

ON THE INTERPOLATION OF SERIES *

Leonhard Euler

§389 A series is said to be interpolated, if its terms are assigned which correspond to fractional or even surd indices. Therefore, if the general term of a series was known, the interpolation will have no difficulty, since, whatever number is substituted instead of the index x , this expression yields the corresponding term. But if the series was of such a nature that its general term cannot be exhibited in any way, then the interpolation of series of such a kind is highly difficult and in most cases the terms corresponding to non-integer indices can only be defined by means of infinite series. Since in the preceding chapter we determined the values corresponding to any indices of expressions of this kind which in the usual manner cannot be expressed finitely, this treatment will have a very great use for interpolations. Therefore, we will prosecute the use, which follows from the superior chapter for this task, more diligently.

§390 Therefore, let any series be propounded

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \cdots & x & \\ A + B + C + D + \cdots + X, & & & & & & \end{array}$$

whose general term X shall be known, but the summatory term S shall not. From this form another series whose general term shall become equal to the summatory term of that series, and this new series will be

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1 2 3 4 5

$A, (A + B), (A + B + C), (A + B + C + D), (A + B + C + D + E)$ etc.

and its general term or the one corresponding to the indefinite index x will be

$$= A + B + C + D + \dots + X = S;$$

since it is not explicitly known, the interpolation of this new series will have the same difficulties those we mentioned before. Therefore, to interpolate this series one needs the values of S which it receives, if instead of x any non-integer numbers are substituted. For, if x would be an integer number, then the corresponding value of S would be found without difficulty, of course by means of addition of as many terms of the series $A + B + C + D + \text{etc.}$ as x contains unities.

§391 Therefore, to use those things, which were treated in the preceding chapter, let us put that x is an integer number such that the value corresponding to it $S = A + B + C + \dots + X$ is known, and let us search for the value Σ into which S is transformed, if instead of x one writes $x + \omega$ while ω is any fraction; and Σ will be the term of the propounded series to be interpolated which term corresponds to the index $x + \omega$; therefore, having found the latter the interpolation of this series will be done. Let Z be the term of the series A, B, C, D, E etc. which corresponds to the index $x + \omega$, and let Z', Z'', Z''' be its consecutive terms having the indices $x + \omega + 1, x + \omega + 2, x + \omega + 3$ etc. And at first let us put that the infinitesimal terms of the series A, B, C, D etc. vanish. Therefore, having put these things the series

1 2 3 4

$A, (A + B), (A + B + C), (A + B + C + D)$ etc.,

whose term corresponding to the index x is $S = A + B + C + \dots + X$, will be interpolated by finding the term Σ which corresponds to the fractional index $x + \omega$. But it will be, as we found,

$$\begin{aligned} \Sigma = S + X' + X'' + X''' + X'''' + \text{etc.} \\ - Z' - Z'' - Z''' - Z'''' - \text{etc.} \end{aligned}$$

and so one will have the infinite term equal to this term Σ in question which because of

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1 \cdot 2dx^2} + \frac{\omega^3 d^3X}{1 \cdot 2 \cdot 3dx^3} + \text{etc.}$$

is transformed into this form that it is

$$\begin{aligned} \Sigma = S - \frac{\omega}{dx} d. (X' + X'' + X''' + X'''' + \text{etc.}) \\ - \frac{\omega^2}{2dx^2} dd. (X' + X'' + X''' + X'''' + \text{etc.}) \\ - \frac{\omega^3}{6dx^3} d^3. (X' + X'' + X''' + X'''' + \text{etc.}) \\ \text{etc.} \end{aligned}$$

of which formulas the one which is more convenient in each case can be applied.

§392 Let us for A, B, C, D etc. take the arbitrary harmonic series

$$\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \text{etc.},$$

whose general term or the one corresponding to the index x is $= \frac{1}{a+(x-1)b} = X$. Hence, let this series be formed

$$\frac{1}{a}, \quad \left(\frac{1}{a} + \frac{1}{a+b}\right), \quad \left(\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b}\right), \quad \left(\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b}\right) \text{ etc.},$$

whose term corresponding to the index x will therefore be

$$S = \frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \dots + \frac{1}{a+(x-1)b}.$$

If now Σ denotes the term corresponding to the index $x + \omega$ of this series, because of $Z = \frac{1}{a+(x+\omega-1)b}$ it will be

$$\begin{aligned}
X' &= \frac{1}{a+bx'}, & Z' &= \frac{1}{a+bx+b\omega} \\
X'' &= \frac{1}{a+b+bx'}, & Z'' &= \frac{1}{a+b+bx+b\omega} \\
X''' &= \frac{1}{a+2b+bx'}, & Z''' &= \frac{1}{a+2b+bx+b\omega} \\
&\text{etc.} & & \text{etc.}
\end{aligned}$$

and hence it will arise

$$\begin{aligned}
\Sigma &= S + \frac{1}{a+bx} + \frac{1}{b+b+bx} + \frac{1}{a+2b+bx} + \text{etc.} \\
&\quad - \frac{1}{a+bx+b\omega} - \frac{1}{a+b+bx+b\omega} - \frac{1}{a+2b+bx+b\omega} - \text{etc.}
\end{aligned}$$

but the other expression will be of this kind

$$\begin{aligned}
\Sigma &= S + b\omega \left(\frac{1}{(a+bx)^2} + \frac{1}{(a+b+bx)^2} + \frac{1}{(a+2b+bx)^2} + \text{etc.} \right) \\
&\quad - b^2\omega^2 \left(\frac{1}{(a+bx)^3} + \frac{1}{(a+b+bx)^3} + \frac{1}{(a+2b+bx)^3} + \text{etc.} \right) \\
&\quad + b^3\omega^3 \left(\frac{1}{(a+bx)^4} + \frac{1}{(a+b+bx)^4} + \frac{1}{(a+2b+bx)^4} + \text{etc.} \right) \\
&\quad \text{etc.}
\end{aligned}$$

EXAMPLE 1

Let this series be propounded

$$1, \quad \overset{1}{\left(1 + \frac{1}{2}\right)}, \quad \overset{2}{\left(1 + \frac{1}{2} + \frac{1}{3}\right)}, \quad \overset{3}{\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)} \quad \text{etc.,}$$

whose terms which correspond to fractional indices are to be found.

Therefore, it will be $a = 1$ and $b = 1$; hence, if the term corresponding to the integer index x is put

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{x}$$

and the term corresponding to the fractional index $x + \omega$ is called $= \Sigma$, it will be

$$\begin{aligned} \Sigma = S + & \frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \frac{1}{4+x} + \frac{1}{5+x} + \text{etc.} \\ & - \frac{1}{1+x+\omega} + \frac{1}{2+x+\omega} + \frac{1}{3+x+\omega} + \frac{1}{4+x+\omega} + \frac{1}{5+x+\omega} + \text{etc.} \end{aligned}$$

But it is to be noted, if the term corresponding to the fractional index ω was found, which we want to put $= T$, that from it the term of the index $x + \omega$ can easily be found; for, it will be, if T', T'', T''' etc. denote the terms corresponding to the indices $1 + \omega, 2 + \omega, 3 + \omega$ etc.,

$$\begin{aligned} T' &= T + \frac{1}{1+\omega} \\ T'' &= T + \frac{1}{1+\omega} + \frac{1}{2+\omega} \\ T''' &= T + \frac{1}{1+\omega} + \frac{1}{2+\omega} + \frac{1}{3+\omega} \\ &\text{etc.,} \end{aligned}$$

whence it suffices to have investigated only the terms which correspond to the indices ω smaller than the unity. For this purpose, let us put $x = 0$; it will also be $S = 0$ and the term T of the series corresponding to the fractional index ω will be expressed this way

$$\begin{aligned} T = & + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} \\ & - \frac{1}{1+\omega} - \frac{1}{2+\omega} - \frac{1}{3+\omega} - \frac{1}{4+\omega} - \text{etc.} \end{aligned}$$

or after having converted these fractions into infinite series another expression will arise

$$\begin{aligned}
T = & +\omega \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} \right) \\
& -\omega^2 \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} \right) \\
& +\omega^3 \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} \right) \\
& -\omega^4 \left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} \right) \\
& \text{etc.,}
\end{aligned}$$

which value is more than apt to find the value of T approximately.

Therefore, let the term corresponding to the index $\frac{1}{2}$ of the propounded series be in question; if it is put = T , it will be

$$T = 1 - \frac{2}{3} + \frac{1}{2} - \frac{2}{5} + \frac{1}{3} - \frac{2}{7} + \frac{1}{4} - \frac{2}{9} + \text{etc.}$$

or

$$T = 2 \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \text{etc.} \right),$$

the value of which series is = $2 - 2 \ln 2$, and so the term of the index = $\frac{1}{2}$ can be expressed finitely. Therefore, the following terms whose indices are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ etc. will be expressed this way

Ind.	1	2	3	4	
Term	2 - 2 ln 2,	2 + $\frac{2}{3}$ - 2 ln 2,	2 + $\frac{2}{3}$ + $\frac{2}{5}$ - 2 ln 2,	2 + $\frac{2}{3}$ + $\frac{2}{5}$ + $\frac{2}{7}$ - 2 ln 2	etc.,

EXAMPLE 2

Let this series be propounded

$$1, \quad \overset{1}{\left(1 + \frac{1}{3} \right)}, \quad \overset{2}{\left(1 + \frac{1}{3} + \frac{1}{5} \right)}, \quad \overset{3}{\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \right)} \quad \text{etc.,}$$

whose general terms corresponding to fractional indices are to be expressed.

Therefore, it will be $a = 1$ and $b = 2$; hence, if the term corresponding to the integer index x is put

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2x-1}$$

and the term corresponding to the fractional index $x + \omega$ is called $= \Sigma$, it will be

$$\Sigma = S + \frac{1}{1+2x} + \frac{1}{3+2x} + \frac{1}{5+2x} + \frac{1}{7+2x} + \text{etc.}$$

$$- \frac{1}{1+2(x+\omega)} - \frac{1}{3+2(x+\omega)} - \frac{1}{5+2(x+\omega)} - \frac{1}{7+2(x+\omega)} - \text{etc.}$$

Since it therefore suffices to have assigned the terms corresponding to indices smaller than unity, let $x = 0$ and $S = 0$; therefore, if the term corresponding to the index ω is put $= T$, it will be

$$T = + \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \text{etc.}$$

$$- \frac{1}{1+2\omega} - \frac{1}{3+2\omega} - \frac{1}{5+2\omega} - \frac{1}{7+2\omega} - \frac{1}{9+2\omega} - \text{etc.}$$

and if ω is put to denote any number, since T is the term corresponding to the index ω , T will be the general term of the propounded series which will also be expressed this way

$$T = \frac{2\omega}{1(1+2\omega)} + \frac{2\omega}{3(3+2\omega)} + \frac{2\omega}{5(5+2\omega)} + \frac{2\omega}{7(7+2\omega)} + \text{etc.}$$

or this way

$$T = 2\omega \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.} \right)$$

$$- 4\omega^2 \left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} + \text{etc.} \right)$$

$$\begin{aligned}
&+ 8\omega^3 \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.}\right) \\
&- 16\omega^4 \left(1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{9^5} + \text{etc.}\right) \\
&\text{etc.}
\end{aligned}$$

Let us put that $\omega = \frac{1}{2}$; therefore, the term corresponding to this index will be

$$T = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.} = \ln 2$$

and we will have

Indices	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	
Terms	$\ln 2,$	$\frac{1}{2} + \ln 2,$	$\frac{1}{2} + \frac{1}{4} + \ln 2,$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ln 2$	etc.,

If $\omega = \frac{1}{4}$, it will be

$$\begin{aligned}
T &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.} \\
&\quad - \frac{2}{3} + \frac{2}{7} - \frac{2}{11} + \frac{2}{15} - \text{etc.}
\end{aligned}$$

or

$$T = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.} - \frac{1}{2} \ln 2 = \frac{\pi}{4} - \ln 2.$$

§393 If the general term corresponding to the index $= \frac{1}{2}$ of this general series

$$\frac{1}{a'} \left(\frac{1}{a} + \frac{1}{a+b} \right), \left(\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} \right) \text{ etc.}$$

is in question, in the expressions of the preceding paragraph put $x = 0$ and $\omega = \frac{1}{2}$ and it will be $S = 0$ and the term corresponding to the index $\frac{1}{2}$ and in question will be

$$\Sigma = \frac{1}{a} - \frac{2}{2a+b} + \frac{1}{a+b} - \frac{2}{2a+3b} + \frac{1}{a+2b} - \frac{2}{2a+5b} + \text{etc.}$$

or having made the terms more uniform it will be

$$\frac{1}{2}\Sigma = \frac{1}{2a} - \frac{1}{2a+b} + \frac{1}{2a+2b} - \frac{1}{2a+3b} + \frac{1}{2a+4b} - \text{etc.};$$

since in this series the signs + and - alternate, by taking the continued differences by means of the method explained above [§ 8] the value of $\frac{1}{2}\Sigma$ will be expressed by means of a more convergent series.

The series of the differences will be

$$\begin{aligned} & \frac{-b}{2a(2a+b)'} \quad \frac{-b}{(2a+b)(2a+2b)'} \quad \frac{-b}{(2a+2b)(2a+3b)} \quad \text{etc.} \\ & \frac{2bb}{2a(2a+b)(2a+2b)'} \quad \frac{2bb}{(2a+b)(2a+2b)(2a+3b)} \quad \text{etc.} \\ & \frac{-6b^3}{2a(2a+b)(2b+2a)(2a+3b)} \quad \text{etc.} \\ & \text{etc.} \end{aligned}$$

From these it is concluded that it will be

$$\frac{1}{2}\Sigma = \frac{1}{4a} + \frac{1b}{8a(2a+b)} + \frac{1 \cdot 2bb}{16a(2a+b)(2a+2b)} + \frac{1 \cdot 2 \cdot 3b^3}{32a(2a+b)(2a+2b)(2a+3b)} + \text{etc.}$$

And hence one will have

$$\Sigma = \frac{1}{2a} + \frac{\frac{1}{2}b}{2a(2a+b)} + \frac{\frac{1}{2} \cdot \frac{2}{2}bb}{2a(2a+b)(2a+2b)} + \frac{\frac{1}{2} \cdot \frac{2}{2} \cdot \frac{3}{2}b^3}{2a(2a+b)(2a+2b)(2a+3b)} + \text{etc.},$$

which series is most convergent and exhibits the value of the term Σ without much work.

§394 But if in general the infinitesimal terms of the series A, B, C, D, E etc. vanish and the term corresponding to the index ω was = Z and its following ones which correspond to the indices $\omega + 1, \omega + 2, \omega + 3$ etc. are Z', Z'', Z''', Z'''' etc., if in the superior expressions (§ 391) put $x = 0$ that it is $S = 0$ and $X' = A, X'' = B, X''' = C$ etc., it will follow, if a series of this kind

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
A, & (A + B), & (A + B + C), & (A + B + C + D) \text{ etc.}
\end{array}$$

and its term corresponding to the index ω is put $= \Sigma$, that it will be

$$\Sigma = (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''') + \text{etc.},$$

from which expression any intermediate terms can be defined. But it will suffice to have investigated the terms which correspond to the indices ω smaller than unity to perform the interpolation. If the term Σ corresponding to an arbitrary index ω of this kind was found and those which correspond to the indices $\omega + 1, \omega + 2, \omega + 3$ etc. are put $\Sigma', \Sigma'', \Sigma'''$ etc., it will be

$$\begin{aligned}
\Sigma' &= \Sigma + Z' \\
\Sigma'' &= \Sigma + Z' + Z'' \\
\Sigma''' &= \Sigma + Z' + Z'' + Z''' \\
&\text{etc.}
\end{aligned}$$

EXAMPLE 1

To interpolate this series

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1, & \left(1 + \frac{1}{4}\right), & \left(1 + \frac{1}{4} + \frac{1}{9}\right), & \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right) \text{ etc.}
\end{array}$$

Let Σ be the term of this series corresponding to the index ω , and because this series is formed from the summation of this one

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.},$$

whose term corresponding to the index ω is $= \frac{1}{\omega^2}$, it will be

$$\Sigma = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$

$$- \frac{1}{(1+\omega)^2} - \frac{1}{(2+\omega)^2} - \frac{1}{(3+\omega)^2} - \frac{1}{(4+\omega)^2} - \text{etc.}$$

If the term corresponding to the index $\frac{1}{2}$ of the propounded series is in question, one will have to put $\omega = \frac{1}{2}$ and it will become

$$\Sigma = 1 - \frac{4}{9} + \frac{1}{4} - \frac{4}{25} + \frac{1}{9} - \frac{4}{49} + \text{etc.}$$

or

$$\Sigma = 4 \left(\frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} + \text{etc.} \right).$$

Since it is

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} + \text{etc.} = \frac{\pi^2}{12},$$

it will be

$$\Sigma = 4 \left(1 - \frac{\pi^2}{12} \right) = 4 - \frac{1}{3}\pi^2,$$

which is the term corresponding to the index $\frac{1}{2}$. Therefore, we will have the correspondences

Indices	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	
Terms	$4 - \frac{1}{3}\pi^2,$	$\frac{4}{1} + \frac{4}{9} - \frac{1}{3}\pi^2,$	$\frac{4}{1} + \frac{4}{9} + \frac{4}{25} - \frac{1}{3}\pi^2$	etc..

EXAMPLE 2

To interpolate this series

1	2	3	4	
1,	$\left(1 + \frac{1}{9}\right),$	$\left(1 + \frac{1}{9} + \frac{1}{25}\right),$	$\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49}\right)$	etc.

Let Σ be the term corresponding to the index ω , and because this series is formed from the summation of this one

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.},$$

from which the term corresponding to the index ω becomes $Z = \frac{1}{(2\omega-1)}$, it will be

$$Z' = \frac{1}{(2\omega+1)^2}, \quad Z'' = \frac{1}{(2\omega+3)^2}, \quad Z''' = \frac{1}{(2\omega+5)^2} \quad \text{etc.}$$

Therefore, one will have

$$\begin{aligned} \Sigma = & 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.} \\ & - \frac{1}{(1+2\omega)^2} - \frac{1}{(3-2\omega)^2} - \frac{1}{(5+2\omega)^2} - \frac{1}{(7+2\omega)^2} - \text{etc.} \end{aligned}$$

Let us put $\omega = \frac{1}{2}$ that we find the term of the propounded series corresponding to the index $= \frac{1}{2}$ which will be

$$\Sigma = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \text{etc.} = \frac{\pi\pi}{12},$$

from which term which fall in the middle between each to given ones will be expressed in the following way. We will have the following correspondences

Indices	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	etc.
Terms	$\frac{\pi\pi}{12},$	$\frac{1}{4} + \frac{\pi\pi}{12},$	$\frac{1}{4} + \frac{1}{16} + \frac{\pi\pi}{12},$	$\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{\pi\pi}{12}$	etc..

EXAMPLE 3

To interpolate this series

$$1, \quad \begin{matrix} 2 \\ \left(1 + \frac{1}{2^n}\right), \end{matrix} \quad \begin{matrix} 3 \\ \left(1 + \frac{1}{2^n} + \frac{1}{3^n}\right), \end{matrix} \quad \begin{matrix} 4 \\ \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n}\right) \end{matrix} \quad \text{etc.}$$

Let, as before, Σ be the term corresponding to the index ω ; it will be $Z = \frac{1}{\omega^n}$ and

$$Z' = \frac{1}{(1+\omega)^n}, \quad Z'' = \frac{1}{(2+\omega)^n}, \quad Z''' = \frac{1}{(3+\omega)^n} \quad \text{etc.}$$

And one will have

$$\Sigma = \begin{array}{ccccccc} 1 & + & \frac{1}{2^n} & + & \frac{1}{3^n} & + & \frac{1}{4^n} & + \text{etc.} \\ - & \frac{1}{(1+\omega)^n} & - & \frac{1}{(2+\omega)^n} & - & \frac{1}{(3+\omega)^n} & - & \frac{1}{(4+\omega)^n} & - \text{etc.} \end{array}$$

If the term corresponding to the index $\frac{1}{2}$ is desired, it will be

$$= 1 - \frac{2^n}{3^n} + \frac{1}{2^n} - \frac{2^n}{5^n} + \frac{1}{3^n} - \frac{2^n}{7^n} + \text{etc.}$$

or

$$= 2^n \left(\frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \text{etc.} \right).$$

If one puts

$$\mathfrak{N} = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \text{etc.},$$

the term of the propounded series which corresponds to the index $\frac{1}{2}$ will be $= 2^n(1 - \mathfrak{N})$; and hence we will have the correspondences

Indices	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	etc.
Terms	$2^n - 2^n \mathfrak{N}$,	$2^n + \frac{2^n}{3^n} - 2^n \mathfrak{N}$,	$2^n + \frac{2^n}{3^n} + \frac{2^n}{5^n} - 2^n \mathfrak{N}$	etc..

EXAMPLE 3

To interpolate this series

$$1, \quad \overset{2}{\left(1 + \frac{1}{3^n}\right)}, \quad \overset{3}{\left(1 + \frac{1}{3^n} + \frac{1}{5^n}\right)}, \quad \overset{4}{\left(1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n}\right)} \text{ etc.}$$

Let Σ be the term which corresponds to the index ω and since it is $Z = \frac{1}{(2\omega-1)^n}$, it will be

$$Z' = \frac{1}{(2\omega+1)^n}, \quad Z'' = \frac{1}{(2\omega+3)^n}, \quad Z''' = \frac{1}{(2\omega+5)^n} \text{ etc.}$$

and

$$\Sigma = \frac{1}{(1+2\omega)^n} + \frac{1}{(3+2\omega)^n} + \frac{1}{(5+2\omega)^n} + \frac{1}{(7+2\omega)^n} + \text{etc.}$$

$$- \frac{1}{(1+2\omega)^n} - \frac{1}{(3+2\omega)^n} - \frac{1}{(5+2\omega)^n} - \frac{1}{(7+2\omega)^n} - \text{etc.}$$

Put $\omega = \frac{1}{2}$ and the term corresponding to the index $\frac{1}{2}$ will arise as

$$= 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \text{etc.} = \mathfrak{N},$$

from which further the remaining terms in the middle between two given ones will be

Indices	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	etc.
Terms	$\mathfrak{N},$	$\frac{1}{2^n} + \mathfrak{N},$	$\frac{1}{2^n} + \frac{1}{4^n} + \mathfrak{N}$	etc..

§395 Now let us put that the infinitesimal terms of the series A, B, C, D, E etc. from which the series to be interpolated is formed do not vanish, but are of such a nature that their differences vanish and let X be the term of this series corresponding to the index x and Z the term corresponding to the exponent $x + \omega$; then let X', X'', X''', X'''' etc. be the terms following X and Z', Z'', Z''', Z'''' etc. the terms following Z . Having put these things let this series to be interpolated be propounded

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
A & (A + B), & (A + B + C), & (A + B + C + D) \quad \text{etc..}
\end{array}$$

whose term corresponding to the index x shall be $= S$, but the term corresponding to the index $x + \omega$ shall be $= \Sigma$, and from the things which were treated in the preceding chapter it will be

$$\begin{aligned}
\Sigma &= S + X' + X'' + X''' + \text{etc.} \\
&\quad - Z' - Z'' - Z''' - \text{etc.} \\
&+ \omega X' + \omega \left\{ \begin{array}{l} X'' + X''' + X'''' + \text{etc.} \\ - X' + X'' + X''' - \text{etc.} \end{array} \right\}
\end{aligned}$$

But because as before it suffices to have investigated the terms corresponding to indices smaller than unity, let us put $x = 0$ that it is $S = 0$, $X' = A$, $X'' = B$ etc., and the term corresponding to the index ω will be

$$\begin{aligned}
\Sigma &= (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''') \quad \text{etc.} \\
&\omega A + \omega((B - A) + (C - B) + (D - C) + (E - D) + \text{etc.})
\end{aligned}$$

But if we want to express these differences in the way we did above, namely as $\Delta A = B - A$, $\Delta B = C - B$ etc., one will have

$$\begin{aligned}
\Sigma &= (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''') \quad \text{etc.} \\
&+ \omega(A + \Delta A + \Delta B + \Delta C + \Delta D + \text{etc.})
\end{aligned}$$

§396 But if the infinitesimal terms of the series A, B, C, D, E etc., from whose summation the series to be interpolated is formed, neither vanish nor have vanishing first differences, then more series have to added to express the value of Σ , until one finally reaches vanishing differences of the infinitesimal terms. For, let as before the term of the series A, B, C, D, E etc. corresponding to the index x be $= X$ and the following ones X', X'', X''' etc., but let the term Z correspond to the index $x + \omega$, which term shall be followed by Z', Z'', Z''' etc. and let this series be propounded

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
A & (A+B), & (A+B+C), & (A+B+C+D) \quad \text{etc.}
\end{array}$$

whose term corresponding to the index x shall be

$$S = A + B + C + D + \dots + X,$$

but let the term Σ correspond to the index $x + \omega$ such that

to the indices	correspond the terms
$X' = \frac{1}{x+1}$	$Z' = \frac{1}{x+1+\omega}$
$X'' = \frac{1}{x+2}$	$Z'' = \frac{1}{x+2+\omega}$
$X''' = \frac{1}{x+3}$	$Z''' = \frac{1}{x+3+\omega}$
etc.	etc.,

If now the differences are expressed in such a way that it is

$$\begin{aligned}
\Delta X' &= X'' - X', & \Delta X'' &= X''' - X'', & \Delta X''' &= X'''' - X''' \quad \text{etc.} \\
\Delta^2 X' &= \Delta X'' - \Delta X', & \Delta^2 X'' &= \Delta X''' - \Delta X'', & \Delta^2 X''' &= \Delta X'''' - \Delta X''' \quad \text{etc.} \\
\Delta^3 X' &= \Delta^2 X'' - \Delta X', & \Delta^3 X'' &= \Delta^2 X''' - \Delta X'' \quad \text{etc.,}
\end{aligned}$$

from § 377 the term Σ will be expressed the following way:

$$\begin{aligned}
\Sigma &= S + X' + X'' + X''' + \text{etc.} \\
&\quad - Z' - Z'' - Z''' - \text{etc.} \\
&\quad + \omega(X' + \Delta X' + \Delta X'' + \Delta X''' + \Delta X'''' + \text{etc.}) \\
&\quad + \frac{\omega(\omega-1)}{1 \cdot 2} (\Delta X' + \Delta^2 X' + \Delta^2 X'' + \Delta^2 X''' + \Delta^2 X'''' + \text{etc.}) \\
&\quad + \frac{\omega(\omega-1)(\omega-2)}{1 \cdot 2 \cdot 3} (\Delta^2 X' + \Delta^3 X' + \Delta^3 X'' + \Delta^3 X''' + \Delta^3 X'''' + \text{etc.}) \\
&\quad \text{etc.}
\end{aligned}$$

§397 It suffices, as we already noted, to have added so many terms until one gets to vanishing differences of the infinitesimal terms; for, if we also want to continue these series to infinity or at least until the differences of the finite terms vanish, then because of

$$Z' = X' + \omega \Delta X' + \frac{\omega(\omega - 1)}{1 \cdot 2} \Delta^2 X' + \frac{\omega(\omega - 1)(\omega - 2)}{1 \cdot 2 \cdot 3} \Delta^3 X' + \text{etc.}$$

the whole found expression will be contracted into this one

$$\Sigma = S + \omega X' + \frac{\omega(\omega - 1)}{1 \cdot 2} \Delta X' + \frac{\omega(\omega - 1)(\omega - 2)}{1 \cdot 2 \cdot 3} \Delta^2 X' + \text{etc.},$$

which involves the summatory term of the series $A + B + C + D + \text{etc.}$; but, if this term would be known, the interpolation would have no difficulty. Nevertheless also this formula can be used which, as often it is truncated, exhibits the term to be interpolated expressed finitely and algebraically; but if it proceeds to infinity, it is in the most cases more convenient to apply the first formula in which the infinitesimal terms are taken into account. But this formula, if one puts $x = 0$ that Σ denoted the term corresponding to the index ω because of $S = 0$ takes this form

$$\begin{aligned} \Sigma &= A + B + C + D + \text{etc.} \\ &\quad - Z' - Z'' - Z''' - Z'''' - \text{etc.} \\ &\quad + \omega(A + \Delta A + \Delta B + \Delta C + \Delta D + \text{etc.}) \\ &\quad + \frac{\omega(\omega - 1)}{1 \cdot 2} (\Delta A + \Delta^2 A + \Delta^2 B + \Delta^2 C + \Delta^2 D + \text{etc.}) \\ &\quad + \frac{\omega(\omega - 1)(\omega - 2)}{1 \cdot 2 \cdot 3} (\Delta^2 A + \Delta^3 A + \Delta^3 B + \Delta^3 C + \Delta^3 D + \text{etc.}) \\ &\quad \text{etc.} \end{aligned}$$

Or if for the sake of brevity one puts

$$\omega = \alpha, \quad \frac{\omega(\omega - 1)}{1 \cdot 2} = \beta, \quad \frac{\omega(\omega - 1)(\omega - 2)}{1 \cdot 2 \cdot 3} = \gamma \quad \text{etc.},$$

it will be

$$\begin{aligned}
\Sigma &= \alpha A + \beta \Delta A + \gamma \Delta^2 A + \delta \Delta^3 A + \text{etc.} \\
&+ A + \alpha \Delta A + \beta \Delta^2 A + \gamma \Delta^3 A + \text{etc.} - Z' \\
&+ B + \alpha \Delta B + \beta \Delta^2 B + \gamma \Delta^3 B + \text{etc.} - Z'' \\
&+ C + \alpha \Delta C + \beta \Delta^2 C + \gamma \Delta^3 C + \text{etc.} - Z''' \\
&\text{etc.,}
\end{aligned}$$

the number of which horizontal series might seem to proceed to infinity, but each single one consists of a finite number of terms.

EXAMPLE

To interpolate this series

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\frac{1}{2}, & \left(\frac{1}{2} + \frac{2}{3}\right), & \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4}\right), & \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}\right) \text{ etc.}
\end{array}$$

Let the term of this series corresponding to the index ω be $= \Sigma$, and since it arises from the summation of this series $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ etc., it will be $Z = \frac{\omega}{\omega+1}$, and since the infinitesimal terms already have vanishing first differences, only the first differences are to be taken, which because of

$$A = \frac{1}{2}, \quad B = \frac{2}{3}, \quad C = \frac{3}{4}, \quad D = \frac{4}{5} \text{ etc.}$$

will be

$$\Delta A = \frac{1}{2 \cdot 3}, \quad \Delta B = \frac{1}{3 \cdot 4}, \quad \Delta C = \frac{1}{4 \cdot 5} \text{ etc.}$$

Hence, one will have

$$\begin{aligned}
\Sigma &= \frac{\omega}{2} + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \text{etc.} \\
&+ \frac{\omega}{2 \cdot 3} + \frac{\omega}{3 \cdot 4} + \frac{\omega}{4 \cdot 5} + \frac{\omega}{5 \cdot 6} + \text{etc.} \\
&- \frac{\omega+1}{\omega+2} - \frac{\omega+2}{\omega+3} - \frac{\omega+3}{\omega+4} - \frac{\omega+4}{\omega+5} - \text{etc.}
\end{aligned}$$

or because of

$$\frac{\omega}{2} + \frac{\omega}{2 \cdot 3} + \frac{\omega}{3 \cdot 4} + \frac{\omega}{4 \cdot 5} + \text{etc.} = \omega$$

it will be

$$\begin{aligned} \Sigma &= \omega + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \text{etc.} \\ &\quad - \frac{\omega+1}{\omega+2} - \frac{\omega+2}{\omega+3} - \frac{\omega+3}{\omega+4} - \frac{\omega+4}{\omega+5} - \text{etc.} \end{aligned}$$

If the term corresponding to the index $\frac{1}{2}$ is in question, it will be

$$\Sigma = \frac{1}{2} + \frac{1}{2} - \frac{3}{5} + \frac{2}{3} - \frac{5}{7} + \frac{3}{4} - \frac{7}{9} + \frac{4}{5} - \frac{9}{11} + \text{etc.}$$

or

$$\Sigma = \frac{1}{2} - \frac{1}{2 \cdot 5} - \frac{1}{3 \cdot 7} - \frac{1}{4 \cdot 9} - \frac{1}{5 \cdot 11} - \frac{1}{6 \cdot 13} - \text{etc.}$$

and hence

$$\frac{1}{2}\Sigma = \frac{1}{4} - \frac{1}{4 \cdot 5} - \frac{1}{6 \cdot 7} - \frac{1}{8 \cdot 9} - \frac{1}{10 \cdot 11} - \frac{1}{12 \cdot 13} - \text{etc.}$$

or

$$\begin{aligned} \frac{1}{2}\Sigma &= \frac{1}{4} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \text{etc.} \\ &\quad + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \text{etc.} \end{aligned}$$

Hence, because it is

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.} = \ln 2,$$

it will be

$$\frac{1}{2}\Sigma = \ln 2 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \ln 2 - \frac{7}{12}$$

and hence

$$\Sigma = 2 \ln 2 - \frac{7}{6}.$$

§398 Now, let us proceed to the interpolation of series whose terms are conflated of factors, and let this most general series be propounded

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \\ A, & AB, & ABC, & ABCD & ABCDE & \text{etc.} \end{array}$$

whose term corresponding to the index ω shall be $= \Sigma$. Therefore, $\ln \Sigma$ will be the term corresponding to index ω in this series

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \ln A, & (\ln A + \ln B), & (\ln A + \ln B + \ln C), & (\ln A + \ln B + \ln C + \ln D) \quad \text{etc.} \end{array}$$

If we put that the infinitesimal terms of this series vanish and the terms of the series A, B, C, D, E etc. corresponding to the index ω is Z and the following ones corresponding to the indices $\omega + 1, \omega + 2, \omega + 3, \omega + 4$ etc. are Z', Z'', Z''', Z'''' etc., it will be from the things demonstrated above

$$\begin{aligned} \ln \Sigma &= \ln A + \ln B + \ln C + \ln D + \text{etc.} \\ &\quad - \ln Z' - \ln Z'' - \ln Z''' - \ln Z'''' - \text{etc.} \end{aligned}$$

Hence, by going back to numbers one will have

$$\Sigma = \frac{A}{Z'} \cdot \frac{B}{Z''} \cdot \frac{C}{Z'''} \cdot \frac{D}{Z''''} \cdot \text{etc.}$$

§399 But if the logarithms of the infinitesimal terms of the series A, B, C, D etc. do not vanish, but have vanishing differences, it will be, as we saw

$$\begin{aligned} \ln \Sigma &= \ln A + \ln B + \ln C + \text{etc.} \\ &\quad - \ln Z' - \ln Z'' - \ln Z''' - \text{etc.} \\ &+ \omega \ln A + \omega \left(\ln \frac{B}{A} + \ln \frac{C}{B} + \ln \frac{D}{C} + \text{etc.} \right) \end{aligned}$$

and hence going from logarithms to numbers it will be

$$\Sigma = A^\omega \cdot \frac{A^{1-\omega} B^\omega}{Z'} \cdot \frac{B^{1-\omega} C^\omega}{Z''} \cdot \frac{C^{1-\omega} D^\omega}{Z'''} \cdot \frac{D^{1-\omega} E^\omega}{Z''''} \cdot \text{etc.}$$

But if just the second differences of those infinitesimal logarithms vanish, it will be

$$\begin{aligned} \ln \Sigma &= \ln A + \ln B + \ln C + \ln D + \text{etc.} \\ &\quad - \ln Z' - \ln Z'' - \ln Z''' - \ln Z'''' - \text{etc.} \\ &\quad + \omega \left(\ln A + \ln \frac{B}{A} + \ln \frac{C}{B} + \ln \frac{D}{C} + \ln \frac{E}{D} + \text{etc.} \right) \\ &\quad + \frac{\omega(\omega-1)}{1 \cdot 2} \left(\ln \frac{B}{A} + \ln \frac{AC}{B^2} + \ln \frac{BD}{C^2} + \ln \frac{CE}{D^2} + \ln \frac{DF}{E^2} + \text{etc.} \right) \end{aligned}$$

From these one will therefore obtain

$$\Sigma = A^{\frac{\omega(3-\omega)}{2}} \cdot B^{\frac{\omega(\omega-1)}{1 \cdot 2}} \cdot \frac{A^{\frac{(\omega-1)(\omega-2)}{1 \cdot 2}} B^{\omega(2-\omega)} C^{\frac{\omega(\omega-1)}{1 \cdot 2}}}{Z'} \cdot \frac{B^{\frac{(\omega-1)(\omega-2)}{1 \cdot 2}} C^{\omega(2-\omega)} D^{\frac{\omega(\omega-1)}{1 \cdot 2}}}{Z''} \cdot \text{etc.}$$

which, if $\omega < 1$, is expressed more conveniently this way

$$\Sigma = \frac{A^{\frac{\omega(3-\omega)}{1 \cdot 2}}}{B^{\frac{\omega(1-\omega)}{1 \cdot 2}}} \cdot \frac{A^{\frac{(1-\omega)(2-\omega)}{1 \cdot 2}} B^{\omega(2-\omega)}}{C^{\frac{\omega(1-\omega)}{1 \cdot 2}} Z'} \cdot \frac{B^{\frac{(1-\omega)(2-\omega)}{1 \cdot 2}} C^{\omega(2-\omega)}}{D^{\frac{\omega(1-\omega)}{1 \cdot 2}} Z''} \cdot \text{etc.}$$

§400 Let us apply this interpolation to this series

1	2	3	4	
$\frac{a}{b'}$	$\frac{a(a+c)}{b(b+c)'}$	$\frac{a(a+c)(a+2c)}{b(b+c)(b+2c)'}$	$\frac{a(a+c)(a+2c)(a+3c)}{b(b+c)(b+2c)(b+3c)}$	etc.

whose factors are taken from this series

1	2	3	4	
$\frac{a}{b'}$	$\frac{a+c}{b+c'}$	$\frac{a+2c}{b+2c'}$	$\frac{a+3c}{b+3c}$	etc.

whose logarithms of the infinitesimal terms are = 0. Therefore, it will be

$$Z = \frac{a-c+c\omega}{b-c+c\omega'}, \quad Z' = \frac{a+c\omega}{b+c\omega} \quad \text{etc.}$$

Hence, if the term of that series corresponding to the index ω is put $= \Sigma$, it will be from § 398

$$\Sigma = \frac{a(b+c\omega)}{b(a+c\omega)} \cdot \frac{(a+c)(b+c+c\omega)}{(b+c)(a+c+c\omega)} \cdot \frac{(a+2c)(b+2c+c\omega)}{(b+2c)(a+2c+c\omega)} \cdot \text{etc.}$$

If the term corresponding to the index $\frac{1}{2}$ is desired having put $\omega = \frac{1}{2}$ it will be

$$\Sigma = \frac{a(2b+c)}{b(2a+c)} \cdot \frac{(a+c)(2b+3c)}{(b+c)(2a+3c)} \cdot \frac{(a+2c)(2b+5c)}{(b+2c)(2a+5c)} \cdot \text{etc.}$$

EXAMPLE

To interpolate this series

1	2	3	4	5	
$\frac{1}{2'}$	$\frac{1 \cdot 3}{2 \cdot 4'}$	$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6'}$	$\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}$	$\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}$	etc.

Since here it is $a = 1, b = 2$ and $c = 2$, if the term corresponding to the index ω is put $= \Sigma$, it will be

$$\Sigma = \frac{1(2+2\omega)}{2(1+2\omega)} \cdot \frac{3(4+2\omega)}{4(3+2\omega)} \cdot \frac{5(6+2\omega)}{6(5+2\omega)} \cdot \frac{7(8+2\omega)}{8(7+2\omega)} \cdot \text{etc.}$$

Hence, if the terms which correspond to the indices $\omega + 1, \omega + 2, \omega + 3$ etc. are put $\Sigma', \Sigma'', \Sigma'''$ etc., it will be

$$\begin{aligned} \Sigma' &= \frac{1+2\omega}{2+2\omega} \cdot \Sigma \\ \Sigma'' &= \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \Sigma \\ \Sigma''' &= \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \frac{5+2\omega}{6+2\omega} \cdot \Sigma \\ &\text{etc.} \end{aligned}$$

If the term corresponding to the index $\frac{1}{2}$ having put $\omega = \frac{1}{2}$ it will be

$$\Sigma = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdot \frac{7 \cdot 9}{8 \cdot 8} \cdot \frac{9 \cdot 11}{10 \cdot 10} \cdot \text{etc.}$$

But having put $\pi =$ the semicircumference of the circle whose radius is $= 1$ we showed above that it is

$$\pi = 2 \cdot \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdot \frac{8 \cdot 8}{7 \cdot 9} \cdot \text{etc.}$$

Therefore, the intermediate terms corresponding to the indices $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ etc. can be expressed by the periphery of the circle this way

Indices:	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	
Terms:	$\frac{2}{\pi}$	$\frac{2 \cdot 2}{3 \cdot \pi}$	$\frac{2 \cdot 4 \cdot 2}{3 \cdot 5 \cdot \pi}$	$\frac{2 \cdot 4 \cdot 6 \cdot 2}{3 \cdot 5 \cdot 7 \cdot \pi}$	etc.

This same interpolation Wallis found in the *Arithmetica infinitorum*.

§401 Now, let us consider this series

1	2	3	4	
$a,$	$a(a+b),$	$a(a+b)(a+2b),$	$a(a+b)(a+2b)(a+3b)$	etc.

whose factors constitute this arithmetic progression

$$a, (a+b), (a+2b), (a+3b), (a+4b), \text{ etc.}$$

whose infinitesimal terms are of such a nature that the differences of their logarithms vanish. Since it therefore is

$$Z = a - b + b\omega$$

and

$$Z' = a + b\omega, \quad Z'' = a + b + b\omega, \quad Z''' = a + 2b + b\omega \quad \text{etc.},$$

if Σ denotes the term of the propounded series whose index is $= \omega$, it will be

$$\Sigma = a^\omega \cdot \frac{a^{1-\omega}(a+b)^\omega}{a+b\omega} \cdot \frac{(a+b)^{1-\omega}(a+2b)^\omega}{a+b+b\omega} \cdot \frac{(a+2b)^{1-\omega}(a+3b)^\omega}{a+2b+b\omega} \cdot \text{etc.}$$

And having found this value, if ω denotes a certain fractional number smaller than unity, the following terms corresponding to the indices $1 + \omega, 2 + \omega, 3 + \omega$ etc. will be determined in such a way that it is

$$\begin{aligned}\Sigma' &= (a + b\omega)\Sigma \\ \Sigma'' &= (a + b\omega)(a + b + b\omega)\Sigma \\ \Sigma''' &= (a + b\omega)(a + b + b\omega)(a + 2b + b\omega)\Sigma \\ &\text{etc.}\end{aligned}$$

If the term corresponding to the index $\frac{1}{2}$ is desired having put $\omega = \frac{1}{2}$ it will be

$$\Sigma = a^{\frac{1}{2}} \cdot \frac{a^{\frac{1}{2}}(a+b)^{\frac{1}{2}}}{a + \frac{1}{2}b} \cdot \frac{(a+b)^{\frac{1}{2}}(a+2b)^{\frac{1}{2}}}{a + \frac{3}{2}b} \cdot \frac{(a+2b)^{\frac{1}{2}}(a+3b)^{\frac{1}{2}}}{a + \frac{5}{2}b} \cdot \text{etc.}$$

and hence having taken the squares

$$\Sigma^2 = a \cdot \frac{a(a+b)}{(a + \frac{1}{2}b)(a + \frac{1}{2}b)} \cdot \frac{(a+b)(a+2b)}{(a + \frac{3}{2}b)(a + \frac{3}{2}b)} \cdot \frac{(a+2b)(a+3b)}{(a + \frac{5}{2}b)(a + \frac{5}{2}b)} \cdot \text{etc.}$$

§402 In the series we treated above [§ 400]

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \frac{f}{g'} & \frac{f(f+h)}{g(g+h)'} & \frac{f(f+h)(f+2h)}{g(g+h)(g+2h)'} & \frac{f(f+h)(f+2h)(f+3h)}{g(g+h)(g+2h)(g+3h)} \end{array} \text{ etc.}$$

the term corresponding to the index $\frac{1}{2} = \theta$; it will be

$$\Theta = \frac{f(g + \frac{1}{2}h)}{g(f + \frac{1}{2}h)} \cdot \frac{(f+h)(g + \frac{3}{2}h)}{(g+h)(f + \frac{3}{2}h)} \cdot \frac{(f+2h)(g + \frac{5}{2}h)}{(g+2h)(f + \frac{5}{2}h)} \cdot \text{etc.};$$

now, set

$$f = a, \quad g = a + \frac{1}{2}b \quad \text{and} \quad h = b;$$

it will be

$$\Theta = \frac{a(a+b)}{(a + \frac{1}{2}b)(a + \frac{1}{2}b)} \cdot \frac{(a+b)(a+2b)}{(a + \frac{3}{2}b)(a + \frac{3}{2}b)} \cdot \text{etc.}$$

and hence it will be $\Sigma^2 = a\Theta$ and $\Sigma = \sqrt{a\Theta}$. Therefore, if the term of this series

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
a, & a(a+b), & a(a+b)(a+2b), & a(a+b)(a+2b)(a+3b) \text{ etc.}
\end{array}$$

corresponding to the index $\frac{1}{2}$ is put $= \Theta$, it will be $\Sigma = \sqrt{a\Theta}$.

Because here the term of the series of the numerators alone corresponding to the index $\frac{1}{2}$ is $= \Sigma$, if in the series of denominators the term corresponding to the index $\frac{1}{2}$ is put $= \Lambda$, it will be $\Theta = \frac{\Sigma}{\Lambda}$; but it is $\Theta = \frac{\Sigma^2}{a}$, whence it will be $\Sigma = \frac{a}{\Lambda}$ or $\Sigma\Lambda = a$ by means of which theorems the interpolation of series of this kind is well illustrated.

EXAMPLE 1

Let this series to be interpolated be propounded

$$1, \quad 1 \cdot 1 \cdot 2, \quad 1 \cdot 2 \cdot 3, \quad 1 \cdot 2 \cdot 3 \cdot 4 \text{ etc.}$$

Since here it is $a = 1$ and $b = 1$, if the term corresponding to the index ω is put $= \Sigma$, it will be

$$\Sigma = \frac{1^{1-\omega} \cdot 2^\omega}{1+\omega} \cdot \frac{2^{1-\omega} \cdot 3^\omega}{2+\omega} \cdot \frac{3^{1-\omega} \cdot 4^\omega}{3+\omega} \cdot \frac{4^{1-\omega} \cdot 5^\omega}{4+\omega} \cdot \text{etc.}$$

Here, one can always take a fraction smaller than unity for ω ; for, nevertheless the interpolation will be extend throughout the whole series. For, if the terms corresponding $1 + \omega, 2 + \omega, 3 + \omega$ etc. are put $\Sigma', \Sigma'', \Sigma'''$ etc., it will be

$$\begin{aligned}
\Sigma' &= (1 + \omega)\Sigma \\
\Sigma'' &= (1 + \omega)(2 + \omega)\Sigma \\
\Sigma''' &= (1 + \omega)(2 + \omega)(3 + \omega)\Sigma \\
&\text{etc.}
\end{aligned}$$

Therefore, the term of the propounded series corresponding to the index $\frac{1}{2}$ will be

$$\Sigma = \frac{1^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}}{1^{\frac{1}{2}}} \cdot \frac{2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}}{2^{\frac{1}{2}}} \cdot \frac{3^{\frac{1}{2}} \cdot 4^{\frac{1}{2}}}{3^{\frac{1}{2}}} \cdot \text{etc.}$$

or

$$\Sigma^2 = \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{8 \cdot 10}{9 \cdot 9} \cdot \text{etc.}$$

Hence, because it is

$$\pi = 2 \cdot \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdot \text{etc.},$$

it will be

$$\Sigma^2 = \frac{\pi}{4} \quad \text{and} \quad \Sigma = \frac{\sqrt{\pi}}{2}$$

and hence we have the correspondences

Indices:	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	etc.
Terms:	$\frac{\sqrt{\pi}}{2}$	$\frac{3}{2} \cdot \frac{\sqrt{\pi}}{2}$	$\frac{3 \cdot 5}{2 \cdot 2} \cdot \frac{\sqrt{\pi}}{2}$	$\frac{3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 2} \cdot \frac{\sqrt{\pi}}{2}$	etc.

EXAMPLE 2

Let this series be propounded to be interpolated

$$1, \quad 1 \cdot 3, \quad 1 \cdot 3 \cdot 5, \quad 1 \cdot 3 \cdot 5 \cdot 7 \quad \text{etc.}$$

Since here it is $a = 1$, $b = 2$, if the term corresponding to the index ω is put $= \Sigma$, it will be

$$\Sigma = \frac{1^{1-\omega} \cdot 3^\omega}{1+2\omega} \cdot \frac{3^{1-\omega} \cdot 5^\omega}{3+2\omega} \cdot \frac{5^{1-\omega} \cdot 7^\omega}{5+2\omega} \cdot \text{etc.}$$

the term following in order will be of such a nature:

$$\begin{aligned} \Sigma' &= (1+2\omega)\Sigma \\ \Sigma'' &= (1+2\omega)(3+2\omega)\Sigma \\ \Sigma''' &= (1+2\omega)(3+2\omega)(5+2\omega)\Sigma \\ &\text{etc.} \end{aligned}$$

Therefore, if the term of the propounded series corresponding to the index $\frac{1}{2}$ is desired and it is called = Σ , it will be

$$\Sigma = \frac{\sqrt{1 \cdot 3}}{2} \cdot \frac{\sqrt{3 \cdot 5}}{4} \cdot \frac{\sqrt{5 \cdot 7}}{6} \cdot \frac{\sqrt{7 \cdot 9}}{8} \cdot \text{etc.},$$

therefore

$$\Sigma^2 = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdot \frac{7 \cdot 9}{8 \cdot 8} \cdot \text{etc.} = \frac{2}{\pi}$$

and one will have $\Sigma = \sqrt{\frac{2}{\pi}}$. But we will have the correspondences

Indices:	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	etc.
Terms:	$\sqrt{\frac{2}{\pi}}$	$2 \cdot \sqrt{\frac{2}{\pi}}$	$2 \cdot 4 \sqrt{\frac{2}{\pi}}$	$2 \cdot 4 \cdot 6 \sqrt{\frac{2}{\pi}}$	etc.

If the first series and this one are multiplied by each other that one has this series

1	2	3	4	5	
$1^2,$	$1^2 \cdot 2 \cdot 3,$	$1^2 \cdot 2 \cdot 3^2 \cdot 5,$	$1^2 \cdot 2 \cdot 3^2 \cdot 4 \cdot 5 \cdot 7$	$1^2 \cdot 2 \cdot 3^2 \cdot 4 \cdot 5^2 \cdot 7 \cdot 9$	etc.,

the term corresponding to the index $\frac{1}{2}$ will be = $\frac{\sqrt{\pi}}{2} \cdot \sqrt{\frac{2}{\pi}} = \frac{1}{\sqrt{2}}$; this is easily seen, if this form is attributed to that series

1	2	3	4	
$\frac{1 \cdot 2}{2},$	$\frac{1 \cdot 2 \cdot 3 \cdot 4}{2^2},$	$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{2^3},$	$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{2^4}$	etc.,

whose term corresponding to the index $\frac{1}{2}$ manifestly is = $\frac{1}{\sqrt{2}}$.

EXAMPLE 3

Let this series be propounded to be interpolated

$$\frac{1}{1}, \quad \frac{2}{1 \cdot 2}, \quad \frac{3}{1 \cdot 2 \cdot 3}, \quad \frac{4}{1 \cdot 2 \cdot 3 \cdot 4} \quad \text{etc.}$$

Consider the numerators and denominators of this series separately, and since the numerators are

$$1, \quad 2, \quad 3, \quad 4 \quad \text{etc.},$$

it will be, after an application of the result before, $a = n$ and $b = -1$, whence the term corresponding to the index ω of this series will be

$$= n^\omega \cdot \frac{n^{1-\omega}(n-1)^\omega}{n-\omega} \cdot \frac{(n-1)^{1-\omega}(n-2)^\omega}{n-1-\omega} \cdot \frac{(n-2)^{1-\omega}(n-3)^\omega}{n-2-\omega} \cdot \text{etc.},$$

which expression because of the factors going over into negatives does not show anything certain. Therefore, transform the propounded series, by putting $1 \cdot 2 \cdot 3 \cdots n = N$ for the sake of brevity, into this one

$$\frac{1}{1 \cdot 1 \cdot 2 \cdot 3 \cdots (n-1)}, \quad \frac{2}{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdots (n-2)}, \quad \frac{3}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3 \cdots (n-3)} \quad \text{etc.}$$

since whose denominators consist of two factors, the ones will constitute this series

$$1 \cdot 2 \cdot 3 \cdots (n-1), \quad 1 \cdot 2 \cdot 3 \cdots (n-2), \quad 1 \cdot 2 \cdot 3 \cdots (n-3) \quad \text{etc.}$$

whose term corresponding to the index ω coincides with term of this series

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & \\
1, & 1 \cdot 2, & 1 \cdot 2 \cdot 3, & 1 \cdot 2 \cdot 3 \cdot 4 & 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 & \text{etc.}
\end{array}$$

corresponding to the index $n - \omega$ which is

$$\frac{1^{1-n-\omega} \cdot 2^{n-\omega}}{1+n-\omega} \cdot \frac{2^{1-n-\omega} \cdot 3^{n-\omega}}{2+n-\omega} \cdot \frac{3^{1-n-\omega} \cdot 4^{n-\omega}}{3+n-\omega} \cdot \text{etc.}$$

But let the term of this series corresponding to the index $1 - \omega$ be $= \Theta$; it will be

$$\Theta = \frac{1^\omega \cdot 2^{1-\omega}}{2-\omega} \cdot \frac{2^\omega \cdot 3^{1-\omega}}{3-\omega} \cdot \frac{3^\omega \cdot 4^{1-\omega}}{4-\omega} \cdot \text{etc.},$$

and because of the correspondences

$$\begin{array}{cccc}
\text{Indices:} & 1 - \omega & 2 - \omega & 3 - \omega & \text{etc.} \\
\text{Terms:} & \Theta, & (2 - \omega)\Theta & (2 - \omega)(3 - \omega)\Theta & \text{etc.},
\end{array}$$

to the index $n - \omega$ this term will correspond

$$(2 - \omega)(3 - \omega)(4 - \omega) \cdots (n - \omega)\Theta.$$

Further, the other factors of those denominators will constitute this series

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & \\
1, & 1 \cdot 2, & 1 \cdot 2 \cdot 3, & 1 \cdot 2 \cdot 3 \cdot 4 & 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 & \text{etc.};
\end{array}$$

if the term corresponding to the index ω is put $= \Lambda$, it will be

$$\Lambda = \frac{1^{1-\omega} \cdot 2^\omega}{1+\omega} \cdot \frac{2^{1-\omega} \cdot 3^\omega}{2+\omega} \cdot \frac{3^{1-\omega} \cdot 4^\omega}{3+\omega} \cdot \text{etc.}$$

Having found these, if the term of the propounded series itself

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\frac{n}{1}, & \frac{n(n-1)}{1 \cdot 2}, & \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}, & \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} & \text{etc.}
\end{array}$$

corresponding to the index ω is put Σ , it will be

$$\Sigma = \frac{N}{\Lambda \cdot (2 - \omega)(3 - \omega)(4 - \omega) \cdots (n - \omega)\Theta}$$

But on the other hand it is

$$\frac{N}{(2 - \omega)(3 - \omega)(4 - \omega) \cdots (n - \omega)} = \frac{2}{2 - \omega} \cdot \frac{3}{3 - \omega} \cdot \frac{4}{4 - \omega} \cdots \frac{n}{n - \omega}$$

and

$$\Lambda\Theta = \frac{1 \cdot 2}{(1 + \omega)(2 - \omega)} \cdot \frac{2 \cdot 3}{(2 + \omega)(3 - \omega)} \cdot \frac{3 \cdot 4}{(3 + \omega)(4 - \omega)} \cdot \text{etc.}$$

From these the term in question corresponding to the index ω will be

$$\Sigma = \frac{2}{2 - \omega} \cdot \frac{3}{3 - \omega} \cdot \frac{4}{4 - \omega} \cdot \frac{5}{5 - \omega} \cdots \frac{n}{n - \omega} \cdot \frac{(1 + \omega)(2 - \omega)}{1 \cdot 2} \cdot \frac{(2 + \omega)(3 - \omega)}{2 \cdot 3} \cdot \frac{(3 + \omega)(4 - \omega)}{3 \cdot 4} \cdot \text{etc. up to infinity.}$$

Therefore, to the index $\frac{1}{2}$ this term will correspond

$$\frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdots \frac{2n}{2n-1} \cdot \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{9 \cdot 9}{8 \cdot 10} \cdot \text{etc.,}$$

which is reduced to

$$\frac{4 \cdot 6 \cdot 8 \cdot 10 \cdots 2n}{3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n - 1)} \cdot \frac{4}{\pi'}$$

or it will be

$$= \frac{2}{\pi} \cdot \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdots 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n - 1)}$$

If $n = 2$, this series to be interpolated will arise

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \\ 1, & 2, & 1, & 0, & 0, & 0, & 0, & \text{etc.} \end{array}$$

whose term corresponding to the index $\frac{1}{2}$ therefore is $= \frac{16}{3\pi}$.

EXAMPLE 4

Let the term corresponding to the index = $\frac{1}{2}$ in this series be in question

$$\begin{array}{cccccc}
 0 & 1 & 2 & 3 & 4 & \\
 1, & +\frac{1}{2}, & -\frac{1 \cdot 1}{2 \cdot 4}, & +\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} & -\frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} & \text{etc.}
 \end{array}$$

This series arises from the proceeding, if one puts $n = \frac{1}{2}$, and therefore the term in question, which shall be = Σ , will be

$$\Sigma = \frac{2}{\pi} \cdot \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n - 1)}$$

having put $n = \frac{1}{2}$. Put

$$\frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n - 1)} = \Theta,$$

if $n = \frac{1}{2}$, and Θ will be the term corresponding to the index $\frac{1}{2}$ in this series

$$\frac{2}{1}, \frac{2 \cdot 4}{1 \cdot 3}, \frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5}, \frac{2 \cdot 4 \cdot 6 \cdot 8}{1 \cdot 3 \cdot 5 \cdot 7} \text{ etc.,}$$

which arises from the superior ones as = $\frac{\pi}{2}$. Therefore, the term of the propounded series corresponding to the index $\frac{1}{2}$ which is in question will be = 1. But since in this series, if the term corresponding to any arbitrary index ω is put = Σ , the one following it will be $\Sigma' = \frac{1-2\omega}{2+2\omega}\Sigma$, the propounded series will be interpolated by terms falling in the respective middles this way:

$$\begin{array}{cccccccccc}
 \text{Indices:} & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & \frac{7}{2} & \text{etc.} \\
 \text{Terms:} & 1, & 1, & \frac{1}{2} & 0, & \frac{-1 \cdot 1}{2 \cdot 4}, & 0, & \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}, & 0 & \text{etc.,}
 \end{array}$$

EXAMPLE 5

If n was an arbitrary fractional number, to find the term corresponding to the index ω in the series

$$\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & \\
1, & \frac{n}{1}, & \frac{n(n-1)}{1 \cdot 2}, & \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}, & \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} & \text{etc.}
\end{array}$$

If we compare the expression

$$\frac{2}{2-\omega} \cdot \frac{3}{3-\omega} \cdot \frac{4}{4-\omega} \cdots \frac{n}{n-\omega}$$

to § 400, it is $a = 1$, $c = 1$, $b = 1 - \omega$ and having put n instead of ω there it will be

$$\frac{1}{1-\omega} \cdot \frac{2}{2-\omega} \cdot \frac{3}{3-\omega} \cdots \frac{n}{n-\omega} = \frac{1(1-\omega+n)}{(1-\omega)(1+n)} \cdot \frac{2(2-\omega+n)}{(2-\omega)(2+n)} \cdot \text{etc.},$$

whence the term in question corresponding to the index ω , if it is put $= \Sigma$, it will be

$$\Sigma = \frac{(1-\omega+n) \cdot 2}{(1+n)(2-\omega)} \cdot \frac{(2-\omega+n)3}{(2+n)(3-\omega)} \cdot \text{etc.} \cdot \frac{(1+\omega)(2-\omega)}{1 \cdot 2} \cdot \frac{(2+\omega)(3-\omega)}{2 \cdot 3} \cdot \text{etc.}$$

and hence

$$\Sigma = \frac{(1+\omega)(1+n-\omega)}{1(1+n)} \cdot \frac{(2+\omega)(2+n-\omega)}{2(2+n)} \cdot \frac{(3+\omega)(3+n-\omega)}{3(3+n)} \cdot \text{etc.};$$

therefore, if $n - \omega$ was an integer number, the value of Σ can be expressed rationally.

$$\begin{array}{ll}
\text{So if it is } n = \omega, & \text{it will be } \Sigma = 1; \\
\text{if } n = 1 + \omega, & \text{it will be } \Sigma = n; \\
\text{if } n = 2 + \omega, & \text{it will be } \Sigma = \frac{n(n-1)}{1 \cdot 2}; \\
\text{if } n = 3 + \omega, & \text{it will be } \Sigma = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \\
& \text{etc.}
\end{array}$$

But if $\omega - n$ was an positive integer, it will always be $\Sigma = 0$.