Investigation of the sum of series from the general Term *

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§103 Let the general term corresponding to the index x of a certain series be $= y$, such that y us an arbitrary function of x. Further, let Sy the sum or the summatory term of the series expressing the aggregate of all terms from the first or another fixed term up to y, inclusively. But we will compute the sum of the series from the first term, whence, if it is $x = 1$, y will give the first term and Sy will exhibited this first term y; but if one puts $x = 0$, the summatory Sy has to go over into nothing, because no terms to be summed are there. Therefore, the summatory term Sy will be a function of x of such a kind which vanished for $x = 0$.

§104 If the general term y consists of several parts, that it is $y = p + q + r +$ etc., then one can consider the series itself as conflated from several other series, whose general terms are p, q, r etc. Hence, if the sums of these series are known, one will also be able to assign the sum of the propounded series; for, it will be the aggregate of all single series. Therefore, if $y = p + q + r +$ etc., it will be $Sy = Sp + Sq + Sq +$ etc. Therefore, because above we exhibited the sums of series, whose general terms are any arbitrary powers of x having positive integer coefficients, hence one will be able to find the summatory term of any series, whose general term is $ax^\alpha + bx^\beta + cx^\gamma +$ etc. while α, β, γ etc. positive integer numbers or whose general term is a polynomial function of x.

*Original title: " Investigatio summae serierum ex termino generali", first published as part of the book „*Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, 1755", reprinted in in „*Opera Omnia*: Series 1, Volume 10, pp. 309 - 336 ", Eneström-Number E212, translated by: Alexander Aycock for the „Euler-Kreis Mainz"

§105 In this series whose general term or the term corresponding to the exponent x is $= y$ let the term preceding this one or the term corresponding to the index $x-1$ be $= v$; since v arises from y, if instead of x one writes $x-1$, it will be

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \text{etc.}$$

Therefore, if y was the general term of this series

$$\begin{array}{cccccc} 1 & 2 & 3 & 4\cdots & \cdots x-1 & x \\ a+ & b+ & c+ & d+ & \cdots + v+ & y \end{array}$$

and the term corresponding to the index 0 of this series was $= A$, v, as it is a function of x, will be the general term of this series

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots\cdots x \\ A+ & a+ & b+ & c+ & d+ & \cdots + \ v, \end{array}$$

whence, if Sv denotes the sum of this series, it will be $Sv = Sy - y + A$. And so having put $x = 0$, since it is $Sy = 0$ and $y = A$, also Sv will vanish.