform function of x, such that, whatever value is attributed to x, for y two values arise, either both real or both imaginary. Let us put that these values of y become imaginary, if one puts $x > f$, but are real, if one sets $x < f$; and having put $x = f$ both values will coalesce into one, which shall be $y = g$. Therefore, because, if one takes $x > f$, the function y has no real value, if it happens that for $x < f$ both values of y become either greater than g or smaller than g, in the first case the value $y = g$ will be a minimum, in the second a maximum, since in that case is is smaller than both preceding ones, but larger in this one. And this maximum or minimum is not found by means of the method treated up to now, because here it is not $\frac{dy}{dx} = 0$. But these maxima or minima are also of a different kind, since such are not maxima or minima with respect to the preceding or following series connected in a series, but with respect to only two distinct either preceding or following values.

§279 This happens, if the propounded equation was of this kind

$$y = p \pm (f-x)\sqrt{f-x}q$$

while p and q are functions of x not divisible by $f-x$; q shall obtain a positive value, if one puts either $x=f$ or a little larger or a little smaller. Let $p=g$ for $x=f$ and it is manifest that it the case $x=f$ the two values of y coalesce into the single one $y=g$; but for $x>f$ both values of y will become imaginary. Therefore, if we put x a little bit smaller than f, say $x=f-\omega$, the function p will go over into

$$g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2dx^2} - \text{etc.}$$

and q into

$$q - \frac{\omega dq}{dx} + \frac{\omega^2 ddq}{2dx^2} - \text{etc.},$$

whence it will be in this case

$$y = g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2dx^2} - \text{etc.} \pm \omega\sqrt{\omega}\left(q - \frac{\omega dq}{dx} + \frac{\omega^2 ddq}{2dx^2} - \text{etc.}\right).$$

Let us put ω very small that with respect to ω its higher powers vanish, and it will be $y = g - \frac{\omega dp}{dx} \pm \omega\sqrt{\omega}q$; these two values of y will both be smaller than g, if $\frac{dp}{dx}$ was positive, but larger, if negative. Hence, the double value of $y=g$ in that case will be a maximum, in this a minimum.

§280 Therefore, these maxima and minima arise, since first having put $x=f$ both values of y become equal, but having put $x>f$ imaginary, but for $x<f$ real; further, since having put $x=f-\omega$ the other irrational term yields higher powers of ω than the rational term. Therefore, this also happens, if it was $y = p \pm (f-x)^n\sqrt{f-x}q$, as long as n is an integer >0. But because not only the square root, but also any other root of even power introduces the same ambiguity of the sign, the same will happen, if it was $y = p \pm (f-x)^{\frac{2n+1}{2m}}q$, as long as it is $2n+1>2m$; therefore, it will be $(y-p)^{2m} = (f-x)^{2n+1}q^{2m}$ or $(y-p)^{2m} = (f-x)^{2n+1}Q$. Therefore, if the function y is expressed by means of such an equation such that it is $2n+1>2m$, having put $x=f$ the value of y will become maximal or minimal; the first, if $\frac{dp}{dx}$ was a positive quantity, the latter, if $\frac{dp}{dx}$ negative for $x=f$. But if in this case it is $\frac{dp}{dx}=0$, then it will be

$$y = g + \frac{\omega^2 ddp}{2dx^2} \pm \omega^{\frac{2n+1}{2m}} q.$$

Therefore, only if $\frac{2n+1}{2m} > 2$, a maximum or a minimum can occur; but if it is indeed $\frac{2n+1}{2m} > 2$, then $y = g$ will be a maximum, if $\frac{ddp}{dx^2}$ had a negative value, a minimum, if it was positive; and so forth, if also $\frac{ddp}{dx^2}$ vanishes, the decision will to be made.

§281 Therefore, if y was a function of x of such a kind, it can happen, that except maxima and minima, which the first method exhibits, also maxima and minima of the other species are given, which can be explored by the method explained here. We will show this in the following examples.

Example 1

To determine the maxima and minima of the function y, which is defined by this equation

$$yy - 2xy - 2xx - 1 + 3x + x^3 = 0.$$

To investigate the maxima and minima of the first kind differentiate the equation and it will be

$$2ydy - 2xdy - 2ydx - 4x + 3dx + 3xxdx = 0$$

and having put $\frac{dy}{dx} = 0$ it will be

$$y = \frac{3}{2} - 2x + \frac{3}{2}xx,$$

which value substituted in the first equation gives

$$9x^4 - 32x^3 + 42xx - 24x + 5 = 0,$$

which is resolved into

$$9xx - 14x + 5 = 0 \quad \text{and} \quad xx - 2x + 1 = 0.$$

The first twice gives $x = 1$ and it is $y = 1$, whence in the case in the fraction

$$\frac{dy}{dx} = \frac{2y - 3 + 4x - 3xx}{2y - 2x}$$

the denominator also vanishes and so a maximum or minimum of the first kind is not given; the first equation $9xx - 14x + 5 = 0$ on the other hand will give $x = 1$ and $x = \frac{5}{9}$, the latter of which values suffers from the same inconvenience as the first. But having put $x = \frac{5}{9}$ it is $y = \frac{3}{2} - \frac{10}{9} + \frac{25}{54} = \frac{23}{27}$. And because it is $\frac{dy}{dx} = \frac{2y-3+4x-3xx}{2y-2x}$, it will be

$$\frac{ddy}{dx^2} = \frac{4-6x}{2y-2x} = \frac{-3x+2}{y-x}$$

because of $dy = 0$ and the numerator $= 0$. Therefore, it will be $\frac{ddy}{dx^2} = \frac{9}{8}$, whence this value $x = \frac{5}{9}$ gives a minimum of the first kind. Further, because it is $(y-x)^2 = (1-x)^3$, it will be

$$y = x \pm (1-x)\sqrt{1-x}$$

and hence having put $x = 1$ a maximum of the second kind arises; for, having put $x = 1 - \omega$ it will be $y = 1 - \omega \pm \omega\sqrt{\omega}$, both of which value are smaller than the unity, if ω is taken very small.

Example 2

To find the maxima and minima of the function $y = 2x - xx \pm (1-x)^2\sqrt{1-x}$.

For the maxima and minima of the first kind differentiate the equation and it will be

$$\frac{dy}{dx} = 2 - 2x \mp \frac{5}{2}(1-x)\sqrt{1-x},$$

which value put $= 0$ arises at first as $x = 1$, and because it is

$$\frac{ddy}{dx^2} = -2 \pm \frac{15}{4}\sqrt{1-x},$$

y in this case will be a maximum of the first case and it is $y = 1$. Having divided the equation $\frac{dy}{dx} = 0$ by $1 - x$ it will be

$$4 \mp 5\sqrt{1-x} = 0 \quad \text{or} \quad 16 = 25 - 25x,$$

whence it is $x = \frac{9}{25}$ and $\frac{ddy}{dx^2} = -2 \pm 3$. Hence, if the superior sign holds, $y = \frac{2869}{3125}$ will be a minimum; but if the inferior sign holds, it will be $y = \frac{821}{3125}$, which might seem to be a maximum; but indeed only the superior sign can hold, since $4 \mp 5\sqrt{1-x}$ can only be $= 0$, if it is $\sqrt{1-x} = +\frac{4}{5}$. Therefore, we find a maximum of the first kind in the case $x = 1$ and $y = 1$ and a minimum in the case $x = \frac{9}{25}$ and $y = \frac{2689}{3125}$. A maximum of the other kind also arise, if $x = 1$, in which case it is $y = 1$. For, having put $x = 1 - \omega$ it will be $y = 1 - \omega\omega \pm \omega^2\sqrt{\omega}$, in both cases < 1. Therefore, here, if $x = 1$, the a maximum of the first and the other kind coalesce and constitute a mixed maximum.

§ 282 From these examples not only the nature of this other kind of maxima and minima becomes clear, but also ad libitum functions of this kind can be formed, which admit either maxima or minima of the second kind. But how, if any function was propounded, it can be explored, whether it has a maximum or minimum of such a kind or not, we will show in the following section, since the nature of curved lines is illustrated in the best way by this investigation. Furthermore, it is easily understood, if y was a function of x of such a kind, which receives a maximum or minimum of the second kind, that then also vice versa x will be a function of such a kind of y. For, since from this equation $(y-x)^2 = (1-x)^3$ for $x = 1$ the function y obtains a maximal value of the second kind, if the variables x and y are permutated, this equation $(x-y)^2 = (1-y)^3$ for y will exhibit a function of such a kind of x, which has a maximum of the second kind. For, having put $x = 1 + \omega$ it will be $(1 + \omega - y)^2 = (1-y)^3$; hence, if we set $y = 1 + \varphi$, it will be $(\omega - \varphi)^2 = (-\varphi)^3 = -\varphi^3$ and hence φ must be negative. Therefore, let $y = 1 - \varphi$; it will be $(\omega + \varphi)^2 = \varphi^3$, because having taken φ very small φ^3 vanishes with respect to φ^2, ω will have to be negative; hence, to the value $x = 1 + \omega$ no real values of y correspond. But having put $x = 1 - \omega$ and $y = 1 - \varphi$ because of $(\varphi - \omega)^2 = \varphi^3$ it will be $\varphi = \omega \pm \omega\sqrt{\omega}$ and hence $y = 1 - \omega \mp \omega\sqrt{\omega}$, whence both values of y corresponding to $x = 1 - \omega$ is smaller than the value $y = 1$, which corresponds to the value $x = 1$; and as a logical consequence this value of y will be maximal.

§283 Up to now we only considered biform functions, whose maxima or minima, since both values can easily be expressed by means of the resolution of the quadratic equation, can be checked for their correctness. But if the function y is expressed by means of an equation of higher order, the method given before, by which we investigated maxima and minima of the first kind, can be applied with the same success. But let us reserve the invention of maxima and minima of the second kind for the following treatise. Therefore, let us show how to treat triform and multiform functions in several examples.

EXAMPLE 1

Define the function y, whose maxima and minima are in question by means of this equation

$$y^3 + x^3 = 3axy.$$

Having differentiated this equation it is

$$3y^2dy + 3xxdy = 3axdy + 3aydx$$

and hence

$$\frac{dy}{dx} = \frac{ay - xx}{yy - ax}.$$

Therefore, a maximum or minimum will be given if it was $ay = xx$ or $y = \frac{xx}{a}$, which value substituted in the propounded equation gives

$$\frac{x^6}{a^3} + x^3 = 3x^3 \quad \text{and} \quad x^6 = 2a^3x^3.$$

Therefore, it will be $x = 0$ trice, in which case also the denominator $yy - ax = 0$ because of $y = \frac{xx}{a} = 0$. Whether in this case a maximum or minimum arises, will become plain, if we attributed a value to x differing hardly from 0. Therefore, let $x = \omega$ and $y = \varphi$; because of $\varphi^3 + \omega^3 = 3a\omega\varphi$ it will be either $\varphi = \alpha\sqrt{\omega}$ or $\varphi = \beta\omega^2$. In the first case it will be $\alpha^3\omega\sqrt{\omega} = 3\alpha a\omega\sqrt{\omega}$ and hence $\alpha = \sqrt{3}a$. Hence, having put $x = \omega$ it will be $y = \pm\sqrt{3}\alpha\omega$. Hence, even though ω cannot be taken negatively, nevertheless one of the two values of y will be greater than 0, the other will be smaller and hence $y = 0$ will be neither a maximum nor a minimum. But if one sets $\varphi = \beta\omega^2$, it will be $\omega^3 = 3aa\beta\omega^3$ and hence $\beta = \frac{1}{3a}$ and $\varphi = \frac{\omega^2}{3a}$. Therefore, in this case, no matter whether x is taken $= +\omega$ or $= -\omega$, the value of $y = \varphi$ will be greater than zero and hence in this case $y = 0$ will be a minimum. Therefore, the third case to be eliminated from the equation $x^3 = 2a^3$ results, which gives $x = a\sqrt[3]{2}$ and $y = a\sqrt[3]{4}$. Whether this is a maximum or a minimum, is to be found from the second differential of the equation $\frac{dy}{dx} = \frac{ay-xx}{yy-ax}$, which differential because of

$dy = 0$ and $ay - xx = 0$ it will be $\frac{ddy}{dx^2} = \frac{-2x}{yy-ax}$, whose value in the present case is $-\frac{2a\sqrt[3]{2}}{2a^2\sqrt[3]{2}-aa\sqrt[3]{2}} = -\frac{2}{a}$, which indicates that the value of y is a maximum.

EXAMPLE 2

If the function y is defined by means of this equation $y^4 + x^4 + ay^3 + ax^3 = b^3x + b^3y$, to find its maximal and minimal values.

Because by means of differentiated it arises

$$4y^3dy + 3ayydy - b^3dy = b^3 - 3axxdx - 4x^3dx,$$

it will be

$$\frac{dy}{dx} = \frac{b^3 - 3axx - 4x^3}{4y^3 + 3ayy - b^3}$$

and one has to put $b^3 = 3axx + 4x^3$. Therefore, the question is reduced to this, that the maxima and minima of the uniform function $b^3 - ax^3 - x^4$ are investigated, which at the same time will be the maxima and the minima of the function y. Let $a = 2$ and $b = 3$ or let this equation be propounded $y^4 + x^4 + 2y^3 + 2x^3 = 27x + 27y$; it will be $\frac{dy}{dx} = \frac{27-6xx-4x^3}{4y^3+6yy-27}$ and $4x^3 + 6xx - 27 = 0$, which divided by $2x - 3 = 0$ gives $2xx + 6x + 9 = 0$; because the roots of the second equation are imaginary, it will be $x = \frac{3}{2}$ and $y^4 + 2y^3 - 27y = \frac{459}{16}$, the single roots of which will be either maxima or minima. But because it is $\frac{dy}{dx} = \frac{27-6xx-4x^3}{4y^3+6yy-27}$, it will be $\frac{ddy}{dx^2} = \frac{-12x-12xx}{4y^3+6yy-27}$, which for $x = \frac{3}{2}$, if it was positive, will indicate a minimum, otherwise a maximum.

EXAMPLE 3

If it was $y^m + ax^n = by^px^q$, to define the maxima and minima of y.

My means of differentiation

$$\frac{dy}{dx} = \frac{qby^px^{q-1} - nax^{n-1}}{my^{m-1} - pby^{p-1}x^q},$$

having put which $= 0$ it will at first be $x = 0$, if n and q ware greater than the unity, and at the same time it was $y = 0$. Whether in this case a maximum or a minimum is given, the closest values are to be investigated, since also the denominator becomes $= 0$; this investigation will depend mainly on the exponents. Furthermore, the equation $\frac{dy}{dx} = 0$ will give $y^p = \frac{na}{qb}x^{n-q}$, which value substituted in the propounded equation by putting $\frac{na}{qp} = g$ will give

$$g^{\frac{m}{p}}x^{\frac{mn-mq}{p}} + ax^n = \frac{na}{q}x^n \quad \text{or} \quad g^{\frac{m}{p}}x^{\frac{mn-mq-np}{p}} = \frac{(n-q)a}{q},$$

whence it is

$$x = \left(\frac{(n-q)a}{q}\right)^{p:(mn-mq-np)} : g^{m:(mn-mq-np)}$$

and at the same time the value of y becomes known. Further, it is to be considered, whether the second differential

$$\frac{ddy}{dx^2} = \frac{q(q-q)by^px^{q-2} - n(n-1)ax^{n-2}}{my^{m-1} - pby^{p-1}x^q}$$

obtains a positive or a negative value, that from the first a minimum will be indicated, from the second a maximum.

EXAMPLE 4

If it was $y^4 + x^4 = 4xy - 2$, to assign the maxima and minima of the function y.

Having performed the differentiation it is

$$\frac{dx}{dy} = \frac{y - x^3}{y^3 - x}$$

and hence it arises $y = x^3$; therefore, it will be $x^{12} = 3x^4 - 2$ or $x^{12} - 3x^4 + 2 = 0$, which equation is resolved into these $x^4 - 1 = 0$ and $x^8 + x^4 - 2 = 0$ and the latter into $x^4 - 1 = 0$ and $x^4 + 2 = 0$. Hence, it will be twice either $x = \pm 1$ or $x = -1$; in both cases the also the denominator of the fraction $\frac{dy}{dx}$ vanishes. Therefore, to investigate, whether in these cases a maximum or a minimum takes place, let us put $x = 1 - \omega$ and $y = 1 - \varphi$; it will be

$$\begin{aligned} 1 - 4\varphi + 6\varphi^2 - 4\varphi^3 + \varphi^4 + 1 - 4\omega + 6\omega^2 - 4\omega^3 + \omega^4 \\ = 4 - 4\omega - 4\varphi + 4\omega\varphi - 2 \end{aligned}$$

and hence

$$4\omega\varphi = 6\varphi^2 + 6\omega^2 - 4\varphi^3 - 4\omega^3 + \varphi^4 + \omega^4$$

and because of the very small ω and φ it is $4\omega\varphi = 6\varphi^2 + 6\omega^2$. Therefore, the value of φ will be imaginary, no matter whether ω is taken positively or negatively. Or if y and x denote the coordinates of a curve, it in the case $x = 1$ and $y = 1$ will have a conjugated point. Therefore, also this value can be neither a maximum nor a minimum, since the preceding and following, to which it would have been compared, become imaginary.

§284 If the equation, by which the relation between x and y is expressed, was of such a nature that the function of y becomes equal to a function of x, say $Y = X$, to find the maxima or minima one will have to put $dX = 0$; therefore, y will be a maximum or a minimum in the same cases, in which X becomes a maximum or a minimum. In similar manner, if x is considered as a function of y, x will become a maximum or a minimum, if $dY = 0$, this means, if Y was a maximum or a minimum. And from this it does nevertheless not follow that y at the same time becomes a maximum or a minimum. For, if it was $2ay - yy = 2bx - xx$, y will be a maximum or a minimum, if $x = b$, and it will be $y = a \pm \sqrt{aa - bb}$. Otherwise x becomes a maximum or a minimum, if $y = a$, and it is $x = b \pm \sqrt{bb - aa}$ and therefore y becomes neither a maximum nor a minimum, if $x = b \pm \sqrt{bb - aa}$, in which case x is nevertheless a maximum or a minimum. Furthermore, in this case, if y has maximal or minimal values, x will have none at all; for, y cannot become a maximum or a minimum, if it was not $a > b$, in which case the maximum or minimum of x becomes imaginary.

§285 Then it can also happen that not all roots of the equation $dX = 0$ yield maximal or minimal values for y; for, if that equations had two equal roots, from this neither a maximum nor a minimum follows; the same happens, if an even number or roots were equal to each other. So, if the equation $b(y - a)^2 = (x - b)^3 + c^3$ is propounded, since having taken the differentials it is $2bdy(y - a) = 3dx(x - b)^2$, the function y will become neither a maximum nor a minimum for $x = b$, since here two equal roots occur. But if x is considered as a function of y, it will become maximal or minimal, if one sets $y = a$, and $x = b - c$ a minimum. Finally, since in equations of this kind $Y = X$ the variables x and y are not mixed, if to x a value is attributed, which is a root of the equation $dX = 0$, all values of y, no matter how many were real, will be maxima or minima; this does not happen, if in the equation the two variables were mixed.

§286 The things, which remain to be explained about the nature of maxima and minima, we will reserve for the following book, since they can be represented and explained in a more convenient way by means of figures. Therefore, let us proceed to functions, which are composited of several variables, and let us investigate the values, which have to be attributed to the single variables, that the function obtains either a maximal or a minimal value. And at first it is certainly clear, if the variables were not mixed among each other, such a function of this kind is propounded $X+Y$, where X is a function only of x and Y is a function only of y, that then the propounded function $X+Y$ will be a maximum, if at the same time X and Y become maximal, and a minimum, if at the same time X and Y become minimal. Therefore, to find the maxima and minima, find the values of x, for which X becomes a maximum, and in similar value the values of y, for which Y becomes a maximum, which analogously is to be said for minima. Therefore, it is to be taken care, that not two values of x and y of different nature are combined, of which one renders X maximal, the other Y maximal, or vice versa. For, if this would happen, the function $X+Y$ would become neither a maximum nor a minimum. But a function of this kind $X-Y$ will become maximal, if X was a maximum and at the same time Y a minimum; on the other hand $X-Y$ will be a minimum, if X was a minimum and Y a maximum. But if both of the functions X and Y would be set either maximal or minimal, their difference $X-Y$ would be neither a maximum nor a minimum; all these things are clear and perspicuous from the nature of maxima and minima explained before.

§287 Therefore, if the maximal or minimal values of a function of two variables is in question, the question is a lot more intricate than the same question for only one variable. For, not only for each variable the cases, in which a maximum or a minimum is produced, are to distinguished diligently, but also from these two of such a kind are to be combined, that the propounded function becomes a maximum or a minimum; this will become more clear from examples.

EXAMPLE 1

Let this function of the two variables x and y be propounded $y^4-8y^3+18y^2-8y+x^3-3xx-3x$ *and let the values for y and x be in question which are to be substituted that this function obtains a maximal or minimal value.*

Since this expression is resolved into two parts of this kind $Y + X$, of which one is a function only of y, the other a function only of x, investigate the cases, in which each of them becomes maximal or minimal. Therefore, because it is

$$Y = y^4 - 8y^3 + 18y^2 - 8y,$$

it will be

$$\frac{dY}{dy} = 4y^3 - 24y^2 + 36y - 8;$$

having put this expression equal to zero it will be having divided by 4

$$y^3 - 6y^2 + 9y - 2 = 0,$$

whose roots are $y = 2$ and $y = 2 \pm \sqrt{3}$. Therefore, because it is $\frac{ddY}{4dy^2} = 3yy - 12y + 9$, in the case $y = 2$ a maximum will arise. For the remaining two roots $y = 2 \pm \sqrt{3}$, which arise from the equation $yy - 4y + 1 = 0$, it will be $\frac{ddY}{12dy^2} = yy - 4y + 3 = 2$, whence both of them give a minimum. But in these cases it will be as follows:

$$\begin{array}{l|ll} y = 2 & Y = +8 & \text{maximum} \\ y = 2 - \sqrt{3} & Y = -1 & \text{minimum} \\ y = 2 + \sqrt{3} & Y = -1 & \text{minimum} \end{array}$$

In similar manner, because it is

$$X = x^3 - 3xx - 3x,$$

it will be

$$\frac{dX}{dx} = 3xx - 6x - 3,$$

whence this equation arises

$$xx = 2x + 1$$

and $x = 1 \pm \sqrt{2}$. But it is $\frac{ddX}{6dx^2} = x - 1 = \pm\sqrt{2}$. Therefore, the root $x = 1 + \sqrt{2}$ gives a minimum, namely $X = -5 - 4\sqrt{2}$, and $x = 1 - \sqrt{2}$ gives a maximum, namely $X = -5 - 4\sqrt{2}$. Therefore, the propounded formula

$$X + Y = y^4 - 8y^3 + 18yy - 8y + x^3 - 3xx - 3x$$

will become a maximum, if one puts $y = 2$ and $x = 1 - \sqrt{2}$, and $X + Y = 3 + 4\sqrt{2}$ will arise. But the same formula $X + Y$ will be minimal, if one takes either $y = 2 - \sqrt{3}$ or $y = 2 + \sqrt{3}$ and $x = 1 + \sqrt{2}$; in both cases it will be $X + Y = -6 - 4\sqrt{2}$.

EXAMPLE 2

If this function of two variables is propounded $y^4 - 8y^3 + 18y^2 - 8y - x^3 + 3xx + 3x$, to investigate in which which cases it becomes either maximal or minimal.

Having put, as we did in the preceding example,

$$Y = y^4 - 8y^3 + 18y^2 - 8y \quad \text{and} \quad X = x^3 - 3xx - 3x$$

the propounded formula will be $Y - X$ and will hence be a maximum, if Y was a maximum and X a minimum. Therefore, because we found these cases before already, it is plain that $Y - X$ obtains a maximal value, if one puts $y = 2$ and $x = 1 + \sqrt{2}$; and it will be $Y - X = 13 + 4\sqrt{2}$. The value of $Y - X$ will be minimal, if Y is a minimum and X a maximum, which happens by putting $y = 2 \pm \sqrt{3}$ and $x = 1 - \sqrt{2}$; but it will be $Y - X = 4 - 4\sqrt{2}$. Furthermore, in each of the two examples it is plain that these values, which we found, are neither the largest nor the smallest of all; for, if one would for the sake of an example put $y = 100$ and $x = 0$, without any doubt a value greater than the one we found would arise; and in similar manner by putting $y = 0$ and either $x = -100$ or $x = +100$ a smaller value than those we found for the case of a minimum would arise. Therefore, the idea explained above is to be kept in mind, which we gave about the nature of maxima and minima, of course that we called a value a maximum, which is larger than the closest preceding and the following values, but this value is a minimum, which was smaller than both the closest preceding and following values. So, in this example the value of $Y - X$, which arises by putting $y = 2$ and $x = 1 + \sqrt{2}$, is larger than those, which result, if one puts $y = 2 \pm \omega$ and $x = 1 + \sqrt{2} \pm \varphi$ having taken sufficiently small quantities for ω and φ.

§288 Having treated these examples the way to find the general solution will be easier. Let V denote any function of the two variables x and y and for x and y let the values to be found, which render the function maximal or minimal. Therefore, because to achieve this to both variables x and y a determined value is to be attributed, let us put that the one y already has the value, which is required to render the function V either maximal or minimal, and having put this it will only be necessary that for the other variable x also an appropriate value is found, which will happen, if the function V is differentiated considering only x as variable and the differential is set equal to zero. In a similar manner, if we assume the variable x to already have the value, which is apt to render the function V either maximal or minimal, the value of y will be found be differentiating with respect to y only and putting this differential equal to zero. Hence, if the differential of the function V was $= Pdx + Qdy$, it will be necessary that $P = 0$ and $Q = 0$, from which two equations the value of both variables x and y can be found.

§289 Since this way without any difference the values for x and y are found, by which the function V is rendered either maximal or minimal, the cases, in which either a maximum or a minimum arises, are to be distinguished carefully from each other. For, that a function V becomes maximal, it is necessary, that both variables conspire for this; for, if the one would exhibit a maximum, the other a minimum, the function itself would become neither a maximum nor a minimum. Therefore, having found the values of x and y from the equations P and Q it is to be investigated, whether both a the same time render the function V either maximal or minimal; and just then, after it is certain that the value of both variables found from this hold for a maximum, we will be able to affirm that the function in this case obtains a maximal value. The same is to be understood for a minimum, such that the function V can only obtain a minimal value, if at the same time both variables x and y produce a minimum. Therefore, all those cases are to be rejected, in which the one variable is detected to indicate a maximum, and the other a minimum. Sometimes it even happens that the values to arise from the equations $P = 0$ and $Q = 0$ of one of the variables or of both exhibit neither a maximum nor a minimum, which cases are to be rejected as inept in exactly the same way.

§290 But whether the values found for x and y are valid for a maximum or a minimum, will be investigated for each of them separately in the same way as above, as if only one variable was there. To make the decision for the variable x consider the other y as constant, and because it is $dV = Pdx$ or $\frac{dV}{dx} = P$, differentiate P again having put y constant that $\frac{ddV}{dx^2} = \frac{dP}{dx}$ arises, and consider, whether the value of $\frac{dP}{dx}$, after instead of x and y the values found before were substituted, becomes positive or negative; for, in the first case a minimum will be indicated, in the second a maximum. Because in similar manner for constant x it is $dV = Qdy$ or $\frac{dV}{dy} = Q$, differentiate Q again having put only y variable and examine the value $\frac{dQ}{dy}$ after having substituted the values for x and y, which were found from the equations $P = 0$ and $Q = 0$; if it was affirmative, it will indicate a minimum, otherwise a maximum. Therefore, it is concluded, if from the values found for x and y the formulas $\frac{dP}{dx}$ and $\frac{dQ}{dy}$ obtain values affected with different signs, the one positive, the other negative, then the function V will become neither maximal nor minimal; but if both of the formulas $\frac{dP}{dx}$ and $\frac{dQ}{dy}$ become affirmative, a minimum will result, and otherwise, if both of them become negative, a maximum.

§291 If one of the formulas $\frac{dP}{dx}$ and $\frac{dQ}{dx}$ or even both, if for x and y the found values are substituted, vanish, then one will have to proceed to the following differentials $\frac{ddP}{dx^2}$ and $\frac{ddQ}{dy^2}$; if they not equally vanish, neither a maximum nor a minimum will take place; but if they vanish, then the decision is to be made from the following differentials $\frac{d^3P}{dx^3}$ and $\frac{d^3Q}{dy^3}$ in the same way it was made for the formulas $\frac{dP}{dx}$ and $\frac{dQ}{dy}$. To be explain the cases, in which this happens, in a more clear way, let the value $x = \alpha$ have been arisen; if it renders the formula $\frac{dP}{dx}$ vanishing, it is necessary that $\frac{dp}{dx}$ has the factor $x - \alpha$; if this factor was the only one of this kind, neither maximum nor a minimum will be indicated; the same happens, if $\frac{dP}{dx}$ had the factor $(x-\alpha)^3$ or $(x-\alpha)^5$ etc. But if the factor was either $(x-a)^2$ or $(x-\alpha)^4$ etc., then a maximum or minimum will be indicated; but furthermore one will have to see, whether it is in agreement with the case indicated by y.

§292 But the work to proceed to higher differentials in these cases can be reduced tremendously; for, if we put, to consider the subject in more generality, that $\alpha x + \beta = 0$ was found and the formula $\frac{dP}{dx}$ has the factor $(\alpha x + \beta)^2$, such that it is $\frac{dP}{dx} = (\alpha x + \beta)^2 T$, since it is $\alpha x + \beta = 0$, it will be $\frac{d^3P}{dx^3} = 2\alpha^2 T$ and hence because of the positive $2\alpha^2$ the decision can be made from the quantity T itself; if it obtains a positive value it will indicate a minimum, otherwise a maximum. And the same auxiliary theorem can be applied in the investigation of maxima and minima, if one single variable is contained, such that it never necessary to ascend to higher differentials. It is not even necessary to ascend to the second differential; for, if from the equation $P = 0$ it is $\alpha x + \beta = 0$, it is necessary, that P has a factor $\alpha x + \beta$; let $P = (\alpha x + \beta)T$, and because it is

$$\frac{dP}{dx} = \alpha T + (\alpha x + \beta)\frac{dT}{dx},$$

because of $\alpha x + \beta = 0$ it will be $\frac{dP}{dx} = \alpha T$ and hence already the factor T itself, depending on whether the value of αT was positive or negative, will immediately indicate a minimum or a maximum.

§293 Therefore, having given these prescriptions, it will not be difficult, if any function involving two variables was propounded, to investigate the cases, in which this function becomes either maximal or minimal. If more is additionally to be said, the expansion of some examples will suggest them, which is why it will be convenient to illustrate the given rules in some examples.

Example 1

Let this function of two variables be propounded $V = xx + xy + yy - ax - by$; inquire, in which cases it becomes either maximal or minimal.

Because it is $dV = 2xdx + ydx + xdy + 2ydy - adx - bdy$, if it is compared to the general formula $dV = Pdx + Qdy$, it will be

$$P = 2x + y - a \quad \text{and} \quad Q = 2y + x - b,$$

whence these equations will be formed

$$2x + y - a = 0 \quad \text{and} \quad 2y + x - b = 0,$$

having combined which by eliminating y it will be $x - b = 4x - 2a$ and hence

$$x = \frac{2a - b}{3} \quad \text{and} \quad y = a - 2x = \frac{2b - a}{3}.$$

Therefore, because it is

$$\frac{dP}{dx} = 2 \quad \text{and} \quad \frac{dQ}{dy} = 2,$$

both of them show a minimum; from this we conclude that the formula

$$xx + xy + yy - ax - by$$

becomes a minimum, if one puts $x = \frac{2a-b}{3}$ and $y = \frac{2b-a}{3}$, and this way it will arise as

$$V = \frac{-3aa + 3ab - 3bb}{9} = \frac{-aa + ab - bb}{3};$$

since this is the only one, it will be the smallest of all. Therefore, it can only in one way be

$$xx + xy + yy - ax - by = \frac{-aa + ab - bb}{3},$$

and since it cannot be smaller, this equation will be impossible

$$xx + xy + yy - ax - by = \frac{-aa + ab - bb}{3} - cc.$$

EXAMPLE 2

If the formula $V = x^3 + y^3 - 3axy$ is propounded, let the cases be in question, in which V obtains a maximal or minimal value.

Because of $dV = 3xxdx + 3yydy - 3aydx - 3axdy$ it will be

$$P = 3xx - 3ay \quad \text{and} \quad Q = 3yy - 3ax,$$

whence it is

$$ay = xx \quad \text{and} \quad ax = yy.$$

Therefore, because if it is $yy = x^4 : aa = ax$, it will be $x^4 - a^3x = 0$ and hence either $x = 0$ or $x = a$. In the first case it is $y = 0$, in the second $y = a$. Therefore, because it is

$$\frac{dP}{dx} = 6x, \quad \frac{ddP}{dx^2} = 6 \quad \text{and} \quad \frac{dQ}{dy} = 6y \quad \text{and} \quad \frac{ddQ}{dy^2} = 6,$$

in the first case, in which it is $x = 0$ and $y = 0$, neither a maximum nor a minimum results. But in the second case, in which both $x = a$ and $y = a$, a minimum arises, if a was a positive quantity, and it will be $V = -a^3$, which value is only smaller than the closest preceding and following ones; for, without any doubt V can obtain a much smaller values, if to both variables x and y negative values are attributed.

EXAMPLE 3

Let this function be propounded $V = x^3 + ayy - bxy + cx$, whose maximal or minimal values are to be investigated.

Since it is $dV = 3xxdx + 2aydy - bydx - bxdy + cdx$, it will be

$$P = 3xx - by + c \quad \text{and} \quad Q = 2ay - bx,$$

having put which values equal to zero it will be $y = \frac{bx}{2a}$ and hence

$$3xx - \frac{bbx}{2a} + c = 0 \quad \text{or} \quad xx = \frac{2bbx - 4ac}{12a},$$

whence it is

$$x = \frac{bb \pm \sqrt{b^4 - 48aac}}{12ac}.$$

Therefore, only if it is $b^4 - 48aac > 0$, a maximum or a minimum takes place. Therefore, let us put that it is $b^4 - 48aac = bbff$ that $c = \frac{bb(bb-ff)}{48aa}$; it will be

$$x = \frac{bb \pm bf}{12a} \quad \text{and} \quad y = \frac{bb(b \pm f)}{24aa}.$$

Since further it is

$$\frac{dP}{dx} = 6x \quad \text{and} \quad \frac{dQ}{dy} = 2a,$$

it will be

$$\frac{dP}{dx} = \frac{b(b \pm f)}{2a}.$$

Therefore, only if $2a$ and $\frac{b(b\pm f)}{2a}$ are quantities of the same sign, either a maximum or a minimum takes place. And if they are indeed both either positive or negative, what happens, if their product $b(b \pm f)$ was positive, then the function V becomes a minimum, if a was a positive quantity, otherwise a maximum, if a is a negative quantity. Hence, if it was $f = 0$ or $c = \frac{b^4}{48aa}$, because of the positive quantity bb the function V will become minimal, if

a was a positive quantity and one sets $x = \frac{bb}{12a}$ and $y = \frac{b^3}{24aa}$; otherwise, if a is negative, these substitutions produce a maximum. If $f < b$, in the two cases either a maximum or a minimum arises; but if $f > b$, then only the case $x = \frac{b(b+f)}{12a}$ and $y = \frac{bb(b+f)}{24aa}$ will yield a maximum or a minimum, depending on whether a was negative or positive. Let $a = 1$, $b = 3$ and $f = 1$, that one has this formula $V = x^3 + yy - 3xy + \frac{3}{2}x$; this will become a minimum because of the positive a, if one puts either $x = 1$ and $y = \frac{3}{2}$ or $x = \frac{1}{2}$ and $y = \frac{3}{4}$. In the first case $V = \frac{1}{4}$ arises, in the second $V = \frac{5}{16}$. It is nevertheless plain that by putting negative numbers instead of x a lot smaller values for V can arise. Therefore, the value of $V = \frac{1}{4}$ must be understood to be smaller than if one puts $x = 1 + \omega$ and $y = \frac{3}{2} + \varphi$, if ω and φ are small numbers, either positive or negative; but the limits, which ω must not pass, is $-\frac{15}{4}$; for if $\omega < -\frac{15}{4}$, it can happen that V becomes smaller than $\frac{1}{4}$.

EXAMPLE 4

To find the maxima or minima of this function

$$V = x^4 + y^4 - axxy.axyy + ccxx + ccyy.$$

Having taken the differential it will be

$$P = 4x^3 - 2axy - ayy + 2ccx \quad \text{and} \quad Q = 4y^3 - axx - 2axy + 2ccy,$$

having put which equal equal to zero, if they are subtracted from each other, it will be

$$4x^3 - 4y^3 + axx - ayy + 2ccx - 2ccy = 0;$$

since it is divisible by $x - y$, it will be $y = x$ at first and then $4x^3 - 3axx + 2ccx = 0$, which gives

$$x = 0 \quad \text{and} \quad 4xx = 3ax - 2cc \quad \text{and} \quad x = \frac{3a \pm \sqrt{9aa - 32cc}}{8}.$$

If we take $x = 0$, it will also be $y = 0$ and because of

$$\frac{dP}{dx} = 12xx - 2ay + 2cc \quad \text{and} \quad \frac{dQ}{dy} = 12yy - 2ax + 2cc$$

the function V becomes minimal $= 0$. If we set $x = y = \frac{3a \pm \sqrt{9aa - 32cc}}{8}$, if it was $9aa > 32cc$, because of $4xx = 3ax - 2cc$ it will be

$$\frac{dP}{dx} = \frac{dQ}{dy} = 12xx - 2ax + 2cc = 7ax - 4cc = \frac{21aa - 32cc \pm 7a\sqrt{9aa - 32cc}}{8};$$

because this value is always positive because of $32cc < 9aa$, the value V will also in this case be minimal and it will be

$$V = -\frac{27}{256}a^4 + \frac{9}{16}aacc - \frac{1}{2}c^4 \mp \frac{a}{256}(9aa - 32cc)^{\frac{3}{2}}.$$

But let us divide the equation $4x^3 - 4y^3 + axx - ayy + 2ccx - 2ccy = 0$ by $x - y$ and it will be $4xx + 4xy + 4yy + ax + ay + 2cc = 0$. But from the equation $P = 0$ it will be $yy = -2xy + \frac{4}{a}x^3 + \frac{2ccx}{a}$, having substituted which value in that equation it is

$$y = \frac{16x^3 + 4axx + aax + 8ccx + 2acc}{4ax - aa}.$$

But this gives

$$y = -x \pm \sqrt{\frac{4x^3 + axx + 2ccx}{a}},$$

whence it is caused

$$16x^3 + 8axx + 8ccx + 2acc = (4x - a)\sqrt{4ax^3 + aaxx + 2accx},$$

which reduced to a rational equation gives

$$\begin{array}{llllll} 256x^6 + 192ax^5 & + 80aa \;\; x^4 & + 4a^3 \quad\;\; x^3 & - a^2 \qquad x^2 & - 2a^3ccx + 4a^2c^4 = 0; \\ & + 256cc & + 160acc & + 48aacc & + 32ac^4 \\ & & & + 64c^4 & \end{array}$$

whose real roots, if it has such, will indicate the maxima or minima of the function V, if $\frac{dP}{dx}$ and $\frac{dQ}{dy}$ become rational quantities affected with the same sign.

EXAMPLE 5

To find the maxima and minima of this expression

$$x^4 + mxxyy + y^4 + aaxx + naaxy + aayy = V.$$

Having done the differentiation it will be

$$\begin{aligned} P &= 4x^3 + 2mxyy + 2aax + naay = 0, \\ Q 4y^3 &\quad + 2mxxy + 2aay + naax = 0, \end{aligned}$$

which equation either subtracted from or add to each other give

$$\begin{aligned} (4xx + 4xy + 4yy - 2mxy + 2aa - naa)(x - y) = 0, \\ (4xx - 4xy + 4yy + 2mxy + 2aa + naa)(x + y) = 0, \end{aligned}$$

which divided by $x - y$ and $x + y$ and either added or subtracted again give

$$4xx + 4yy + 2aa = 0 \quad \text{and} \quad 4xy - 2mxy - naa = 0.$$

From the latter it is $y = \frac{naa}{2(2-m)x}$; but the first does not admit real values. Therefore, we have three cases.

I. Let $y = x$ and it will be $4x^3 + 2mx^3 + 2aax + naax = 0$, whence it is either $x = 0$ or $2(2 + m)xx + (2 + n)aa = 0$. Let $x = 0$; it will be $y = 0$ and because of

$$\frac{dP}{dx} = 12xx + 2myy + 2aa \quad \text{and} \quad \frac{dQ}{dy} = 12yy + 2mxx + 2aa$$

in this case $V = 0$ will become a minimum, if the coefficient aa was positive. The other case gives $xx = -\frac{(n+2)aa}{2(m+2)}$, which can only be real, if $\frac{n+2}{m+2}$ is a negative number. Let $\frac{n+2}{m+2} = -2kk$ or $n = -2kkm - 4kk - 2$; it will be $x = \pm ka$ and $y = \pm ka$. But

$$\frac{dP}{dx} = 12kkaa + 2mkkaa + 2aa \quad \text{and} \quad \frac{dQ}{dy} = 12kkaa + 2mkkaa + 2aa;$$

since they are equal, V will be either a maximum or a minimum, depending on whether these quantities were positive or negative.

II. Let $y = -x$ and it will be $2(m+2)x^3 = (n-2)aax$, therefore either $x = 0$ or $xx = \frac{(n-2)aa}{2(m+2)}$. The first root $x = 0$ recedes to the preceding. The second on the other hand will be real, if $\frac{(n-2)aa}{2(m+2)}$ was a positive quantity, and because it is $\frac{dP}{dx} = \frac{dQ}{dy}$, either maximum or a minimum will arise.

III. Let $y = \frac{naa}{2(2-m)x}$; it will be

$$4x^3 + \frac{mn^2a^4}{2(2-m)^2x} + 2aax + \frac{nna^4}{2(2-m)x} = 0 \quad \text{or} \quad 4x^4 + 2aaxx + \frac{nna^4}{(2-m)^2} = 0,$$

no root of which equation is real, if aa is not a negative quantity.

Example 6

Let this determined function be propounded $V = x^4 + y^4 - xx + xy - yy$, whose maximal or minimal values are to be investigated.

Since therefore it is $P = 4x^3 - 2x + y = 0$ and $Q = 4y^3 - 2y + x = 0$, it will be from the first $y = 2x - 4x^3$, which substituted in the other gives

$$256x^9 - 384x^7 + 192x^5 - 40x^3 + 3x = 0.$$

One of its roots is $x = 0$, whence it also is $y = 0$. Therefore, in this case because of

$$\frac{dP}{dx} = 12xx - 2 \quad \text{and} \quad \frac{dQ}{dy} = 12yy - 2$$

a maximum $V = 0$ arises. But having divided the found equation by x it will be

$$256x^8 - 384x^6 + 192x^4 - 40xx + 3 = 0,$$

which has the factor $4xx - 1$, whence it is $4xx = 1$ and $x = \pm\frac{1}{2}$ and $y = \pm\frac{1}{2}$; then it will be $\frac{dP}{dx} = \frac{dQ}{dy} = 1$; therefore, in each of the two cases a maximum $V = -\frac{1}{8}$ arises. Divide that equation by $4xx - 1$ and one will obtain

$$64x^6 - 80x^4 + 28xx - 3 = 0,$$

which again which contains $4xx - 1 = 0$ twice such that the preceding case arises. Furthermore, from this it is $4xx - 3 = 0$ and $x = \frac{\pm\sqrt{3}}{2}$; to this corresponds $y = \frac{\mp\sqrt{3}}{2}$. Therefore, it will also be $\frac{dP}{dx} = \frac{dQ}{dy} = 7$ and therefore V will be a minimum $= -\frac{9}{8}$; this is the smallest value of all, which the function V can receive, and therefore, this equation $V = -\frac{9}{8} - cc$ is always impossible. But hence the way to determined maxima and minima of functions containing three or more variables is plain.