On the expression of integrals by means of factors*

Leonhard Euler

Geometers until now have mainly studied the reduction of integrals to infinite series for two reasons: First, to understand the nature of series more clearly and secondly because of the immense use, which series have to find the values of integrals approximately. But now in Volume 11 of the Novi Commentarii academiae scientiarum Petropolitanae[1], motivated by the same reasons, I showed that the reduction of integrals to infinte products is not less worthy to be developed carefully, and there I already gave many specimens of these reductions, which seem to have a use not to be contemned in the whole field of Analysis, even though the treatment was not sufficiently fleshed out and structured very well at that time. Therefore, it is advisable to resume this subject here; first, I will explain the foundations, on which it is based, more diligently, but then expand many cases, which seem to be especially memorable, more accurately.

But it should especially be noted that it is not possible to express integrals this way in general in such a way that it equally extends to all values, for which task infinite series are more appropriate, but products can only be used conveniently then, if the value of the integral is investigated only, if a certain determined value is attributed to the variable. And it is indeed not possible to assume this value arbitrarily, but it must rather be of such a nature, that it already enjoys a special property in the differential, namely, that the

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[1]Euler refers to his paper "De productis ex infinitis factoribus ortis". This is paper E122 in the Eneström-Index.

differential becomes either zero or infinity for this case.

But cases of this kind are already especially remarkable with respect to the remaining ones and are usually sought after in the practical applications, since in most cases the question is already about calculating the integrals for a certain values of this kind. As if the quadrature of the circle is in consideration, or the value of this formula $\int \frac{dx}{\sqrt{1-xx}}$ is in question in the case $x = 1$, or the value of the formula $\frac{dx}{1+xx}$ is in question in the case $x = \infty$; but in the last formula the differential becomes infinite, in the first on the other hand it vanishes for the given value of x.

Therefore, to cover the subject in more generality, I will expand formulas of two kinds here, which are

$$\int x^{m-1}dx(1-x^n)^k \quad \text{and} \quad \int \frac{x^{m-1}dx}{(1+x^n)^k},$$

both of which I assume to be integrated in such a way that they vanish for $x = 0$. But then it is the idea to determine the only value of the first integral $\int x^{m-1}dx(1-x^n)^k$ for the value $x = 1$; but then I will only investigate that value of the integral $\int \frac{x^{m-1}dx}{(1+x^n)^k}$ for the case $x = \infty$. But it is evident that these cases of the integrals with respect to the remaining ones enjoy such an eminent prerogative that they especially deserve it to be expanded.

Although seeking for elegance here I omitted the coefficients, it is nevertheless plain that these formulas extend equally far as if coefficients would have been added. For, a formula of this kind $\int \gamma y^{m-1}dx(\alpha - \beta y^n)^k$ having put $\frac{\beta y^n}{\alpha} = x^n$ is obviously reduced to the one under consideration, $\int x^{m-1}dx(1-x^n)^k$, and therefore is seen not to extend further and by a similar reduction this formula $\int \frac{\gamma y^{m-1}dy}{(\alpha+\beta y^n)^k}$ is contained in the other $\int \frac{x^{m-1}dx}{(1+x^n)^k}$, whence it would be completely superfluous to want to use these more complicated formulas instead of our formulas expressed in simpler manner.

But even the one of the formulas we want to consider here is contained in the other, such that it suffices to have treated only the one of them.

For, if one puts $x = \frac{y}{(1+y^n)^{\frac{1}{n}}}$, it will be

$$1 - x^n = \frac{1}{1+y^n}, \quad x^m = \frac{y^m}{(1+y^n)^{\frac{m}{n}}} \quad \text{and} \quad \frac{dx}{x} = \frac{dy}{y()1+y^n)};$$

having substituted these values one will obtain

$$\int x^{m-1}dx(1-x^n)^k = \int \frac{y^{m-1}dy}{(1+y^n)^{k+1+\frac{m}{n}}}$$

having taken these integrals in such a way that they vanish having put $x=0$ and $y=0$ which condition is always to be understood to be fulfilled in the following. Therefore, because having put $y=\infty$ it is $x=1$, we will have the following theorem.

THEOREM 1

§1 *The value of this integral formula*

$$\int x^{m-1}dx(1-x^n)^k$$

in the case $x=1$ is equal to the value of this integral formula

$$\int \frac{y^{m-1}dy}{(1+y^n)^{k+1+\frac{m}{n}}}$$

in the case $y=\infty$.

The reason for this this equality is that first form is actually transformed into the second, if one puts $x=\frac{y}{(1+y^n)^{\frac{1}{n}}}$.

The following theorem, which results from a similar reduction, also has a lot of utility; therefore, I state it together with its proof.

THEOREM 2

§2 *The value of this integral formula*

$$\int x^{m-1}dx(1-x^n)^k$$

in the case $x=1$ is equal to the value of this integral formula

$$\int y^{nk+n-1}dy(1-y^n)^{\frac{m-n}{n}}$$

in the case $y=1$.

Put $x=(1-y^n)^{\frac{1}{n}}$ that it is

$$1 - x^n = y^n, \quad x^m = (1-y^n)^{\frac{m}{n}} \quad \text{and} \quad \frac{dx}{x} = \frac{-y^{n-1}dy}{1-y^n};$$

having substituted these values one will have

$$x^{m-1}dx(1-x^n)^k = -y^{nk+n-1}dy(1-y^n)^{\frac{m-n}{n}}.$$

Let

$$Y = \int y^{nk+n-1}dy(1-y^n)^{\frac{m-n}{n}}$$

having taken the integral in such a way that it vanishes having put $y = 0$; but then having put $y = 1$ let Y go over into A. Since now those formulas must be integrated in such a way that they vanish having put $x = 0$, in which case it is $y = 1$, it will be

$$\int x^{m-1}dx(1-x^n)^k = A - Y.$$

Now put $x = 1$, in which case it is $y = 0$ and hence $Y = 0$, and our integral formula will become $= A$ or the integral $\int x^{m-1}dx(1-x^n)^k$ in the case $x = 1$ will be equal to the integral $\int y^{nk+n-1}dy(1-y^n)^{\frac{m-n}{n}}$ in the case $y = 1$. Q.E.D.

COROLLARY 1

§3 Therefore, since these three formulas

$$\text{I.} \quad \int x^{m-1}dx(1-x^n)^k, \qquad \text{II.} \quad \int \frac{y^{m-1}dy}{(1+y^n)^{k+1+\frac{m}{n}}}, \qquad \text{III.} \quad \int z^{nk+n-1}dz(1-z^n)^{\frac{m-n}{n}}$$

depend on each other in such a way that the first goes over into the second having put $x = \frac{y}{(1+y^n)^{\frac{1}{n}}}$, but having put $x = (1-z^n)^{\frac{1}{n}}$ it goes over into the third one taken negatively, it is obvious that, if one of these forms was integrable absolutely, also the remaining two will be absolutely integrable.

COROLLARY 2

§4 But the first is absolutely integrable, as it is perspicuous per se, if k is a positive integer number, whatever number is put for m. Nevertheless, only the cases are excluded, in which m becomes equal to a certain number of this progression

$$0, \quad -n, \quad -2n, \quad -3n, \cdots, -kn;$$

for, in these cases a part of the integral will depend on logarithms. Therefore, these cases to be excluded reduce to this that the absolute integration succeeds while k is a positive integer number, if $-\frac{m}{n}$ is a positive integer number either smaller than k or equal to k, or if $k + \frac{m}{n}$ is not a positive integer not greater than k.

COROLLARY 3

§5 In like manner the second form will be integrable, if $-k-1-\frac{m}{n}$ was a positive integer number, say i; but here the cases, in which $-\frac{m}{n}$ equally is a positive integer number not greater than i, are excluded. Or if ω denotes an arbitrary positive integer number of this series $0, 1, 2, \cdots i$, the cases, in which it is $-\frac{m}{n} = \omega$, are excluded.

COROLLARY 4

§6 But the third formula will be absolutely integrable, if $\frac{m-n}{n}$ was a positive integer number, say i; but the cases, in which it is $-k-1=\omega$, while ω denotes any arbitrary positive integer not greater than i, are excluded.

COROLLARY 5

§7 Therefore, having noted this the formula $\int x^{m-1}dx(1-x^n)^k$ will be integrable absolutely in the following cases, in which i denotes an arbitrary positive number, but ω denotes an arbitrary positive integer number not greater than i:

I. If $k = i$ and nevertheless not $-\dfrac{m}{n} = \omega$.

II. If $-k-1-\dfrac{m}{n} = i$ and nevertheless not $-\dfrac{m}{n} = \omega \quad (\text{or} \quad -k-1=\omega)$.

III. If $\dfrac{m-n}{n} = i$ and nevertheless not $-k-1=\omega$.

Corollary 6

§8 But it is obvious that the cases of integrability will be the same in this further extending formula $\int x^{m-1}dx(a+bx^n)^k$, for which the proof is equally valid. And from these three conditions the cases of integrability of all formulas of this kind can be distinguished.

Although these things are not the main subjects I want to study in this paper, it nevertheless, since they easily follow from the two theorems mentioned in advance, did not seem to be out of place to add them here. Therefore, now I proceed to the true foundation of the subject, which is based on the reduction of the integrals to other forms. To explain this more distinctly, I contemplate this algebraic form

$$x^\alpha(1-x^n)^\gamma = P;$$

having differentiated this form I obtain

$$dP = \alpha x^{\alpha-1}dx(1-x^n)^\gamma - \gamma n x^{\alpha+n-1}dx(1-x^n)^{\gamma-1},$$

which can still in other ways be split into two terms, as, e.g.,

$$dP = \alpha x^{\alpha-1}dx(1-x^n)^{\gamma-1} - (\alpha+\gamma n)x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}.$$

But then if in the last term one writes $1-(1-x^n)$ for x^n, the first form will give

$$dP = (\alpha+\gamma n)x^{\alpha-1}dx(1-x^n)^\gamma - \gamma n x^{\alpha-1}dx(1-x^n)^{\gamma-1},$$

but the second on the other hand reduces to the same. Hence by integrating we will obtain

$$\begin{aligned}
P &= \alpha\int x^{\alpha-1}dx(1-x^n)^\gamma - \gamma n\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1},\\
P &= \alpha\int x^{\alpha-1}dx(1-x^n)^{\gamma-1} - (\alpha+\gamma n)\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1},\\
P &= (\alpha+\gamma n)\int x^{\alpha-1}dx(1-x^n)^\gamma - \gamma n\int x^{\alpha-1}dx(1-x^n)^{\gamma-1}.
\end{aligned}$$

Since these integrals must vanish having put $x=0$, it is necessary that in the same case $P = x^\alpha(1-x^n)^\gamma$ vanishes, what certainly always happens, if α is an arbitrary positive integer. For, if γ also was a positive integer, it is evident that having put $x=1$ it also is $P=0$ in this case; hence we find the following theorems.

THEOREM 3

§9 *If α and γ were positive numbers and after the integration one puts $x = 1$, one will have the following equalities of the integral formulas*

$$\begin{aligned}
&\text{I.} \quad \alpha \int x^{\alpha-1}dx(1-x^n)^{\gamma} = \gamma n \int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1},\\
&\text{II.} \quad \alpha \int x^{\alpha-1}dx(1-x^n)^{\gamma-1} = (\alpha+\gamma n) \int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1},\\
&\text{III.} \quad (\alpha+\gamma n) \int x^{\alpha-1}dx(1-x^n)^{\gamma} = \gamma n \int x^{\alpha-1}dx(1-x^n)^{\gamma-1}.
\end{aligned}$$

PROOF

For, since after the integration one puts $x = 1$, for this case in the above formulas it is $P = 0$ and hence clearly the equations propounded here follow. Q.E.D.

COROLLARY 1

§10 Each of these three equations is already contained in the two remaining ones, whence they will be comprehended in this form

$$\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1} = \frac{\alpha}{\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma} = \frac{\alpha}{\alpha+\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma-1},$$

or the three following integral formulas will be equal to each other

$$\frac{1}{\alpha}\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1} = \frac{1}{\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma} = \frac{1}{\alpha+\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma-1},$$

if certainly α and γ are positive numbers.

COROLLARY 2

§11 Since by Theorem 2 it is

$$\int x^{m-1}dx(1-x^n)^k = \int x^{nk+n-1}dx(1-x^n)^{\frac{m-n}{n}}$$

having likewise put $x = 1$, one will obtain an equality between these six integral formulas

$$\begin{array}{llll}
\text{I.} & \dfrac{1}{\alpha}\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1}, & \text{II.} & \dfrac{1}{\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma}, \\
\text{III.} & \dfrac{1}{\alpha+\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma-1}, & \text{IV.} & \dfrac{1}{\alpha}\int x^{n\gamma-1}dx(1-x^n)^{\frac{\alpha}{n}}, \\
\text{V.} & \dfrac{1}{\gamma n}\int x^{n\gamma+n-1}dx(1-x^n)^{\frac{\alpha-n}{n}}, & \text{VI.} & \dfrac{1}{\alpha+\gamma n}\int x^{n\gamma-1}dx(1-x^n)^{\frac{\alpha-n}{n}},
\end{array}$$

as long as the exponents α and γ were positive.

COROLLARY 3

§12 If α was an infinite number, it will be

$$\int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1} = \int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$$

and for the same reason it will be

$$\int x^{\alpha+2n-1}dx(1-x^n)^{\gamma-1} = \int x^{\alpha+n-1}dx(1-x^n)^{\gamma-1} = \int x^{\alpha-1}dx(1-x^n)^{\gamma-1},$$

whence it is concluded in general that it will be

$$\int x^{\alpha+\mu-1}dx(1-x^n)^{\gamma-1} = \int x^{\alpha-1}dx(1-x^n)^{\gamma-1},$$

as long as μ was a finite number and α was infinite.

COROLLARY 4

§13 In like manner, if γ was an infinite number, it will be

$$\int x^{\alpha-1}dx(1-x^n)^{\gamma} = \int x^{\alpha-1}dx(1-x^n)^{\gamma-1}$$

and in the same way it will be

$$\int x^{\alpha-1}dx(1-x^n)^{\gamma+1} = \int x^{\alpha-1}dx(1-x^n)^{\gamma},$$

whence it is concluded in general that it will be

$$\int x^{\alpha-1}dx(1-x^n)^{\gamma\pm\mu} = \int x^{\alpha-1}dx(1-x^n)^{\gamma},$$

if ν was a finite number and γ is infinite.

Problem 1

§14 *If m and n are positive numbers and i denotes an arbitrary positive integer, to define the ratio of the formula*

$$\int x^{m-1}dx(1-x^n)^{k-1}$$

to the formula

$$\int x^{m-1}dx(1-x^n)^{k+1}$$

in the case $x = 1$.

Solution

Since it is [§ 9, III]

$$\int x^{\alpha-1}dx(1-x^n)^{\gamma-1} = \frac{\alpha+\gamma n}{\gamma n}\int x^{\alpha-1}dx(1-x^n)^{\gamma},$$

by putting m and k for α and γ it will be

$$\int x^{m-1}dx(1-x^n)^{k-1} = \frac{m+kn}{kn}\int x^{m-1}dx(1-x^n)^{k};$$

if now, while $\alpha = m$, one puts $\gamma = k+1$, γ will be a lot greater positive number, since k is such a one, and hence in like manner one will have

$$\int x^{m-1}dx(1-x^n)^{k} = \frac{m+(k+1)n}{(k+1)n}\int x^{m-1}dx(1-x^n)^{k+1}$$

and by proceeding in the same way it will be

$$\int x^{m-1}dx(1-x^n)^{k+1} = \frac{m+(k+2)n}{(k+2)n}\int x^{m-1}dx(1-x^n)^{k+2}.$$

Therefore, hence it is concluded in general, while i denotes an arbitrary integer number, that it will be

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{m-1}dx(1-x^n)^{k+i}} = \frac{m+kn}{kn}\cdot\frac{m+kn+n}{kn+n}\cdot\frac{m+kn+2n}{kn+2n}\cdot\frac{m+kn+3n}{kn+3n}\cdots\frac{m+kn+in}{kn+in}.$$

Q.E.I.

COROLLARY 1

§ 15 Since it is [§ 11]

$$\int x^{m-1}dx(1-x^n)^{k-1} = \int x^{nk-1}dx(1-x^n)^{\frac{m-n}{n}}$$

and hence even

$$\int x^{m-1}dx(1-x^n)^{k+i} = \int x^{kn+in+n-1}dx(1-x^n)^{\frac{m-n}{n}},$$

it will also be

$$\frac{\int x^{km-1}dx(1-x^n)^{\frac{m-n}{n}}}{x^{kn+in+n-1}dx(1-x^n)^{\frac{m-n}{n}}} = \frac{m+kn}{kn}\cdot\frac{m+kn+n}{kn+n}\cdot\frac{m+kn+2n}{kn+2n}\cdots\frac{m+kn+in}{kn+in}.$$

COROLLARY 2

§16 If here one puts $kn=\mu$ and $\frac{m}{n}=\varkappa$ or $m=\varkappa n$, such that now μ and $\varkappa$ are positive numbers, one will have this reduction

$$\frac{\int x^{\mu-1}dx(1-x^n)^{\varkappa-1}}{\int x^{\mu+in+n-1}dx(1-x^n)^{\varkappa-1}} = \frac{\mu+\varkappa n}{\mu}\cdot\frac{\mu+\varkappa n+n}{\mu+n}\cdot\frac{\mu+\varkappa+2n}{\mu+2n}\cdots\frac{\mu+\varkappa n+in}{\mu+in};$$

but having written m and k for the letters μ and $\varkappa$ it will be

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{m+in+n-1}dx(1-x^n)^{k-1}} = \frac{m+kn}{m}\cdot\frac{m+kn+n}{m+n}\cdot\frac{m+k+2n}{m+2n}\cdots\frac{m+kn+in}{m+in}.$$

Corollary 3

§17 If this expression is divided by the expression found in the problem, this equation will result

$$\frac{\int x^{m-1}dx(1-x^n)^{k+i}}{\int x^{m+in+n-1}dx(1-x^n)^{k-1}} = \frac{kn}{m}\cdot\frac{kn+n}{m+n}\cdot\frac{kn+2n}{m+2n}\cdots\frac{kn+in}{m+in},$$

in which factors so the numerators as the denominators proceed in an arithmetic progression whose difference is $= n$.

Problem 2

§18 *To express the value of the formula*

$$\int x^{m-1}dx(1-x^n)^{k-1},$$

which it obtains in the case $x=1$, by means of infinite products, if the exponents m and k are positive.

Solution

In the form of the preceding problem set the number i to be infinite and one will have

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{m-1}dx(1-x^n)^{k+i}} = \frac{m+kn}{kn}\cdot\frac{m+kn+n}{kn+n}\cdot\frac{m+kn+2n}{kn+2n}\cdot\frac{m+kn+3n}{kn+3n}\cdot\text{etc. to infinity.}$$

Now, while i still is the same infinite number, take another finite number $\varkappa$ instead of k and in like manner one will have

$$\frac{\int x^{m-1}dx(1-x^n)^{\varkappa-1}}{\int x^{m-1}dx(1-x^n)^{\varkappa+i}} = \frac{m+\varkappa n}{\varkappa n}\cdot\frac{m+\varkappa n+n}{\varkappa n+n}\cdot\frac{m+\varkappa n+2n}{\varkappa n+2n}\cdot\frac{m+\varkappa n+3n}{\varkappa n+3n}\cdot\text{etc.},$$

where the number of factors is equal to the number of factors of the preceding expression. Of course, in both cases $= i+1$. But because of the infinite i it is, as we noted in § 13,

$$\int x^{m-1}dx(1-x^n)^{k+i} = \int x^{m-1}dx(1-x^n)^{\varkappa+i},$$

hence having divided the first form by the second this equation will result

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{m-1}dx(1-x^n)^{\varkappa-1}} = \frac{\varkappa(m+kn)}{k(m+\varkappa n)} \cdot \frac{(\varkappa+1)(m+kn+n)}{(k+1)(m+\varkappa n+n)} \cdot \frac{(\varkappa+2)(m+kn+2n)}{(k+2)(m+\varkappa n+2n)} \cdot \text{etc..}$$

Now set $\varkappa = 1$ and it will be $\int x^{m-1}dx(1-x^n)^{\varkappa-1} = \frac{x^m}{m} = \frac{1}{m}$ having put $x = 1$, whence it will be

$$\int x^{m-1}dx(1-x^n)^{k-1} = \frac{1}{m} \cdot \frac{1(m+kn)}{k(m+n)} \cdot \frac{2(m+kn+n)}{(k+1)(m+2n)} \cdot \frac{3(m+kn+2n)}{(k+2)(m+3n)} \cdot \frac{4(m+kn+3n)}{(k+3)(m+4n)} \cdot \text{etc.}$$

Q.E.I.

Another Proof

Proceed the same way as in § 16 by setting i to be an infinite number and it will be

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{m+in-1}dx(1-x^n)^{k-1}} = \frac{m+kn}{m} \cdot \frac{m+kn+n}{m+n} \cdot \frac{m+kn+2n}{m+2n} \cdot \frac{m+kn+3n}{m+3n} \cdot \text{etc.}$$

Now having put another finite number μ for m it will in like manner be

$$\frac{\int x^{\mu-1}dx(1-x^n)^{k-1}}{\int x^{\mu+in-1}dx(1-x^n)^{k-1}} = \frac{\mu+kn}{\mu} \cdot \frac{\mu+kn+n}{\mu+n} \cdot \frac{\mu+kn+2n}{\mu+2n} \cdot \frac{\mu+kn+3n}{\mu+3n} \cdot \text{etc.}$$

But since because of the infinite number i it is

$$\int x^{m+in-1}dx(1-x^n)^{k-1} = \int x^{\mu+in-1}dx(1-x^n)^{k-1} = \int x^{in}dx(1-x^n)^{k-1}$$

while the finite quantities vanish with respect to the infinite ones, and since in both of them one has the same number of factors, by dividing the first form by the second this equation will result

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{\mu-1}dx(1-x^n)^{k-1}} = \frac{\mu(m+kn)}{m(\mu+kn)} \cdot \frac{(\mu+n)(m+kn+n)}{(m+n)(\mu+kn+n)} \cdot \frac{(\mu+2n)(m+kn+2n)}{(m+2n)(\mu+kn+2n)} \cdot \text{etc.}$$

Now set $\mu = n$; it will be

$$\int x^{n-1}dx(1-x^n)^{k-1} = \frac{1-(1-x^n)^k}{kn}$$

having done the integration in such a way that the integral vanishes for $x = 0$. Now having put $x = 1$ this value goes over into $\frac{1}{kn}$, whence one will obtain

$$\int x^{m-1}dx(1-x^n)^{k-1} = \frac{1}{kn} \cdot \frac{1(m+kn)}{m(1+k)} \cdot \frac{2(m+kn+n)}{(m+n)(2+k)} \cdot \frac{3(m+kn+2n)}{(m+2n)(3+k)} \cdot \text{etc.}$$

Therefore, lo and behold another product consisting of infinitely many factors not very different from the first and even equal to it, by which the value of the propounded integral formula is expressed. Q.E.I.

COROLLARY 1

§19 But that these two infinite forms are equal to each other is clear per se; for, having divided the second by the first because of the equal numerators of the single terms this quotient results

$$1 = \frac{m}{kn} \cdot \frac{k(m+n)}{m(k+1)} \cdot \frac{(k+1)(m+2n)}{(m+n)(k+2)} \cdot \frac{(k+2)(m+3n)}{(m+2n)(k+3)} \cdot \text{etc.}$$

But the first two factors give $\frac{m+n}{n(k+1)}$, the first three $\frac{m+2n}{n(k+2)}$, four $\frac{m+3n}{n(k+3)}$ and infinitely many give $\frac{m+in}{n(k+i)} = \frac{in+m}{in+kn} = 1$.

COROLLARY 2

§20 One can form infinitely many infinite products of a form of this kind, whose value is $= 1$. For, since it is

$$\frac{p}{p+q} \cdot \frac{p+q}{p+2q} \cdot \frac{p+2q}{p+3q} \cdot \frac{p+3q}{p+4q} \cdots = \frac{p}{p+iq} = \frac{p}{iq}$$
$$\frac{r+s}{r} \cdot \frac{r+2s}{r+s} \cdot \frac{r+3s}{r+2s} \cdot \frac{r+4s}{r+3s} \cdots = \frac{r+is}{r} = \frac{is}{r},$$

by multiplying these two forms we will have

$$1 = \frac{qr}{ps} \cdot \frac{p(r+s)}{r(p+q)} \cdot \frac{(p+q)(r+2s)}{(r+s)(p+2q)} \cdot \frac{(p+2q)(r+3s)}{(r+2s)(p+3q)} \cdot \text{etc.}$$

COROLLARY 3

§21 Therefore, if the found value of the integral formula is multiplied by this expression $= 1$, the following further extending expression equal to it will result, namely

$$\int x^{m-1}dx(1-x^n)^{k-1}$$
$$= \frac{qr}{knps} \cdot \frac{1(m+kn)p(r+s)}{m(k+1)r(p+q)} \cdot \frac{2(m+kn+n)(p+q)(r+2s)}{(m+n)(k+2)(r+s)(p+2s)} \cdot \frac{3(m+kn+2n)(p+2q)(r+3s)}{(m+2n)(k+3)(r+2s)(p+3q)} \cdot \text{etc.},$$

where it is possible to assume any arbitrary numbers for p, q, r, s. Therefore, they can be assumed in many ways that each arbitrary factor is reduced to a simpler form.

COROLLARY 4

§22 Let $p = m$ and $q = n$ and it will be

$$\int x^{m-1}dx(1-x^n)^{k-1}$$
$$= \frac{r}{mks} \cdot \frac{1(m+kn)(r+s)}{(m+n)(k+1)r} \cdot \frac{2(m+kn+n)(r+2s)}{(m+2n)(k+2)(r+s)} \cdot \frac{3(m+kn+2n)(r+3s)}{(m+3n)(k+3)(r+2s)} \cdot \text{etc.}$$

which is the expression found first. But if it is $r = m+kn$ and $s = n$, it will be

$$\int x^{m-1}dx(1-x^n)^{k-1} = \frac{m+kn}{mkn} \cdot \frac{1(m+kn+n)}{(m+n)(k+1)} \cdot \frac{2(m+kn+2n)}{(m+2n)(k+2)} \cdot \frac{3(m+kn+3n)}{(m+3n)(k+3)} \cdot \text{etc.}$$

COROLLARY 5

§23 If one puts $p = k+1$ and $q = 1$, it will be

$$\int x^{m-1}dx(1-x^n)^{k-1}$$
$$= \frac{r}{k(k+1)ns} \cdot \frac{1(m+kn)(r+s)}{mr(k+2)} \cdot \frac{2/m+kn+n)(r+2s)}{(m+n)(r+s)(k+3)} \cdot \frac{3(m+kn+2n)(r+3s)}{(m+2n)(r+2s)(k+4)} \cdot \text{etc.};$$

further, let $r = 1$ and $s = 1$; it will be

$$\int x^{m-1}dx(1-x^n)^{k-1} = \frac{1}{k(k+1)n} \cdot \frac{2(n+kn)}{m(k+2)} \cdot \frac{3(m+kn+n)}{(m+n)(k+3)} \cdot \frac{4(m+kn+2n)}{(m+2n)(k+4)} \cdot \text{etc.};$$

but if one puts $r = m + kn$ and $s = n$, it will be

$$\int x^{m-1}dx(1-x^n)^{k-1} = \frac{m+kn}{k(k+1)nn} \cdot \frac{1(m+kn+n)}{m(k+2)} \cdot \frac{2(m+kn+2n)}{(m+n)(k+3)} \cdot \frac{3(m+kn+3n)}{(m+2n)(k+4)} \cdot \text{etc.}$$

COROLLARY 6

§24 If, while the exponent k remains the same, we change the remaining exponents m and n to μ and ν, we will have

$$\int x^{\mu-1}dx(1-x^\nu)^{k-1} = \frac{1}{\mu} \cdot \frac{1(\mu+k\nu)}{(\mu+\nu)k} \cdot \frac{2(\mu+k\nu+\nu)}{(\mu+2\nu)(k+1)} \cdot \frac{3(\mu+k\nu+2\nu)}{(\mu+3\nu)(k+2)} \cdot \text{etc.},$$

as long as μ, ν and k are positive numbers. Therefore, having divided that form [§ 17] by this one we will have

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{\mu-1}dx(1-x^\nu)^{k-1}} = \frac{\mu}{m} \cdot \frac{(\mu+\nu)(m+kn)}{(m+n)(\mu+k\nu)} \cdot \frac{(\mu+2\nu)(m+kn+n)}{(m+2n)(\mu+k\nu+\nu)} \cdot \frac{(\mu+3\nu)(m+kn+2n)}{(m+3n)(\mu+k\nu+2\nu)} \cdot \text{etc.}$$

COROLLARY 7

§25 But if even in the other form k is changed to $\varkappa$, one will have

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{\mu-1}dx(1-x^\nu)^{\varkappa-1}}$$
$$= \frac{\mu}{m} \cdot \frac{\varkappa(\mu+\nu)(m+kn)}{k(m+n)(\mu+\varkappa\nu)} \cdot \frac{(\varkappa+1)(\mu+2\nu)(m+kn+n)}{(k+1)(m+2n)(\mu+\varkappa\nu+\nu)} \cdot \frac{(\varkappa+2)(\mu+3\nu)(m+kn+2n)}{(k+2)(m+3n)(\mu+\varkappa\nu+2\nu)} \cdot \text{etc.}$$

having put $x = 1$ after the integration and while all exponents m, n, k and μ, ν, $\varkappa$ are positive.

SCHOLIUM

§26 Having explained these conversions of integral formulas into infinite products let us see vice versa, how a propounded infinite product of this kind must be reduced to integrations of differential formulas in the case $x = 1$. Here it is especially to be considered, how many factors the terms, which constitute this infinite product, are composited of; these terms first must only be of such a nature that the infinitesimal terms 1. Hence they will be fractions and will consist of a certain number so of numerators as of denominators and in both of them the factors will proceed in an arithmetic progression; for, even though the different parts obtain different differences, they can nevertheless easily be reduced to the same. Therefore, because there is no obstruction that this difference is set equal to 1, we will have the following orders of infinite products of this kind for the different numbers of factors of each fraction in the product

$$\frac{a}{b}\cdot\frac{a+1}{b+1}\cdot\frac{a+2}{b+2}\cdot\frac{a+3}{b+3}\cdot\frac{a+4}{b+4}\cdot\frac{a+5}{b+5}\cdot\text{etc.},$$

$$\frac{ac}{be}\cdot\frac{(a+1)(c+1)}{(b+1)(e+1)}\cdot\frac{(a+2)(c+2)}{(b+2)(e+2)}\cdot\frac{(a+3)(c+3)}{(b+3)(e+3)}\cdot\text{etc.},$$

$$\frac{acf}{beg}\cdot\frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)}\cdot\frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)}\cdot\text{etc.},$$

$$\frac{acfh}{begk}\cdot\frac{(a+1)(c+1)(f+1)(h+1)}{(b+1)(e+1)(g+1)(k+1)}\cdot\frac{(a+2)(c+2)(f+2)(h+2)}{(b+2)(e+2)(g+2)(k+2)}\cdot\text{etc.}$$

Therefore, let us see, how the value of each of these products is to be expressed by integral formulas.

PROBLEM 3

§27 *To define the value of the following infinite product consisting of simple terms by means of integral formulas*

$$P=\frac{a}{b}\cdot\frac{a+1}{b+1}\cdot\frac{a+2}{b+2}\cdot\frac{a+3}{b+3}\cdot\frac{a+4}{b+4}\cdot\frac{a+5}{b+5}\cdot\text{etc.}$$

SOLUTION

While i denotes an infinite number we saw [§ 16] that it is

$$\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{in}dx(1-x^n)^{k-1}}=\frac{m+kn}{m}\cdot\frac{m+kn+n}{m+n}\cdot\frac{m+kn+2n}{m+2n}\cdot\text{etc.},$$

which form will be reduced to the propounded one by putting $n=1$, $m+k=a$ and $m=b$, whence it is $k=a-b$. Therefore, because k must be a positive number, if it was $a>b$, it will be

$$P = \frac{\int x^{b-1}dx(1-x)^{a-b-1}}{x^i dx(1-x)^{a-b-1}} = \frac{\int x^{a-b-1}dx(1-x)^{b-1}}{\int x^{a-b-1}dx(1-x)^i},$$

but if it is $b > a$, it will be

$$P = \frac{\int x^i dx(1-x)^{b-a-1}}{\int x^{a-1}dx(1-x)^{b-a-1}} = \frac{\int x^{b-a-1}dx(1-x)^i}{\int x^{b-a-1}dx(1-x)^{a-1}}.$$

Q.E.I.

COROLLARY 2

§28 But it is obvious, if $a > b$, that the value P will be infinite, but if $b > a$, that it will be $P = 0$. But in the case $a = b$ it is $P = 1$; since the case equally extends to both we explained, it is evident that it is $\int \frac{x^{a-1}dx}{1-x} = \int \frac{x^i dx}{1-x}$, which integrals in the case $x = 1$ certainly become infinite in such a way that their ratio is 1. But in general it is

$$\int \frac{x^{a-1}dx}{1-x} = \int \frac{x^{b-1}dx}{1-x}.$$

PROBLEM 4

§29 *To define the value of the following infinite product consisting of two factors in the denominator and the numerator of each fraction by means of integral formulas*

$$P = \frac{ac}{be} \cdot \frac{(a+1)(c+1)}{(b+1)(e+1)} \cdot \frac{(a+2)(c+2)}{(b+2)(e+2)} \cdot \frac{(a+3)(c+3)}{(b+3)(e+3)} \cdot \text{etc.}$$

SOLUTION

Since by § 24, while m, n, k, μ, ν denote positive numbers, it is

$$\frac{(\mu+\nu)(m+kn)}{(m+n)(\mu+k\nu)} \cdot \frac{(\mu+2\nu)(m+kn+n)}{(m+2n)(\mu+k\nu+\nu)} \cdot \text{etc.} = \frac{m}{\mu} \cdot \frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{\mu-1}dx(1-x^\nu)^{k-1}},$$

put $n = 1$, $\nu = 1$, $\mu + 1 = a$, $m + k = c$, $m + 1 = b$, $\mu + k = e$; it will be $\mu = a - 1$, $m = b - 1$ and $k = c - b + 1 = c - a + 1$. Hence, that this form can be reduced to the propounded form, it is necessary that it is $c - b = e - a$;

for, if this condition is not satisfied, the value of the propounded product P would be either infinite or vanishing. That this inconvenience does not occur, let $c-b=e-a$ or $a+c=b+e$, and as long as $a-1$, $b-1$ and $c-b$ or $c-1$ are positive numbers, it will be

$$P=\frac{b-1}{a-1}\cdot\frac{\int x^{b-2}dx(1-x)^{c-b}}{\int x^{a-2}dx(1-x)^{c-a}}.$$

Or consider this form

$$\frac{\mu(m+kn-n)}{m(\mu+k\nu-\nu)}\cdot\frac{(\mu+\nu)(m+kn)}{(m+n)(\mu+k\nu)}\cdot\text{etc.}=\frac{m+kn+-n}{\mu+k\nu-\nu}\cdot\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{\mu-1}dx(1-x^\nu)^{k-1}},$$

which clearly follows from that one, and put $n=1$, $\nu=1$, $\mu=a$, $m=b$, $c=m+k+1$ and $e=\mu+k-1$ and it will be $k-1=c-b=e-a$; therefore, it must again be $a+c=b+e$. Therefore, now, as long as a, b and $c-b+1$ or $e-a+1$ are positive numbers, it will be

$$P=\frac{c}{e}\cdot\frac{\int x^{b-1}dx(1-x)^{c-b}}{\int x^{a-1}dx(1-x)^{c-a}}.$$

Therefore, if it was $a+c=b+e$, the value in question P is finite and is expressed by these integral formulas in the case $x=1$. Q.E.I.

Corollary 1

§30 Since it is $a+c=b+e$, if it is $c>b$, it will also be $e>a$ and a and b in the first term $\frac{ac}{be}$ denote the smaller factors of the numerator and the denominator. But it is required that $c-b+1$ is a positive number. Hence, if also $c-e+1$ is a positive number, the value P can additionally be expressed in another ways, namely by permuting b and e this way

$$P=\frac{c}{b}\cdot\frac{\int x^{e-1}dx(1-x)^{c-e}}{\int x^{a-1}dx(1-x)^{b-a}}.$$

Corollary 2

§31 And each of these formulas will hold

$$P=\frac{c}{b}\cdot\frac{\int x^{e-1}dx(1-x)^{c-e}}{\int x^{a-1}dx(1-x)^{b-a}}=\frac{c}{e}\cdot\frac{\int x^{b-1}dx(1-x)^{c-b}}{\int x^{a-1}dx(1-x)^{c-a}}=\frac{a}{b}\cdot\frac{\int x^{e-1}dx(1-x)^{a-e}}{\int x^{c-1}dx(1-x)^{b-c}}$$
$$=\frac{a}{e}\cdot\frac{\int x^{b-1}dx(1x)^{a-b}}{\int x^{c-1}dx(1-x)^{e-c}}.$$

The first of them holds, if $c-e+1=b-a+1$ is >0, the second, if $c-b+1=e-a+1>0$, the third, if $a-e+1=b-c+1>0$, and the fourth, if $a-b+1=e-c+1>0$.

COROLLARY 3

§32 The first and the second form will hold at the same time, if the difference of a and b is smaller than 1 and hence also the difference of c and e is smaller than 1. And all four will hold at the same time, if additionally the difference of a and c was greater than 1.

COROLLARY 4

§33 Therefore, if one puts $a=p+m$, $b=p+n$, $c=p-m$ and $e=p-n$, that it is $a+c=b+e=2p$ and it was $m+n<1$, it will be

$$P=\frac{p-m}{p+n}\cdot\frac{\int x^{p-n-1}dx(1-x)^{n-m}}{\int x^{p+m-1}dx(1-x)^{m-n}}=\frac{p+m}{p+n}\cdot\frac{\int x^{p-n-1}dx(1-x)^{m+n}}{\int x^{p-m-1}dx(1-x)^{n+m}},$$
$$P=\frac{p-m}{p-n}\cdot\frac{\int x^{p+n-1}dx(1-x)^{-n-m}}{\int x^{p+m-1}dx(1-x)^{-m-n}}=\frac{p+m}{p-n}\cdot\frac{\int x^{p+n-1}dx(1-x)^{m-n}}{\int x^{p-m-1}dx(1-x)^{n-m}}.$$

And these four formulas will be equal to each other.

PROBLEM 5

§34 *To express the value of the following infinite product consisting of three factors in the numerator and denominator of each fraction by means of integral formulas*

$$P=\frac{acf}{beg}\cdot\frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)}\cdot\frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)}\cdot\text{etc.}$$

SOLUTION

Since in § 25 we found

$$\frac{\varkappa(\mu+\nu)(m+kn)}{k(m+n)(\mu+\varkappa\nu)}\cdot\frac{(\varkappa+1)(\mu+2\nu)(m+kn+n)}{(k+1)(m+2n)(\mu+\varkappa\nu+\nu)}\cdot\text{etc.}=\frac{m}{\mu}\cdot\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{\mu-1}dx(1-x^\nu)^{\varkappa-1}},$$

by also adding the term of the preceding it will be

$$\frac{(\varkappa-1)\mu(m+kn-n)}{(k-1)m(\mu+\varkappa\nu-\nu)}\cdot\frac{\varkappa(\mu+\nu)(m+kn)}{k(m+n)(\mu+\varkappa\nu)}\cdot\text{etc.}=\frac{(\varkappa-1)(m+kn-n)}{(k-1)(\mu+\varkappa\nu-\nu)}\cdot\frac{\int x^{m-1}dx(1-x^n)^{k-1}}{\int x^{\mu-1}dx(1-x^\nu)^{\varkappa-1}};$$

in order to reduce this form to the propounded one, set

$$\varkappa-1=a,\quad k-1=b,\quad \mu=c,\quad m=e,\quad n=1,\quad \nu=1$$

and

$$m+k-1=e+b=f,\quad \mu+\varkappa-1=c+a=g.$$

Therefore, since this reduction does only succeed under this condition, let $f=b+e$ and $g=a+c$, that one has this infinite product

$$P=\frac{ac(b+e)}{be(a+c)}\cdot\frac{(a+1)(c+1)(b+e+1)}{(b+1)(e+1)(a+c+1)}\cdot\frac{(a+2)(c+2)(b+e+2)}{(b+2)(e+2)(a+c+2)}\cdot\text{etc.}$$

Hence, because in this case it is $m=e$, $k=b+1$, $\mu=c$ and $\varkappa=a+1$ while $n=\nu=1$, it will be

$$P=\frac{a(b+e)}{b(a+c)}\cdot\frac{\int x^{a-1}dx(1-x)^b}{\int x^{a-1}dx(1-x)^a},$$

if c, e, $b+1$ and $a+1$ are positive numbers. Q.E.I.

COROLLARY 1

§35 Since by means of § 9 it is

$$\int x^{\alpha-1}dx(1-x)^{\gamma-1}=\frac{\alpha+\gamma}{\alpha}\int x^{\alpha}dx(1-x)^{\gamma-1},$$

it will be

$$\int x^{e-1}dx(1-x)^b = \frac{b+e+1}{e}\int x^e dx(1-x)^b$$

and hence

$$P = \frac{ac(b+e)(b+e+1)}{be(a+c)(a+c+1)} \cdot \frac{\int x^e dx(1-x)^b}{\int x^c dx(1-x)^a}.$$

And because it is

$$\int x^{\alpha-1}dx(1-x)^\gamma = \frac{\gamma}{\alpha+\gamma}\int x^{\alpha-1}dx(1-x)^{\gamma-1},$$

it will be

$$\int x^{e-1}dx(1-x)^b = \frac{b}{b+e}\int c^{e-1}dx(1-x)^{b-1};$$

one will also have

$$P = \frac{\int x^{e-1}dx(1-x)^{b-1}}{\int x^{c-1}dx(1-x)^{a-1}}.$$

Corollary 2

§36 But this formula holds, if a, b, c and e are positive numbers, and since now a and c, likewise b and c, can be permuted, it will also be

$$P = \frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}},$$

which conversion is also obvious from Theorem 2 per se.

Scholium 1

§37 Therefore, the propounded problem is not solved in general, but only in the case, in which $f = b+e$ and $g = a+c$, and so our solution is restricted by two conditions. But indeed only one restriction is necessary, for the value of P to not become infinite or vanishing; for this it is required that it is $a+c+f = b+e+g$. But to solve the problem for this one restriction, it is necessary to introduce more integral formulas into the calculation, which can be done this way. Therefore, since, having put $a+c+f = b+e+g$, it is

$$P=\frac{acf}{beg}\cdot\frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)}\cdot\frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)}\cdot\text{etc.},$$

set $P=QR$ and let

$$Q=\frac{(p+q)(p-1)}{(p+r)(p-r)}\cdot\frac{(p+q+1)(p-q+1)}{(p+r+1)(p-r+1)}\cdot\text{etc.}=\frac{p+q}{p+r}\cdot\frac{\int x^{p-r-1}dx(1-x)^{q+r}}{\int x^{p-q-1}dx(1-x)^{q+r}}$$

and

$$R=\frac{\alpha\gamma(\beta+\varepsilon)}{\beta\varepsilon(\alpha+\gamma)}\cdot\frac{(\alpha+1)(\gamma+1)(\beta+\varepsilon+1)}{(\beta+1)(\varepsilon+1)(\alpha+\gamma+1)}\cdot\text{etc.}=\frac{\int x^{\beta-1}dx(1-x)^{\varepsilon-1}}{\int x^{\alpha-1}dx(1-x)^{\gamma-1}}.$$

Now let the first term of the product QR become equal to the first term of the propounded form P, namely

$$\frac{\alpha\gamma(\beta+\varepsilon)(p+q)(p-q)}{\beta\varepsilon(\alpha+\gamma)(p+r)(p-r)}=\frac{acf}{beg},$$

which can be done in many ways. For, the first terms can be split into three factors in several ways; of course, put $\beta+\varepsilon=p+r$ and $\alpha+\gamma=p+q$, that one has $q=\alpha+\gamma-p$ and $r=\beta+\varepsilon-p$, and it will be

$$\frac{\alpha\gamma(2p-\alpha-\gamma)}{\beta\varepsilon(2p-\beta-\varepsilon}=\frac{acf}{beg}.$$

Therefore, if one sets

$$\alpha=a,\quad\beta=b,\quad\gamma=c,\quad\varepsilon=e\quad\text{and}\quad 2p=a+c+f=b+e+g,$$

it will be $q=a+c-p$ and $r=b+e-p$. And so no other restriction is necessary here than that it is $a+c+f=b+e+g=2p$. Therefore, for this case the value of the propounded infinite product will be

$$P=\frac{a+c}{b+e}\cdot\frac{\int x^{2p-b-e-1}dx(1-x)^{a+b+c+e-2p}}{\int x^{2p-a-c-1}dx(1-x)^{a+b+c+e-2p}}\cdot\frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}},$$

where now so the letters a and c as b and c can be permuted arbitrarily.

To give another proof set $\gamma = p + r$ and $\varepsilon = p - q$ that it is

$$\frac{\alpha(\beta+\varepsilon)(p+q)}{\beta(\alpha+\gamma)(p-r)} = \frac{acf}{beg}.$$

Now let

$$\alpha = a, \quad \beta = b, \quad \varepsilon = c - b, \quad \gamma = e - a;$$

it will be

$$q = p - c + b \quad \text{and} \quad r = e - a - p$$

and hence

$$f = 2p - c + b \quad \text{and} \quad g = 2p - e + a.$$

But if the sum is put $a + c + f = b + e + g = s$, it will be

$$a + b + 2p = s \quad \text{and} \quad 2p = s - a - b$$

and so

$$p + q = s - a - c = f, \quad p - q = c - b, \quad p + r = e - a,$$
$$p - r = s - b - e = g \quad \text{and} \quad q + r = b + e - a - c.$$

And hence this expression results

$$P = \frac{s-a-c}{e-a} \cdot \frac{\int x^{s-b-e-1}dx(1-x)^{b+e-a-c}}{\int x^{c-b-1}dx(1-x)^{b+c-a-c}} \cdot \frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{c-a-1}},$$

where again so the letters a and c as b and e can be permuted. Or because of the many values of Q it will also be

$$P = \frac{c-b}{e-a} \cdot \frac{\int x^{g-1}dx(1-x)^{c+e-s}}{\int x^{f-1}dx(1-x)^{a+e-s}} \cdot \frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{e-a-1}}.$$

But the formula found first by putting s for $2p$ goes over into this one

$$P = \frac{a+c}{b+e} \cdot \frac{\int x^{g-1}dx(1-x)^{b+e-f}}{\int x^{f-1}dx(1-x)^{b+e-f}} \cdot \frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}}.$$

SCHOLIUM 2

§38 Now, if all these permutations are applied, which are obtained for the formula Q, and the following formula was propounded

$$P = \frac{acf}{beg} \cdot \frac{(a+1)(c+1)(f+1)}{(b+1)(e+1)(g+1)} \cdot \frac{(a+2)(c+2)(f+2)}{(b+2)(e+2)(g+2)} \cdot \text{etc.}$$

and it was $a+c+f = b+e+g$, one will find the following values for the value P, namely

$$\begin{aligned}
P &= \frac{f}{g} \cdot \frac{\int x^{e-a-1}dx(1-x)^{a+f-e}}{\int x^{c-b-1}dx(1-x)^{a+f-e}} \cdot \frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{c-a-1}},\\
P &= \frac{f}{e-a} \cdot \frac{\int x^{g-1}dx(1-x)^{f-g}}{\int x^{c-b-1}dx(1-x)^{f-g}} \cdot \frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{e-a-1}},\\
P &= \frac{c-b}{g} \cdot \frac{\int x^{e-a-1}dx(1-x)^{g-f}}{\int x^{f-1}dx(1-x)^{g-f}} \cdot \frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{e-a-1}},\\
P &= \frac{c-b}{e-a} \cdot \frac{\int x^{g-1}dx(1-x)^{e-a-f}}{\int x^{f-1}dx(1-x)^{e-a-f}} \cdot \frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{e-a-1}},\\
P &= \frac{f}{g} \cdot \frac{\int x^{b+e-1}dx(1-x)^{f-b-e}}{\int x^{a+c-1}dx(1-x)^{f-b-e}} \cdot \frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}},\\
P &= \frac{f}{b+e} \cdot \frac{\int x^{g-1}dx(1-x)^{f-g}}{\int x^{a+c-1}dx(1-x)^{f-g}} \cdot \frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}},\\
P &= \frac{a+c}{g} \cdot \frac{\int x^{b-e-1}dx(1-x)^{g-f}}{\int x^{f-1}dx(1-x)^{g-f}} \cdot \frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}},\\
P &= \frac{a+c}{b+e} \cdot \frac{\int x^{g-1}dx(1-x)^{b+f-e}}{\int x^{f-1}dx(1-x)^{b+e-f}} \cdot \frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}}.
\end{aligned}$$

But further it is possible to permute so the three letters a, c, f as b, e, g arbitrarily here, from which a very large amount of formulas, which are all equal to the same value P, will result.

SCHOLIUM 3

§39 Hence even for the simpler product

$$P = \frac{ac}{be} \cdot \frac{(a+1)(c+1)}{(b+1)(e+1)} \cdot \frac{(a+2)(c+2)}{(b+2)(e+2)} \cdot \text{etc.},$$

if it was $a+c=b+e$, except for the values found above one will even be able to exhibit many others. For, first, since it is $a+c=b+e$, the value found in problem 5 extends to this

$$P=\frac{\int x^{e-1}dx(1-x)^{b-1}}{\int x^{c-1}dx(1-x)^{a-1}}.$$

Further, if in the series of the preceding paragraph one of the letters a, c, f is set equal to one of b, e, g, either this same expression or others will be obtained, which together with the preceding ones will be

$$P=\frac{\int x^{e-1}dx(1-x)^{a-e-1}}{\int x^{c-1}dx(1-x)^{b-c-1}},\qquad P=\frac{\int x^{b-1}dx(1-x)^{a-b-1}}{\int x^{c-1}dx(1-x)^{e-c-1}},$$

$$P=\frac{\int x^{e-1}dx(1-x)^{c-e-1}}{\int x^{a-1}dx(1-x)^{b-a-1}},\qquad P=\frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{e-a-1}},$$

$$P=\frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{c-1}},\qquad P=\frac{\int x^{b-1}dx(1-x)^{c-b-1}}{\int x^{a-1}dx(1-x)^{e-a-1}},$$

where it is

$$e-a=c-b \quad \text{and} \quad c-e=b-a.$$

In the following n expressions is an arbitrary number:

$$P=\frac{\int x^{e-n-1}dx(1-x)^{n+a-e-1}}{\int x^{c-n-1}dx(1-x)^{n+b-c-1}}\cdot\frac{\int x^{n-1}dx(1-x)^{c-n-1}}{\int x^{n-1}dx(1-x)^{e-n-1}},$$

$$P=\frac{\int x^{n+b-1}dx(1-x)^{c-b-n-1}}{\int x^{n+a-1}dx(1-x)^{c-b-n-1}}\cdot\frac{\int x^{n-1}dx(1-x)^{b-1}}{\int x^{n-1}dx(1-x)^{a-1}},$$

$$P=\frac{\int x^{e-1}dx(1-x)^{n+b-c-1}}{\int x^{c-1}dx(1-x)^{n+b-c-1}}\cdot\frac{\int x^{n-1}dx(1-x)^{b-1}}{\int x^{n-1}dx(1-x)^{a-1}},$$

$$P=\frac{\int x^{n-1}dx(1-x)^{a-1}}{\int x^{c-1}dx(1-x)^{a-1}}\cdot\frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{a-1}dx(1-x)^{n-1}}=\frac{\int x^{b-1}dx(1-x)^{e-1}}{\int x^{c-1}dx(1-x)^{a-1}},$$

which last expression is already contained in the preceding ones. But here it is to be noted that it would be superfluous to define the ratio of the exponents here as it was done above. For, since P certainly is a finite value for $a+c=b+e$, if a certain of the integral formulas has negative exponents smaller thn -1, then it is possible to reduce them to greater one and then the true value of P will be obtained. But simpler formulas are contained in this theorem.

THEOREM 4

§40 *If it was $a+c=b+e=s$, then it will be*

$$\frac{\int x^{a-1}dx(1-x)^{c-1}}{\int x^{b-1}dx(1-x)^{e-1}}=\frac{\int x^{a-1}dx(1-x)^{s-a-b-1}}{\int x^{b-1}dx(1-x)^{s-a-b-1}},$$

if after the integration one sets $x=1$.

PROOF

For, from the preceding formulas it is

$$\frac{\int x^{a-1}dx(1-x)^{c-1}}{\int x^{b-1}dx(1-x)^{c-1}}=\frac{\int x^{a-1}dx(1-x)^{c-b-1}}{\int x^{b-1}dx(1-x)^{e-a-1}}.$$

But because of $a+c=b+e=s$ it is $c=s-a$ and $e=s-b$, whence it will be

$$c-b=e-a=s-a-b,$$

whence the propounded formula is constructed. Q.E.D.

COROLLARY 1

§41 Here it is possible to permute so the numbers a and c as b and e, whence one obtains four integral formulas equal to the first, namely each single one of these formulas

$$\frac{\int x^{a-1}dx(1-x)^{s-a-b-1}}{\int x^{b-1}dx(1-x)^{s-a-b-1}},\quad \frac{\int x^{a-1}dx(1-x)^{s-a-e-1}}{\int x^{e-1}dx(1-x)^{s-a-e-1}}$$
$$\frac{\int x^{c-1}dx(1-x)^{s-b-c-1}}{\int x^{b-1}dx(1-x)^{s-b-c-1}},\quad \frac{\int x^{c-1}dx(1-x)^{s-c-e-1}}{\int x^{e-1}dx(1-x)^{s-c-e-1}}$$

is equal to this form

$$\frac{\int x^{a-1}dx(1-x)^{c-1}}{\int x^{b-1}dx(1-x)^{e-1}}.$$

COROLLARY 2

§42 But the value of each of these formulas is equal to this infinite product

$$\frac{be}{ac}\cdot\frac{(b+1)(e+1)}{(a+1)(c+1)}\cdot\frac{(b+2)(e+2)}{(a+2)(c+2)}\cdot\text{etc.}$$

COROLLARY 3

§43 If it is $e=1$ and hence $b=s-1$, $a=s-c$, having put

$$P=\frac{1(s-1)}{c(s-c)}\cdot\frac{2\cdot s}{(c+1)(s-c+1)}\cdot\frac{3(s+1)}{(c+2)(s-c+2)}\cdot\text{etc.}$$

because of

$$\int x^{b-1}dx(1-x)^{e-1}=\int x^{s-2}dx=\frac{1}{s-1}$$

it will be

$$P=(s-1)\int x^{s-c-1}dx(1-x)^{c-1},$$
$$P=(c-1)\int x^{s-c-1}dx(1-x)^{c-2}=(s-1)\int x^{s-c-1}dx(1-x)^{c-1},$$
$$P=(s-c-1)\int x^{c-1}dx(1-x)^{s-c-2},$$
$$P=\frac{\int x^{s-c-1}dx(1-x)^{c-s}}{\int x^{s-2}dx(1-x)^{c-s}}=\frac{\int x^{c-1}dx(1-x)^{-c}}{\int x^{s-2}dx(1-x)^{-c}}=(s-1)\int x^{s-c-1}dx(1-x)^{c-1}.$$

SCHOLIUM

§44 But since I exhibited many comparisons of integral formulas of this kind, here I want to persecute some - with respect to the others - more notable cases and want to show, how they can be expressed by means of integral formulas. But mainly those infinite products expressing the sines and cosines of a certain angle are remarkable. For, while ρ denotes the right angle and φ an arbitrary angle it is known that it is

$$\sin\varphi=\varphi\left(1-\frac{\varphi\varphi}{4\rho\rho}\right)\left(1-\frac{\varphi\varphi}{16\rho\rho}\right)\left(1-\frac{\varphi\varphi}{36\rho\rho}\right)\left(1-\frac{\varphi\varphi}{64\rho\rho}\right)\cdot\text{etc.}$$

and

$$\cos\varphi = \left(1-\frac{\varphi\varphi}{\rho\rho}\right)\left(1-\frac{\varphi\varphi}{9\rho\rho}\right)\left(1-\frac{\varphi\varphi}{25\rho\rho}\right)\left(1-\frac{\varphi\varphi}{49\rho\rho}\right)\cdot\text{etc.}$$

Now, if one puts $\varphi = \frac{m}{n}\rho$, it will be

$$\left(1-\frac{mm}{4nn}\right)\left(1-\frac{mm}{16nn}\right)\left(1-\frac{mm}{36nn}\right)\cdot\text{etc.} = \frac{n}{m\rho}\sin\frac{m}{n}\rho,$$

$$\left(1-\frac{mm}{nn}\right)\left(1-\frac{mm}{9nn}\right)\left(1-\frac{mm}{25nn}\right)\cdot\text{etc.} = \cos\frac{m}{n}\rho.$$

Or if the angle equal to two right ones π is introduced and because of $\rho = \frac{1}{2}\pi$ one writes $2m$ for m, by expanding the factors it will be

$$\frac{(m-n)(n+m)}{n\cdot n}\cdot\frac{(2n-m)(2n+m)}{2n\cdot 2n}\cdot\frac{(3n-m)(3n+m)}{3n\cdot 3n}\cdot\text{etc.} = \frac{n}{m\pi}\sin\frac{m}{n}\pi,$$

$$\frac{(n-2m)(n+2m)}{n\cdot n}\cdot\frac{(3n-2m)(3n+2m)}{3n\cdot 3n}\cdot\frac{(5n-2m)(5n+2m)}{5n\cdot 5n}\cdot\text{etc.} = \cos\frac{m}{n}\pi,$$

But by reducing the differences to 1 it will be

$$\frac{\left(1-\frac{m}{n}\right)\left(1+\frac{m}{n}\right)}{1\cdot 1}\cdot\frac{\left(2-\frac{m}{n}\right)\left(2+\frac{m}{n}\right)}{2\cdot 2}\cdot\frac{\left(3-\frac{m}{n}\right)\left(3+\frac{m}{n}\right)}{3\cdot 3} = \frac{n}{m\pi}\sin\frac{m}{n}\pi,$$

$$\frac{\left(\frac{1}{2}-\frac{m}{n}\right)\left(\frac{1}{2}+\frac{m}{n}\right)}{\frac{1}{2}\cdot\frac{1}{2}}\cdot\frac{\left(\frac{3}{2}-\frac{m}{n}\right)\left(\frac{3}{2}+\frac{m}{n}\right)}{\frac{3}{2}\cdot\frac{3}{2}}\cdot\text{etc.} = \cos\frac{m}{n}\pi.$$

PROBLEM 6

§46 *To find the integral formula, whose value in the case $x = 1$ yields $\sin\frac{m}{n}\pi$.*

SOLUTION

Because it is

$$\frac{n}{m\pi}\sin\frac{m}{n}\pi = \frac{\left(1-\frac{m}{n}\right)\left(1+\frac{m}{n}\right)}{1\cdot 1}\cdot\frac{\left(2-\frac{m}{n}\right)\left(2+\frac{m}{n}\right)}{2\cdot 2}\cdot\text{etc.},$$

compare this infinite product to the general form

$$P = \frac{be}{ac} \cdot \frac{(b+1)(e+1)}{(a+1)(c+1)} \cdot \frac{(b+2)(e+2)}{(a+2)(c+2)} \cdot \text{etc.},$$

whose value was exhibited in several ways by integrals formulas in § 41. Therefore, one has to set

$$a = 1, \quad c = 1, \quad b = 1 - \frac{m}{n} \quad \text{and} \quad e = 1 + \frac{m}{n}$$

and it will be $s = a + c = b + e = 2$, but then

$$\begin{aligned} s-a-b-1 &= -1+\frac{m}{n}, \quad s-a-e-1 = -1-\frac{m}{n}, \\ s-b-c-1 &= -1+\frac{m}{n}, \quad s-c-e-1 = -1-\frac{m}{n}. \end{aligned}$$

Therefore, hence for P the following expression follows

$$P = \frac{\int dx(1-x)^0}{\int x^{\frac{-m}{n}} dx (1-x)^{\frac{m}{n}}} = \frac{1}{\int x^{\frac{-m}{n}} dx (1-x)^{\frac{m}{n}}},$$

to which all remaining ones are easily reduced. Therefore, this formula gives

$$\int x^{\frac{m}{n}} dx (1-x)^{-\frac{m}{n}} = \int \frac{x^{\frac{m}{n}} dx}{(1-x)^{\frac{m}{n}}} = \frac{m\pi}{n \sin \frac{m}{n}\pi}$$

and having put $x = y^n$ one will have

$$\int \frac{y^{m+n-1} dy}{(1-y^n)^{\frac{m}{n}}} = \frac{m\pi}{nn \sin \frac{m}{n}\pi} \quad \text{or} \quad \int \frac{y^{m-1} dy}{(1-y^n)^{\frac{m}{n}}} = \frac{\pi}{n \sin \frac{m}{n}\pi}.$$

Therefore, we find

$$\sin \frac{m}{n}\pi = \frac{\pi}{n} : \int \frac{y^{m-1} dy}{(1-y^n)^{\frac{m}{n}}}.$$

Q.E.I.

Corollary 1

§46 Therefore, by means of Theorem 1 this form $\int y^{m-1} dy (1-y^n)^{\frac{-m}{n}}$ because of $k = -\frac{m}{n}$ is converted into this one $\int \frac{y^{m-1} dy}{1+y^n}$ and hence one will have

$$\int \frac{y^{m-1}dy}{1+y^n} = \frac{\pi}{n\sin\frac{m}{n}\pi}$$

in the case $y=\infty$, which form because of its simplicity is especially remarkable.

COROLLARY 2

§47 Therefore, we will find these two very remarkable equalities

$$\frac{m\pi}{n\sin\frac{m}{n}\pi} = \int \frac{my^{m-1}dy}{(1-y^n)^{\frac{m}{n}}}$$

having put $y=1$ and

$$\frac{m\pi}{n\sin\frac{m}{n}\pi} = \int \frac{my^{m-1}dy}{1+y^n}$$

having put $y=\infty$, in which cases therefore the integral of each of both formulas can be exhibited conveniently.

COROLLARY 3

§48 Therefore, since having put $x=1$ and $y=\infty$ it is

$$\frac{\pi}{n\sin\frac{m}{n}\pi} = \int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}} = \int \frac{y^{m-1}dy}{1+y^n},$$

if one writes $2in+m$ for m, because of $\sin\frac{2in+m}{n}\pi = \sin\frac{m}{n}\pi$ it will also be

$$\int \frac{x^{2in+m-1}dx}{(1-x^n)^{\frac{2in+m}{n}}} = \int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}} = \frac{\pi}{n\sin\frac{m}{n}\pi}$$

and

$$\int \frac{y^{2in+m-1}dy}{1+y^n} = \int \frac{y^{m-1}dy}{1+y^n} = \frac{\pi}{n\sin\frac{m}{n}\pi}$$

while i denotes an arbitrary integer number.

COROLLARY 4

§49 Since further while i denotes an arbitrary integer number, if one writes $2in - m$ for m, it is $\sin \frac{2in-m}{n}\pi = -\sin \frac{m}{n\pi}$, it will be

$$\int \frac{x^{2in-m-1}dx}{(1-x^n)^{\frac{2in-m}{n}}} = -\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}} = -\frac{\pi}{n\sin\frac{m}{n}\pi}$$

and

$$\int \frac{y^{2in-m-1}dy}{1+y^n} = -\int \frac{y^{m-1}dy}{1+y^n} = -\frac{\pi}{n\sin\frac{m}{n}\pi}.$$

Further, if one writes $(2i-1) - m$ for m, because of $\sin \frac{(2i-1)n-m}{n}\pi = \sin \frac{m}{n}\pi$ it will be

$$\int \frac{x^{(2i-1)n-m-1}dx}{(1-x^n)^{\frac{(2i-1)n-m}{n}}} = -\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}} = \frac{\pi}{n\sin\frac{m}{n}\pi}$$

$$\int \frac{y^{(2i-1)n-m-1}dy}{1+y^n} = \int \frac{y^{m-1}dy}{1+y^n} = \frac{\pi}{n\sin\frac{m}{n}\pi}.$$

Finally, in the same way it will be

$$\int \frac{x^{(2i-1)n+m-1}dx}{(1-x^n)^{\frac{(2i-1)n+m}{n}}} = -\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}} = -\frac{\pi}{n\sin\frac{m}{n}\pi}$$

$$\int \frac{y^{(2i-1)n-m-1}dy}{1+y^n} = -\int \frac{y^{m-1}dy}{1+y^n} = -\frac{\pi}{n\sin\frac{m}{n}\pi}.$$

COROLLARY 5

§50 Since the integral formulas $\int \frac{y^{m-1}dy}{1+y^n}$ occur more often, it will be worth one's while to list its values for the principal cases having put $y = \infty$. Therefore, it will be

$$\int \frac{dy}{1+y^2} = \frac{\pi}{2\sin\frac{\pi}{2}} = \frac{\pi}{2} \quad \text{because of} \quad \sin\frac{\pi}{2} = 1,$$
$$\int \frac{dy}{1+y^3} = \frac{\pi}{3\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}} \quad \text{because of} \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},$$
$$\int \frac{ydy}{1+y^3} = \frac{\pi}{3\sin\frac{2\pi}{3}} = \frac{2\pi}{3\sqrt{3}} \quad \text{because of} \quad \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2},$$
$$\int \frac{dy}{1+y^4} = \frac{\pi}{4\sin\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}},$$
$$\int \frac{y^2dy}{1+y^4} = \frac{\pi}{4\sin\frac{3\pi}{4}} = \frac{\pi}{2\sqrt{2}},$$
$$\int \frac{dy}{1+y^5} = \int \frac{y^3dy}{1+y^5} = \frac{\pi}{5\sin\frac{\pi}{5}},$$
$$\int \frac{ydy}{1+y^5} = \int \frac{y^2dy}{1+y^5} = \frac{\pi}{5\sin\frac{2\pi}{5}},$$
$$\int \frac{dy}{1+y^6} = \int \frac{y^4dy}{1+y^6} = \frac{\pi}{6\sin\frac{\pi}{6}} = \frac{\pi}{3}$$

and so forth.

Problem 7

§51 *To find the integral formula, whose value in the case $x = 1$ yields $\cos\frac{m}{n}\pi$.*

Solution

Since it is

$$\cos\frac{m}{n}\pi = \frac{\left(\frac{1}{2}-\frac{m}{n}\right)\left(\frac{1}{2}+\frac{m}{n}\right)}{\frac{1}{2}\cdot\frac{1}{2}} \cdot \frac{\left(\frac{3}{2}-\frac{m}{n}\right)\left(\frac{3}{2}+\frac{m}{n}\right)}{\frac{3}{2}\cdot\frac{3}{2}} \cdot \text{etc.},$$

compare the general form

$$P = \frac{be}{ac} \cdot \frac{(b+1)(e+1)}{(a+1)(c+1)} \cdot \text{etc.}$$

to this infinite product and hence set

$$a = \frac{1}{2}, \quad c = \frac{1}{2}, \quad b = \frac{1}{2} - \frac{m}{n}, \quad e = \frac{1}{2} + \frac{m}{n},$$

such that it is $s = a + c = b + e = 1$ and

$$\begin{aligned} s - a - b - 1 &= -1 + \frac{m}{n}, \qquad & s - a - e - 1 &= -1 - \frac{m}{n}, \\ s - b - c - 1 &= -1 + \frac{m}{n}, \qquad & s - c - e - 1 &= -1 - \frac{m}{n}, \end{aligned}$$

And therefore it will be

$$P = \frac{x^{-\frac{1}{2}}dx(1-x)^{-\frac{1}{2}}}{\int x^{-\frac{1}{2}-\frac{m}{n}}dx(1-x)^{-\frac{1}{2}-\frac{m}{n}}} = \frac{\int dx : \sqrt{x-xx}}{\int \frac{x^{\frac{m}{n}-\frac{1}{2}}dx}{(1-x)^{\frac{1}{2}+\frac{m}{n}}}}$$

But it is $\int \frac{dx}{\sqrt{x-xx}} = \pi$ having put $x = 1$, whence it is

$$P = \cos\frac{m}{n}\pi = \frac{\pi}{\int \frac{x^{\frac{m}{n}-\frac{1}{2}}dx}{(1-x)^{\frac{1}{2}+\frac{m}{n}}}}.$$

But by means of the remaining formulas of P one will have

$$P = \cos\frac{m}{n}\pi = \frac{\int x^{-\frac{1}{2}}dx(1-x)^{-1+\frac{m}{n}}}{\int x^{-\frac{1}{2}-\frac{m}{n}}dx(1-x)^{-1+\frac{m}{n}}} = \frac{\int x^{-\frac{1}{2}}dx(1-x)^{-1-\frac{m}{n}}}{\int x^{-\frac{1}{2}+\frac{m}{n}}dx(1-x)^{-1-\frac{m}{n}}}.$$

Q.E.I.

COROLLARY 1

§52 Put $x = y^2$ and the first form will go over into this one

$$\cos\frac{m}{n}\pi = \frac{\pi}{2\int \frac{y^{\frac{2m}{n}}dy}{(1-yy)^{\frac{1}{2}+\frac{m}{n}}}}$$

such that it is

$$\int \frac{x^{\frac{2m}{n}}dx}{(1-xx)^{\frac{1}{2}+\frac{m}{n}}} = \frac{\pi}{2\cos\frac{m}{n}\pi}.$$

COROLLARY 2

§53 But on the other hand by means of Theorem 1 it is

$$\int x^{\frac{m}{n}-\frac{1}{2}}dx(1-x)^{-\frac{1}{2}-\frac{m}{n}} = \int \frac{y^{\frac{m}{n}-\frac{1}{2}}dy}{1+y}$$

having put $y=\infty$. Therefore, because it is

$$\int \frac{y^{\frac{m}{n}-\frac{1}{2}}dy}{1+y} = \frac{\pi}{\cos\frac{m}{n}\pi},$$

put y^n for y and it will be

$$\int \frac{y^{m+\frac{1}{2}n-1}dy}{1+y^n} = \frac{\pi}{n\cos\frac{m}{n}\pi} = \int \frac{y^{\frac{1}{2}n-m-1}dy}{1+y^n}.$$

COROLLARY 3

§54 If also the remaining formulas are converted by means of Theorem 1, these equations will result

$$\int x^{-\frac{1}{2}}dx(1-x)^{-1+\frac{m}{n}} = \int \frac{y^{-\frac{1}{2}}dy}{(1+y)^{\frac{1}{2}+\frac{m}{n}}} = \int \frac{y^{\frac{m}{n}-1}dy}{(1+y)^{\frac{1}{2}+\frac{m}{n}}},$$

$$\int x^{-\frac{1}{2}-\frac{m}{n}}dx(1-x)^{-1+\frac{m}{n}} = \int \frac{y^{-\frac{1}{2}-\frac{m}{n}}dy}{\sqrt{1+y}} = \int \frac{y^{\frac{m}{n}-1}dy}{\sqrt{1+y}}$$

having put $y=\infty$. Therefore, having put y^n for y it will be

$$\cos\frac{m\pi}{n} = \frac{\int \frac{y^{m-1}dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}{\int \frac{y^{m-1}dy}{\sqrt{1+y^n}}} = \frac{\int \frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}{\int \frac{y^{\frac{1}{2}-m-1}dy}{\sqrt{1+y^n}}}.$$

COROLLARY 4

§54[a] If one writes $\frac{1}{2}n-m$ for m, because of

$$\cos\left(\frac{1}{2}n-m\right)\frac{\pi}{n} = \sin\frac{m}{n}\pi$$

one will at first obtain

$$\frac{\pi}{n\sin\frac{m}{n}\pi}=\int\frac{y^{m-1}dy}{1+y^n}$$

as before; but the remaining formulas will give

$$\sin\frac{m\pi}{n}=\frac{\int\frac{y^{\frac{1}{2}n-m-1}dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int\frac{y^{\frac{1}{2}n-m-1}dy}{\sqrt{1+y^n}}}=\frac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int\frac{y^{m-1}dy}{\sqrt{1+y^n}}},$$

and since for the cosine it is possible to take a negative m, it will also be

$$\sin\frac{m\pi}{n}=\frac{\int\frac{y^{\frac{1}{2}n-m-1}dy}{(1+y^n)^{\frac{m}{n}}}}{\int\frac{y^{m-\frac{1}{2}n-1}dy}{\sqrt{1+y^n}}}=\frac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{m}{n}}}}{\int\frac{y^{n-m-1}dy}{\sqrt{1+y^n}}}.$$

COROLLARY 5

§55 But on the other hand it is also possible to find another formula for the cosine from the preceding problem. For, since having put $2m$ for m it is

$$\frac{\pi}{n\sin\frac{2m}{n}\pi}=\frac{\pi}{2n\sin\frac{m}{n}\pi\cos\frac{m}{n}\pi}=\int\frac{y^{2m-1}dy}{1+y^n}$$

and

$$\int\frac{y^{m-1}dy}{1+y^n}=\frac{\pi}{n\sin\frac{m}{n}\pi},$$

if this form is divided by that one, we will have

$$2\cos\frac{m}{n}\pi=\frac{\int\frac{y^{m-1}dy}{1+y^n}}{\int\frac{y^{2m-1}dy}{1+y^n}}\quad\text{and}\quad\cos\frac{m}{n}\pi=\frac{\frac{1}{2}\int\frac{y^{m-1}dy}{1+y^n}}{\int\frac{y^{2m-1}dy}{1+y^n}}.$$

COROLLARY 6

§56 Therefore, lo and behold the many integral formulas, which in the case $y=\infty$ yield $\sin\frac{m}{n}\pi$:

$$\text{I.}\quad \frac{\pi}{n\int\frac{y^{m-1}dy}{1+y^n}},\qquad \text{II.}\quad \frac{\int\frac{y^{\frac{1}{2}n-m-1}dy}{1+y^n}}{2\int\frac{y^{n-2m-1}dy}{1+y^n}},\qquad \text{III.}\quad \frac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{m}{n}}}}{\int\frac{y^{n-m-1}dy}{\sqrt{1+y^n}}},$$

$$\text{IV.}\quad \frac{\int\frac{y^{m-\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{m}{n}}}}{\int\frac{y^{m-\frac{1}{2}n-1}dy}{\sqrt{1+y^n}}},\qquad \text{V.}\quad \frac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int\frac{y^{m-1}dy}{\sqrt{1+y^n}}},\qquad \text{VI.}\quad \frac{\int\frac{y^{\frac{1}{2}n-m-1}dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int\frac{y^{\frac{1}{2}n-m-1}dy}{\sqrt{1+y^n}}},$$

where it is to be noted that in the forms III and IV, likewise in V and VI the numerators and the denominators are equal to each other.

COROLLARY 7

§57 In like manner we will have as many formulas for $\cos\frac{m}{n}\pi$ which are:

$$\text{I.}\quad \frac{\pi}{n\int\frac{y^{\frac{1}{2}n-m-1}dy}{1+y^n}},\qquad \text{II.}\quad \frac{\int\frac{y^{m-1}dy}{1+y^n}}{2\int\frac{y^{2m-1}dy}{1+y^n}},\qquad \text{III.}\quad \frac{\int\frac{y^{m-1}dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}{\int\frac{y^{m-1}dy}{\sqrt{1+y^n}}},$$

$$\text{IV.}\quad \frac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}{\int\frac{y^{\frac{1}{2}n-m-1}dy}{\sqrt{1+y^n}}},\qquad \text{V.}\quad \frac{\int\frac{y^{-m-1}dy}{(1+y^n)^{\frac{1}{2}-\frac{m}{n}}}}{\int\frac{y^{-m-1}dy}{\sqrt{1+y^n}}},\qquad \text{VI.}\quad \frac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{1}{2}-\frac{m}{n}}}}{\int\frac{y^{\frac{1}{2}n+m-1}dy}{\sqrt{1+y^n}}}.$$

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§58 Hence it is possible to deduce even formulas for the tangent of the angle $\frac{m}{n}\pi$; I will exhibit the simpler ones of them here:

$$\tan\frac{m}{n}\pi=\frac{\int\frac{y^{\frac{1}{2}-m-1}dy}{1+y^n}}{\int\frac{y^{m-1}dy}{1+y^n}},\qquad \tan\frac{m}{n}\pi=\frac{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int\frac{y^{m-1}dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}=\frac{\int\frac{y^{\frac{1}{2}n-m-1}dy}{(1+y^n)^{1-\frac{m}{n}}}}{\int\frac{y^{\frac{1}{2}n-1}dy}{(1+y^n)^{\frac{1}{2}+\frac{m}{n}}}}.$$

More extraordinary properties will be found from the combination of these formulas; as if it was $n=4$ and $m=1$, it will be

$$\frac{1}{\sqrt{2}} = \frac{\pi}{4\int\frac{dy}{1+y^4}} = \frac{\int\frac{dy}{1+y^4}}{2\int\frac{ydy}{1+y^4}} = \frac{\int\frac{ydy}{\sqrt[4]{1+y^4}}}{\int\frac{yydy}{\sqrt{1+y^4}}} = \frac{\int\frac{ydy}{\sqrt[4]{(1+y^4)^3}}}{\int\frac{dy}{\sqrt{1+y^4}}} = \frac{\int\frac{dy}{\sqrt[4]{(1+y^4)^3}}}{\int\frac{dy}{\sqrt{1+y^4}}},$$

whence it is concluded that it will be

$$\int\frac{ydy}{\sqrt[4]{(1+y^4)^3}} = \int\frac{dy}{\sqrt[4]{(1+y^4)^3}}$$

in the case $y = \infty$ or that it is

$$\int\frac{(1-y)dy}{\sqrt[4]{(1+y^4)^3}} = 0;$$

but I will not spend more time on finding such properties here.