Investigation of Functions from a given condition of the differentials *

Leonhard Euler

§1 If V denotes an arbitrary function of the two variables x and y, and it is differentiated that its differential arises as

$$dV = Pdx + Qdy,$$

but then these two quantities P and Q are differentiated again, and so it arises

$$dP = pdx + rdy \quad \text{and} \quad dQ = sdx + 1dy,$$

it is known that it will always be $r = s$. I usually also express this property this way that I say that it is

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right).$$

Of course, by an expression of this kind $\left(\frac{dP}{dy}\right)$ I indicate that the function P is differentiated in such a way that only the quantity y is treated as a variable and the resulting differential is divided by dy, having done which a finite quantity free from differentials necessarily arises.

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§2 Therefore, if the formula $Pdx + Qdy$ was of such a nature that according to this notation in it it is $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, we hence vice versa conclude that this formula $Pdx + Qdy$ indeed arises from the differentiation of a certain function V of x and y. But because this property extends very far, hence it is not possible to conclude more than that the formula $Pdx + Qdy$ is integrable, and is not defined in general by this condition alone, whence a peculiar property of the function, from whose differentiation it arose, could be deduced.

§3 But whenever the function V is referred to a certain class, then having put $dV = Pdx + Qdy$, among the quantities P and Q, except for this general property, another particular relation intercedes. So we know, if V is a function of no dimension of the two variables x and y, that then except for that general property by which it is $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$ additionally this particular one holds that it is $Px + Qy = 0$. Further, in similar manner it was demonstrated, if V was a function of such a kind of x and y, whose number of dimensions was $= n$, that its differential $dV = Pdx + Qdx$ is always of such a nature that it is

$$nV = Px + Qy.$$

Therefore, in this case it is especially remarkable that the integral of the differential formula $Pdx + Qdy$ can be assigned immediately, since it is

$$V = \frac{1}{n}(Px + Qy).$$

§4 Since these things are demonstrated, recently it came into my mind, to treat questions of this kind in inverse manner, and investigate a method, by means of which, if having put $dV = Pdx + Qdy$ it was already found that it is either $Px + Qy = 0$ or $Px + Qy = nV$, it can be found vice versa that V is either a function of no dimension or a homogeneous function, in which the two variables x and y constitute n dimensions everywhere. Of course, not having taken into account those already known properties, from that alone that it is either $Px + Qy = 0$ or $Px + Qy = nV$ by means of legitimate analytical reasons it has to be found that the function V has this property that it is either of no dimension or homogeneous of n dimensions. But it is to be understood that the general property $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$ has to hold always, without which the equation $dV = Pdx + Qdy$ would even be absurd.

§5 These questions, which seem to be hardly touched until now, open a very vast field to further extend the boundaries of Analysis. For, having propounded the equation $dV = Pdx + Qdy$, one can ask for the nature of the function V, if any relation between the two quantities P and Q or even between the three P, Q and V is propounded. Even though questions of this kind seem to be almost new, there is nevertheless no doubt that the method to resolve them correctly will have the highest utility throughout the whole of Mathematics. For, in the problem of vibrating strings the whole power of the solution is to be referenced to this kind, since it is contained in a certain relation between the quantities P and Q. Further, I even captured the whole science of the motion of fluids in differential formulas of this kind, where a certain relation between parts of the differential formulas are prescribed, from which because of the lack of such a method hardly anything can be concluded.

§6 But questions of this kind can also be propounded in another way that only the function V whose nature is in question occurs. For, because having put $dV = Pdx + Qdy$ it is $P = \left(\frac{dV}{dx}\right)$ and $Q = \left(\frac{dV}{dy}\right)$, the first question can be enunciated as this:

To find the nature of the function V that it is

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = 0,$$

the second on the other hand in this way:

To find the nature of the function V that it is

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = nV;$$

and in the first it must be shown that V is a function of no dimension of x any y; in the second on the other hand that the same is a homogeneous function of n dimensions. But it is to be made a note that in our notation it is

$$dV = dx\left(\frac{dV}{dx}\right) + dy\left(\frac{dV}{dy}\right).$$

§7 Therefore, a further extending function will behave in such a way that having propounded any arbitrary relation between the quantities V, $\left(\frac{dV}{dx}\right)$, and $\left(\frac{dV}{dy}\right)$ has to be defined, how the function V depends on x and y. Further, questions of this kind can also be extended to functions of three or more variables; even quantities to arise from double or triple differentiation can be introduced; of this kind are $\left(\frac{ddV}{dx^2}\right)$, $\left(\frac{ddV}{dxdy}\right)$, $\left(\frac{ddV}{dy^2}\right)$, $\left(\frac{d^3V}{dx^3}\right)$, $\left(\frac{d^3V}{dx^2dy}\right)$ etc., whose meaning is that for the sake of an example $\left(\frac{d^3V}{dx^2dy}\right)$ arises, if first V is differentiated having taken only x as a variable, and the differential is divided by dx that $\left(\frac{dV}{dx}\right)$ arises; but then this quantity is differentiated again, having taken only x as a variable, that its differential divided by dx yields $\left(\frac{ddV}{dx^2}\right)$, which finally differentiated once more, having taken only y as a variable, and divided by dy will give $\left(\frac{d^3V}{dx^2dy}\right)$.

§8 But while using this notation it is to be noted that it is

$$\left(\frac{ddV}{dxdy}\right)=\left(\frac{ddV}{dydx}\right)$$

and

$$\left(\frac{d^3V}{dx^2dy}\right)=\left(\frac{d^3V}{dxdydx}\right)=\left(\frac{d^3V}{dydx^2}\right).$$

It does not matter, in which order the quantities x and y, both of which is assumed as the only variable in each differentiation, are distributed, as long as only the prescribed number of differentiations is done. Even if V is a function of three variables x, y, z a similar convenience takes place, for it will be

$$\left(\frac{d^3V}{dxdydz}\right)=\left(\frac{d^3V}{dxdzdy}\right)=\left(\frac{d^3V}{dydzdx}\right)=\left(\frac{d^3V}{dzdxdy}\right)=\left(\frac{d^3V}{dzdydx}\right);$$

but here I will restrict my investigations only to functions of two variables.

§9 This way we not only led to unusual still hardly treated questions but also to new signs, which we are still hardly used to; hence this method, whose cultivation we must desire that much, is justly to be considered as a completely new part of Analysis. Therefore, I think it will be quite useful, if I constitute only the first foundations of this method, and because of the novelty of the subject I do not dare to promise hardly the smallest part of the construction to be built on them. But in time without any doubt a lot of and tireless work will be necessary, before this part of Analysis, in itself certainly most difficult, can, I do not say, be completed, but only accommodated to a broader use. Therefore, I decided to explain the things which I found until now in this subject in order and carefully, that for others, which the dignity of the subject encourages to invest the same amount of work, I clear the first obstacles out of the way and prepare the minds for this new kind of investigations.

§10 But before I provide this service, I think certain per se perspicuous principles are to be mentioned in advance. At first, of course, if it was $\left(\frac{dV}{dx}\right) = 0$, it is understood that the function V does not depend on x at all, but is conflated of the other variable y alone and constants, and so the form $\left(\frac{dV}{dy}\right)$ will also be a function of y only. But if vice verse $\left(\frac{dV}{dy}\right)$ was a function of y only, the quantity V will be an aggregate of a function only of y and of a function only of x; therefore, in this case the form $\left(\frac{dV}{dx}\right)$ will be a function only of x. Further, if it was $dV = RdS$, it is necessary that R is a function of S or S is one of R, whence also V will be a function of S, or of R; since otherwise the integral $\int RdS$ is not determined. Therefore, having set these principles I will resolve the two initially mentioned questions, which marked the starting point of this investigation, afterwards I will proceed to others.

PROBLEM 1

§11 While it is $dV = Pdx + Qdy$, if it was $Px + Qy = 0$, to find, what a function of x and y V is that this condition is satisfied.

SOLUTION

Therefore, since between the quantities P and Q this condition is prescribed that it is $Px + Qy = 0$, it will be $Q = -P\frac{x}{y}$, which value substituted in the equality $dV = Pdx + Qdy$ will give

$$dV = P\left(dx - \frac{xdy}{y}\right) = \frac{P(ydx - xdy}{y};$$

therefore, it is necessary that the formula

$$\frac{P(ydx - xdy)}{y}$$

is integrable. To reduce it to the form RdS, represent it this way

$$dV = Py \cdot \frac{ydx - xdy}{yy}.$$

For, having taken

$$\frac{ydx - xdy}{yy} = dS$$

and $S = \frac{x}{y}$, since it is $dV = PydS$, it is necessary that Py is a function of S and hence V will be a function of S, this means of $\frac{x}{y}$. Therefore, the prescribed property, by which it is $Px + Qy = 0$, declares a form of the function of this kind that V is any arbitrary function of $\frac{x}{y}$; but hence it is manifest that for V a function of no dimension of x and y arises.

COROLLARY 1

§12 Therefore, if because of

$$P = \left(\frac{dV}{dx}\right) \quad \text{and} \quad Q = \left(\frac{dV}{dy}\right)$$

this property of the function V is propounded that it is

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = 0,$$

we hence safely conclude that V is a function of the formula $\frac{x}{y}$ or, what reduces to the same, it is a function of no dimension of x and y.

§13 Therefore, hence vice versa it is confirmed, what is known for a long time already, as often as V was a function of no dimension of x and y, that so often it will also be:

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = 0.$$

But as this is proven very easily its converse required a singular proof.

Scholium

§14 The essence of this solution lies at this that I reduced the differential of the function V to this form $dV = RdS$, from which, since it contains the one single differential dS, it manifestly follows that V is a function of the quantity S only; but it was $S = \frac{x}{y}$, and it is known that all functions of $\frac{x}{y}$ at the same time are functions of no dimension and vice versa. Therefore, it is understood that the same principle is to be applied in the solution of the following problems. Furthermore, without the letters P and Q the problem could have been so propounded as resolved this way: Of course, if one has to find the form of the function V that it is

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = 0,$$

because it is

$$dV = dx\left(\frac{dV}{dx}\right) + dy\left(\frac{dV}{dy}\right),$$

because of

$$\left(\frac{dV}{dy}\right) = -\frac{x}{y}\left(\frac{dV}{dx}\right),$$

it will be

$$dV = \left(dx - \frac{xdy}{y}\right)\cdot\left(\frac{dV}{dx}\right) = y\left(\frac{dV}{dx}\right)\cdot d.\frac{x}{y},$$

it is manifest that V necessarily must be a function of this one quantity $\frac{x}{y}$. But how, if it was $dV = Rdr$, it is rightly concluded that V is a function of r only, so further, if is was

$$dV = Rdr + Sds,$$

we have to conclude that V is a function of the two variables r and s; this principle will have use in the investigation of the nature of function of three variables, while a certain condition of the differentials is propounded.

PROBLEM 2

§15 While it is

$$dV = Pdx + Qdy,$$

to define the nature of the function V that it is

$$Px + Qy = nV,$$

where n denotes an arbitrary number.

SOLUTION

From the prescribed condition

$$Px + Qy = nV$$

found the one of the quantities P and Q, say

$$Q = \frac{nV}{y} - \frac{Px}{y},$$

which value substituted in the differential equality will give

$$dV = Pdx + \frac{nVdy}{y} - \frac{Pxdy}{y},$$

which immediately gives this form:

$$dV = -\frac{nVdy}{y} = Py\frac{(ydx - xdy)}{yy} = Pyd.\frac{x}{y},$$

which, that the first side is rendered integrable, shall be multiplied by y^{-n}, and so it will arise

$$d.y^{-n}V = Py^{1-n}d.\frac{x}{y},$$

whence we conclude that $y^{-n}V$ is a function of the quantity $\frac{x}{y}$ or a function of no dimension of the two variables x and y. Therefore, let Z denote an arbitrary function of no dimension, and because it is $y^{-n}V = Z$, it will be $V = y^nZ$; but such an expression contains all homogeneous functions of x and y, whose number of dimensions is $= n$.

COROLLARY 1

§16 Therefore, whenever we know that V is a function of such a nature that it is

$$nV = x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right),$$

we will able to affirm with certainty that V is a homogeneous function in which the two variables constitute n dimensions everywhere.

COROLLARY 2

§17 Therefore, if one puts

$$dV = Pdx + Qdy,$$

also P and Q will be homogeneous functions of x and y, but whose number of dimensions is $n-1$, lower by one order, of course.

COROLLARY 3

§18 Hence, if P and Q were homogeneous functions of the two variables x and y, whose number of dimensions is the same, say $= n-1$, and if it furthermore was

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right),$$

such that $Pdx + Qdy$ is a complete differential formula, then its integral will be assigned most easily. It will be

$$\int (Pdx + Qdy) = \frac{Px + Qy}{n}.$$

Therefore, as long as n does not vanish, an integral of formulas of this kind can be exhibited without any other operation.

SCHOLIUM

§19 Therefore, lo and behold the solution of both questions which I mentioned initially; since they already contain a specimen of this method, which is to be used in this kind of questions, it will be possible to apply the same to the solution of other similar problems. Therefore, because here a certain relation between the function V and the quantities hence derived $P = \left(\frac{dV}{dx}\right)$ and $Q = \left(\frac{dV}{dy}\right)$, from which the nature of the function V must be defined, that I preserve a certain order in the questions of this kind, since so P and Q as V are functions of x and y, at first I will assume that the form of the one of the letters P and Q is given; further, I will proceed to problems of such a kind, in which a certain relation between P and Q is prescribed; but then to the others, where between P and V or between Q and V a certain relation must intercede. Finally, I will assume that the given relation is extended to all three quantities V, P and Q, as it was done in the second problem. But because here we only take into account the quantities V, $\left(\frac{dV}{dx}\right)$, $\left(\frac{dV}{dy}\right)$, it is evident that investigations of this kind can be extended a lot further, while the prescribed relation contains other quantities derived from V, as for example $\left(\frac{ddV}{dx^2}\right)$, $\left(\frac{ddV}{dxdy}\right)$ and $\left(\frac{ddV}{dy^2}\right)$. But so much is missing that I dare to promise solutions of all problems of this kind, that I will rather expand only those which are easier. For, it will soon be plain that innumerable problems of such a kind can be propounded whose solutions supersede these first attempts by far, and, before this new branch of Analysis was further developed, cannot be hoped for.

Problem 3

§20 While it is

$$dV = Pdx + Qdy,$$

if P was a function only of x to define the nature of the function V.

Solution

That from the part Pdx of the differential the function V can already be found is known, while having considered y as a constant the differential Pdx is integrated and the arbitrary constant to be added is assumed to involve the other variable y somehow. Therefore, because P is a function of x only, $\int Pdx$ will also be a function of such a kind, which shall be $= X$, and represent the constant to be added by an arbitrary function Y of y only. Therefore, hence $V = X + Y$ will arise, or the nature in question of the function V will consist is this that V is the aggregate of two functions, the one of x only, the other of y only.

COROLLARY

§21 Therefore, because hence it is

$$dV = dX + dY,$$

it is manifest, if P was a function of x only, that then Q will be a function of y only, which is certainly a per se very well known property.

PROBLEM 4

§22 While it is

$$dV = Pdx + Qdy,$$

if P was a function of y only, to define the nature of the function V.

SOLUTION

Since P is a function of y only, from the part Pdx of the differential alone, having considered y as a constant, the function V is defined in such a way that it is $V = Px + Y$, where Y denotes an arbitrary function of y. Hence the nature in question of the function V will consist in this that, while P and Y denote arbitrary functions of y, the form of the function V always is of this kind $V = Px + Y$.

COROLLARY 1

§23 Therefore, if $P = \left(\frac{dV}{dx}\right)$ is a function of y only, since it is

$$dV = Pdx + xdP + dY,$$

it will be

$$Q = \frac{xdP + dY}{dy},$$

or because of $\frac{dP}{dy} = \left(\frac{ddV}{dxdy}\right)$ it will be

$$Q = x\left(\frac{ddV}{dxdy}\right) + \frac{dY}{dy}.$$

COROLLARY 2

§24 Therefore, as often as $\left(\frac{dV}{dx}\right)$ was a function of y only, so often it is necessary that it is

$$\left(\frac{dV}{dy}\right) = x\left(\frac{ddV}{dxdy}\right) + \frac{dY}{dy},$$

where $\frac{dY}{dy}$ can denote an arbitrary function only of y. Hence one can vice versa conclude, if it was

$$() = x\left(\frac{ddV}{dxdy}\right) + f \cdot y,$$

that $\left(\frac{dV}{dx}\right)$ will be a function only of y.

COROLLARY 3

§25 In similar manner it will be shown, if $Q = \left(\frac{dV}{dy}\right)$ was a function only of x, that it will be

$$V = Qy + X,$$

where X denotes an arbitrary function only of x; but then also, if it was

$$\left(\frac{dV}{dx}\right) = y\left(\frac{ddV}{dxdy}\right) + f \cdot x,$$

that $\left(\frac{dV}{dy}\right)$ will be a function only of x.

PROBLEM 5

§26 While $dV = Pdx + Qdy$, if P was a homogeneous function of x and y, whose number of dimensions is $= n$, to define the nature of the function V.

SOLUTION

Since P is a homogeneous function of n dimensions, if the part Pdx of the differential is integrated having considered y as a constant, the integral will be a homogeneous function of $n+1$ dimensions, let Z be such an arbitrary homogeneous function, and it will be $V = Z + Y$, while Y is an arbitrary function of y only, in which the nature in question of the function V consists.

COROLLARY

§27 Therefore, in similar manner it will be shown, if Q was a homogeneous function of n dimensions, that it will be $V = Z + X$, where, as before, Z denotes an arbitrary homogeneous function of $n+1$ dimensions and X one of x only.

PROBLEM 6

§28 While $dv = Pdx + Qdy$, if it was $Q = nP$, to define the nature of the function V.

SOLUTION

Because it is $Q = nP$, it will be $dV = P(dx + ndy)$; hence having put $x + ny = s$, it will be $dV = Pds$. From this V can only have a certain value, if P is a function of s, from which also V will be a function of s. As a logical consequence, if it was $Q = nP$, the nature of the quantity V will consist in this that V is an arbitrary function of the the formula $x + ny$; or if the character Φ is used to denote an arbitrary function of the quantity, which it prefixed to, it will be $V = \Phi : (x + ny)$.

PROBLEM 7

§29 While $dV = Pdx + Qdy$, if it was

$$Py + Qx = 0,$$

to find the nature of the function V.

SOLUTION

Since it is $Py + Qx = 0$, it will be $Q = -\frac{Py}{x}$, and hence

$$dV = Pdx - \frac{Pydy}{x} = \frac{P}{x}(xdx - ydy).$$

Therefore, having put $xx - yy = s$, because of $xdy - ydy = \frac{1}{2}ds$, it is $dV = \frac{P}{2x}ds$. Since this formula by assumption is integrable, it is necessary that $\frac{P}{2x}$ is a function of s, whence also V will arise as a function of $s = xx - yy$. Therefore, the nature in question will consist in this that V is an arbitrary function of the quantity $xx - yy$.

PROBLEM 8

§30 While it is $dV = Pdx + Qdy$, if it was $Q = Pp$, while p expresses an arbitrary given function of x and y, to define the nature of the function V.

SOLUTION

Therefore, we will have $dV = Pdx + Ppdy = P(dx + pdy)$. Now, consider the formula $dx + pdy$, if which was not integrable per se, a multiplicator q will be given, which renders is integrable. Therefore, let $qdx + pqdy = ds$, and s will also be a given function of x and y, and because of $dx + pdy = \frac{ds}{q}$, one will have $dV = \frac{P}{q}ds$. Therefore, it is necessary that this formula is integrable and hence the nature in question will consist in this that V is an arbitrary function of the quantity s, how which is composited of x and y, must be concluded from the condition of the given quantity p.

COROLLARY 1

§31 This problem extends sufficiently far, since in it an arbitrary ratio between the quantities P and Q or $\left(\frac{dV}{dx}\right)$ and $\left(\frac{dV}{dy}\right)$ is propounded. For, if it is $P : Q = 1 : p$, whatever function of x and y was given for p, it can be defined, what kind of function V is.

COROLLARY 2

§32 But although always a multiplicator q exists, which renders the formula $dx + pdy$ integrable, it can nevertheless often happen that because of the missing Analysis this multiplicator cannot be assigned. And in these cases the solution of the problem is impeded.

COROLLARY 3

§33 But at another occasion I showed that multiplicators of this kind can always be exhbited, if the equation $dx + pdy = 0$ can be resolved. Hence if p was not a function of x and y of such kind that the equation $dx + pdy = 0$ can be resolved, this is to be attributed to the missing Analysis, if the problem cannot be resolved.

PROBLEM 9

§34 If X and Y are given functions, the first only of x, the other on the other hand only of y, but then p is also a given function of x and y, to define the nature of the function V, that having put $dV = Pdx + Qdy$ it is $Q = (P + X)p + Y$.

SOLUTION

Therefore, because it is $Q = (P + X)p + Y$, the differential equation will be $dV = Pdx + Ppdy + XPdy + Ydy$, which is reduced to this form:

$$dV = (P + X)(dx - dy) - Xdx + Ydy,$$

where the parts Xdx and Ydy are integrable per se. Therefore, again find the multiplicator q rendering the formula $dx + pdy$ integrable, and let $q = (dx + pdy) = ds$, and it will be

$$dV = \frac{P + X}{q} ds - Xdx + Ydy,$$

since which formula is integrable by assumption, it is necessary that $\frac{P+X}{q}$ is a function of s only, if which is put $= S$, it will arise $V = \int Sds - \int Xdx + \int Ydy$, which is the desired nature of the function V.

COROLLARY 1

§35 Therefore, because from the given function p the function s is defined, if for Σ an arbitrary function of this quantity s is taken, the function V in question must be of such a nature that it is

$$V = \Sigma - \int X dx + \int Y dy.$$

COROLLARY 2

§36 Therefore, this addition of the functions X and Y does not render the solution of the problem more difficult; as long as X is function only of x, and Y one only of y. But the solution, as before, depends on the resolution of the differential equation $dx + pdy = 0$, if which supersedes our powers, we also cannot resolve the problem.

COROLLARY 3

§37 One can instead of X and Y also assume other functions of the variables x and y, say M and N, as long as the formula $Mdx + Ndy$ admits an integration. For, if this condition is propounded that it is $Q - N = (P - M)p$, the propounded function V will arise expressed as this:

$$V = \Sigma + \int (Mdx + Ndy).$$

PROBLEM 10

§38 While $dV = Pdx + Qdy$, to find, what kind of function V is of x and y that it is

$$Q = \frac{Py}{x} + nx.$$

SOLUTION

Because it is

$$Q = \frac{Py}{x} + nx,$$

it will be

$$dV = \frac{P}{x}(xdx + ydy) + nxdy.$$

Set $xx + yy = ss$ that it is $x = \sqrt{ss - yy}$ and it will be

$$dV = \frac{Ps}{x}ds + nxdy = \frac{Ps}{x}ds + ndy\sqrt{ss - yy},$$

whence it is plain that V is function of y and s, and of course such a one that having put s to be constant it differentiated yields $ndy\sqrt{ss-yy}$. Hence vice verse the function V will be found, if the formula $ndy\sqrt{ss-yy}$, having considered s as a constant, is integrated, and additionally any function of s is added. Therefore, because it is

$$\int ndy\sqrt{ss-yy} = \frac{1}{2}ny\sqrt{ss-yy} + \frac{1}{2}nss \arcsin\frac{y}{s},$$

by assuming Φ for the sign of an arbitrary function it will be

$$V = \frac{1}{2}nxy + \frac{1}{2}n(xx+yy)\arctan\frac{y}{x} + \Phi : (xx+yy),$$

in which form the function V in question is always contained.

Scholium

§39 This example, no matter how particular, is nevertheless not contained in the preceding problem, and also not in its generalization made in the last corollary, since in the reduced formula the term

$$nxdy = ndy\sqrt{ss-yy}$$

is not integrable. Hence, carefully note the artifice I used here and which consists in this that I reduced the value of the differential dV to the two other variables s and y, of course $dV = Rds + Tdy$, whose first term Tdy is given absolutely, whence the problem belongs to the first class, in which the one of the two quantities P and Q is known. And by means of this artifice the following problem extending a lot further can be resolved.

Problem 11

§40 If p and t are given arbitrary functions of the two variables x and y, to define the nature of the function V that having put $dV = Pdx + Qdy$ it is $Q = Pp + t$.

SOLUTION

Therefore, one will have, having substituted that value for Q

$$dV = P(dx + pdy) + tdy, \qquad (1)$$

where for the differential formula $dx + pdy$ again a suitable multiplicator q is to be found, which renders it integrable. Therefore, let $q(dx + pdy) = ds$, and now introduce the quantity s as a new variable, by means of which and y the other variable x is defined. This way x will become equal to a certain given function of s and y, which shall be written in t instead of x everywhere, and so t will also be a given function of s and y. Therefore, because of $dx + pdy = \frac{ds}{q}$ it is $dV = \frac{P}{q}ds + tdy$, V will be a function of s and y of such a kind which having considered s as a constant differentiated yields tdy, whence vice versa for finding the function V integrate the differential formula tdy, having considered s as a constant this time; let the integral arising this way be $\int tdy = T$, which is therefore also given, then, since the quantity P is not given, it will be $V = T + \Phi : s$. Finally, here for s and in T resubstitute the value of s via x and y, and it will become plain, how the function V is composited of x and y.

EXAMPLE

§41 *Let the nature of the function V be in question that having put $dV = Pdx + Qdy$, it is $Px + Qy = n(xx + yy)$.*

Therefore, because it is

$$Q = -\frac{Px}{y} + \frac{n(xx+yy)}{y}, \quad \text{it will be} \quad p = -\frac{x}{y} \quad \text{and} \quad t = \frac{n(xx+yy)}{y},$$

whence

$$dx + pdy = dx - \frac{xdy}{y}.$$

Take $q = \frac{1}{y}$, it will be

$$ds = \frac{dx}{y} - \frac{xdy}{yy} \quad \text{and} \quad s = \frac{x}{y}$$

and hence

$$x = sy \quad \text{and} \quad t = ny(ss+1).$$

Hence having considered s as a constant one will have

$$\int tdy = \frac{1}{2}nyy(ss+1) = \frac{1}{2}n(xx+yy) = T;$$

and so it finally arises

$$V = \frac{1}{2}n(xx+yy) + \Phi : \frac{x}{y},$$

where it is to be noted that $\Phi : \frac{x}{y}$ expressed an arbitrary function of no dimension of the variables x and y.

Scholium

§42 Hence, luckily we covered the case, in which the relation between P and Q is expressed by means of an arbitrary equation of first degree, in which the quantities P and Q do not raise higher than the first dimension. For, from such an equation Q will always be defined in such a way that it is $Q = Pp + t$, while p and t are arbitrary given functions of x and y. But here it is again in short supply, as often as the equation $dx + pdy = 0$ cannot be resolved, since then the quantity s cannot be found. But then, even though this quantity s was found, since it is highly transcendental, from it it will nevertheless in most cases be very difficult to define the variable x such that only the two s and y remain in the calculation. One will be able to think of auxiliary tools and theorems, although from the given function t the variable x is not eliminated, one can nevertheless find the integral T of the formula t, which has to arise, having considered s as a constant. But no matter how large these difficulties are, they are not to be attributed to this method, which I started to sketch. Therefore, let us see how far we can proceed, if a relation between P and Q either of second or higher degrees is given.

Problem 12

§43 While it is $dV = Pdx + Qdy$, to define the nature of the function V that it is $PQ = \alpha$.

SOLUTION

Because it is $Q = \frac{\alpha}{P}$, it will be $dV = Pdx + \frac{\alpha dy}{P}$, and it is in question, what kind of function P must be that this differential formula $Pdx + \frac{\alpha dy}{P}$ becomes integrable. But let us apply an obvious transformation of integrals here, according to which it is $\int zdu = zu - \int udz$, and one will find:

$$V = Px + \frac{\alpha y}{P} - \int xdP + \int \frac{\alpha ydP}{PP}.$$

Hence it is necessary that this differential formula $dP\left(\frac{\alpha y}{PP} - x\right)$ is integrable; this can only happen, if $\frac{\alpha y}{PP} - x$ is a function of P; in this case also the integral $\int dP\left(\frac{\alpha y}{PP} - x\right)$ will be a function of P. Therefore, let Π denote an arbitrary function of P, and put $\frac{\alpha y}{PP} - x = \Pi$, from the resolution of which equation the quantity P is understood to be defined by x and y. But having found this function P we will have

$$V = Px + \frac{\alpha y}{P} + \int \Pi dP.$$

COROLLARY 1

§44 Therefore, the simplest case, by which this problem is satisfied, is, if it is $\Pi = 0$, in which case it is

$$P = \sqrt{\frac{\alpha y}{x}} \quad \text{and} \quad \int \Pi dP = \text{Const.}$$

Therefore, we will have

$$V = 2\sqrt{\alpha xy} + \text{Const.},$$

for because of

$$dV = \frac{dx\sqrt{\alpha y}}{\sqrt{x}} + \frac{dy\sqrt{\alpha x}}{\sqrt{y}}$$

it will be $PQ = \alpha$, of course.

COROLLARY 2

§45 Then having taken $\Pi = \beta$, it will be

$$P = \sqrt{\frac{\alpha y}{x+\beta}} \quad \text{and} \quad \int \Pi dP = \beta P.$$

Therefore, in this case we will obtain the following satisfying function:

$$V = x\sqrt{\frac{\alpha}{x+\beta}} + \sqrt{\alpha y(x+\beta)} + \beta\sqrt{\frac{\alpha y}{x+\beta}} = 2\sqrt{\alpha y(x+\beta)},$$

and more general it is manifest to satisfy:

$$V = 2\sqrt{\alpha(x+\beta)(y+\gamma)}.$$

COROLLARY 3

§46 If we want higher composite functions, which can nevertheless be exhibited, let

$$\Pi = \beta PP \quad \text{and hence} \quad \int \Pi dP = \frac{1}{3}\beta P^3.$$

But because we have

$$\frac{\alpha y}{PP} - x = \beta PP \quad \text{and} \quad P^4 = \frac{-xPP + \alpha y}{\beta},$$

it will be

$$PP = \frac{-x + \sqrt{xx + 4\alpha\beta y}}{2\beta},$$

whence because of

$$V = \frac{PPy + \alpha y + \frac{1}{3}\beta P^4}{P} = \frac{2PPx + 4\alpha y}{3P}$$

and having done the substitution

$$V = \sqrt{\frac{2}{9\beta}\left((xx + 4\alpha\beta y)^{\frac{3}{2}} + x(12\alpha\beta y - x^2)\right)}.$$

SCHOLIUM 1

§47 It is plain that one can resolve the problem the same way, if Q must be an arbitrary function of P. For, put $dQ = RdP$, and it will be

$$V = Px + Qy - \int (x + Ry)dP.$$

Therefore, it is necessary that $x + Ry$ is a function of P, of which also R is a given function. Hence, if one puts, as before, $x + Ry = \Pi$, from this equation P will be defined by means of x and y; having afterwards substituted which value in Q, R and Π one will obtain a function $V = Px + Qy - \int \Pi dP$ expressed by the two variables x and y only.

EXAMPLE

§48 *Let the function V be in question that having put $dV = Pdx + Qdy$ it is $PP + QQ = aa$ or $Q = \sqrt{aa - PP}$.*

and hence it is

$$R = \frac{-P}{\sqrt{aa - PP}} \quad \text{and} \quad x - \frac{Py}{\sqrt{aa - PP}} = \Pi,$$

whence P must be defined. But now take $\Pi = 0$, it will be

$$P = \frac{ax}{\sqrt{xx + yy}}, \quad Q = \frac{ay}{\sqrt{xx + yy}} \quad \text{and} \quad V = a\sqrt{xx + yy}.$$

SCHOLIUM 2

§49 But if Q is not only given by means of P but also the variables x and y somehow go into its determination, then this way the task cannot be completed. But in these cases it is to be taken into account, how P and Q are treated as functions of x and y, that so any two of the four P, Q,x and y can be considered as functions of the two remaining ones. Hence in each case we are restricted to this formula $Pdx + Qdy$, which must be rendered integrable, but we will complete the task equally, if we render either this one $-ddP + Qdy$ or this one $Pdx - ydQ$ or this one $-xdP - yqQ$ integrable; one can even introduce new variables by means of substitutions, how the method of solving is immensely increases; it will be helpful to have given several examples of this.

PROBLEM 13

§50 While it is $dV = Pdx + Qdy$, if Q is given somehow by x and P, to find the nature of the function V.

SOLUTION

Since Q is put to be given by x and P, one will have an equation between the three quantities x, P and Q, from which one will also be able to define Q by x and Q such that P becomes equal to a certain function of x and Q. Therefore, take x and Q for the two variables form which all remaining ones are to be determined, and because it is

$$V = Qy + \int (Pdx - ydQ),$$

it is necessary that the differential formula $Pdx - ydQ$ is integrable, whose integral is to be considered a function of x and Q. Therefore, because P is given by x and Q, but the first variable y left undetermined, this integral $\int (Pdx - ydQ)$ will be found, if having considered Q as constant the formula Pdx is integrated and to the integral any arbitrary function of Q is added. Therefore, let the integral taken this way be $\int Pdx = R$, and R will be a given function of x and Q, whence it is $dR = Pdx + SdQ$, having assumed both quantities x and Q as variable, of course. Having put this we will have $\int (Pdx - ydQ) = R + \Phi : Q$ and $V = Qy + R + \Phi : Q$. Now denote the differential of $\Phi : Q$ by $dQ.\Phi' : Q$, and it will be

$$Pdx - ydQ = Pdx + SdQ + dQ.\Phi' : Q,$$

whence it is $y = -S - \Phi' : Q$. Finally, from this equation $y = -S - \Phi' : Q$ together with the relation which intercedes between Q, P and x by means of the variables x and y define the other two P and Q, and their resubstituted values will show, what kind of function V must be of x and y; in this that what is in question consists.

COROLLARY 1

§51 In similar manner, if Q is given by y and P such that x does not go into this relation, one will have to use this form $V = Px + \int (Qdy - xdP)$, which, since Q must be considered as a given function of x and P, by the same operations will be led to integrability.

COROLLARY 2

§52 But if either P or Q is determined by x and y, the question has no difficulty. For, if P is a given function of x and y, let the integral $\int Pdx$ be in question, having considered y as a constant, and having put $\int Pdx = R$, it will be $V = R + \Phi : y$.

COROLLARY 3

§53 Therefore, as often as a relation between the quantities P, Q, x and y is given by an equation of this kind, into which only three of these three quantities enter, the nature of the function V can be defined by the already treated problems.

SCHOLIUM

§54 Therefore, from this class the cases remain, in which the given relation contains all four letters P, Q, x and y. But for this we expanded the case, in which it was $Q = Pp + t$, while p and t are arbitrary functions of x and y, whose solution was given in Problem 11. But instead of the two variables x and y the following pairs of variables can be treated:

I. If it is $Q = xM + N$,
while M and N are arbitrary functions of P and y.

II. If it is $P = yM + N$,
while M and N are arbitrary functions of Q and x.

III. If it is $y = xM + N$,
while M and N are arbitrary functions of P and Q. In these cases the solution can of course also be found by means of the prescription given in § 40.

EXAMPLE

§55 *While it is $dV = Pdx + Qdy$, the found function V shall to be defined that it is $xyPQ = \alpha$.*

Therefore, because it is $Q = \frac{\alpha}{Pxy}$, it will be

$$dV = Pdx + \frac{\alpha dy}{Pdx},$$

which case is contained in none of the treated ones. But having put $\log y = u$, because it is $dV = Pdx + \frac{\alpha du}{Pdx}$, if we consider u instead of y and compare this form to $Pdx + Qdy$, it will be this $Q = \frac{\alpha}{Pdx}$, and hence given only by x and P, such that this example is reduced to present problem. Therefore, to not confound this Q with the principal one, because it is $P = \frac{\alpha}{Qx}$, we will have by writing y instead of u

$$V = Qy + \int \left(\frac{\alpha dx}{Qx} - ydQ \right),$$

therefore, having assumed Q to be constant it will be

$$\int \frac{\alpha dx}{Qx} = R = \frac{\alpha}{Q} \log x,$$

and hence $S = \frac{-\alpha \log x}{QQ}$, whence it is

$$u = \log y = \frac{\alpha \log x}{QQ} - \Phi' : Q$$

and

$$V = Q \log y + \frac{\alpha \log x}{Q} + \Phi : Q,$$

while $\Phi : Q = \int dQ \cdot dQ\Phi' : Q$, where for $\Phi' : Q$ any function of Q can be taken.

Therefore, for the simplest case let it be $\Phi' : Q = 0$, and it will be

$$Q = \sqrt{\frac{\alpha \log x}{\log y}} \quad \text{and} \quad V = 2\sqrt{\alpha \log x \cdot \log y} + \text{Const.}$$

And if one takes $\Phi' : Q = n - \frac{\alpha m}{QQ}$, it will be

$$\Phi : Q = nQ + \frac{\alpha m}{Q} + C \quad \text{and} \quad \log y + n = \frac{\alpha(\log x + m)}{QQ},$$

and hence

$$Q = \sqrt{\alpha \frac{\log x + m}{\log y + n}}$$

and

$$V = 2\sqrt{\alpha(\log x + m)(\log y + n)} + \text{Const.}$$

PROBLEM 14

§56 While it is $dV = Pdx + Qdy$, to define the nature of the function V that it is $PP + QQ = xx + yy$.

SOLUTION

This problem is contained in none of the cases treated up to this point; but nevertheless by a suitable transformation it can be reduced to a very simple case. For, put $PP + QQ = xx + yy = tt$, and by introducing two indefinite angles Φ and θ let it be:

$$P = t\sin\Phi, \quad Q = t\cos\Phi, \quad x = t\sin\theta \quad \text{and} \quad y = t\cos\theta,$$

because of

$$dx = dt\sin\theta + td\theta\cos\theta \quad \text{and} \quad dy = dt\cos\theta - td\theta\sin\theta$$

it will be

$$dV = tdt(\sin\Phi\sin\theta + \cos\Phi\cos\theta) - ttd\theta(\cos\Phi\sin\theta - \sin\Phi\cos\theta)$$

or

$$dV = tdt\cos(\theta - \Phi) - ttd\theta\sin(\theta - \Phi).$$

But it is

$$\int tdt\cos(\theta - \Phi) = \frac{1}{2}tt\cos(\theta - \Phi) + \frac{1}{2}\int tt(d\theta - d\Phi)\sin(\theta - \Phi),$$

whence it is

$$V = \frac{1}{2}tt\cos(\theta - \Phi) - \frac{1}{2}\int tt(d\theta + d\Phi)\sin(\theta - \Phi).$$

Therefore, because this formula must be integrable it is necessary that it is $tt\sin(\theta - \Phi) = \text{funct.}(\theta + \Phi)$. Hence, because it is $tt = xx + yy$ and $\tan\theta = \frac{x}{y}$, hence the angle Φ will be determined, whose value substituted will give the function V expressed by x and y. Let, that we find simpler algebraic functions,

$$tt\sin(\theta - \Phi) = \alpha\sin(\theta + \Phi) + \beta\cos(\theta + \Phi),$$

and it will be

$$V = \frac{1}{2}tt\cos(\theta - \Phi) + \frac{1}{2}\alpha\cos(\theta + \Phi) - \frac{1}{2}\beta\sin(\theta + \Phi)$$

whence, if tt is eliminated, it arises

$$2V\sin(\theta - \Phi) = \alpha\sin 2\alpha + \beta\cos 2\theta = \frac{2\alpha xy - \beta(xx - yy)}{xx + yy}.$$

But having expanded those angles it is

$$ttx\cos\Phi - tty\sin\Phi = \alpha x\cos\Phi + \alpha y\sin\Phi + \beta y\cos\Phi - \beta x\sin\Phi,$$

and hence

$$\tan\Phi = \frac{ttx - \alpha x - \beta y}{tty + \alpha y - \beta x}$$

and

$$\sec\Phi = \frac{\sqrt{t^6 - 2\alpha tt(xx - yy) - 4\beta ttxy + \alpha\alpha tt + \beta\beta tt}}{tty + \alpha y - \beta x},$$

let

$$T = t\sqrt{t^4 - 2\alpha(xx - yy) - 4\beta xy + \alpha\alpha + \beta\beta},$$

it will be

$$\sin\Phi = \frac{ttx - \alpha x - \beta y}{T} \quad \text{and} \quad \cos\Phi = \frac{tty + \alpha y - \beta x}{T},$$

and hence

$$\sin(\theta - \Phi) = \frac{2\alpha xy - \beta(xx - yy)}{Tt},$$

having substituted which value $V = \frac{T}{2t}$ will arise and hence

$$V = \frac{1}{2}\sqrt{(xx+yy)^2 - 2\alpha(xx-yy) - 4\beta xy + \alpha\alpha + \beta\beta},$$

which function can also be represented this way:

$$V = \frac{1}{2}\sqrt{(\alpha - xx + yy)^2 + (\beta - 2xy)^2}.$$

The simplest case arise by taking $\alpha = 0$ and $\beta = 0$, in which it is

$$V = \frac{1}{2}(xx+yy), \quad \text{and} \quad dV = xdx + ydy.$$

SCHOLIUM

§57 From this problem it is understood, how questions of this kind, which, while all four letters go into the prescribed relation, seem have a most difficult solution, nevertheless by means of a suitable substitution sometimes can be reduced to cases already treated. But on the other hand I still do not see a way, by which in general, no matter of what nature the relation between the for quantities P, Q, x and y was, the solution can be obtained; I could give a lot of other examples of this kind, in which the reduction to already treated cases can be done; but since I am certain to have exhausted this subject by no means, I proceed to the following chapters, whenever the prescribed relation except for the quantities P, Q, x and y also contains the function in question V itself; here, it is perspicuous per se, if only a relation between V, x and y would be propounded, that it would not even be a question, since the function V would be given immediately by x and y. Hence I will begin with problems of such a kind, where the prescribed relation except for the function V contains the one of the two quantities P and Q or even both, while the variables x and y either go in at the same time or not. But it is easily understood that these problems are a lot more difficult than the preceding ones.

PROBLEM 15

§58 While it is $dV = Pdx + Qdy$, to define the nature of the function V that it is $P = nV$.

SOLUTION

Because it is $P = nV$, it will be $dV = nVdx + Qdy$ or $DV - nVdx = Qdy$. Multiply the first side by e^{-nx} that it becomes integrable, and its integral $e^{-nx}V = \int e^{-nx}Q$ will have to become equal to an arbitrary function of y, which shall be $= Y$. Hence the function in question will be $V = e^{nx}Y$.

ANOTHER SOLUTION

Since V must be a function of x and y that its differential is $dV = nVdx + Qdy$ it is perspicuous, if the function V is differentiated having put y to be constant, that $dV = nVdx$ will arise. Hence vice versa from the integration of the formula $dV = nVdx$ the function V will be found, if y is considered as a constant, but then the constant introduced by integration can involve the quantity y somehow. But the equation $dV = nVdx$ integrated yields

$$\log V = nx + \log Y \quad \text{or} \quad V = e^{nx}Y,$$

as before.

COROLLARY 1

§59 The same way one will be able to resolve the further extending question, if P must be an arbitrary function of V. For, consider, having treated y as a constant, this differential equation $dV = Pdx$, which, because it contains only the two variables V and x, is to be integrated and then somehow involves the quantity y in the introduced integration constant.

COROLLARY 2

§60 Since the two variables x and y are interchangeable, the problem is resolved the same way, if Q must be an arbitrary function of V.

PROBLEM 16

§61 While it is $dV = Pdx + Qdy$, to define the nature of the function V that P becomes an arbitrary function of V and x.

Solution

Therefore, because P is given by means of V and x, if we consider y as a constant, we will have this equation $dV = Pdx$ between the two variables x and V. Therefore, integrate it, and instead of the constant introduce an arbitrary function of y into the integral equation; this way one will obtain an equation between V, x and y, by which the nature of the function V will be defined by means of x and y.

Corollary 1

§62 Therefore, whatever relation between the three quantities V, P and x is propounded, whether from it V is defined by x and P or P by V and x or x by V and P, the solution will always be easy.

Corollary 2

§63 Because of the interchangeability of the variables x and y, the problem will be solved in the same way, if an arbitrary relation between Q, V and y is propounded, and it is not necessary that we expand this case separately.

Example 1

§64 *Having put $dV = Pdx + Qdy$ is shall have to be*

$$P = \frac{mV}{x} + nx.$$

Therefore, having considered y as a constant, it will be

$$dV = \frac{mVdx}{x} + nxdx \quad \text{or} \quad dV - \frac{mVdx}{x} = nxdx,$$

whose integral is

$$\frac{V}{x^m} = \frac{nx^{2-m}}{2-m} + Y$$

where Y is an arbitrary function of y. Hence it will be

$$V = x^m Y + \frac{n}{2-m}xx,$$

for $m = 2$, it would be $V = xxY + nxx\log x$.

Example 2

§65 *Having put $dV = Pdx + Qdy$ is shall have to be $aV = P(aa - xx)$.*

Therefore, because it is

$$P = \frac{aV}{aa - xx},$$

having assumed y to be a constant, it will be

$$dV = \frac{aVdx}{aa - xx} \quad \text{or} \quad \frac{dV}{V} = \frac{adx}{aa - xx},$$

whose integral is

$$\log V = \frac{1}{2}\log\frac{a+x}{a-x} + \log Y,$$

whence one will have

$$V = Y\sqrt{\frac{a+x}{a-x}},$$

where Y denotes an arbitrary function of y.

Problem 17

§66 While $dV = Pdx + Qdy$, to define the nature if the function V, if P is an arbitrary given function of x, y and V.

Solution

Here, I assume that the propounded relation is contained in a certain equation between the quantities x, y, V and P; therefore, from this P can be defined by x, y and V. Therefore, consider y as a constant quantity, and because it is $dV = Pdx$, this equation will already involve only the two variables x and V. Therefore, integrate it, and instead of the constant introduce an arbitrary function of y, and this way an equation will arise showing the nature of the function V.

COROLLARY

§67 Therefore, the problem will be solved in similar manner, if an arbitrary relation between the four quantities x, y, Q and V is propounded, in which case only this difference is to be observed, that at first the quantity x is considered as a constant.

EXAMPLE

§68 *Having put $dV = Pdx + Qdy$ it shall have to be $V = \frac{Px}{y}$.*

Therefore, because it is $P = \frac{Vy}{x}$, having assumed y to be constant it will be

$$dV = \frac{Vydx}{x} \quad \text{and hence} \quad \log V = y \log x + \log Y,$$

whence the function is question arises as $V = x^y V$.

SCHOLIUM

§69 Therefore, if only one of the quantities P and Q go into the propounded relation, the problems are easily solvable. But if both quantities P and Q are in it, a major difficulty occurs, which is always so great that it cannot be overcome. Therefore, since in this case a general solution cannot be expected, let us run through several examples, which already extend sufficiently far.

PROBLEM 18

§70 While $dV = Pdx + Qdy$ to find the nature of the function V that it is $V = mPx + nQy$.

SOLUTION

Since hence it is

$$Q = \frac{V - mPx}{ny},$$

it will be

$$dV - \frac{Vdy}{ny} = Pdx - \frac{mPxdy}{ny} = \frac{P}{ny}(nydx - mxdy).$$

Therefore, find the multiplicator which renders the formula $nydx - mxdy$ integrable, since which is $\frac{1}{xy}$ and hence

$$dV - \frac{Vdy}{ny} = \frac{Px}{n}\left(\frac{ndx}{x} - \frac{mdy}{y}\right),$$

put

$$n\log x - m\log y = \log z \quad \text{or} \quad z = \frac{x^n}{y^m};$$

hence it is $x = y^{\frac{m}{n}} z^{\frac{1}{n}}$ which value is to be assumed to substituted instead of x. Hence because it is

$$dV = \frac{Vdy}{ny} + \frac{Pxdz}{nz},$$

the quantity V can be considered as function of the two quantities y and z, which must therefore be such a one that having assumed z to be constant it is $dV = \frac{Vdy}{ny}$. Therefore, hence by integration it will arise:

$$\log V = \frac{1}{n}\log y + \log Z \quad \text{or} \quad V = y^{\frac{1}{n}} Z$$

having taken an arbitrary function of $z = \frac{x^n}{y^m}$ for Z; and so one will have

$$V = y^{\frac{1}{n}} \Phi : \frac{x^n}{y^m}.$$

COROLLARY 1

§71 Because $\frac{x^{\frac{1}{m}}}{y^{\frac{1}{n}}}$ is a function of $\frac{x^n}{y^m}$, it will also be

$$V = x^{\frac{1}{m}} \Phi : \frac{x^n}{y^m}.$$

But then it can also exhibited this way

$$V = x^{\frac{1}{m}} \Phi : \frac{x^{\lambda n}}{y^{\lambda m}} \quad \text{or} \quad V = y^{\frac{1}{n}} \Phi : \frac{x^{\lambda n}}{y^{\lambda m}},$$

having assumed an arbitrary number for λ.

COROLLARY 2

§72 If it is $m = n$, one will have the case of homogeneous functions covered above. For, having taken $\lambda = \frac{1}{n}$ $\Phi : \frac{x}{y}$ will denote an arbitrary function of no dimension of x and y; and V will be a homogeneous function of the same, the number of dimension of which function is $= \frac{1}{n}$.

COROLLARY 3

§73 If we put in general

$$x^{\frac{1}{m}} = t \quad \text{and} \quad y^{\frac{1}{n}} = u,$$

but then take $\lambda = \frac{1}{mn}$, we will have

$$V = t\Phi : \frac{t}{u},$$

or V will be a homogeneous function of one dimension of the two quantities t and u.

SCHOLIUM

§74 If it is desired that it is $V = mP + nQ$ the solution will equally have hardy any difficulty. For, because of

$$Q = \frac{V}{n} - \frac{mP}{n}$$

it will be

$$dV - \frac{Vdy}{n} = P\left(dx - \frac{mdy}{n}\right).$$

Set $x - \frac{my}{n} = z$ that it is

$$dV = \frac{Vdy}{n} + Pdz:$$

now having considered z as a constant it will be

$$\log V = \frac{y}{n} + \Phi : z,$$

and hence

$$V = e^{\frac{y}{n}}\Phi : (nx - my).$$

But if it must be $V = mPy + nQx$, because of

$$Q = \frac{V - mPy}{nx}$$

it will be

$$dV = P\left(dx - \frac{mydy}{nx}\right) + \frac{Vdy}{nx}.$$

Now set $nxx - myy = zz$ that it is

$$x = \sqrt{\frac{zz + myy}{n}},$$

and because it is

$$dV = \frac{Vdy}{\sqrt{n(zz + myy)}} + \frac{P}{nx}dz,$$

consider the quantity z as constant and because of

$$\frac{dV}{V} = \frac{dy}{\sqrt{nzz + mnyy}},$$

it will be

$$\log V = \frac{1}{\sqrt{mn}}\log(y\sqrt{mn} + \sqrt{n(zz + myy)}) + \log Z,$$

and hence because of $\sqrt{n(zz + myy)} = nx$, it will arise

$$V = (y\sqrt{m} + x\sqrt{n})^{\frac{1}{\sqrt{mn}}}\Phi : (nxx - myy).$$

Hence, if it must be $V = Py + Qx$, it will be

$$V = (x + y)\Phi : (xx - yy).$$

But the following problem contains all cases of this kind in it.

Problem 19

§75 If p is an arbitrary given function of x and y, but M even an arbitrary given function of x, y and V, to define the nature of the function V that having put $dV = Pdx + Qdy$ it is $Q = Pp + M$.

Solution

Having substituted this value instead of Q we have

$$dV = Mdy + P(dx + pdy).$$

Find the multiplicator q rendering the formula $dx + pdy$ integrable and let $\int q(dx + pdy) = z$; hence define the value of x by y and z, and substitute it in M, if x is in it, for x, having done which it will be

$$dV = Mdy + \frac{Pdz}{q},$$

and so V can be considered as a function of y and z. Now consider z as a constant quantity, and because it is $dV = Mdy$, where only the two variables y and V are to understood to be contained in it, integrate this equation and instead of the constant introduce an arbitrary function of z; if in it for z the value in x and y, of course $\int q(dx + pdy)$ is resubstituted, that equation $dV = Mdy$ integrated will exhibit the nature of the function V, how is must depend on the two variables x and y.

Example

§76 *Having put $dV = Pdx + Qdy$ it shall have to be $V = pyy + Qxx$.*

Therefore, it is

$$Q = -\frac{Pyy}{xx} + \frac{V}{xx},$$

whence it is

$$dV = \frac{Vdy}{xx} + P\left(dx - \frac{yydy}{xx}\right).$$

Now take $q = xx$, it will be

$$\int (xxdx - yydy) = z = \frac{1}{3}x^3 - \frac{1}{3}y^3$$

or $x^3 = y^3 + 3z$ and hence $xx = (y^3 + 3z)^{\frac{2}{3}}$. Therefore, having assumed z to be constant, one has

$$\frac{dV}{V} = \frac{dy}{(y^3 + 3z)^{\frac{2}{3}}}.$$

Therefore, let S be the integral of the formula $\frac{dy}{(y^3+3z)^{\frac{2}{3}}}$, while z is assumed to be constant, and one will obtain

$$V = e^S \Phi : z = e^S \Phi : (x^3 - y^3),$$

of course in S one has to substitute its value $\frac{1}{3}x^3 - \frac{1}{3}y^3$ for z everywhere.

PROBLEM 20

§77 While it is $dV = Pdx + Qdy$, to define the nature of the function V that it is $V = nPQ$.

SOLUTION

Therefore, because of $Q = \frac{V}{nP}$ it will be

$$dV = Pdx + \frac{Vdy}{nP}.$$

That now V can be separated from the second term, put $P = R\sqrt{V}$ and it will arise

$$\frac{dV}{\sqrt{V}} = Rdx + \frac{dy}{nR},$$

whence by converting one obtains

$$2\sqrt{V} = Rx + \frac{y}{nR} - \int dR \left(x - \frac{y}{nRR} \right).$$

Hence it is necessary that $x - \frac{y}{nRR}$ is a function of R only; and having assumed such a function one will be able to define R by x and y, whence also the function in question V will be found expressed by x and y.

Another Solution

Since it is $V = nPQ$, eliminate V that one has this equation

$$nPdQ + nQdP = Pdx + Qdy,$$

from which it is

$$dy = -\frac{Pdx}{Q} + \frac{nPdQ}{Q} + ndP,$$

and hence

$$y = nP + \int \frac{P}{Q}(ndQ - dx).$$

Therefore, it is necessary that $\frac{P}{Q}$ is a function of the quantity $nQ - x$. Put $nQ - x = z$; and let $\int \frac{P}{Q}dz = \Phi : z$, it will be $\frac{P}{Q} = \Phi' : z$ and

$$y = nP + \Phi : z = nQ\Phi' : z + \Phi : z.$$

But it is $V = nQQ\Phi' : z$, whence $Q = \sqrt{\frac{V}{n\Phi' : z}}$; and so one will have these equations:

$$\sqrt{\frac{nV}{\Phi' : z}} = x + \quad \text{and} \quad y = \Phi : z + \sqrt{nV\Phi' : z},$$

from which one concludes $nV = (x+z)(y - \Phi : z)$, and if the quantity z is eliminated, the function V will arise expressed by x and y.

Corollary 1

§78 Assume z as a constant or $ndQ - dx = 0$, it will be

$$Q = \frac{x+a}{n} \quad \text{and} \quad y = nP - B \quad \text{or} \quad P = \frac{y+b}{n},$$

whence it arises

$$V = \frac{(x+a)(y+b)}{n},$$

which is the simplest case.

COROLLARY 2

§79 If one sets $\Phi' : z = a$, it will be $\phi : z = az + b$, whence it is

$$\sqrt{\frac{nV}{a}} = x + z \quad \text{and} \quad nV = (y - az - b)\sqrt{\frac{nV}{a}} \quad \text{and} \quad \sqrt{naV} = y - b - az,$$

from which combined we obtain $2\sqrt{naV} = ax + y - b$, and hence

$$V = \frac{(ax + y - b)^2}{4na},$$

which is another very simple case.

COROLLARY 3

§80 Let $\Phi' : z = \frac{1}{(az+b)^2}$ that it is $\Phi : z = -\frac{1}{a(az+b)} + c$, and it will be

$$(az + b)\sqrt{nV} = x + z \quad \text{and} \quad y - c + \frac{1}{a(az + b)} = \frac{\sqrt{nV}}{az + b}$$

or

$$a(az + b)(y - c) = a\sqrt{nV} - 1;$$

but hence it is

$$z = \frac{x - b\sqrt{nV}}{a\sqrt{nV} - 1} \quad \text{and hence} \quad az + b = \frac{ax - b}{a\sqrt{nV} - 1};$$

and having substituted this value

$$a(ax - b)(y - c) = (a\sqrt{nV} - 1)^2,$$

which expansion yields

$$V = \frac{1 + a(ax - b)(y - c) \pm 2m\sqrt{a(ax - b)(y - c)}}{naa}.$$

PROBLEM 21

§81 While it is $dV = Pdx + Qdy$, if V is given somehow by P and Q, to define the nature of the function V or how V is determined by x and y.

Therefore, because V is a function of the two quantities P and Q, put its differential $dV = MdP + NdQ$, and also M and N will be given functions of P and Q. Hence, because it is

$$MdP + NdQ = Pdx + Qdy, \tag{2}$$

it will be

$$dy = -\frac{Pdx}{Q} + \frac{MdP + NdQ}{Q},$$

and hence

$$y = -\frac{Px}{Q} + \int \left(xd.\frac{P}{Q} + \frac{MdP}{Q} + \frac{NdQ}{Q} \right).$$

Put $P = QS$, which value is to be understood to be substituted for P in M and N, such that the variables Q and S are now to be considered, and it will be

$$y = -Sx + \int (dS(x + M) + \frac{dQ}{Q}(N + MS)).$$

Since here M and N are given functions of Q and S, assume S to be constant, and put the integral

$$\int \frac{dQ}{Q}(N + MS) = R + \Phi : S,$$

therefore, it will be

$$x + M = \left(\frac{dR}{dS}\right) + \Phi' : S,$$

with $\Phi : s = \int dS\Phi' : S$ and

$$y = MS - S\left(\frac{dR}{dS}\right) - S\Phi' : S + R + \Phi : S.$$

Since now R and M are given by Q and S and because of $P = QS$ also V is given by Q and S. If this relation is combined with these two

$$x = -M + \left(\frac{dR}{dS}\right) + \Phi' : S \quad \text{and} \quad y = -Sx + R + \Phi : S,$$

one will hence be able to eliminate the two quantities S and Q, having done which an equation will arise, by which V will be determined by x and y.

EXAMPLE 1

§82 *While it is $dV = Pdx + Qdy$, it shall have to be $V = mPP + nQQ$.*

Therefore, because it is $dV = 2mPdP + 2nQdQ$, it will be $M = 2mP$ and $N = 2nQ$ or $M = 2mQS$ because of $P = QS$ such that it is $V = QQ(mSS + n)$. Therefore, we will have

$$N + MS = 2Q(mSS + n),$$

and hence having considered S as a constant

$$R)\int \frac{dQ}{Q}(N + MS) = 2Q(mSS + n),$$

and hence $\left(\frac{dR}{dS}\right) = 4mQS$.

Hence we obtain these three equations:

I. $\quad V = QQ(mSS + n),$

II. $\quad x + 2mQS = 4mQS + \Phi' : S$

III. $\quad y + Sx = 2Q(mSS + n) + \Phi : S \quad \text{or} \quad y = 2nQ + \Phi : S - S\Phi' : S.$

Hence, if Q is eliminated from II and III, it will be

$$\text{IV.} \quad nx - mSy = (mSS + n)\Phi' : S - mS\Phi : S,$$

but from the same combined it is $Q = \frac{Sx+y-\Phi:S}{2(mSS+n)}$, which with the first gives

$$\text{V.} \quad 2\sqrt{V(mSS + n)} = Sx + y - \Phi : S.$$

Hence it remains that S is eliminated from IV and V, and so the function V will arise expressed by x and y.

Let $\Phi' : S = a$, it will be $\Phi = S = aS + B$ and

$$\text{IV.} \quad nx - mSy = na - mbS$$

$$\text{V.} \quad 2\sqrt{V(mSS + n)} = Sx + y - aS - b.$$

Hence it is $S = \frac{n(x-a)}{m(y-b)}$, having substituted which value, it will be

$$2\sqrt{mnV} = \sqrt{n(x-a)^2 + m(y-b)^2},$$

and hence

$$V = \frac{n(x-a)^2 + m(y-b)^2}{4mn}.$$

EXAMPLE 2

§83 *While it is $dV = Pdx + Qdy$ is shall have to be $V = \frac{P}{Q}$.*

Therefore, it will be

$$M = \frac{1}{Q}, \quad N = -\frac{P}{QQ} = -\frac{S}{Q}$$

because of $P = QS$ and $V = S$ and $N + MS = 0$, whence it is $R = 0$. Hence it arises

$$x + \frac{1}{Q} = \Phi' : S \quad \text{and} \quad y + Sx = \Phi : s,$$

and because it is $S = V$, so the function V is determined by x and y that it is $y + Vx = \Phi : V$.

Put $\Phi : V = \frac{\alpha + 2\beta V + \gamma VV}{2\delta + 2\varepsilon V}$ that it is

$$2\delta y + 2\varepsilon Vy + 2\delta Vx + 2\varepsilon VVx = \alpha + 2\beta V + \gamma VV,$$

and hence

$$VV = \frac{2V(\delta x + \varepsilon y - \beta) + 2\delta y - \alpha}{\gamma - 2\varepsilon x}$$

and

$$V = \frac{\delta x + \varepsilon y - \beta \pm \sqrt{(\delta x - \varepsilon y)^2 + 2(\alpha\varepsilon - \beta\delta) + 2(\gamma\delta - \beta\varepsilon)y + \beta\beta - \alpha\gamma}}{\gamma - 2\varepsilon x};$$

if it is $\gamma = 0$ and $\varepsilon = 0$, it will be

$$V = \frac{2\delta y - \alpha}{2\beta - 2\delta x} \quad \text{or} \quad V = \frac{y - m}{n - x}.$$

SCHOLIUM

§84 Many other questions of this kind could be propounded and resolved, but since their solution is founded on the same principles we used until now, I will not spend more them on multiplying them, since the mentioned ones already seem to suffice, to develop the elements of this new method. One could still add several things for the cases, in which even the formulas of this kind $\left(\frac{ddV}{dx^2}\right)$, $\left(\frac{ddV}{dxdy}\right)$, $\left(\frac{ddV}{dy^2}\right)$ etc. go into the propounded relation, and similarly for the cases, in which the function to be found must be defined by three or more variables; but that this treatise does not become too long, I will retain it for another occasion.