On the Use of Differential Calculus in the Formation of Series *

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§198 Until now we only considered one single use of differential calculus in the doctrine of series which consists in the formation of series itself and to which we above referred already above, when there was the question how to expand the fraction whose denominator is any power of a certain function into a series. But this method is similar to that one we already used several times, where the fraction to be converted into a series is set equal to a certain series having undetermined coefficients in the single terms which coefficients are then determined from the constituted equality. But this determination is often simplified tremendously, if, before it is actually done, the equation both is differentiated once and sometimes also twice. Since this method is of widest use in integral calculus let us explain it here more diligently.

§199 Therefore, at first let us repeat what we discussed above about the expansion of fractions into series without the application of differential calculus. Let any fraction be propounded

$$\frac{A + Bx + Cx^2 + Dx^3 + \text{etc.}}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}} = s,$$

which is to be converted into a powers series in x. Assume an undetermined series for s

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$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \text{etc.}$$

Therefore, since having removed the fraction by multiplication it is

$$\begin{aligned} &A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + \text{etc.} \\ &= s(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \eta x^6 + \text{etc.}), \end{aligned}$$

if for s the assumed series is substituted, the following equation arises

$$\begin{array}{rrrrrrr} & A + Bx & + Cx^2 & + Dx^3 & + Ex^4 & + Fx^5 & + \text{etc.} \\ \hline = & \mathfrak{A}\alpha + \mathfrak{B}\alpha x & + \mathfrak{C}\alpha x^2 & + \mathfrak{D}\alpha x^3 & + \mathfrak{E}\alpha x^4 & + \mathfrak{F}\alpha x^5 & + \text{etc.} \\ & + \mathfrak{A}\beta & + \mathfrak{B}\beta & + \mathfrak{C}\beta & + \mathfrak{D}\beta & + \mathfrak{E}\beta & + \text{etc.} \\ & & + \mathfrak{A}\gamma & + \mathfrak{B}\gamma & + \mathfrak{C}\gamma & + \mathfrak{D}\gamma & + \text{etc.} \\ & & & + \mathfrak{A}\delta & + \mathfrak{B}\delta & + \mathfrak{C}\delta & + \text{etc.} \\ & & & & + \mathfrak{A}\varepsilon & + \mathfrak{B}\varepsilon & + \text{etc.} \\ & & & & & + \mathfrak{A}\zeta & + \text{etc.} \end{array}$$

Therefore, having equated the single terms which contain the same powers of x, it will be

$$\begin{aligned} &\mathfrak{A}\alpha - A = 0 \\ &\mathfrak{B}\alpha + \mathfrak{A}\beta - B = 0 \\ &\mathfrak{C}\alpha + \mathfrak{B}\beta + \mathfrak{A}\gamma - C = 0 \\ &\mathfrak{D}\alpha + \mathfrak{C}\beta + \mathfrak{B}\gamma + \mathfrak{A}\delta - D = 0 \\ &\mathfrak{E}\alpha + \mathfrak{D}\beta + \mathfrak{C}\gamma + \mathfrak{B}\delta + \mathfrak{A}\varepsilon - E = 0 \\ &\qquad\qquad\qquad \text{etc.} \end{aligned}$$

from which equations the assumed coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. are determined, and so the infinite series

$$\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.}$$

equal to the propounded fraction s is found. And in this form, if both the numerator and the denominator of the fraction s consist of a finite number of terms, all recurring series will be contained, which were treated in a lot greater detail above.

§200 But if either the numerator or the denominator or both were raised to any power, then this way the series is obtained rather difficultly, since the task, if not a binomial function was raised, becomes more than laborious. But by means of differential calculus this work can be avoided. At first, let only the numerator be there and let be

$$s = (A + Bx + Cxx)^n,$$

whence the power series in x equal to this trinomial power is to be found; it is plain that it will be finite, if the exponent n was an positive integer. Again assume an indefinite series for s

$$S = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \text{etc.}$$

whose first term $\mathfrak{A}$ is known to be $= A^n$; for, if one puts $x = 0$, from the first propounded form it is $s = A^n$, but from the assumed series it is $s = \mathfrak{A}$. But this determination of the first term is to be derived from the nature of the subject itself, if we want to descend to differentials, since hence the first term is not determined, as it will become plain soon.

§201 Because it is $S = (A + Bx + Cx^2)^n$, it will be by taking logarithms

$$\ln s = n \ln(A + Bx + Cx^2)$$

and hence having taken the differentials one will have

$$\frac{ds}{s} = \frac{nBdx + 2nCxdx}{A + Bx + Cx^2} \quad \text{or} \quad (A + Bx + Cx^2)\frac{ds}{dx} = ns(B + 2Cx).$$

But from the assumed series it is

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \text{etc.}$$

Therefore, if this series is substituted for $\frac{ds}{dx}$ and for s the assumed series itself is substituted, the following equation will arise

$$\begin{array}{rrrrrr}
A\mathfrak{B} + & 2A\mathfrak{C}x + & 3A\mathfrak{D}x^2 + & 4A\mathfrak{E}x^3 + & 5A\mathfrak{F}x^4 & + \text{etc.} \\
+ & B\mathfrak{B} & + \; 2B\mathfrak{C} & + \; 3B\mathfrak{D} & + \; 4B\mathfrak{E} & + \text{etc.} \\
 & + & C\mathfrak{B} & + \; 2C\mathfrak{C} & + \; 3C\mathfrak{D} & + \text{etc.} \\
\hline
= nB\mathfrak{A} + & nB\mathfrak{B} & + \; nB\mathfrak{C} & + \; nB\mathfrak{D} & + \; nB\mathfrak{E} & + \text{etc.} \\
+ \; 2nC\mathfrak{A} & + \; 2nC\mathfrak{B} & + \; 2nC\mathfrak{C} & + \; 2nC\mathfrak{D} & & + \text{etc.}
\end{array}$$

Therefore having equated the terms of the same power of x here it will be

$$\begin{aligned}
\mathfrak{B} &= \frac{nB\mathfrak{A}}{A}\\
\mathfrak{C} &= \frac{(n-1)B\mathfrak{B}+2nC\mathfrak{A}}{2A}\\
\mathfrak{D} &= \frac{(n-2)B\mathfrak{C}+(2n-1)C\mathfrak{B}}{3A}\\
\mathfrak{E} &= \frac{(n-3)B\mathfrak{D}+(2n-2)C\mathfrak{C}}{4A}\\
\mathfrak{F} &= \frac{(n-4)B\mathfrak{E}+(2n-3)C\mathfrak{D}}{5A}\\
&\text{etc.}
\end{aligned}$$

Therefore, since, as we saw before, it is $\mathfrak{A} = A^n$, it will be $\mathfrak{B} = nA^{n-1}B$ and hence the remaining coefficients will successively defined. But the law which they follow is plain most easily from these formulas which would have remained immensely obscure, if we would have wanted to actually expand the trinomial.

§202 The same method succeeds, if any polynomial function has to raised up to a certain power. Let

$$s = (A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.})^n$$

and assume

$$s + \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.};$$

it will be $\mathfrak{A} = A^n$ which value is concluded, if one puts $x = 0$. Now, having taken the logarithms their differentials as before one will found

$$\frac{ds}{s} = \frac{nBdx + 2nCxdx + 3nDx^2dx + 4nEx^3dx + \text{etc.}}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}$$

or

$$\begin{aligned}
&(A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.})\frac{ds}{dx}\\
&= s(nB + 2nCx + 3nDx^2 + 4nEx^3 + \text{etc.}).
\end{aligned}$$

Therefore, since it is

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \text{etc.},$$

having substituted these series for s and $\frac{ds}{dx}$ it will be

$$\begin{array}{rrrrrr}
A\mathfrak{B} + & 2A\mathfrak{C}x + & 3A\mathfrak{D}x^2 + & 4A\mathfrak{E}x^3 + & 5A\mathfrak{F}x^4 & + \text{etc.} \\
+ & B\mathfrak{B} \ + & 2B\mathfrak{C} \ + & 3B\mathfrak{D} \ + & 4B\mathfrak{E} & + \text{etc.} \\
 & + & C\mathfrak{B} \ + & 2C\mathfrak{C} \ + & 3C\mathfrak{D} & + \text{etc.} \\
 & & + & D\mathfrak{B} \ + & 2D\mathfrak{C} & + \text{etc.} \\
 & & & + & E\mathfrak{B} & + \text{etc.} \\
\hline
= nB\mathfrak{A} + & nB\mathfrak{B} \ + & nB\mathfrak{C} \ + & nB\mathfrak{D} \ + & nB\mathfrak{E} & + \text{etc.} \\
 & + 2nC\mathfrak{A} & + 2nC\mathfrak{B} & + 2nC\mathfrak{C} & + 2nC\mathfrak{D} & + \text{etc.} \\
 & & + 3nD\mathfrak{A} & + 3nD\mathfrak{B} & + 3nD\mathfrak{C} & + \text{etc.} \\
 & & & + 4nE\mathfrak{A} & + 4nE\mathfrak{B} & + \text{etc.} \\
 & & & & + 5nF\mathfrak{A} & + \text{etc.}
\end{array}$$

Hence, the following determinations are derived

$$\begin{aligned}
A\mathfrak{B} &= nB\mathfrak{A} \\
2A\mathfrak{C} &= (n-1)B\mathfrak{B} + 2nC\mathfrak{A} \\
3A\mathfrak{D} &= (n-2)B\mathfrak{C} + (2n-1)C\mathbb{B} + 3nD\mathfrak{A} \\
4A\mathfrak{E} &= (n-3)B\mathfrak{D} + (2n-2)C\mathfrak{C} + (3n-1)D\mathfrak{B} + 4nE\mathfrak{A} \\
5A\mathfrak{F} &= (n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D} + (3n-2)D\mathfrak{C} + (4n-1)E\mathfrak{B} + 5nF\mathfrak{A} \\
&\text{etc.,}
\end{aligned}$$

whence it becomes very clear, how these assumed coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. depend on each other and are hence determined, because it is $\mathfrak{A} = A^n$.

§203 Since, if the quantity $A + Bx + Cx^2 + Dx^3 + \text{etc.}$ consists of a finite number of terms and the number n was a positive integer, any power also has to consist of a finite number of terms, it is manifest that in this case the formulas just found must finally vanish and, since all finite terms must be there, until the first vanishes, at the same time all following ones must vanish.

Let us put that the propounded formula $A + Bx + Cx^2$ is a trinomial and its cube is in question that it is $n = 3$; it will be

$$\begin{array}{rcll}
\mathfrak{A} & = & 2A^3 & \text{and hence} \quad \mathfrak{A} = A^3 \\
A\mathfrak{B} & = & 3B\mathfrak{A} & \mathfrak{B} = 3A^2B \\
2A\mathfrak{C} & = & 2B\mathfrak{B} + 6C\mathfrak{A} & \mathfrak{C} = 3AB^2 + 3A^2C \\
3A\mathfrak{D} & = & 1B\mathfrak{C} + 5C\mathfrak{B} & \mathfrak{D} = B^3 + 6ABC \\
4A\mathfrak{E} & = & 0 \quad + 4C\mathfrak{C} & \mathfrak{E} = 3B^2C + 3AC^2 \\
5A\mathfrak{F} & = & - \ B\mathfrak{E} + 3C\mathfrak{D} & \mathfrak{F} = 3BC^2 \\
6A\mathfrak{G} & = & -2B\mathfrak{F} + 2C\mathfrak{E} & \mathfrak{G} = C^3 \\
7A\mathfrak{H} & = & -3B\mathfrak{G} + 2C\mathfrak{F} & \mathfrak{H} = 0 \\
8A\mathfrak{I} & = & -4B\mathfrak{H} + 0 & \mathfrak{I} = 0
\end{array}$$

Therefore, since already two are $= 0$ and any arbitrary of the following depends on the two preceding ones, it is plain that all following ones must vanish in similar manner. And because of this reason the law according to which these coefficients were found to depend on each other is even more note worthy.

§204 If n was a negative number such that s becomes equal to the fraction the series will continue to infinity. Therefore, let

$$s = \frac{1}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varphi x^3 + \varepsilon x^4 + \text{etc.})^n};$$

for its value assumes this series

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \text{etc.}$$

And if in the superior formulas for the letters A, B, C, D etc., one puts α, β, γ, δ etc. and at the same time n becomes negative, the following determinations of the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. will arise

$$\mathfrak{A} = \alpha^{-n} = \frac{1}{\alpha^n}$$
$$\alpha\mathfrak{B} + n\beta\mathfrak{A} = 0$$

$$2\alpha\mathfrak{C} + (n+1)\beta\mathfrak{B} + 2n\gamma\mathfrak{A} = 0$$
$$3\alpha\mathfrak{D} + (n+2)\beta\mathfrak{C} + (2n+1)\gamma\mathfrak{B} + 3n\delta\mathfrak{A} = 0$$
$$4\alpha\mathfrak{E} + (n+3)\beta\mathfrak{D} + (2n+2)\gamma\mathfrak{C} + (3n+1)\delta\mathfrak{B} + 4n\varepsilon\mathfrak{A} = 0$$
$$5\alpha\mathfrak{F} + (n+4)\beta\mathfrak{E} + (2n+3)\gamma\mathfrak{D} + (3n+2)\delta\mathfrak{C} + (4n+1)\varepsilon\mathfrak{B} + 5n\zeta\mathfrak{A} = 0$$
$$\text{etc.}$$

These formulas contain the same law of these coefficients of numbers we already observed above in the *Introductio* and whose validity could therefore be demonstrated rigidly just now.

§205 These behave this way, if the numerator of the fraction was the unity or also any power of x, say x^m; for, in the latter case it will only be necessary to multiply the series found first $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \text{etc.}$ by x^m. But if the denominator consists of two or more terms, then we did not observe the law of progression above: therefore, let us investigate it there by means of differentiation. Therefore, let

$$s = \frac{A + Bx + Cx^2 + Dx^3 + \text{etc.}}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.})^n}$$

and assume the following series for the value of this fraction

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \text{etc.};$$

to define its first term $\mathfrak{A}$ put $x = 0$ and from the first expression it will be $s = \frac{A}{\alpha^n}$, from the assumed one on the other hand $s = \mathfrak{A}$, whence it is necessary, that it is $\mathfrak{A} = \frac{A}{\alpha^n}$. Having determined this term the remaining ones will become known by means of differentiation.

§206 Having taken logarithms it will be

$$\ln s = \ln(A + Bx + Cx^2 + Dx^3 + \text{etc.})$$

$$-n\ln(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.})$$

and hence by differentiation it will arise

$$\frac{ds}{s}=\frac{Bdx+2Cdx+3Dx^2dx+\text{etc.}}{A+Bx+Cx^2+Dx^3+\text{etc.}}-\frac{n\beta dx+2n\gamma xdx+3n\delta x^2dx+\text{etc.}}{\alpha+\beta x+\gamma x^2+\delta x^3+\text{etc.}}$$

and having got rid of the fractions by multiplication it will be

$$\left\{\begin{array}{lllll} A\alpha & +\ A\beta x & +\ A\gamma x^2 & +\ A\delta x^3 & +\ \text{etc.} \\ & +\ B\alpha & +\ B\beta & +\ B\gamma & +\ \text{etc.} \\ & & +\ C\alpha & +\ C\beta & +\ \text{etc.} \\ & & & +\ D\alpha & +\ \text{etc.} \end{array}\right\}\frac{ds}{dx}$$

$$=\left\{\begin{array}{lllll} B\alpha+ & B\beta x+ & B\gamma x^2+ & B\delta x^3 & +\ \text{etc.} \\ & +\ 2C\alpha & +\ 2C\beta & +\ 2C\gamma & +\ \text{etc.} \\ & & +\ 3D\alpha & +\ 3D\beta & +\ \text{etc.} \\ & & & +\ 4E\alpha & +\ \text{etc.} \end{array}\right\}s$$

$$-\left\{\begin{array}{lllll} A\beta & +\ 2A\gamma x & +\ 3A\delta x^2 & +\ 4A\varepsilon x^3 & +\ \text{etc.} \\ & +\ B\beta & +\ 2B\gamma & +\ 3B\delta & +\ \text{etc.} \\ & & +\ C\beta & +\ 2C\gamma & +\ \text{etc.} \\ & & & +\ D\beta & +\ \text{etc.} \end{array}\right\}ns.$$

Since now it is $\frac{ds}{dx}=\mathfrak{B}+2\mathfrak{C}x+3\mathfrak{D}x^2+4\mathfrak{E}x^3+\text{etc.}$, it will be having done the substitutions

$$\left.\begin{array}{rr} A\alpha\mathfrak{B} & +\ nA\beta\mathfrak{A} \\ & -\ \ B\alpha\mathfrak{A} \end{array}\right\}=0$$

$$\left.\begin{array}{rrr} 2A\alpha\mathfrak{C} & +\ (n+1)A\beta\mathfrak{B} & +\ 2nA\gamma\mathfrak{A} \\ & +\ 0B\alpha\mathfrak{B} & +\ (n-1)B\beta\mathfrak{A} \\ & & -\ 2C\alpha\mathfrak{A} \end{array}\right\}=0$$

$$\left.\begin{array}{r}3A\alpha\mathfrak{D} + (n+2)A\beta\mathfrak{C} + (2n+1)A\gamma\mathfrak{B} + 3nA\delta\mathfrak{A} \\ + B\alpha\mathfrak{C} + nB\beta\mathfrak{B} + (2n-1)B\gamma\mathfrak{A} \\ - C\alpha\mathfrak{B} + (n-2)C\beta\mathfrak{A} \\ - 3D\alpha\mathfrak{A}\end{array}\right\} = 0$$

$$\left.\begin{array}{r}4A\alpha\mathfrak{E} + (n+3)A\beta\mathfrak{D} + (2n+2)A\gamma\mathfrak{C} + (3n+1)A\delta\mathfrak{B} + 4nA\varepsilon\mathfrak{A} \\ + B\alpha\mathfrak{D} + (n+1)B\beta\mathfrak{C} + 2nB\gamma\mathfrak{B} + (3n-1)B\delta\mathfrak{A} \\ + 0C\alpha\mathfrak{C} + (n-1)C\beta\mathfrak{B} + (3n-2)C\gamma\mathfrak{A} \\ - 2D\alpha\mathfrak{B} + (n-3)D\beta\mathfrak{A} \\ - 4E\alpha\mathfrak{A}\end{array}\right\} = 0.$$

etc.

Hence, the law according to which these formulas proceed is easily seen; for, the first line of each equation follows the same law we had in § 204. But then the coefficients of the second lines arise, if from the superior coefficients $n+1$ is subtracted, and in similar manner from the second line the third line is formed and the following from the superior ones always by subtracting $n+1$; but the letters building each term are most easily formed by inspection alone.

§207 But if also the numerator of a fraction was a certain power, namely

$$s = \frac{(A + Bx + Cx^2 + Dx^3 + \text{etc.})^m}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.})^n},$$

and one assumes

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.},$$

it will be $\mathfrak{A} = \frac{A^m}{\alpha^n}$; but the remaining coefficients will be determined from the following formulas

$$\left.\begin{array}{r}A\alpha\mathfrak{B} + nA\beta\mathfrak{A} \\ - mB\alpha\mathfrak{A}\end{array}\right\} = 0$$

$$\left.\begin{array}{r}2A\alpha\mathfrak{C} + (n+1)A\beta\mathfrak{B} + 2nA\gamma\mathfrak{A} \\ - (m-1)B\alpha\mathfrak{B} + (n-m)B\beta\mathfrak{A} \\ - 2mC\alpha\mathfrak{A}\end{array}\right\} = 0$$

$$\left.\begin{array}{r}3A\alpha\mathfrak{D} + (n+2)A\beta\mathfrak{C} + \quad (2n+1)A\gamma\mathfrak{B} + \qquad 3nA\delta\mathfrak{A}\\ -(m-2)B\alpha\mathfrak{C} + (n-m+1)B\beta\mathfrak{B} + (2n-m)B\gamma\mathfrak{A}\\ - \quad (2m-1)C\alpha\mathfrak{B} + (n-2m)C\beta\mathfrak{A}\\ - \qquad 3mD\alpha\mathfrak{A}\end{array}\right\} = 0$$

$$\left.\begin{array}{r}4A\alpha\mathfrak{E} + (n+3)A\beta\mathfrak{D} + \quad (2n+2)A\gamma\mathfrak{C} + \quad (3n+1)A\delta\mathfrak{B} + \qquad 4nA\varepsilon\mathfrak{A}\\ -(m-3)B\alpha\mathfrak{D} + (n+m-2)B\beta\mathfrak{C} + (2n-m+1)B\gamma\mathfrak{B} + (3n-m)B\delta\mathfrak{A}\\ - \quad (2m-2)C\alpha\mathfrak{C} + (n-2m+1)C\beta\mathfrak{B} + (n-3m)C\gamma\mathfrak{A}\\ - \quad (3m-1)D\alpha\mathfrak{B} + (n-3m)D\beta\mathfrak{A}\\ - \qquad 4mE\alpha\mathfrak{A}\end{array}\right\} = 0.$$

etc.

The law how these formulas are continued is clear easier from inspection than it can be described by words. But by descending the coefficients are diminished by the difference $m+n$; but by proceeding horizontally the differences will be augmented continuously by the difference $n-1$.

§208 Therefore, this way the doctrine of recurring series is amplified, as we cured this defect and defined the law of the coefficients, if not only the denominator of the fraction was any power, but also the numerator consists of any arbitrary number of terms, to detect which law induction alone did not suffice. But except the many uses of recurring series which we explained, they have the greatest use to find the sums of certain series approximately; we exhibited a specimen of this already in the first chapter of this section, where we transformed the series into another by means of the substitution $x = \frac{y}{1+ny}$ which often consists of a finite number of terms. And the method could have been extended further, if for x other functions were substituted. Since then the law of progression of series, which had to be substituted for the power of x, is not sufficiently clear, it seemed advisable to reserve this amplification up to this point, after the mentioned law had already been completely discovered. Nevertheless, considering this with more attention, we learn that the same task can be done without this law of progression only by using the method we used here to investigate the law itself here.

§209 Therefore, let any series be propounded

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

which shall be transformed into another one whose single terms shall be fractions whose denominators proceed according to powers of a formula of this kind

$$\alpha + \beta x + \gamma x^2 + \delta x^3 + \text{etc.}$$

To start from simpler ones, let us put that it is

$$s = \frac{\mathfrak{A}}{\alpha + \beta x} + \frac{\mathfrak{B}x}{(\alpha + \beta x)^2} + \frac{\mathfrak{C}x^2}{(\alpha + \beta x)^3} + \frac{\mathfrak{D}x^3}{(\alpha + \beta x)^4} + \text{etc.};$$

Having equated the series to this expression multiply by $\alpha + \beta x$ everywhere and it will be

$$\left.\begin{array}{r} A\alpha + B\alpha\, x + C\alpha x^2 + D\alpha x^3 + +\text{etc.} \\ A\beta x + b\beta \quad + C\beta \quad + +\text{etc.} \end{array}\right\} = \mathfrak{A} + \frac{\mathfrak{B}x}{\alpha + \beta x} + \frac{\mathfrak{C}x^2}{(\alpha + \beta x)^2} + \text{etc.}$$

Put $\mathfrak{A} = A\alpha$ and let

$$\begin{aligned} A\beta + B\alpha &= A' \\ B\beta + C\alpha &= B' \\ C\beta + D\alpha &= C' \\ D\beta + E\alpha &= D' \\ &\text{etc.}; \end{aligned}$$

having divided by x it will be

$$A' + B'x + C'x^2 + D'x^3 + \text{etc.} = \frac{\mathfrak{B}}{\alpha + \beta x} + \frac{\mathfrak{C}x}{(\alpha + \beta)^2} + \frac{\mathfrak{D}x^2}{(\alpha + \beta x)^3} + \text{etc.}$$

Multiply by $\alpha + \beta x$ again and having put

$$\begin{aligned} A'\beta + B'\alpha &= A'' \\ B'\beta + C'\alpha &= B'' \\ C'\beta + D'\alpha &= C'' \\ &\text{etc.} \end{aligned}$$

it will be

$$A'\alpha + A''x + B''x^2 + C''x^3 + \text{etc.} = \mathfrak{B} + \frac{\mathfrak{C}x}{\alpha + \beta x} + \frac{\mathfrak{D}x^2}{(\alpha + \beta x)^2} + \text{etc.}$$

Therefore, let $\mathfrak{B} = A'\alpha$; and arguing exactly as before, if it is

$$\begin{aligned} A''\beta + B''\alpha &= A''' \qquad & A'''\beta + B'''\alpha &= A'''' \\ B''\beta + C''\alpha &= B''' \qquad & B'''\beta + C'''\alpha &= B'''' \\ C''\beta + D''\alpha &= C''' \qquad & C'''\beta + D'''\alpha &= C'''' \\ &\text{etc.} & &\text{etc.,} \end{aligned}$$

it will be $\mathfrak{C} = A''\alpha$, $\mathfrak{D} = A'''\alpha$, $\mathfrak{E} = A''''\alpha$; hence, the sum of the propounded series will be expressed this way that it is

$$s = \frac{A\alpha}{\alpha + \beta x} + \frac{A'\alpha x}{(\alpha + \beta x)^2} + \frac{A'\alpha x^2}{(\alpha + \beta x)^3} + \frac{A'''\alpha x^3}{(\alpha + \beta x)^4} + \text{etc.}$$

This same series would have arisen from the substitution

$$\frac{x}{\alpha + \beta x} = y \quad \text{oder} \quad x = \frac{\alpha y}{1 - \beta y}.$$

§210 This transformation is used with the greatest success, if the propounded series $A + Bx + Cx^2 + \text{etc.}$ was of such a nature that it is finally confounded with a recurring series or better a geometric series arising from the fraction $\frac{P}{\alpha+\beta x}$. For, then the values A', B', C', D' etc. will finally vanish and hence the letters A'', A''', A'''' etc. will even more constitute a highly converging series.

But in similar manner we will be able to use trinomial and any polynomial denominators which will have an extraordinary use, if the propounded series is finally confounded with a recurring series. Therefore, having propounded the series

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

set

$$s = \frac{\mathfrak{A} + \mathfrak{B}x}{\alpha + \beta x + \gamma x^2} + \frac{\mathfrak{A}'x^2 + \mathfrak{B}'x^3}{(\alpha + \beta x + \gamma x^2)^2} + \frac{\mathfrak{A}''x^4 + \mathfrak{B}''x^5}{(\alpha + \beta x + \gamma x^2)^3} + \frac{\mathfrak{A}'''x^6 + \mathfrak{B}'''x^7}{(\alpha + \beta x + \gamma x^2)^4} + \text{etc.}$$

Multiply by $\alpha + \beta x + \gamma x^2$ everywhere and having put

$$\begin{aligned} A\gamma + B\beta + C\alpha &= A' \quad \text{and} \quad \mathfrak{A} = A\alpha \\ B\gamma + C\beta + D\alpha &= B' \quad \text{and} \quad \mathfrak{B} = A\beta + B\alpha \\ C\gamma + D\beta + E\alpha &= C' \end{aligned}$$

an equation similar to the first will arise having divided by xx

$$A' + B'x + C'x^2 + D'x^3 + E'x^4 + \text{etc.}$$
$$= \frac{\mathfrak{A}' + \mathfrak{B}'x}{\alpha + \beta x + \gamma xx} + \frac{\mathfrak{A}'' + \mathfrak{B}''x}{(\alpha + \beta x + \gamma xx)^2} + \frac{\mathfrak{A}''' + \mathfrak{B}'''x}{(\alpha + \beta x + \gamma xx)^3} + \text{etc.}$$

Therefore, if the operation is done as before by putting

$$\begin{aligned} A'\gamma + B'\beta + C'\alpha &= A'' \quad \text{and} \quad \mathfrak{A}' = A'\alpha \\ B'\gamma + C'\beta + D'\alpha &= B'' \quad \text{and} \quad \mathfrak{B} = A'\beta + B'\alpha \\ C'\gamma + D'\beta + E'\alpha &= C'' \\ &\text{etc.} \end{aligned}$$

and further

$$\begin{aligned} A''\gamma + B''\beta + C''\alpha &= A''' \quad \text{and} \quad \mathfrak{A}'' = A''\alpha \\ B''\gamma + C''\beta + D''\alpha &= B''' \quad \text{and} \quad \mathfrak{B} = A''\beta + B''\alpha \\ C''\gamma + D''\beta + E''\alpha &= C''' \\ &\text{etc.} \end{aligned}$$

and by investigating the further values in this manner, it will be

$$s = \frac{A\alpha + (A\beta + b\alpha)x}{\alpha + \beta x + \gamma xx} + \frac{(A'\alpha + (A'\beta + B'\alpha))x^2}{(\alpha + \beta x + \gamma xx)^2} + \frac{(A''\alpha + (A''\beta + B''\alpha))x^4}{(\alpha + \beta x + \gamma xx)^3} + \text{etc.}$$

§211 If one puts $x = 1$, which can be put without loss of generality, because α, β, γ can be taken ad libitum, and it was

$$s = A + B + C + D + E + F + G + \text{etc.},$$

having successively put the following values

$$\begin{array}{ll} A\gamma + B\beta + C\alpha = A' & A'\gamma + B'\beta + C'\alpha = A'' \\ B\gamma + C\beta + D\alpha = B' & B'\gamma + C'\beta + D'\alpha = B'' \qquad \text{and so fourth} \\ C\gamma + D\beta + E\alpha = C' & C'\gamma + D'\beta + E'\alpha = C'' \\ \text{etc.} & \text{etc.} \end{array}$$

but one puts for the sake of brevity

$$\alpha + \beta + \gamma = m,$$

one will obtain the sum of the propounded series expressed this way

$$s = \left\{ \begin{array}{r} (\alpha+\beta)\left(\dfrac{A}{m} + \dfrac{A'}{m^2} + \dfrac{A''}{m^3} + \dfrac{A'''}{m^4} + \text{etc.}\right) \\ +\alpha\left(\dfrac{B}{m} + \dfrac{B'}{m^2} + \dfrac{B''}{m^3} + \dfrac{B'''}{m^4} + \text{etc.}\right) \end{array} \right\}$$

§212 The same denominators consisting of more terms can be taken, and since the operation is easily understood from the preceding, let us only expand the case for the quadrinomial. Therefore, let

$$s = A + B + C + D + E + F + G + \text{etc.}$$

Find the following values

$$\begin{aligned} A\delta + B\gamma + C\beta + D\alpha &= A' \\ B\delta + C\gamma + D\beta + E\alpha &= B' \\ C\delta + D\gamma + E\beta + F\alpha &= C' \\ &\text{etc.} \end{aligned}$$

$$\begin{aligned} A'\delta + B'\gamma + C'\beta + D'\alpha &= A'' \\ B'\delta + C'\gamma + D'\beta + E'\alpha &= B'' \\ C'\delta + D'\gamma + E'\beta + F'\alpha &= C'' \\ &\text{etc.} \end{aligned}$$

$$\begin{aligned} A''\delta + B''\gamma + C''\beta + D''\alpha &= A''' \\ B''\delta + C''\gamma + D''\beta + E''\alpha &= B''' \\ C''\delta + D''\gamma + E''\beta + F''\alpha &= C''' \\ &\text{etc.} \end{aligned}$$

But then let $\alpha + \beta + \gamma + \delta = m$ and it will be

$$s = \begin{Bmatrix} (\alpha+\beta+\gamma)\left(\dfrac{A}{m} + \dfrac{A'}{m^2} + \dfrac{A''}{m^3} + \dfrac{A'''}{m^4} + \text{etc.}\right) \\ (\alpha+\beta)\left(\dfrac{B}{m} + \dfrac{B'}{m^2} + \dfrac{B''}{m^3} + \dfrac{B'''}{m^4} + \text{etc.}\right) \\ +\alpha\left(\dfrac{C}{m} + \dfrac{C'}{m^2} + \dfrac{C''}{m^3} + \dfrac{C'''}{m^4} + \text{etc.}\right) \end{Bmatrix}$$

whence at the same time the progression, if even more parts are attributed to the denominator m, is most clearly seen.

§213 And it is not necessary at all that the denominators of the fractions, to which we reduced the sum of the series, are powers of the same formula

$$\alpha + \beta x + \gamma x^2 + \text{etc.},$$

but this can be varied in the single terms. That this becomes clear, let us at first only take two terms and assume that the series

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

is converted into this series of fractions

$$s = \frac{\mathfrak{A}}{\alpha + \beta x} + \frac{\mathfrak{A}' x}{(\alpha + \beta x)(\alpha' + \beta' x)} + \frac{\mathfrak{A}'' x^2}{(\alpha + \beta x)(\alpha' + \beta' x)(\alpha'' + \beta'' x)} + \text{etc.}$$

At first, multiply both sides by $\alpha + \beta x$ and put

$$\begin{aligned} A\beta + B\alpha &= A' \\ B\beta + C\alpha &= B' \quad \text{and} \quad \mathfrak{A} = A\alpha \\ C\beta + D\alpha &= C' \\ &\text{etc.} \end{aligned}$$

and having divided by x it will be

$$A' + B' + C'x^2 + D'x^3 + \text{etc.} = \frac{\mathfrak{A}}{\alpha' + \beta' x} + \frac{\mathfrak{A}'' x}{(\alpha' + \beta' x)(\alpha'' + \beta'' x)} + \text{etc.}$$

Further, in the same manner by multiplying by $\alpha' + \beta' x$ and then by $\alpha'' + \beta'' x$ and so forth, if one sets

$$\begin{array}{lll} A'\beta' + B'\alpha' = A'' & A''\beta'' + B''\alpha'' = A''' & A'''\beta''' + B'''\alpha''' = A'''' \\ B'\beta' + C'\alpha' = B'' & B''\beta'' + C''\alpha'' = B''' & B'''\beta''' + C'''\alpha''' = B'''' \\ C'\beta' + D'\alpha' = C'' & C''\beta'' + D''\alpha'' = C''' & C'''\beta''' + D'''\alpha''' = C'''' \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array} \quad \text{etc.}$$

it will be

$$\mathfrak{A}' = A'\alpha', \quad \mathfrak{A}'' = A''\alpha'', \quad \mathfrak{A}''' = A'''\alpha''' \quad \text{etc.}$$

and hence the propounded series will be converted into this one

$$s = \frac{A\alpha}{\alpha + \beta x} + \frac{A'\alpha' x}{(\alpha + \beta x)(\alpha' + \beta' x)} + \frac{A''\alpha'' x}{(\alpha + \beta x)(\alpha' + \beta' x)(\alpha'' + \beta'' x)} + \text{etc.},$$

where the values α, β, α', β', α'', β'' etc. are arbitrary, but can be taken in such a way for each case that this new series is highly convergent.

§214 Let us apply this also to trinomial factors and having propounded any series $s = A + B + C + D + E + F + G + \text{etc.}$ let

$$\begin{array}{ll} A\gamma + B\beta + C\alpha = A' & A'\gamma' + B'\beta' + C'\alpha' = A'' \\ B\gamma + C\beta + D\alpha = B' & B'\gamma' + C'\beta' + D'\alpha' = B'' \\ C\gamma + D\beta + E\alpha = C' & C'\gamma' + D'\beta' + E'\alpha' = C'' \\ \text{etc.} & \text{etc.} \end{array}$$

$$\begin{array}{ll} A''\gamma' + B''\beta'' + C''\alpha'' = A''' & A'''\gamma''' + B'''\beta''' + C'''\alpha''' = A'''' \\ B''\gamma'' + C''\beta'' + D''\alpha'' = B''' & B'''\gamma''' + C'''\beta''' + D'''\alpha''' = B'''' \\ C''\gamma'' + D''\beta'' + E''\alpha'' = C''' & C'''\gamma''' + D'''\beta''' + E'''\alpha''' = C'''' \\ \text{etc.} & \text{etc.} \end{array}$$

Further, for the sake of brevity put

$$\begin{aligned}
\alpha \quad + \beta \quad + \gamma \quad &= m \\
\alpha' \quad + \beta' \quad + \gamma' \quad &= m' \\
\alpha'' \quad + \beta'' \quad + \gamma'' \quad &= m'' \\
\alpha''' + \beta''' + \gamma''' &= m''' \\
&\text{etc.}
\end{aligned}$$

and the sum of the propounded series will be

$$s = \frac{\alpha(A+B)}{m} + \frac{\alpha'(A'+B')}{mm'} + \frac{\alpha''(A''+B'')}{mm'm''} + \frac{\alpha'''(A'''+B''')}{mm'm''m'''} + \text{etc.}$$
$$+ \frac{\beta A}{m} + \frac{\beta' A'}{mm'} + \frac{\beta'' A''}{mm'm''} + \frac{\beta''' A'''}{mm'm''m'''} + \text{etc.}$$

§215 Since these extend so far that their use can be seen less clearly, let us restrict ourselves to the case of the transformation given in § 213 and let be $x = -1$ that one has this series

$$s = A - B + C - D + E - F + G - \text{etc.}$$

and set

$$\begin{array}{llll}
B - A = A' & B' - 2A' = A'' & B'' - 3A'' = A''' & B''' - 4A''' = A'''' \\
C - B = B' & C' - 2B' = B'' & C'' - 3B'' = B''' & C''' - 4B''' = B'''' \\
D - C = C' & D' - 2C' = C'' & D'' - 3C'' = C''' & D''' - 4C''' = C'''' \\
\text{etc.} & \text{etc.} & \text{etc.} & \text{etc.}
\end{array}$$

Having found these values the sum of the propounded series will be equal to the following series

$$s = \frac{A}{2} - \frac{A'}{2 \cdot 3} + \frac{A''}{2 \cdot 3 \cdot 4} - \frac{A'''}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{A''''}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \text{etc.}$$

Therefore, any propounded series can be transformed into innumerable other ones equal to it, between which without any doubt a most convergent series will be found by means of which the propounded sum can be found approximately.

§216 But let us return to the invention of series whose law of progression the differential calculus declares. Therefore, because this was already archived for algebraic quantities, let us proceed to transcendental functions and let the series equal to this logarithm be in question

$$s = \ln(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.});$$

assume that the series in question is this one

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \mathfrak{F}x^6 + \text{etc.}$$

Therefore, because from the differentiation of the first equation it follows

$$\frac{ds}{dx} = \frac{\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \text{etc.}}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}},$$

it will be

$$(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \text{etc.})\frac{ds}{dx} = \alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + \text{etc.}$$

But since from the assumed equation it is

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \text{etc.},$$

having done the substitution this equation arises

$$\begin{array}{rrrrrr}
\mathfrak{A} & + 2\mathfrak{B}x + & 3\mathfrak{C}x^2 + & 4\mathfrak{D}x^3 + & 5\mathfrak{E}x^4 & + \text{etc.} \\
 & + \mathfrak{A}\alpha & + 2\mathfrak{B}\alpha & + 3\mathfrak{C}\alpha & + 4\mathfrak{D}\alpha & + \text{etc.} \\
 & & + \mathfrak{A}\beta & + 2\mathfrak{B}\beta & + 3\mathfrak{C}\beta & + \text{etc.} \\
 & & & + \mathfrak{A}\gamma & + 2\mathfrak{B}\gamma & + \text{etc.} \\
 & & & & + \mathfrak{A}\delta & + \text{etc.} \\
\hline
= \alpha + & 2\beta x + & 3\gamma x^2 + & 4\delta x^3 + & 5\varepsilon x^4 & + \text{etc.}
\end{array}$$

From it one obtains the following determinations

$$\mathfrak{A} = \alpha$$
$$\mathfrak{B} = -\frac{1}{2}\mathfrak{A}\alpha + \beta$$
$$\mathfrak{C} = -\frac{2}{3}\mathfrak{B}\alpha - \frac{1}{3}\mathfrak{A}\beta + \gamma$$
$$\mathfrak{D} = -\frac{3}{4}\mathfrak{C}\alpha - \frac{2}{4}\mathfrak{B}\beta - \frac{1}{4}\mathfrak{A}\gamma + \delta$$
$$\mathfrak{E} = -\frac{4}{5}\mathfrak{D}\alpha - \frac{3}{5}\mathfrak{C}\beta - \frac{2}{5}\mathfrak{B}\gamma - \frac{1}{5}\mathfrak{A}\delta + \text{etc.}$$
$$\text{etc.}$$

§217 Now, let this exponential quantity be propounded

$$s = e^{\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}}$$

in which e denotes the number whose hyperbolic logarithm is $= 1$, and assume this series in question

$$s = 1 + \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \text{etc.}$$

For, from the case $x = 0$ it is plain that the first term must be the unity. Therefore, since it is by taking logarithms

$$\ln s = \alpha + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \text{etc.},$$

it will be having taken the differentials

$$\frac{ds}{dx} = s(\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \text{etc.})$$

But from the assumed equation it will be

$$\begin{array}{rcccccc}
\frac{ds}{dx} = \mathfrak{A} & + 2\mathfrak{B}x & + 3\mathfrak{C}x^2 & + 4\mathfrak{D}x^3 & + 5\mathfrak{E}x^4 & + \text{etc.} \\
\hline
= \alpha & + \mathfrak{A}\alpha x & + \mathfrak{B}\alpha x^2 & + \mathfrak{C}\alpha x^3 & + \mathfrak{D}\alpha x^4 & + \text{etc.} \\
 & + 2\beta & + 2\mathfrak{A}\beta & + 2\mathfrak{C}\beta & + 2\mathfrak{C}\beta & + \text{etc.} \\
 & & + 3\gamma & + 3\mathfrak{A}\gamma & + 3\mathfrak{B}\gamma & + \text{etc.} \\
 & & & + 4\delta & + 4\mathfrak{A}\delta & + \text{etc.} \\
 & & & & + 5\varepsilon & + \text{etc.},
\end{array}$$

from which the following determination of the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. arise

$$\begin{aligned}
\mathfrak{A} &= \alpha \\
\mathfrak{B} &= \beta + \frac{1}{2}\mathfrak{A}\beta \\
\mathfrak{C} &= \gamma + \frac{2}{3}\mathfrak{A}\beta + \frac{1}{3}\mathfrak{B}\alpha \\
\mathfrak{D} &= \delta + \frac{3}{4}\mathfrak{A}\gamma + \frac{2}{4}\mathfrak{B}\beta + \frac{1}{4}\mathfrak{C}\alpha \\
\mathfrak{E} &= \varepsilon + \frac{4}{5}\mathfrak{A}\delta + \frac{3}{5}\mathfrak{B}\gamma + \frac{2}{5}\mathfrak{C}\beta + \frac{1}{5}\mathfrak{D}\alpha \\
&\text{etc.}
\end{aligned}$$

§218 If also the arc, whose sine or cosine is in question, is expressed by a binomial or polynomial or even an infinite series, this way one can even express its sine and cosine by means of an infinite series. But to do this the most convenient way, it does not suffice to proceed to the first differentials, but it is necessary, that the differentials of the second grade are used. Therefore, let

$$s = \sin(\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.})$$

and assume the series in question to be

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \text{etc.}$$

For, it is plain that the first term vanishes; but since one has to descend to the second differentials, the coefficient $\mathfrak{A}$ also has to be determined from elsewhere, which will happen, if we put x infinitely small. For then, because of the arc $= \alpha x$ the sine itself will become equal to it and it will therefore be $\mathfrak{A} = \alpha$. Now, for the sake of brevity let us put

$$z = \alpha x + \beta x^2 + \gamma x^3 + \text{etc.},$$

that it is $s = \sin z$; by differentiating it will be $ds = dz \cos z$ and by differentiating again it will be $dds = ddz \cos z - dz^2 \sin z$. Therefore, since it is $\sin z = s$ and $\cos z = \frac{ds}{dz}$, it will be

$$dds = \frac{dsddz}{dz} - sdz^2 \quad \text{and} \quad dzdds + sdz^3 = dsddz.$$

§219 Let us put that the arc z is only expressed by a binomial and it is

$$z = \alpha x + \beta x^2;$$

it will be

$$dz = (\alpha + 2\beta x)dx$$

and having put dx constant

$$ddz = 2\beta dx^2$$

and

$$dz^3 = (\alpha^3 + 6\alpha^2\beta x + 12\alpha\beta^2 x^2 + 8\beta^3 x^3)dx^3.$$

Further, because of $s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \text{etc.}$ it will be

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + \text{etc.}$$

and

$$\frac{dds}{dx^2} = 2\mathfrak{B} + 6\mathfrak{C}x + 12\mathfrak{D}x^2 + \text{etc.}$$

Having substituted these values in the differential equation it will be

$$\begin{array}{rcrcrcrcrcrl}
\dfrac{dzdds}{dx^3} & = 1\cdot 2\mathfrak{B}\alpha & + & 2\cdot 3\mathfrak{C}\alpha x & + & 3\cdot 4\mathfrak{D}\alpha x^2 & + & 4\cdot 5\mathfrak{E}\alpha x^3 & + & 5\cdot 6\mathfrak{F}\alpha x^4 & + \text{etc.} \\
 & & + & 2\cdot 1\cdot 2\mathfrak{B}\beta & + & 2\cdot 2\cdot 3\mathfrak{C}\beta & + & 2\cdot 3\cdot 4\mathfrak{D}\beta & + & 2\cdot 4\cdot 5\mathfrak{E}\beta & + \text{etc.} \\
\dfrac{sdz^3}{dx^3} & = & + & \mathfrak{A}\alpha^3 & + & \mathfrak{B}\alpha^3 & + & \mathfrak{C}\alpha^3 & + & \mathfrak{D}\alpha^3 & + \text{etc.} \\
 & & & & + & 6\mathfrak{A}\alpha^2\beta & + & 6\mathfrak{B}\alpha^2\beta & + & 6\mathfrak{C}\alpha^2\beta & + \text{etc.} \\
 & & & & & & + & 12\mathfrak{A}\alpha\beta^2 & + & 12\mathfrak{B}\beta^2 & + \text{etc.} \\
 & & & & & & & & + & 8\mathfrak{A}\beta^3 & + \text{etc.} \\
\hline
\dfrac{dsddz}{dx^3} & = \quad 2\mathfrak{A}\beta & + & 4\mathfrak{B}\beta & + & 6\mathfrak{C}\beta & + & 8\mathfrak{D}\beta & + & 10\mathfrak{E}\beta & + \text{etc.}
\end{array}$$

Hence, the coefficients will be defined the following way:

$$\begin{array}{lcl}
\mathfrak{B} & = & \dfrac{2\mathfrak{A}\beta}{2\alpha} \\
\mathfrak{C} & = & 0 - \dfrac{\mathfrak{A}\alpha^2}{2\cdot 3} \\
\mathfrak{D} & = & -\dfrac{2\mathfrak{C}\beta}{4\alpha} - \dfrac{6\mathfrak{A}\alpha\beta}{3\cdot 4} - \dfrac{\mathfrak{B}\alpha^2}{3\cdot 4} \\
\mathfrak{E} & = & -\dfrac{4\mathfrak{D}\beta}{5\alpha} - \dfrac{12\mathfrak{A}\beta^2}{4\cdot 5} - \dfrac{6\mathfrak{B}\alpha\beta}{4\cdot 5} - \dfrac{\mathfrak{C}\alpha^2}{4\cdot 5} \\
\mathfrak{F} & = & -\dfrac{6\mathfrak{E}\beta}{6\alpha} - \dfrac{8\mathfrak{A}\beta^3}{5\cdot 6\alpha} - \dfrac{12\mathfrak{B}\beta\beta}{5\cdot 6} - \dfrac{6\mathfrak{C}\alpha\beta}{5\cdot 6} - \dfrac{\mathfrak{D}\alpha^2}{5\cdot 6} \\
\mathfrak{G} & = & -\dfrac{8\mathfrak{F}\beta}{7\alpha} - \dfrac{8\mathfrak{B}\beta^3}{6\cdot 7\alpha} - \dfrac{12\mathfrak{C}\beta\beta}{6\cdot 7} - \dfrac{6\mathfrak{D}\alpha\beta}{6\cdot 7} - \dfrac{\mathfrak{E}\alpha^2}{6\cdot 7} \\
 & & \text{etc.}
\end{array}$$

Having found these values it will be

$$\sin(\alpha x + \beta x^2) = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \text{etc.}$$

while $\mathfrak{A} = \alpha$.

§220 In similar manner the cosine of any angle is converted into a series; but since an arc is very rarely expressed by a polynomial, let us show the use of differential equations for the invention of the series for the cosine of the arc x. Therefore, let $s = \cos x$ and assume

$$s = 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \mathfrak{D}x^8 - \text{etc.}$$

Since it is $ds = -dx \sin x$ and $dds = -dx^2 \cos x = -sdx^2$, it will be

$$dds + sdx^2 = 0;$$

having done the substitution it will be

$$\begin{array}{rl}
\dfrac{dds}{dx^2} = & -1\cdot 2\mathfrak{A} + 3\cdot 4\mathfrak{B}x^2 - 5\cdot 6\mathfrak{C}x^4 + 7\cdot 8\mathfrak{D}x^6 - \text{etc.} \\
s = & \quad 1 - \quad \mathfrak{A}x^2 \quad - \mathfrak{B}x^4 - \quad \mathfrak{C}x^6 + \text{etc.}
\end{array}$$

and by comparing the coefficients it follows

$$\mathfrak{A} = \frac{1}{1\cdot 2}$$
$$\mathfrak{B} = \frac{\mathfrak{A}}{3\cdot 4} = \frac{1}{1\cdot 2\cdot 3\cdot 4}$$
$$\mathfrak{C} = \frac{\mathfrak{B}}{5\cdot 6} = \frac{1}{1\cdot 2\cdot 3\cdots 6}$$
$$\mathfrak{D} = \frac{\mathfrak{C}}{7\cdot 8} = \frac{1}{1\cdot 2\cdot 3\cdots 8}$$
etc.

Therefore, it is plain, what we demonstrated in more detail above already, that it is

$$\cos x = 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdots 6} + \frac{x^8}{1\cdot 2\cdot 3\cdots 8} - \text{etc.};$$

the first series for the sine having put $\beta = 0$ and $\alpha = 1$ will give

$$\sin x = \frac{x}{1} - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\cdots 7} + \frac{x^9}{1\cdot 2\cdot 3\cdots 9} - \text{etc.}$$

§221 From the well-known series for the sine and cosine the series for the tangent, cotangent, secant, cosecant of a certain angle are deduced. For, the tangent arises, if the sine is divided by the cosine, the cotangent, if the cosine is divided by the sine, the secant, if the radius 1 by the cosine, and the cosecant, if the radius by the sine. But the series arising from these divisions seem to be most irregular; but, with the exception of the series exhibiting the secant, all remaining ones by means of the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. can be reduced to a simple law of progression. For, since we found above (§ 127) that it is

$$\frac{\mathfrak{A}u^2}{1\cdot 2} + \frac{\mathfrak{B}u^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\mathfrak{C}u^6}{1\cdot 2\cdot 3\cdots 6} + \frac{\mathfrak{D}u^8}{1\cdot 2\cdot 3\cdots 8} + \text{etc.} = 1 - \frac{u}{2}\cot\frac{1}{2}u,$$

having put $\frac{1}{2}u = x$ it will be

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1\cdot 2} - \frac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^6\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \frac{2^8\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} - \text{etc.},$$

and if one puts $\frac{1}{2}x$ for x, it will be

$$\cot\frac{1}{2}x=\frac{2}{x}-\frac{2\mathfrak{A}x}{1\cdot 2}-\frac{2\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4}-\frac{2\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6}-\frac{2^{\mathfrak{D}}x^7}{1\cdot 2\cdot 3\cdots 8}-\text{etc.},$$

§222 But hence the tangent of any arc will be expressed by means of an infinite series the following way. Because it is

$$\tan 2x=\frac{2\tan x}{1-\tan x},$$

it will be

$$\cot 2x=\frac{1}{2\tan x}-\frac{\tan x}{2}=\frac{1}{2}\cot x-\frac{1}{2}\tan x$$

and hence

$$\tan x=\cot x-2\cot 2x.$$

Therefore, because it is

$$\cot x=\frac{1}{x}-\frac{2^2\mathfrak{A}x}{1\cdot 2}-\frac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4}-\frac{2^6\mathfrak{C}x^5}{1\cdot 2\cdots 6}-\frac{2^8\mathfrak{D}x^7}{1\cdot 2\cdots 8}-\text{etc.},$$
$$2\cot 2x=\frac{1}{x}-\frac{2^4\mathfrak{A}x}{1\cdot 2}-\frac{2^8\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4}-\frac{2^{12}\mathfrak{C}x^5}{1\cdot 2\cdots 6}-\frac{2^{16}\mathfrak{D}x^7}{1\cdot 2\cdots 8}-\text{etc.},$$

by subtracting this series from the first it will be

$$\tan x=\frac{2^2(2^2-1)\mathfrak{A}x}{1\cdot 2}+\frac{2^4(2^4-1)\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4}+\frac{2^6(2^6-1)\mathfrak{C}x^5}{1\cdot 2\cdots 6}+\frac{2^8(2^8-1)\mathfrak{D}x^7}{1\cdot 2\cdots 8}+\text{etc.}$$

Therefore, if here the numbers A, B, C, D etc. found in § 182 are introduced, it will be

$$\tan x=\frac{2Ax}{1\cdot 2}+\frac{2^3Bx^3}{1\cdot 2\cdot 3\cdot 4}+\frac{2^5Cx^5}{1\cdot 2\cdot 3\cdot 6}+\frac{2^7Dx^7}{1\cdot 2\cdots 8}+\text{etc.}$$

§223 But the cosecant will be found the following way. Since it is

$$\cot x = \tan x + 2\cot 2x = \frac{1}{\cot x} + 2\cot 2x,$$

it will be

$$\cot^2 x = 2\cot x \cot 2x + 1$$

and having extracted the root

$$\cot x = \cot 2x + \csc 2x,$$

whence it is

$$\csc 2x = \cot x - \cot 2x$$

and having put x for $2x$

$$\csc x = \cot\frac{1}{2}x - \cot x.$$

Hence, because we have the cotangent, namely

$$\begin{aligned}
\cot\frac{1}{2}x &= \frac{2}{x} - \frac{2\mathfrak{A}x}{1\cdot 2} - \frac{2\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2\mathfrak{C}x^5}{1\cdot 2\cdots 6} - \text{etc.}\\
\cot x &= \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1\cdot 2} - \frac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^5\mathfrak{C}x^5}{1\cdot 2\cdots 6} - \text{etc.},
\end{aligned}$$

having subtracted this series from the first, it will be

$$\csc x = \frac{1}{x} + \frac{2(2-1)\mathfrak{A}x}{1\cdot 2} + \frac{2(2^3-1)\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} + \frac{2(2^5-1)\mathfrak{C}x^5}{1\cdot 2\cdots 6} + \text{etc.}$$

§224 But the secant cannot be expressed by means of these Bernoulli numbers, but it requires other numbers which go into the sums of the odd powers of the reciprocals. For, if one puts

$$
\begin{aligned}
1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.} &= \alpha \cdot \frac{\pi}{2^2}\\
1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\text{etc.} &= \frac{\beta}{1\cdot 2} \cdot \frac{\pi^3}{2^4}\\
1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\text{etc.} &= \frac{\gamma}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{\pi^5}{2^6}\\
1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\text{etc.} &= \frac{\delta}{1\cdot 2\cdots 6} \cdot \frac{\pi^7}{2^8}\\
1-\frac{1}{3^9}+\frac{1}{5^9}-\frac{1}{7^9}+\frac{1}{9^9}-\text{etc.} &= \frac{\delta}{1\cdot 2\cdots 8} \cdot \frac{\pi^9}{2^{10}}\\
1-\frac{1}{3^{11}}+\frac{1}{5^{11}}-\frac{1}{7^{11}}+\frac{1}{9^{11}}-\text{etc.} &= \frac{\delta}{1\cdot 2\cdots 10} \cdot \frac{\pi^{11}}{2^{12}}\\
&\text{etc.,}
\end{aligned}
$$

it will be

$$
\begin{aligned}
\alpha &= 1\\
\beta &= 1\\
\gamma &= 5\\
\delta &= 61\\
\varepsilon &= 1385\\
\zeta &= 50521\\
\eta &= 2702765\\
\theta &= 199360981\\
\iota &= 19391512145\\
\varkappa &= 2040487967544 1\text{etc.}
\end{aligned}
$$

and from these values one will obtain

$$
\sec x = \alpha + \frac{\beta}{1\cdot 2}xx + \frac{\gamma}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{\delta}{1\cdot 2\cdots 6}x^6 + \frac{\varepsilon}{1\cdot 2\cdots 8}x^8 + \text{etc.}
$$

§225 To show the connection of this series to the numbers α, β, γ, δ etc., let us consider the series treated above [§ 33]

$$\frac{\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{m+n} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.}$$

Put $m = \frac{1}{2}n - k$ and it will be

$$\frac{\pi}{2n\cos\frac{k}{n}} = \frac{1}{n-2k} + \frac{1}{n+2k} - \frac{1}{3n-2k} - \frac{1}{3n+2k} + \frac{1}{5n-2k} + \text{etc.}$$

Let $\frac{k\pi}{n} = x$ or $k\pi = nx$; it will be

$$\frac{\pi}{2n}\sec x = \frac{\pi}{n\pi-2nx} + \frac{\pi}{n\pi+2nx} - \frac{\pi}{3n\pi-2nx} - \frac{\pi}{3n\pi+2nx} + \frac{\pi}{5n\pi-2nx} + \text{etc.}$$

or

$$\sec x = \frac{2}{\pi-2x} + \frac{2}{\pi+2x} - \frac{2}{3\pi-2x} - \frac{2}{3\pi+2x} + \frac{2}{5\pi-2x} + \text{etc.}$$

or

$$\sec x = \frac{4\pi}{\pi^2-4x^2} - \frac{4\cdot 3\pi}{9\pi^2-4xx} + \frac{4\cdot 5\pi}{25\pi^2-4xx} - \frac{4\cdot 7\pi}{49\pi^2-x^2} + \text{etc.}$$

If now the single terms are converted into series, it will be

$$\begin{aligned} \sec x = {} & \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.}\right) \\ & + \frac{2^4x^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.}\right) \\ & + \frac{2^6x^4}{\pi^5}\left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.}\right) \\ & \text{etc.;} \end{aligned}$$

if instead of these series the values assigned above are substituted, the same series we gave will arise for the secant.

§226 Hence, at the same time the law is plain, according to which the numbers α, β, γ, δ etc., by means of which the sums of the odd powers are constituted, proceed. For, because it is

$$\sec x = \frac{1}{\cos x} = \alpha + \frac{\beta}{1\cdot 2}x^2 + \frac{\gamma}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{\delta}{1\cdot 2\cdots 6}x^6 + \text{etc.},$$

it is necessary that this series is equal to the fraction

$$\frac{1}{1 - \frac{xx}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdots 6} + \frac{x^6}{1\cdot 2\cdots 8} - \text{etc.}};$$

therefore, having equated the two expressions it will be

$$\begin{aligned} 1 = \alpha &+ \frac{\beta}{1\cdot 2}x^2 + \frac{\gamma}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{\delta}{1\cdot 2\cdots 6}x^6 + \frac{\varepsilon}{1\cdot 2\cdots 8}x^8 + \text{etc.} \\ &- \frac{\alpha}{1\cdot 2} \quad - \frac{\beta}{1\cdot 2\cdot 1\cdot 2} \quad - \frac{\gamma}{1\cdot 2\cdot 1\cdots 4} \quad - \frac{\delta}{1\cdot 2\cdot 1\cdots 6} \quad - \text{etc.} \\ &\qquad + \frac{\alpha}{1\cdot 2\cdot 3\cdot 4} \quad + \frac{\beta}{1\cdots 4\cdot 1\cdot 2} \quad + \frac{\gamma}{1\cdots 4\cdot 1\cdots 4} \quad - \text{etc.} \\ &\qquad\qquad - \frac{\alpha}{1\cdot 2\cdots 6} \quad + \frac{\beta}{1\cdots 6\cdot 1\cdots 2} \quad - \text{etc.} \\ &\qquad\qquad\qquad + \frac{\alpha}{1\cdot 2\cdots 8} \quad + \text{etc.}, \end{aligned}$$

whence these equations follow

$$\begin{aligned} \alpha &= 1 \\ \beta &= \frac{2\cdot 1}{1\cdot 2}\alpha \\ \gamma &= \frac{4\cdot 3}{1\cdot 2}\beta - \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}\alpha \\ \delta &= \frac{6\cdot 5}{1\cdot 2}\gamma - \frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}\beta + \frac{6\cdots 1}{1\cdots 6}\alpha \\ \varepsilon &= \frac{8\cdot 7}{1\cdot 2}\delta - \frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\gamma + \frac{8\cdots 3}{1\cdots 6}\beta - \frac{8\cdots 1}{1\cdots 8}\alpha \\ &\text{etc.} \end{aligned}$$

And from these formulas the values of these letters were found which we exhibited in § 224 and by means of which the sums of the series contained in this form

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \text{etc.},$$

if n was an odd number, can be expressed.