# On Criteria for imaginary roots *

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**§313** In the preceding chapter we exhibited a method to explore the nature of roots of any equation such that by means of it, if any arbitrary equation is propounded, one can find out, how many real and imaginray roots it has. In most cases, this investigation is certainly most difficult, if the differential equation is of such a nature that its roots cannot be exhibited. But although in these cases the same operation could be applied to the differential equation itself and the nature of its roots could be explored from its differential and hence the roots of the original equation could be assigned approximately, the work would nevertheless be too cumbersome in almost every case. Therefore, in this case it often suffices to know criteria of such a kind, from which, if the their conditions are met, one can conclude with safety that imaginary roots are contained in the propounded equation, even though, if the conditions are not met, one can vice versa not infer that all roots are real. Even if then the knowledge is not complete, it will be useful very often; therefore, we dedicated the present chapter to the explanation of these criteria.

**§314** So, in the preceding chapter we saw, if any arbitrary equation

$$z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

has only real roots that then its differential

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$$\frac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.} = 0$$

will have also only real roots. But at the same time we showed, even though the differential equation only has real roots, hence it nevertheless does not follow that all roots of the propounded equation are real. Nevertheless, if the differential equation has imaginary roots, then we will always correctly conclude that the propounded equation itself must at least have as many imaginary roots. I say at least; for, it can happen that the equation has more imaginary roots. Therefore, this way from the differential equation one cannot conclude more than, if it has imaginary roots, that the propounded equation must also have roots of such a kind, and at least as many.

**§315** If the propounded equation is multiplied by any power $x^m$, while $m$ denotes a positive integer, because this new equation will have only real roots, if all roots of the propounded one were real, then also the roots of its differential, after having divided by $x^{m-1}$, will all be real. Hence, if this equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

has only real roots, then also this equation

$$(m+n)x^n - (m+n-1)Ax^{n-1} + (m+n-2)Bx^{n-2} - \text{etc.} = 0$$

will have only real roots. For the same reason, if this equation is multiplied by $x^k$ and differentiated again, the resulting equation

$$(m+n)(k+n)x^n - (m+n+-1)(k+n-1)Ax^{n-1} + (m+n-2)(k+n-2)Bx^{n-2} - \text{etc.} = 0$$

will have still real roots and so one can continue as far as one wishes to. But if an equation of this kind is detected to have imaginary roots, then it will be certain at the same time that the propounded equation will have at least as many imaginary roots.

**§316** If the propounded equation, before it is differentiated, is multiplied by no power of $x$, then the decision is to be made for an equation of one degree lower. So, if the propounded equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \text{etc.} = 0$$

has only real roots, then also all roots of its differentials of all orders will be real. Hence also the roots of all the following equations will be real

$$nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.} = 0,$$
$$n(n-1)x^{n-2} - (n-1)(n-2)Ax^{n-3} + (n-2)(n-3)Bx^{n-4} - \text{etc.} = 0,$$
$$n(n-1)(n-2)x^{n-3} - (n-1)(n-2)(n-3)Ax^{n-4} + \text{etc.} = 0,$$
$$n(n-1)(n-2)(n-3)x^{n-4} - (n-1)(n-2)(n-3)(n-4)AX^{n-5} + \text{etc.} = 0$$
$$\text{etc.},$$

which equations are reduced to the following forms

$$x^{n-1} - \frac{n-1}{n}Ax^{n-2} + \frac{(n-1)(n-2)}{n(n-1)}Bx^{n-3} - \frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)}Cx^{n-4} + \text{etc.} = 0,$$
$$x^{n-2} - \frac{n-2}{n}Ax^{n-3} + \frac{(n-2)(n-3)}{n(n-1)}Bx^{n-4} - \frac{(n-2)(n-3)(n-4)}{n(n-1)(n-2)}Cx^{n-5} + \text{etc.} = 0,$$
$$x^{n-3} - \frac{n-3}{n}Ax^{n-4} + \frac{(n-3)(n-4)}{n(n-1)}Bx^{n-5} - \frac{(n-3)(n-4)(n-5)}{n(n-1)(n-2)}Cx^{n-6} + \text{etc.} = 0,$$
$$x^{n-4} - \frac{n-4}{n}Ax^{n-5} + \frac{(n-4)(n-5)}{n(n-1)}Bx^{n-6} - \frac{(n-4)(n-5)(n-6)}{n(n-1)(n-2)}Cx^{n-7} + \text{etc.} = 0$$
$$\text{etc.}$$

**§317** Therefore, this way the decision can be reduced to an equation of given lower degree than the propounded one itself. So, if $m$ was any arbitrary number smaller than $n$, then, if the propounded equation has only real roots, then also all roots of this equation of degree $m$ will be real

$$x^m - \frac{m}{n}Ax^{m-1} + \frac{m(m-1)}{n(n-1)}Bx^{m-2} - \frac{m(m-1)(m-2)}{n(n-1)(n-2)}Cx^{m-3} + \text{etc.} = 0.$$

Hence, if one puts $m = 2$, this equation will arise

$$x^2 - \frac{2}{n}Ax + \frac{2 \cdot 1}{n(n-1)}B = 0,$$

whose roots will have to be real, if the propounded equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \text{etc.} = 0$$

has only real roots. But because this quadratic equation can only have real roots, if it is $\frac{AA}{nn} > \frac{2 \cdot 1}{n(n-1)}B$, it follows that all roots of the propounded equation can only be real, if it is $AA > \frac{2n}{2n-1}B$. Therefore, if it was $AA < \frac{2n}{2n-1}B$, this will be a certain sign that at least two roots of the propounded equation will be imaginary.

**§318** Hence we obtained a necessary condition, which the coefficients of the three first terms have to satisfy, if all roots of the propounded equation were real. And this is a criterion of such kind as we mentioned at the beginning: Even though in the case $AA > \frac{2n}{n-1}B$ nothing follows for the realness of the roots, but if it is $AA < \frac{2n}{n-1}B$, it will nevertheless be a certain sign for at least two imaginary roots. So that all roots are real, by successively substituting the numbers 2, 3, 4, 5 etc. for $n$ it is required as follows:

$$\begin{array}{ll} x^2 - Ax + B = 0 & A^2 > 4B \\ x^3 - Ax^2 + Bx - C = 0 & A^2 > \frac{6}{2}B \\ x^4 - Ax^3 + Bx^2 - Cx + D = 0 & A^2 > \frac{8}{3}B \\ x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0 & A^2 > \frac{10}{4}B. \end{array}$$

Hence, if the second term is missing and the coefficient $B$ of the third is positive that the equation is of this kind

$$x^n + Bx^{n-1} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0,$$

all roots cannot be real, but at least two will be imaginary.

**§319** Criteria of this kind can indeed be found four the coefficients of the following terms, if we consider that this equation

$$1 - Ay + By^2 - Cy^3 + Dy^4 - \text{etc.} = 0$$

has as many real and imaginary roots as the propounded equation itself. For, this equation arises from the given one, if one puts $x = \frac{1}{y}$ such that from the roots of this equation one at the same time knows the roots of the latter. Hence, if the propounded equation has only real roots, then also all roots of the differential equation of the reciprocal equation, namely of this one

$$-A + 2By - 3Cy^2 + 4Dy^3 - \text{etc.} = 0,$$

will be real. In this again substitute $x$ for $\frac{1}{y}$ and this equation will emerge

$$Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3} - 4Dx^{n-4} + \text{etc.} = 0,$$

whose roots will therefore be real, if the roots of the propounded equation were such. Hence it is now plain, if it was $n = 3$, that it is necessary that it is $BB > 3AC$.

**§320** But now differentiate this equation further and these will arise

$$\begin{aligned}
Ax^{n-2} - \frac{2(n-2)}{n-1}Bx^{n-3} + \frac{3(n-2)(n-3)}{(n-1)(n-2)}Cx^{n-4} - \text{etc.} = 0 \\
Ax^{n-3} - \frac{2(n-3)}{n-1}Bx^{n-4} + \frac{3(n-3)(n-4)}{(n-1)(n-2)}Cx^{n-5} - \text{etc.} = 0 \\
Ax^{n-4} - \frac{2(n-4)}{n-1}Bx^{n-5} + \frac{3(n-4)(n-5)}{(n-1)(n-2)}Cx^{n-6} - \text{etc.} = 0 \\
\text{etc.}
\end{aligned}$$

Therefore, in general, if the number $m$ is smaller than $n$, it will be

$$Ax^m - \frac{2m}{n-1}Bx^{m-1} + \frac{3m(m-1)}{(n-1)(n-2)}Cx^{m-2} - \text{etc.} = 0.$$

If one now puts $m = 2$, one will have this equation

$$Ax^2 - \frac{4}{n-1}Bx + \frac{6}{(n-1)(n-2)}C = 0;$$

that its roots are real it is necessary that it is $\frac{4BB}{(n-1)^2} > \frac{6AC}{(n-1)(n-2)}$. Hence, if the propounded equation has only real roots, it will be

$$BB > \frac{3(n-1)}{2(n-2)}AC.$$

And if it was $BB < \frac{3(n-1)}{2(n-2)}AC$, this is a certain sign that the propounded equation has at least two imaginary roots. Therefore, if it is $n = 3$, the criterion will be $BB > 3AC$; but if $n = 4$, it will be $BB > \frac{3 \cdot 3}{2 \cdot 2}AC$; if $n = 5$, it will be $BB > \frac{3 \cdot 4}{2 \cdot 3}AC$ and so forth.

**§321** To transfer these criteria to the following coefficients, let us resume the differential equation found in $y$

$$-A + 2By - 3Cy^2 + 4Dy^3 - 5Ey^4 + \text{etc.} = 0$$

and let us differentiate this again that we have

$$2B - 6Cy + 12Dy^2 - 20Ey^3 + \text{etc.} = 0,$$

which having resubstituted $\frac{1}{x}$ for $y$ will give

$$Bx^{n-2} - 3Cx^{n-3} + 6Dx^{n-4} - 10Ex^{n-5} + \text{etc.} = 0,$$

from whose further differentiation these equations follow

$$Bx^{n-3} - \frac{3(n-3)}{n-2}Cx^{n-4} + \frac{6(n-3)(n-4)}{(n-2)(n-3)}Dx^{n-5} - \text{etc.} = 0$$

and in general

$$Bx^{m} - \frac{3m}{n-2}Cx^{m-1} + \frac{6m(m-1)}{(n-2)(n-3)}Dx^{m-3} - \text{etc.} = 0.$$

Therefore, if we put $m = 2$ in general, this quadratic equation will arise

$$Bx^2 - \frac{2 \cdot 3}{n-2}Cx + \frac{6 \cdot 2}{(n-2)(n-3)}D = 0,$$

whose roots will be real, if it was $\frac{)CC}{(n-2)^2} > \frac{6 \cdot 2BD}{(n-2)(n-3)}$ or

$$CC > \frac{4(n-2)}{3(n-3)}BD.$$

Hence, if the propounded equation has only real roots, it will be $CC > \frac{4(n-2)}{3(n-3)}BD$, and if this condition is not met, the equation will certainly have at least two imaginary roots.

**§322** If we differentiate the superior equation $2B - 6Cy + 12Dy^2 - \text{etc.} = 0$ one again, it will arise

$$-6C + 24Dy - 60Ey^2 + \text{etc.} = 0$$

or

$$C - 4Dy + 10Ey^2 - 20Fy^3 + \text{etc.} = 0,$$

which having resubstituted $x$ for $\frac{1}{y}$ will go over into this one

$$Cx^{n-3} - 4Dx^{n-4} + 10Ex^{n-5} - 20Fx^{n-6} + \text{etc.} = 0,$$

from whose further differentiation these follow

$$Cx^{n-4} - \frac{4(n-4)}{n-3}Dx^{n-5} + \frac{10(n-4)(n-5)}{(n-3)(n-4)}Ex^{n-6} - \text{etc.} = 0,$$
$$Cx^{n-5} - \frac{4(n-5)}{n-3}Dx^{n-6} + \frac{10(n-5)(n-6)}{(n-3)(n-4)}Ex^{n-7} - \text{etc.} = 0$$

and in general

$$Cx^m - \frac{4m}{n-3}Dx^{m-1} + \frac{10m(m-1)}{(n-3)(n-4)}Ex^{m-2} - \text{etc.} = 0.$$

Let us put $m = 2$ and it will be

$$Cx^2 - \frac{2 \cdot 4}{n-3}Dx + \frac{2 \cdot 10}{(n-3)(n-4)}E = 0,$$

from which, if its roots are real, it follows that it will be

$$\frac{4 \cdot 4}{(n-3)^2} > \frac{2 \cdot 10}{(n-3)(n-4)}CE \quad \text{or} \quad DD > \frac{5(n-3)}{4(n-4)}CE.$$

**§323** From these one already clearly sees the relation for all coefficients. Therefore, in general, if this equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \text{etc.} = 0$$

only has real roots, it will be

$$
\begin{aligned}
AA &> \frac{2n}{1(n-1)}B \\
BB &> \frac{3(n-1)}{2(n-2)}AC \\
CC &> \frac{4(n-2)}{3(n-3)}BD \\
DD &> \frac{5(n-3)}{4(n-4)}CE \\
EE &> \frac{6(n-4)}{5(n-5)}DF \\
&\text{etc.}
\end{aligned}
$$

If one of these conditions is missing, the equation will have at least two imaginary roots. And if these criteria do not depend on each other, it is easily seen that as many pairs of imaginary roots are given as the number of non-satisfied conditions. But even if these conditions all hold in one single equation, it hence nevertheless does not follow that no imaginary roots are given; it can even happen that, because there is no reason against it, all roots are imaginary. Therefore it is to be taken care that not more is attributed to these criteria than it can actually attributed to them via the principles whence they were deduced.

**§324** But it is easily clear that not every single condition, which is not met, can indicate imaginary roots; for, in an equation of $n$ dimensions, since one has $n+1$ terms and from the single ones except the first and the last the criterion can be taken, one will in total have $n-1$ conditions; and nevertheless, if the some are not satisfied, the equation cannot have $2n-2$ imaginary roots, since it in total has only $n$ roots. But one condition alone always reveals two imaginary roots, and since it can happen that two conditions of this kind do not show more roots, one has to consider, whether these two conditions are contiguous or not; in the first case the number of imaginary roots will not be augmented, in the second on the other hand, since the conditions involve completely different letters, each one of them will show two imaginary roots. So, even though it was

$$AA < \frac{2n}{1(n-1)}B \quad \text{and} \quad BB < \frac{3(n-1)}{2(n-2)}AC,$$

hence nevertheless not necessarily four imaginary roots are indicated, but both of them might indicate the same two roots. On the other hand, if it was

$$AA < \frac{2n}{1(n-1)B} \quad \text{and} \quad CC < \frac{4(n-2)}{3(n-3)}BD$$

while $BB > \frac{3(n-1)}{2(n-2)}AC$, four imaginary roots will be indicated.

**§325** Therefore, from the criteria immediately following each other for imaginary roots it does not follow more than from one; but if they proceed in an interrupted order that between each two one or more contrary criteria fall, then from each one of them one can conclude two imaginary roots. This consideration yields the following rule. Except for the first and the last term write the coefficients found before of the criteria over the single terms of the propounded equation this way

$$\begin{array}{cccccc} & \frac{2n}{1(n-1)} & \frac{3(n-1)}{2(n-2)} & \frac{4(n-2)}{3(n-3)} & \frac{5(n-3)}{4(n-4)} & \text{etc.} \\ x^n & - Ax^{n-1} + & Bx^{n-2} - & Cx^{n-3} + & Dx^{n-4} - & \text{etc.} = 0 \\ + & \cdots & \cdots & \cdots & \cdots & \text{etc.} \end{array}$$

Then examine the square of each coefficient, whether it is larger or smaller than the fraction written above it multiplied by the product of the corresponding coefficients; in the first case give the term the sign $+$, in the second the sign $-$; but always give the first and the last term the sign $+$. Having done this, the equation will have at least as many imaginary roots as variations in the given signs occur.

**§326** This is the rule found by Newton to explore the imaginary roots of each equation; but one has to pay attention, as we already mentioned, as it can happen often that this equation has more imaginary roots than one detects by means of this method. Hence others tried to find other similar rules which then would yield the number of imaginary roots more exactly, such the the true number of roots would exceed the number, which the rule gives, less often. In this regard, especially the rule of Campbellus added to Newtons universal Arithmetic stands out, which will therefore will be conveniently explained here, even if it is not perfect. It is founded on this lemma: If $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ etc. were the quantities and the number is $m$, put the sum of these quantities

$$\alpha + \beta + \gamma + \delta + \text{etc.} = S,$$

the sum of the squares

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \text{etc.} = V,$$

and it will be $V > 0$. But because the product of each two is

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \text{etc.} = \frac{SS - V}{2},$$

it will be $(m-1)V > SS - V$ or $mV > SS$. For, if the squares of the differences of two quantities are taken, their sum will be

$$\begin{aligned}
&= (\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\alpha-\delta)^2 + (\beta-\gamma)^2 + (\beta-\delta)^2 + \text{etc.}\\
&= (m-1)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \text{etc.}) - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \text{etc.})\\
&= (m-1)V - 2\frac{SS - V}{2} = mV - SS.
\end{aligned}$$

Therefore, since the sum of real squares is always positive, it will be

$$mV - SS > 0 \quad \text{and hence} \quad mV > SS.$$

**§327** Having given this lemma in advance, if one has this equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + Fx^{n-6} - \text{etc.} = 0$$

and all its $n$ roots were real, which shall be $a$, $b$, $c$, $d$, $e$ etc., it will be, as it is known from the nature of equations,

| | numbers of terms |
|---|---|
| $A = a + b + c + d + \text{etc.}$ | $n$ |
| $B = ab + ac + ad + bc + bd + \text{etc.}$ | $\frac{n(n-1)}{1 \cdot 2}$ |
| $C = abc + abd + abe + acd + bcd + \text{etc.}$ | $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$ |
| $B = abcd + abce + abde + \text{etc.}$ | $\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$ |

etc.

Now take the squares of these single terms and put

$$\begin{aligned}
P &= a^2+b^2+c^2+d^2+\text{etc.},\\
Q &= a^2b^2+a^2c^2+a^2d^2+b^2c^2+\text{etc.},\\
R &= a^2b^2c^2+a^2b^2d^2+a^2b^2e^2+a^2c^2d^2+\text{etc.},\\
S &= a^2b^2c^2d^2+a^2b^2c^2e^2+a^2b^2d^2e^2+\text{etc.}\\
&\text{etc.};
\end{aligned}$$

it will be from the nature of combinations

$$\begin{aligned}
P &= A^2-2B,\\
Q &= B^2-2AC+2D,\\
R &= C^2-2BD+2AE-2F,\\
S &= D^2-2CE+2BF-2AG+2H\\
&\text{etc.}
\end{aligned}$$

**§328** By means of the lemma stated in advance we will therefore

$$\begin{gathered}
nP > AA,\\
\frac{n(n-1)}{1\cdot 2}Q > BB,\\
\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}R > CC,\\
\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}S > DD\\
\text{etc.}
\end{gathered}$$

Therefore, if for the values $P$, $Q$, $R$ etc. the values found before are substituted, we will obtain the following properties of the real roots

$$nAA-2nB > AA \quad \text{or} \quad AA > \frac{2n}{n-1}B,$$

$$\frac{n(n-1)}{1\cdot 2}BB - \frac{2n(n-1)}{1\cdot 2}AC + \frac{2n(n-1)}{1\cdot 2}D > BB$$

or

$$BB > \frac{\frac{2n(n-1)}{1\cdot 2}}{\frac{n(n-1)}{1\cdot 2} - 1}(AC - D)$$

and in similar manner the following equations yield

$$CC > \frac{\frac{2n(n-1)(n-2)}{1\cdot 2\cdot 3}}{\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - 1}(BD - AE + F),$$

$$DD > \frac{\frac{2n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}}{\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} - 1}(CE - BF + AG - H).$$

Therefore, the square of each coefficient is not only compared to the product of the closest terms, but also to the rectangles of two equally distant ones, nevertheless in such a way that the signs of these rectangles alternate.

**§329** Therefore, except for the first and the last terms one has to write the fractions, whose numerators are twice the binomial coefficients of the same power, but whose denominators are the binomial coefficients diminished by the unity, above the single terms of the equation. So, by considering quadratic, cubic, biquadratic equations etc., if their roots all were real, it will be

$$x^2 - \overset{\frac{4}{1}}{A}x + B = 0; \quad A^2 > 4B;$$

For the cubic equation

$$x^3 - \overset{\frac{6}{2}}{A}x^2 + \overset{\frac{6}{2}}{B}x - C = 0$$

it will be

$$A^2 > 3B \quad \text{and} \quad B^2 > 3AC.$$

For the biquadratic equation

$$x^4 - \overset{\frac{8}{3}}{A}x^3 + \overset{\frac{12}{5}}{B}x^2 - \overset{\frac{8}{3}}{C}x + \ = 0$$

it will be

$$A^2 > \frac{8}{3}B, \quad B^2 > \frac{12}{5}(AC - D), \quad C^2 > \frac{8}{3}BD.$$

For the equation of the fifth power

$$x^5 - \overset{\frac{10}{4}}{A}x^4 + \overset{\frac{20}{9}}{B}x^3 - \overset{\frac{20}{9}}{C}x^2 + \overset{\frac{10}{4}}{D}x - E = 0$$

it will be

$$AA > \frac{10}{4}B, \quad B^2 > \frac{20}{9}(AC - D), \quad C^2 > \frac{20}{9}(BD - AE) \quad \text{and} \quad D^2 > \frac{10}{4}CE.$$

For the equation of the sixth power

$$x^6 - \overset{\frac{12}{5}}{A}x^5 + \overset{\frac{30}{14}}{B}x^4 - \overset{\frac{40}{19}}{C}x^3 + \overset{\frac{30}{14}}{D}x^2 - \overset{\frac{12}{5}}{E}x + F = 0$$

it will be

$$A^2 > \frac{12}{5}B, \quad B^2 > \frac{30}{14}(AC - D), \quad C^2 > \frac{40}{19}(BD - AE + F),$$
$$D^2 > \frac{30}{14}(CE - BF), \quad E^2 > \frac{12}{5}DF.$$

etc.

**§330** Therefore, if a certain condition is not satisfied, it will be a sign that at least two imaginary roots are contained in the propounded equation. But because, if the single ones are not met, the equation cannot have twice as many imaginary roots, in similar manner a decision is to be made in these cases, as we mentioned before for Newton's rule. If the square of a certain term was larger then the fraction written above it multiplied by the product of the closest and equally distant terms, then to this term ascribe the sign $+$, otherwise the sign $-$; but to the first and the last term always ascribe the sign $+$. Having done this check the order of the signs, and as often as a variation occurs, an imaginary root will be indicated. Therefore, if this rule indicates more imaginary roots than Newton's rule, it will be closer to the truth. Nevertheless it can happen that the equation has more imaginary roots than each of the rules indicates.

**§331** Therefore, we would make a mistake, if we wanted the use these criteria as perfect sign for real and imaginary roots, since it can happen that the equation has more imaginary roots than these criteria indicate; and the error could be the greater, the higher the degree of the propounded equation was. For, in the case of quadratic equations these criteria are true in such a way that, if they do not indicate any imaginary roots, the equation will also have none. But the cubic equation can have two imaginary roots, even though both rules (they coincide in this case) exhibit them. Therefore, to someone wanting to investigate these cases let this general cubic equation be propounded

$$x^3 - \overset{3}{Ax^2} + \overset{3}{Bx} - C = 0;$$

if in this it was $AA > 3B$ and $BB > 3AC$, none of both rules indicates imaginary roots. But above (§ 306) we saw that for this, that there are no imaginary roots, at first it is required that it is $B < \frac{1}{3}AA$, which condition also both rules require. Therefore, let $B = \frac{1}{3}AA - \frac{1}{3}FF$ and it is necessary that $C$ is contained within these limits

$$\frac{1}{27}A^3 - \frac{1}{9}AFF - \frac{2}{27}f^2 \quad \text{and} \quad \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3.$$

But both rules only demand that it is $C < \frac{BB}{3A}$, this means

$$C < \frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A}.$$

This condition can hold, even though $C$ is not contained within the mentioned limits.

**§332** For, let

$$C = \frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg$$

and the rules will indicate no imaginary roots. There will nevertheless be imaginary roots, if it was either

$$\frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg < \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$$

or

$$\frac{1}{27}A^3 - \frac{2}{27}Aff + \frac{f^4}{27A} - gg > \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3.$$

Therefore, if it was either

$$gg > \frac{(ff+Af)^2}{27A} \quad \text{or} \quad gg < \frac{(Af-ff)^2}{27A},$$

the cubic equation will have two imaginary roots, even though none of both rules indicates them. But here we assumed that $A$ is a positive quantity; for, if it was negative, by putting $x = -y$ the equation would be transformed into a form of such kind, in which $A$ would be positive. Hence one can form infinitely many cubic equations, which have two imaginary roots, even though they are not indicated by the rule. For, let $gf = \frac{(ff+Af)^2}{27A} + hh$; it will be

$$C = \frac{(ff-AA)^2}{27A} - gg = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 - hh \quad \text{and} \quad B = \frac{1}{3}AA - \frac{1}{3}ff.$$

Or let it be $gg = \frac{(Af-ff)^2}{27A} - hh$ with $hh < \frac{(Af-ff)^2}{27A}$; it will be

$$C = \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3 + hh \quad \text{and} \quad B = \frac{1}{3}AA - \frac{1}{3}ff.$$

In both cases an equation having two imaginary roots which none of both rules indicate will arise. For the sake of an example let us put $A = 4$, $f = 1$; it will be $B = 5$ and because of $gg = \frac{25}{108} + hh$ it will be

$$C = \frac{225}{108} - \frac{25}{108} - hh = \frac{50}{27} - hh.$$

Hence, if it is $C < \frac{50}{27}$, the equation $x^3 - 4x^2 + 5x - C = 0$ will always have two imaginary roots. But having taken $gg = \frac{1}{12} - hh$ it must be $hh < \frac{1}{12}$ and it will be

$$C = \frac{25}{12} - \frac{1}{12} + hh = 2 + hh.$$

Let $hh = \frac{1}{16}$ and the equation $x^3 - 4x^2 + 5x - \frac{33}{16} = 0$ will have two imaginary roots, even though none is revealed by the rules.

**§333** One is even able to form general equations of such a kind, in which none of both rules exhibits imaginary roots, even though in most cases two or more are contained in the equation. This happens, if two equal signs alternate, as in

$$x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} - Ex^{n-5} - Fx^{n-6} + \text{etc.} = 0$$

or

$$x^n + Ax^{n-1} - Bx^{n-2} - Cx^{n-3} + Dx^{n-4} + Ex^{n-5} - Fx^{n-6} - \text{etc.} = 0;$$

here, both rules do not reveal an imaginary root. But that they most often can contain roots of this kind, is also clear from the cubic equation $x^3 - Ax^2 - Bx + C = 0$, which for $ff = AA + 3B$ always has two imaginary roots, if it was either

$$-C < \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 \quad \text{or} \quad -C > \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3.$$

Nevertheless, also these cases can be found from the rules, if the equation by means of a substitution is transformed into another form. Put $x = y + k$ and it will be

$$\left.\begin{array}{rrr} y^3 + 3ky^2 3kky & & +k^3 \\ -Ayy & -2Aky & -Akk \\ & -By & -Bk \\ & & +C \end{array}\right\} = 0,$$

which examined according to the rule will first immediately give

$$(3k - A)^2 > 3(3kk - 2Ak - B);$$

but that it is

$$(3kk - 2Ak - B)^2 > 3(3k - A)(k^3 - Akk - Bk + C),$$

which is the other criterion, it is necessary that it is

$$BB + 3AC + (AB - 9C)k + (AA + 3B)kk > 0,$$

whichever value is attributed to $k$. Therefore, take $k$ in such a way that this expression obtains a minimal value, what will happen by putting $k = \frac{9C - AB}{2(AA+3B)}$, and if this expression was still $> 0$, it will be probable that the propounded equation has no imaginary roots. But it will be

$$BB + 3AC - \frac{(AB - 9C)^2}{2(AA + 3B)} + \frac{(AB - 9C)^2}{4(AA + 3B)} > 0$$

or

$$BB + 3AC > \frac{(AB - 9C)^2}{4(AA + 3B)}.$$

Therefore, because it is $B = \frac{1}{3}ff - \frac{1}{3}AA$, it will be

$$4ff\left(\frac{1}{9}f^4 - \frac{2}{9}AAff + \frac{1}{9}A^4 + 3AC\right) > \left(\frac{1}{3}Aff - \frac{1}{3}A^3 - 9C\right)^2$$

or

$$4f^6 - (A^2f^4 + 4A^4ff + 108ACff > A^2f^4 - 2A^4f^2 - 54ACff + A^6 + 54A^3C + 729CC$$

or

$$4f^6 > 9A^2f^4 - 6A^4ff - 162ACff + A^6 + 54A^3C + 729CC,$$

whence having taken factors it will have to be

$$(2f^3 + A^3 - 3Af^2 + 27C)(2f^3 - A^3 + 54A^3C + 27C) > 0.$$

And hence the rules will show imaginary roots, if it was either

$$\begin{aligned} C &> -\frac{1}{27}A^3 + \frac{1}{9}Af^2 - \frac{2}{27}f^3 \quad \text{and} \quad C > -\frac{1}{27}A^3 + \frac{1}{9}Af^2 + \frac{2}{27}f^3 \\ C &< -\frac{1}{27}A^3 + \frac{1}{9}Af^2 - \frac{2}{27}f^3 \quad \text{and} \quad C < -\frac{1}{27}A^3 + \frac{1}{9}Af^2 + \frac{2}{27}f^3. \end{aligned}$$

These are the same conditions which we found above [§ 306]. Therefore, it is plain that by means of a suitable transformation the rules given in this chapter can be stated in such a way that they are always true, even though they are converted.

**§334** From these principles also Harriot's rule can be demonstrated, by which any arbitrary equation is predicted to have as many positive roots as there are variations in the signs, but as many negative as there are successions of the same sign; but this rule only holds for real roots. Therefore, let us put that the equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

has only real and positive roots and its differential

$$nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - \text{etc.} = 0$$

will have not only also have only real and positive roots, but the roots of this one will also constitute the limits of the roots of the propounded equation. Furthermore, having put $x = \frac{1}{y}$ this equation

$$1 - Ay + By^2 - Cy^3 + Dy^4 - \text{etc.} = 0$$

will have only real positive roots, but they are the reciprocals of the others, such that the roots, which are the maxima in that equation, are the minima in this one. Having put these things, if that propounded equation is continuously differentiated until one gets to an equation of first order, which will be $x^n - \frac{1}{n}A = 0$ (§ 317), the root of this one will still be positive and hence the coefficient of the second term will have the sign $-$, as we assumed. But if this coefficient would have the sign $+$, then it would certainly follow that the propounded equation has not only positive real roots, but at least one will be negative, and of course the one, which corresponds to the mentioned limits.

**§335** If the propounded equation is converted into its reciprocal and is differentiated, then $x$ is resubstituted and the differentiations are continued until one gets to a simple equation, which from § 320 will be of this kind $Ax - \frac{2}{n-1}B = 0$, its roots must therefore also be positive, if the propounded equation has only real and positive roots, and hence the second and the third term will have different signs. Therefore, if these two terms have the same signs, at least one negative root will be indicated corresponding to the limit assigned in this equation, which will different from the limit indicated by the equation, since here they were once converted into its reciprocals; hence one concludes, if the three initial terms of the equation have equal signs, that then two negative roots will be indicated.

**§336** In similar manner, if the conversions and differentiations are done according to § 320 and are continued until one gets to the simple equation $bx - \frac{3}{n-2}C = 0$, also the roots of this equation must be positive, if all roots of the propounded equation were positive, of course; hence, if the third and the fourth term have equal signs, one negative root will be indicated. And so forth, if any two contiguous terms have the same sign, one negative root will be indicated; and hence, no matter how many successions of the same sign occur, the propounded equation will have at least as many negative roots, since these single criteria refer to different limits. But if the propounded equation is put to have only negative roots, then, because the roots of all differential equations deduced from it equally must be negative, all terms need to have the same signs. Hence, if two contiguous terms have different signs, from them at least one positive root will be concluded. And in similar manner, no matter how many variations of the signs of two terms occur in the propounded equation, at least as many positive roots are to be said to be in the equation. Therefore, since each equation has as many roots as there are combinations of two contiguous signs, and not more, it follows that each equation, whose roots are all real, has as many positive roots as there were variations of contiguous signs, but has many negative roots as there were successions of the same sign.