On the motion of bodies attracted to two fixed centres of force *

Leonhard Euler

§1 Since there is certainly no doubt anymore that celestial bodies are moved as if they would mutually attract each other in the squared reciprocal ratio of the distances, the theory of Astronomy would be elevated to the highest ground, if any motion of bodies attracting each other in that ratio could be determined. Hence the perfection of Astronomy is to be expected from Mechanics, since from its principles that motions are easily reduced to differential equations, the whole task depends on Analysis, and on that part of it which concerns the resolution of differential equations. Therefore, what has not been explored sufficiently in Astronomy, the understanding of it is to be concluded from Analysis alone, the significant advance of which is still desired, before we can give the perfect explanation of even one single phenomenon.

§2 But what has been able to be achieved in this task until now, contains such a small part of the whole theory that is almost to be considered as insignificant; for, not more could be achieved by the authors who have treated this subject than that they taught to define the motion of just two bodies which attract each other in the squared reciprocal ratio of the distances. But as soon as three bodies attracting each other according to this law are propounded, which case is nevertheless still far away from the designated goal, since the number of bodies attracting each other on earth is very large, all artifices

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which have been found in Analysis up to this point, still do not suffice to solve it at all. And who has tackled this problem, has not achieved more than that they assigned the motion just approximately in the case in which the force of one body is very small compared to the other two.

§3 From this source everything is taken, what has been explored about lunar motion and about perturbations affecting the motions of planets, where it conveniently happened that the force by which the moon is pulled towards the earth supersedes the force directed towards the sun by a lot, but on planets the force directed towards the sun is a lot greater than the force by which they act on each other. If it were not for this circumstance, all effort in the determination of the motions would be in vain. Concerning lunar motion, whose motion has still been able to be defined sufficiently exactly by approximations, if it would be farther away from earth, I do not see, how we could obtain hardly any notion of its motion, if it would be so far away from earth, that it would be free from its destiny as earth's satellite, i.e. it would go over into the constellation of primary planets. For, then its motion would follow a certain average law of the motion of a satellite of earth and the motion of primary planets, whose nature could hardly be understood in any way, since the approximations can not be applied anymore.

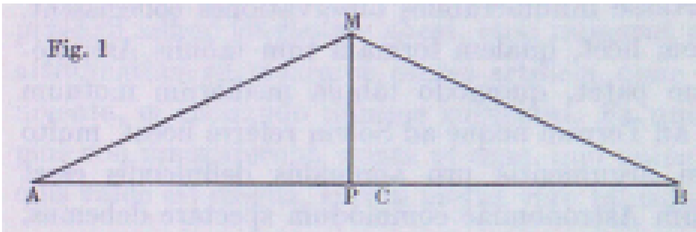
§4 If the moon would be closer to earth than it actually is, the perturbing force exerted by the sun would be weakened, and hence the moon in its motion around the earth would follow Kepler's law more exactly; but the aberration would be defined more easily and more certain; but the further away we assume the moon to be from earth, the greater aberrations will enter into its motion, until it reaches a region of such a kind where the force pulling towards the sun exceeds the force of the earth by a lot, - and at this point it essentially leaves earth -, it will begin to follow the nature of the motion the primary planets, but it will nevertheless still be subject to perturbations caused by the force of earth, which perturbations again, but in another way, will be able to be investigated by approximations, precisely as perturbations in the motions of the primary planets are usually represented.

§5 But the motion of the moon would be quite impossible to determine, if it would be away from earth four or five times as much than it actually is, and if it would have pleased God to assign the motion in such a region to the moon, astronomers would certainly be exhausted by the mere ways for its investigation, and maybe even without any success, since which without the help of theory they then could not determine the position of the moon at a given time without notable error, in this case they would always commit huge mistakes in the assignation of the position of the moon, even though they might would have had collected innumerable observational data. Yes, it is not even possible to suspect what kind of form would then be convenient to give the astronomical tables, and it is not clear, how the table of the general motion can be constructed, since it is not possible to refer them to earth or to the sun, and it is understood a lot less, which arguments would to made for the definition of anomalies. So we have to consider it as an extraordinary convenience for Astronomy that at least in our solar system there are no bodies of such a kind on which there is doubt whether they must be counted to the primary or secondary planets.

§6 But our ignorance about celestial motion would be most severe, if earth itself would be positioned among the other bodies in such a way that it would follow neither the law of primary nor secondary planets; since then the apparent motion of the sun, on which the cognition of the remaining motions is based, would be completely inexplicable to us, even though we would have collected the observational data for many centuries. The one single way to get to the notion of Astronomy would be by Analysis, by means of which the problems on the motion of three or more bodies attracting each other would have to be resolved, and not equipped with this tool we could not achieve anything in this science. But the solution of this problem would still by of highest utility, since we could assign the celestial motions accurately, given that we can only find them only approximately; such that just then the study of Astronomy is to be considered to be elevated to the highest degree of perfection.

§7 Therefore, since the expansion of the case of many bodies it not even to be expected before the case of three bodies was solved, this must be considered as the foundation of a more complete astronomical understanding, which is therefore to be seen to be completely worth one's while that all Geometers join forces for its resolution. Certainly very huge difficulties occur which the Geometers have unsuccessfully tried to overcome, but the results to be hoped for from the attempts are too precious than that it is convenient to be intimidated by a further investigation. And if we tackling this problem try every dead-end, this has been often be a help in other tasks, while the treatment of other related questions has led to the desired goal eventually, let us do the same here, and focus all efforts on other similar questions, even though they do not seem to have any use per se, trusting that hence some light will be thrown on those shadows which we want to remove.

§8 Therefore, following this outline, I undertook to treat that problem that given two fixed bodies I investigate the motion of a third body that is attracted to each of them. Of course, let the bodies be fixed at A and B (Fig. 1)



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the masses of which shall also be indicated by A and B, but let the third body, the motion of which I assume to take place in the same plane the points A and B live in and the motion of which must be assigned, be located at the point M after the time t. This problem, even though in the real world a similar case does not occur, is nevertheless detected to be impacted by the same difficulties as the case on which whole Astronomy is based, which difficulties seem to be able to be overcome more easily since here the two bodies A and B are assumed to be immobile; for, it contains some per se perspicuous cases, whose consideration seems to lead to a general expansion.

§9 For, first I observe, if the mass of the one of the bodies A and B vanishes, that the motion of the body M will follow Kepler's laws, such that it would then describe a conic section around one of the points A or B. The same will happen approximately, if the body M was positioned in such a way that it remains close to the one body but keeps so much distance from the other one that the force pulling it in that direction is very small compared to the other one. Hence it is plain that the motion deviates from Kepler's laws the more the less equal the distances from the points A and B are; and in this case the motion of the body M even seems to be similar to the one it would follow, if the bodies A and B would not be fixed, that hence nothing for the understanding of the problem could be hoped for. Indeed, then here also the case deserves to be mentioned in which both bodies A and B are equal to each other and the body M is moved in such a way that its orbit is referred to both of them equally; for, in this case the motion will also be detected to take place on a conic section.

§10 Therefore, having observed these things let us set the constant distance $AB = a$ and the variable distances $AM = v$, $BM = u$; but then having dropped the perpendicular MP from M to AB, let $AP = x$ and $PM = y$, and hence $BP = a - x$ and

$$v = \sqrt{xx + yy} \quad \text{and} \quad u = \sqrt{(a-x)^2 + yy}.$$

Because now the accelerating force, by which the body M is attracted to A, is $\frac{A}{vv}$, and the force, by which is attracted to B, is as $\frac{B}{uu}$, hence the force in the direction PA will result

$$= \frac{Ax}{v^3} - \frac{B(a-x)}{u^3}$$

and in the direction MP

$$= \frac{Ay}{v^3} + \frac{By}{u^3};$$

from these, having assumed the temporal element dt to be constant, the principles of Mechanics yield these formulas:

$$\text{I.} \quad ddx = -2gdt^2\left(\frac{Ax}{v^3} - \frac{B(a-x)}{u^3}\right),$$

$$\text{II.} \quad ddy = -2gdt^2\left(\frac{Ay}{v^3} + \frac{By}{u^3}\right),$$

which contain the determination of motion, where g is a certain constant quantity introduced for absolute measures.

§11 Since none of these equations admits an integration, one has to see, whether they can be combined that hence an integrable equation results, and it is necessary to achieve this in two ways. And one combination is certainly obvious; for, having multiplied the first by dx and the other by dy the sum results as

$$dxddx + dyddy = -2gdt^2 \left(\frac{A(xdx+ydy)}{v^3} + \frac{B(ydy-(a-x)dx)}{u^3} \right),$$

which, because of

$$vdv = xdx + ydy \quad \text{and} \quad udu = ydy - (a-x)dx,$$

goes over into this one

$$dxddx + dyddy = -2gdt^2 \left(\frac{Adv}{vv} + \frac{Bdu}{uu} \right),$$

the integral of which, having introduced a new constant, is

$$dx^2 + dy^2 = 4gdt^2 \left(\frac{A}{v} + \frac{B}{u} + \frac{C}{a} \right);$$

since here $\sqrt{dx^2+dy^2}$ expresses the element of the curve described by the body M in the infinitely small time interval dt,

$$\frac{dx^2+dy^2}{dt}$$

will be the velocity of the body M which is therefore conveniently determined by the distances v and u.

§12 Having done one integration, to find another one additionally, let us throw out the other mass from the formulas found first, and so we will obtain these equations

$$(a-x)ddy + yddx = -2gdt^2 \cdot \frac{Aay}{v^3},$$

$$xddy - yddx = -2gdt^2 \cdot \frac{Bay}{u^3},$$

from which we seem to get hardly anything lucrative. But if we consider that

$$d\frac{x}{v} = \frac{(xx+yy)dx - x(xdx+ydy)}{v^3} = \frac{y(ydx-xdy)}{v^3}$$

and

$$d\frac{a-x}{u} = \frac{-dx((a-x)^2+yy) - (a-x)(ydy-(a-x)dx)}{u^3} = -\frac{y(ydx+(a-x)dy)}{u^3},$$

let us multiply the first by $xdy - ydx$ and the second by $(a-x)dy + ydx$, and we will have

$$(xdy-ydx)((a-x)ddy+yddx) = 2gAadt^2 \cdot d\frac{x}{v},$$

$$((a-x)dy+ydx)(xddy-yddx) = 2gBadt^2 \cdot \frac{a-x}{u}.$$

§13 Now since

$$(a-x)ddy + yddx = d((a-x)dy+ydx)$$

and

$$xddy - yddy = d(xdy-ydx),$$

it conveniently happens that the sum of these equations is integrable, while the integral results as

$$(xdy-ydx)((a-x)dy+ydx) = 2gadt^2\left(\frac{Ax}{v} + \frac{B(a-x)}{u} + D\right),$$

and so we already deduced the problem to differential equations of first order, to which point it has not been possible to get in the solution of the problem on three mobile bodies. If we now throw out the time element dt from this, we get to this simple differential equation

$$a(dx^2+dy^2)\left(\frac{Ax}{v} + \frac{B(a-x)}{u} + D\right)$$
$$= 2(xdy-ydx)((a-x)dy+ydx)\left(\frac{A}{v} + \frac{B}{u} + \frac{C}{a}\right)$$

between the two variables x and y, by which the nature of the curve in question is determined; such that now the whole task has been reduced to the differential equation of first order, for the solution of which Analysis provides a lot of auxiliary tools.

§14 But here two obstacles occur, the one that the differentials dx and dy ascend to two dimensions, the other consists in the irrational constants v and u. To be able to remove these obstacles more easily, let us put the angles $BAM = \zeta$, $ABM = \eta$, and it will be $x = v\cos\zeta$, $y = v\sin\zeta = u\sin\eta$ and $a - x = u\cos\eta$, whence one concludes

$$dx^2 + dy^2 = dv^2 + vvd\zeta^2 = du^2 + uud\eta^2,$$
$$xdy - ydx = vvd\zeta$$

and

$$(a-x)dy + ydx = uud\eta,$$

by which values our equation is reduced to this simpler form:

$$a(dv^2 + vvd\zeta^2)(A\cos\zeta + B\cos\eta + D) = 2vvuud\zeta d\eta\left(\frac{A}{v} + \frac{B}{u} + \frac{C}{a}\right).$$

Further, because of

$$v = \frac{a\sin\eta}{\sin(\zeta+\eta)} \quad \text{and} \quad u = \frac{a\sin\zeta}{\sin(\zeta+\eta)}, \quad \text{it will be} \quad x = \frac{a\cos\zeta\sin\eta}{\sin(\zeta+\eta)}$$

and

$$y = \frac{a\sin\zeta\sin\eta}{\sin(\zeta+\eta)}, \quad \text{and hence} \quad dx = -\frac{-ad\zeta\sin\eta\cos\eta + ad\eta\sin\zeta\cos\zeta}{\sin^2(\zeta+\eta)}$$

and

$$dy = \frac{ad\zeta\sin^2\eta + ad\eta\sin^2\zeta}{\sin^2(\zeta+\eta)},$$

whence

$$dx^2 + dy^2 = \frac{aa(d\zeta^2\sin^2\eta + d\eta^2\sin^2\zeta - 2d\zeta d\eta\sin\zeta\sin\eta\cos(\zeta+\eta))}{\sin^4(\zeta+\eta)} = dv^2 + vvd\zeta^2.$$

§15 If we reduce our equations to just the two angles ζ and η by means of these values, we will obtain:

$$(d\zeta^2 \sin^2\eta + d\eta^2 \sin^2\zeta - 2d\zeta d\eta \sin\zeta \sin\eta \cos(\zeta+\eta))(A\cos\zeta + B\cos\eta + D)$$
$$= 2d\zeta d\eta \sin^2\zeta \sin^2\eta \left(\frac{A\sin(\zeta+\eta)}{\sin\eta} + \frac{B\sin(\zeta+\eta)}{\sin\zeta} + C\right)$$
$$= 2d\zeta d\eta \sin\zeta \sin\eta (A\sin\zeta\sin(\zeta+\eta) + B\sin\eta\sin(\zeta+\eta) + C\sin\zeta\sin\eta),$$

which is reduced to this much simpler form:

$$(d\zeta^2 \sin^2\eta + d\eta^2 \sin^2\zeta)(A\cos\zeta + B\cos\eta + D)$$
$$= 2d\zeta d\eta \sin\zeta\sin\eta(A\cos\eta + B\cos\zeta + C\sin\zeta\sin\eta + D\cos(\zeta+\eta)).$$

Or, because of

$$\cos(\zeta+\eta) = \cos\zeta\cos\eta - \sin\zeta\sin\eta,$$

let us set $C - D = E$ that we have:

$$d\zeta^2\sin^2\eta + d\eta^2\sin^2\zeta = \frac{2d\zeta d\eta\sin\zeta\sin\eta(A\cos\eta + B\cos\zeta + D\cos\zeta\cos\eta + E\sin\zeta\sin\eta)}{A\cos\zeta + B\cos\eta + D},$$

whence, if for the sake of the brevity we put

$$A\cos\eta + B\cos\zeta + D\cos\zeta\cos\eta + E\sin\zeta\sin\eta = P$$

and

$$A\cos\zeta + B\cos\eta + D = Q,$$

extracting the root we deduce:

$$\frac{d\zeta\sin\eta}{d\eta\sin\zeta} = \frac{P \pm \sqrt{PP - QQ}}{Q}.$$

§16 Since there is no way to resolve equations of this kind, let us contemplate cases, in which the resolution is possible, which are, whenever $A = 0$ or $B = 0$; for, even though in these cases the last equation seems hardly manageable, nevertheless from the principal formulas the solution is easily deduced. For, if we put $B = 0$, the first integration yields:

$$dx^2 + dy^2 = 4gdt^2 \left(\frac{A}{v} + \frac{C}{a} \right),$$

but then from paragraph 12, because of $B = 0$, we obtain

$$xddy - yddx = 0 \quad \text{and hence} \quad xdy - ydx = \text{Const. } dt,$$

therefore, let us put

$$(xdy - ydx)^2 = 4gFadt^2,$$

and it will be

$$Fa(dx^2 + dy^2) = (xdy - ydx)^2 \left(\frac{A}{v} + \frac{C}{a} \right) = v^4 d\zeta^2 \left(\frac{A}{v} + \frac{C}{a} \right)$$

and after the substitution indicated above:

$$\begin{aligned} F(d\zeta^2 \sin^2 \eta + d\eta^2 \sin^2 \zeta - 2d\zeta d\eta \sin \zeta \sin \eta \cos(\zeta + \eta)) \\ = d\zeta^2 \sin^4 \eta \left(\frac{A \sin(\zeta + \eta)}{\sin \eta} + C \right) \end{aligned}$$

or

$$\begin{aligned} d\zeta^2 \sin^2 \eta \left(1 - \frac{A \sin \eta \sin(\zeta + \eta) + C \sin^2 \eta}{F} \right) + d\eta^2 \sin^2 \zeta \\ = 2d\zeta d\eta \sin \zeta \sin \eta \cos(\zeta + \eta), \end{aligned}$$

the resolution of which certainly seems to be hardly simpler than of the preceding; but having extracted the square root it becomes sufficiently obvious.

§17 But before we got to this last equation among ζ and η, we had already obtained an equation containing only the two letters v and ζ, i.e.:

$$Fa(dv^2 + vvd\zeta^2) = v^4 d\zeta^2 \left(\frac{A}{v} + \frac{C}{a}\right)$$

from which immediately this one is found

$$Fadv^2 = vvd\zeta^2 \left(Av + \frac{Cvv}{a} - Fa\right)$$

or

$$Fadv^2 = v^4 d\zeta^2 \left(\frac{C}{a} + \frac{A}{v} - \frac{Fa}{vv}\right),$$

whence

$$\frac{dv}{vv}\sqrt{Fa} = d\zeta\sqrt{\frac{C}{a} + \frac{A}{v} - \frac{Fa}{vv}},$$

from which the nature of conic sections is found as usual. For, having put $\frac{1}{v} = \frac{z}{a}$,

$$-dz = d\zeta\sqrt{\frac{C + Az}{F} - zz},$$

whence it follows

$$\zeta + \alpha = \arccos\frac{2Fz - A}{\sqrt{AA + 4CF}}$$

and hence

$$2Fz = A + \cos(\zeta + \alpha)\sqrt{AA + 4CF},$$

such that

$$v = \frac{2Fa}{A + \cos(\zeta + \alpha)\sqrt{AA + 4CF}} = \frac{a \sin\eta}{\sin(\zeta + \eta)}.$$

§18 Therefore, hence the integral equation among the angles ζ and η is expressed in such a way that

$$\frac{2F\sin(\zeta + \eta)}{\sin\eta} = A + \cos(\zeta + \alpha)\sqrt{AA + 4CF}$$

or, that form given the angle ζ the angle η can be found more easily, because of

$$\sin(\zeta+\eta) = \sin\zeta\cos\eta + \cos\zeta\sin\eta,$$

it will be

$$2F\sin\zeta\cot\eta + 2F\cos\zeta = A + \cos(\zeta+\alpha)\sqrt{AA+4CF}$$

or, having changed the form of the constants,

$$\cot\eta + \cot\zeta = \frac{A + M\cos\zeta + N\sin\zeta}{2F\sin\zeta}.$$

And hence we understand at the same time, if we put $A=0$, that it will be

$$\cot\zeta + \cot\eta = \frac{B + M'\cos\eta + N'\sin\eta}{2F'\sin\eta}.$$

Hence now the forms are known to which the integral of the differential equation given in paragraph 15 is reduced in the cases in which either $A=0$ or $B=0$, such that the way to get to these integrals can be investigated.

For the Case $B = 0$

§19 In this case the principal equation found in paragraph 15 goes over into this form:

$$d\zeta^2\sin^2\eta + d\eta^2\sin^2\zeta = \frac{2d\zeta d\eta\sin\zeta\sin\eta(A\cos\eta + D\cos\zeta\cos\eta + E\sin\zeta\sin\eta)}{A\cos\zeta + D},$$

the integral of which we therefore know to have a form of this kind:

$$\cot\eta + \cot\zeta = \frac{A + M\cos\zeta + N\sin\zeta}{2F\sin\zeta}$$

or more briefly

$$\cot\eta = \alpha + \frac{\beta + \gamma\cos\zeta}{\sin\zeta},$$

how which is to be found from the differential must thus be investigated. Even though this is easily seen by a calculation accommodated immediately from the beginning to this case, nevertheless the consideration of the body B changes the calculation in such a way that this conclusion seems that it can be arrived at only in a non intuitive way. But first we understand that instead of the angle η its cotangent will not be introduced without use; therefore, having divided the equation by $\sin^4\eta$, we have

$$\frac{d\zeta^2}{\sin^2\eta}+\sin\zeta^2(d\cot\eta)^2=-\frac{2d\zeta\sin\zeta d\cot\eta(A\cot\eta+D\cos\zeta\cot\eta+E\sin\zeta)}{A\cos\zeta+D}.$$

§20 Let us put $\cot\eta=z$, because of

$$\sin\eta=\frac{1}{\sqrt{1+zz}},$$

it will be:

$$d\zeta^2(1+zz)+dz^2\sin^2\zeta=-\frac{2d\zeta dz\sin\zeta(z(A+D\cos\zeta)+E\sin\zeta)}{A\cos\zeta+D},$$

whence by extracting the root

$$\frac{dz\sin\eta}{d\eta}=$$

$$\frac{-z(A+D\cos\zeta)-E\sin\zeta+\sqrt{(AA-DD)zz\sin^2\zeta+2E(A+D\cos\zeta)z\sin\zeta+EE\sin^2\zeta-(A\cos\zeta+D)^2}}{A\cos\zeta+D},$$

where it is to be noted that the quantity under the square root sign can be represented this way:

$$\left(z\sin\zeta\sqrt{AA-DD}+\frac{E(A+D\cos\zeta)}{\sqrt{AA-DD}}\right)^2-\frac{(A-DD+EE)(A\cos\zeta+D)^2}{AA-DD}.$$

Hence having put

$$z\sin\zeta\sqrt{AA-DD}+\frac{E(A+D\cos\zeta)}{\sqrt{AA-DD}}=\frac{s(A\cos\zeta+D)\sqrt{AA-DD+EE)}}{AA-DD},$$

the quantity under the square root sign will be

$$\frac{(A\cos\zeta+D)\sqrt{AA-DD+EE}}{\sqrt{AA-DD}}\sqrt{ss-1}.$$

§21 For the sake of brevity put this quantity equal to the irrational quantity $= V$, and since our equation is

$$dz\sin\zeta(A\cos\zeta + D) + zd\zeta(A + D\cos\zeta) + Ed\zeta\sin\zeta = Vd\zeta,$$

divide it by $(A\cos\zeta + D)^2$ and it will be possible to represent it this way

$$d\frac{Az\sin\zeta + E}{A(A\cos\zeta + D)} = \frac{Vd\zeta}{(A\cos\zeta + D)^2}.$$

But by our substitution

$$Az\sin\zeta + E = \frac{-DE(A\cos\zeta + D) + As(A\cos\zeta + D)\sqrt{AA - DD + EE}}{AA - DD},$$

having substituted which value and the same time having substituted the value of V again it will be

$$d\cdot\frac{-DE + As\sqrt{AA - DD + EE}}{A(AA - DD)} = \frac{d\zeta\sqrt{AA - DD + EE}}{(A\cos\zeta + D)\sqrt{AA - DD}}\sqrt{ss - 1}$$

or

$$\frac{ds}{\sqrt{AA - DD}} = \frac{d\zeta\sqrt{ss - 1}}{A\cos\zeta + D} \quad \text{or} \quad \frac{ds}{\sqrt{ss - 1}} = \frac{d\zeta(AA - DD)}{A\cos\zeta + D},$$

which can also be represented this way:

$$-\frac{ds}{\sqrt{1 - ss}} = \frac{d\zeta\sqrt{DD - AA}}{A\cos\zeta + D},$$

the integral of which is

$$\arccos s = \arccos\frac{A + D\cos\zeta}{A\cos\zeta + D} + \alpha.$$

§22 Since now

$$\arccos\frac{A + D\cos\zeta}{A\cos\zeta + D} = \arcsin\frac{\sin\zeta\sqrt{DD - AA}}{A\cos\zeta + D},$$

it will be

$$s = \frac{(A + D\cos\zeta)\cos\alpha - \sin\alpha\sin\zeta\sqrt{DD - AA}}{A\cos\zeta + D}$$

or this way:

$$s = \frac{n(A + D\cos\zeta) - \sin\zeta\sqrt{(1 - nn)(DD - AA)}}{A\cos\zeta + D},$$

if here $D < A$, the number n must be taken > 1.

Therefore, having substituted this value the integral equation in question will be:

$$\sin\zeta\cot\eta = \frac{E(A + D\cos\zeta)}{DD - AA}$$
$$+ \frac{-n(A + D\cos\zeta) + \sin\zeta\sqrt{(1 - nn)(DD - AA)}}{DD - AA}\sqrt{AA - DD + EE}$$

and having put

$$n = \frac{E - F}{\sqrt{AA - DD + EE}}$$

it will be

$$\sin\zeta\cot\eta = \frac{F(A + D\cos\zeta)}{DD - AA} + \frac{\sin\zeta\sqrt{AA - DD + 2EF - FF}}{\sqrt{DD - AA}},$$

where F is an arbitrary constant quantity introduced by the new integration. But having changed it it will be

$$\sin\zeta\cot\eta = \frac{A + D\cos\zeta}{G} + \sin\zeta\sqrt{\frac{2E}{G} + \frac{AA - DD}{GG} - 1}.$$

For the Case $A = B$ and $D = E = 0$

§23 In like manner, the case $A = 0$ is covered, and the integral equation does not differ from the preceding, except that the letters A and B as the angles ζ and η are permuted. But in this case, in which $A = B$ and $D = E = 0$, our equation becomes

$$d\zeta^2\sin^2\eta + d\eta^2\sin^2\zeta = 2d\zeta d\eta\sin\zeta\sin\eta,$$

which manifestly yields $d\zeta \sin\eta = d\eta \sin\zeta$, and hence by integration

$$\log\tan\frac{1}{2}\zeta = \text{Const.} + \log\tan\frac{1}{2}\eta,$$

whence

$$m\tan\frac{1}{2}\zeta = n\tan\frac{1}{2}\eta \quad \text{or} \quad m(1-\cos\zeta)\sin\eta = n(1-\cos\eta)\sin\zeta,$$

such that the tangents of the half of the angles BAM and ABM always have the same ratio. Since now having introduced the coordinates x and y we have $\cos\zeta = \frac{x}{v}$, $\sin\zeta = \frac{y}{v}$, $\cos\eta = \frac{a-x}{u}$ and $\sin\eta = \frac{y}{u}$, it will be

$$\frac{m(v-x)y}{vu} = \frac{n(u-a+x)y}{vu}$$

or

$$m(v-x) = n(u-a+x)$$

such that $m(AM - AP) = n(BM - BP)$.

§24 Therefore, since $m(v-x) = n(u-a+x)$, note that

$$x = \frac{aa+vv-uu}{2a} \quad \text{and} \quad a-x = \frac{aa+uu-vv}{2a},$$

whence

$$m(uu-(a-v)^2) = n(vv-(a-u)^2)$$

or

$$m(u+v-a)(u+a-v) = n(v+u-a)(v+a-u),$$

which divided by $(u+v-a)$ yields

$$m(a+u-v) = n(a+v-u) \quad \text{or} \quad (m+n)(u-v) = (n-m)a,$$

such that

$$u-v = \frac{(n-m)a}{m+n},$$

which shall be compared to this one:

$$nu - mv = na - (m+n)x,$$

whence one concludes

$$(n-m)u = \frac{(mm+nn)a}{m+n} - (m+n)x$$

and

$$(n-m)v = \frac{2mna}{m+n} - (m+n)x,$$

which squared yields

$$(n-m)^2yy + (n-m)^2xx = \frac{4mmnnaa}{(m+n)^2} - 4mnax + (m+n)^2xx$$

or

$$(n-m)^2yy = \frac{4mmnnaa}{(m+n)^2} - 4mnax + 4mnxx.$$

§25 Let us take the abscissas from the middle point C, and let $CA = CB = b$ and hence $a = 2b$ and put $CP = z$; but then set $m + n = b$ and $n - m = c$, and, because of $x = b - z$, we will have

$$cv = bz - cc \quad \text{and} \quad cu = bz + cc$$

and hence

$$yy = \frac{bb-cc}{cc}(zz-cc),$$

whence it is plain that this curve is a hyperbola described around the center C, whose semiaxis is $= c$ and the distance of the focal point is $CA = CB = b$, and that it will be

$$\tan\frac{1}{2}\zeta : \tan\frac{1}{2}\eta = b + c : b - c.$$

Further, since

$$dy = \frac{zdz}{\sqrt{zz-cc}} \cdot \frac{\sqrt{bb-cc}}{c},$$

it will be

$$dx^2 + dy^2 = dy^2 + dz^2 = \frac{dz^2(bbzz - c^4)}{cc(zz - cc)};$$

hence, since because of $C = D + E = 0$ and $B = A$ we have

$$dx^2 + dy^2 = 4Agdt^2 \left(\frac{c}{bz - cc} + \frac{c}{bz + cc} \right) = \frac{8Abcgzdt^2}{bbzz - c^4},$$

the velocity in M will be

$$= \frac{\sqrt{dx^2 + dy^2}}{dt} = \frac{2\sqrt{2Abcgz}}{\sqrt{bbzz - c^4}};$$

and having put $z = c$ the velocity at the vertex of the hyperbola results as

$$= \frac{2\sqrt{2Abg}}{\sqrt{bb - cc}}.$$

Therefore, even though the hyperbola goes over into an ellipse for $c > b$, nevertheless it is evident that the motion can not describe an ellipse, since the velocity would be imaginary such that in this case the body M can not be moved in a hyperbola.

§26 But of what nature this motion on the hyperbola will be, will be concluded from the property of time. Of course, since

$$\sqrt{dx^2 + dy^2} = \frac{dz}{c} \sqrt{\frac{bbzz - c^4}{zz - cc}},$$

it will be

$$2dt\sqrt{2Abcg} = \frac{dz(bbzz - c^4)}{c\sqrt{z(zz - cc)}}$$

and hence

$$2ct\sqrt{2Abcg} = \int \frac{dz(bbzz - c^4)}{\sqrt{z(zz - cc)}}.$$

But by integration by parts it is known to be

$$\int \frac{zzdz}{\sqrt{z(zz-cc)}} = \frac{2}{3}\sqrt{z(zz-cc)} + \frac{1}{3}cc\int \frac{dz}{\sqrt{z(zz-cc)}},$$

whence the time t is determined in such a way that

$$2ct\sqrt{2Abcg} = \frac{2}{3}\sqrt{z(zz-cc)} + \frac{1}{3}cc(bb-3cc)\int \frac{dz}{\sqrt{z(zz-cc)}}.$$

Therefore, the determination of time depends on the integration of the formula

$$\int \frac{dz}{\sqrt{z(zz-cc)}},$$

which is known to be not reducible to the quadrature of the circle or the hyperbola.

§27 Let us reduce this determination to the angle $BAM = \zeta$, too, and since

$$\tan\frac{1}{2}\zeta = \frac{v-x}{y} = \frac{v-b+z}{y},$$

we will have

$$\tan\frac{1}{2}\zeta = \sqrt{\frac{b+c}{b-c}\cdot\frac{z-c}{z+c}}$$

and hence

$$z = \frac{c(b-c\cos\zeta)}{c-b\cos\zeta},$$

whence

$$v = \frac{bb-cc}{c-b\cos\zeta}.$$

Further, hence we obtain

$$\sqrt{zz-cc} = \frac{c\sin\zeta\sqrt{bb-cc}}{c-b\cos\zeta}$$

and

$$dz = -\frac{c(bb-cc)d\zeta\sin\zeta}{(c-b\cos\zeta)^2}.$$

Therefore,

$$\frac{dz}{\sqrt{zz-cc}} = -\frac{\sqrt{bb-cc}}{c-b\cos\zeta}d\zeta$$

and

$$\frac{dz}{\sqrt{z(zz-cc)}} = -\frac{d\zeta\sqrt{(bb-cc)(c-b\cos\zeta)}}{\sqrt{c(b-c\cos\zeta)}},$$

whence we conclude

$$2ct\sqrt{2Abcg} = \frac{2bbc\sin\zeta\sqrt{(bb-cc)c(b-c\cos\zeta)}}{3(c-b\cos\zeta)\sqrt{c-b\cos\zeta}},$$
$$-\frac{1}{3}cc(bb-3cc)\int\frac{d\zeta\sqrt{(bb-cc)(c-b\cos\zeta)}}{\sqrt{c(b-\cos\zeta)}}$$

or

$$\frac{2t\sqrt{2Abg}}{\sqrt{bb-cc}} = \frac{2bb\sin\zeta\sqrt{b-c\cos\zeta}}{3(c-b\cos\zeta)^{\frac{3}{2}}} - \frac{1}{3}(bb-3cc)\int\frac{d\zeta}{\sqrt{(b-c\cos\zeta)(c-b\cos\zeta)}}.$$

§28 Here a remarkable case occurs in which $bb = 3cc$, since then the time can be assigned. But then the velocity in the vertex of the hyperbola will be

$$= 2\sqrt{\frac{Ag\sqrt{3}}{c}} = 2\sqrt{\frac{3Ag}{b}}.$$

If this velocity is called $= k$, it will be

$$kt = \frac{2cc\sin\zeta}{c-b\cos\zeta}\sqrt{\frac{b-c\cos\zeta}{c-b\cos\zeta}} = \frac{2c\sin\zeta}{1-\cos\zeta\cdot\sqrt{3}}\sqrt{\frac{\sqrt{3}-\cos\zeta}{1-\cos\zeta\cdot\sqrt{3}}},$$

or shorter this way:

$$\frac{t}{c}\sqrt{2ABcg} = \sqrt{z(zz-cc)},$$

to define the position of the body M from this one has to solve this cubic equation:

$$z^3 - czz = \frac{2Abcgtt}{cc} = 2Agtt\sqrt{3}.$$

But for other cases than the ones treated it will hardly be possible to define the motion; indeed, this provides the opportunity to apply artifices of such kind which might can have some use in the further treatment of this subject. Let us nevertheless add the case in which the body will be moved on an ellipse, both focal points of which are at the points A and B.

ON THE MOTION OF THE BODY M ON AN ELLIPSE

§29 Since we saw in the preceding case that the body can be moved on a hyperbola, there is no doubt that in a certain case the motion can happen on an ellipse, which case will be different from the preceding one in which it was $D = 0$ and $E = 0$ while $A = B$. But for an ellipse to result it is necessary that

$$\tan\frac{1}{2}\zeta\tan\frac{1}{2}\eta = m,$$

or, keeping the values $CP = z$ and $CA = CB = b$, that

$$(v - b + z)(u - b - z) = myy.$$

But since either

$$yy = vv - (b - z)^2 = (v - b + z)(v + b - z),$$

or

$$yy = uu - (b + z)^2 = (u - b - z)(u + b + z),$$

having used each of them separately it will be either

$$u - b - z = m(v + b - z)$$

or

$$v - b + z = m(u + b + z),$$

having added which it results

$$u + v - 2b = m(u + v + 2b),$$

such that

$$u + v = \frac{2(1+m)b}{1-m} = 2c,$$

or

$$m = \frac{c-b}{c+b},$$

while $2c$ denotes the transverse axis. Therefore, since $uu - vv = 4bz$, having divided this equation by the last it will be

$$u - v = \frac{2bz}{c} \quad \text{and} \quad v = c - \frac{bz}{c}$$

and $u = c + \frac{bz}{c}$, and hence

$$yy = \frac{cc - bb}{cc}(cc - zz).$$

§30 Since now

$$\log\tan\frac{1}{2}\zeta + \log\tan\frac{1}{2}\eta = \log m,$$

by differentiation it will be

$$\frac{d\zeta}{\sin\zeta} + \frac{d\eta}{\sin\eta} = 0,$$

and hence

$$\frac{d\zeta\sin\eta}{d\eta\sin\zeta} = -1.$$

Hence from paragraph 15 it is necessary that

$$\frac{P - \sqrt{PP - QQ}}{Q} = -1,$$

and hence $P + Q = 0$, whence it must be

$$(A+B)(\cos\zeta + \cos\eta) + D + D\cos\zeta\cos\eta + E\sin\zeta\sin\eta = 0,$$

where the constants are to be defined in such a way that this equation agrees with the nature of the ellipse

$$\tan\frac{1}{2}\zeta\tan\frac{1}{2}\eta = m = \frac{c-b}{c+b}$$

or this one

$$\frac{(1-\cos\zeta)(1-\cos\eta)}{\sin\zeta\sin\eta} = m = \frac{c-b}{c+b}.$$

Therefore, since

$$\sin\zeta\sin\eta = \frac{1-\cos\zeta-\cos\eta+\cos\zeta\cos\eta}{m},$$

having substituted this value there

$$m(A+B)(\cos\zeta+\cos\eta)+mD+mD\cos\zeta\cos\eta-E(\cos\zeta+\cos\eta)$$
$$+E+E\cos\zeta\cos\eta=0,$$

for which reason these conditions are required that

$$E = m(A+B) \quad \text{and} \quad D = -\frac{E}{m} = -A-B$$

and hence

$$E = \frac{c-b}{c+b}(A+B) \quad \text{and} \quad C = D+E = -\frac{2b}{c+b}(A+B).$$

§31 To determine the nature of the motion on this ellipse, because of

$$dy = -\frac{zdz}{\sqrt{cc-zz}}\cdot\frac{\sqrt{cc-bb}}{c},$$

it will be

$$dz^2+dy^2 = \frac{dz^2(c^4-bbzz)}{c\sqrt{cc-zz}}$$

and

$$\sqrt{dz^2+dy^2} = \frac{dz\sqrt{c^4-bbzz}}{c\sqrt{cc-zz}},$$

but above we found that

$$dx^2 + dy^2 = 4gdt^2 \left(\frac{A}{v} + \frac{B}{u} + \frac{C}{2b} \right)$$

or

$$dx^2 + dy^2 = 4gdt^2 \left(\frac{Ac}{cc - bz} + \frac{Bc}{cc + bz} - \frac{A + B}{c + b} \right),$$

which goes over into this form:

$$dx^2 + dy^2 = \frac{4bgdt^2(A(c+z)(cc+bz) + B(c-z)(cc-bz))}{(c+b)(c^4 - bbzz)},$$

whence we conclude

$$\frac{4bccgdt^2}{b+c} = \frac{dz^2(c^4 - bbzz)^2}{(cc-zz)(A(c+z)(cc+bz) + B(c-z)(cc-bz)}$$

and hence by integration

$$2ct\sqrt{\frac{bg}{b+c}} = \int \frac{(c^4 - bbzz)dz}{\sqrt{(cc-zz)(A(c+z)(cc+bz) + B(c-z)(cc-bz)}}.$$

§32 If we put $B = 0$, the case is reduced to one single centre of force A, the calculation of which we discussed above; but this solution does not agree with that one at all, whence the method used here is rendered quite suspect. I, going to investigate the reason for this extraordinary phenomenon, observe that by the above equation (paragraph 30) the two letters D and E are not even determined. For, from the equation

$$(1 - \cos\zeta)(1 - \cos\eta) = m \sin\zeta \sin\eta$$

squared we conclude

$$(1 - \cos\zeta)(1 - \cos\eta) = mm(1 + \cos\zeta)(1 + \cos\eta),$$

whence

$$(1 - mm)(1 + \cos\zeta\cos\eta) = (1 + mm)(\cos\zeta + \cos\eta).$$

Since now it must be

$$(E + mD)(1 + \cos\zeta\cos\eta) = (E - m(A + B))(\cos\zeta + \cos\eta),$$

it suffices that

$$E(1 + mm) + Dm(1 + mm) = E(1 - mm) - (A + B)m(1 - mm) = 0,$$

whence

$$2Em + D(1 + mm) + (A + B)(1 - mm) = 0,$$

or

$$E = -\frac{(A + B)(1 - mm)}{2m} - \frac{D(1 + mm)}{2m}$$

and hence

$$C = D + E = -\frac{(A + B)(1 - mm)}{2m} - \frac{D(1 - m)^2}{2m}.$$

§33 Hence the deficiency of the method we used here becomes a lot more clear. For, since the quantity D remains undetermined, even though the curve described by the body M is given, the velocity of the body M would not be determined at each point of its orbit, but would rather be arbitrary. For, for the vertex of the ellipse closer to the focal point A, in which the distance is $v = c - b$ and $u = c + b$, or because of

$$\frac{c - b}{c + b} = m, \quad v = \frac{2mb}{1 - m} \quad \text{and} \quad u = \frac{2b}{1 - m},$$

the square of the velocity will be

$$\begin{aligned} \frac{dx^2 + dy^2}{dt^2} &= \frac{2g}{b}\left(\frac{A(1 - m)}{2m} + \frac{B(1 - m)}{1} - \frac{(A + B)(1 - mm)}{2m} - \frac{D(1 - m)^2}{2m}\right) \\ &= \frac{g}{mb}\left(-Am(1 - m)^2 - B(1 - m)^2 - D(1 - m)^2\right) \end{aligned}$$

and hence the velocity itself

$$\frac{\sqrt{dx^2 + dy^2}}{dt} = \sqrt{\frac{(1 - m)^2 g}{mb}(A - B - D)},$$

since which can not be undetermined in any way, it is obvious hat the method used in paragraph 30 is erroneous, what is seen even more clearly, if we assume both bodies A and B as vanishing, in which case the body M would certainly travel on a straight line, and therefore not an ellipse as we assumed here, even though our calculation indicates differently. Therefore, it will be very interesting to know the error of this method, that we, using a similar method on another occasion, do not commit any mistakes.

§34 Since in the calculation no error is detected, the reasoning itself we gave here is necessarily false, which is based on that foundation that the differential equation

$$\frac{d\zeta \sin \eta}{d\eta \sin \zeta} = \frac{P + \sqrt{PP - QQ}}{Q}$$

from paragraph 15 is satisfied by this finite equation

$$(1 - \cos \zeta)(1 - \cos \eta) = m \sin \zeta \sin \eta$$

(which is for an ellipse, of course), if one of the constants C and D is assumed in a certain way. But on another occasion I observed that it can happen that a differential equation is satisfied by a finite equation which is nevertheless not contained in the equation deduced from it by integration at all. As, e.g., this equation

$$ds\sqrt{1 - zz} = dz$$

is obviously satisfied by the value $z = 1$, which is nevertheless not contained in the integral equation

$$s = \alpha + \arcsin z$$

or $z = \sin(s - \alpha)$ by any means, whatever value is attributed to the constant α. Therefore, there is no doubt that for a similar reason the method applied here led to an error.

§35 When I inquired the origin of this error more accurately, against all expectation I stumbled upon the complete solution of the propounded problem, from which everything that had been desired in this subject until now will be seen clearly, and at the same time the origin of the error made here will be understand so lucidly that we can easily avoid errors in other similar cases of this kind. And this way the treatment of the mechanical problem provided such great explanations in Analysis which we might would have been searching for without any success in other circumstances, what is not to be considered as unusual, since the most artifices which have been found in Analysis are to be referred to be gained from questions in Mechanics and Physics. For, in these often investigations of such a kind occur which provide us with the opportunity to explore the nature of the equation more accurately, and hence very often the commission of an mistake was compensated by beautiful findings, as it happened to me treating this problem, the solution of which, if I had not made the mentioned error, I would certainly never had found.

Complete Solution of the Propounded Problem

§36 Since the propounded problem depends on the integration of this differential equation:

$$\frac{d\zeta \sin\eta}{d\eta \sin\zeta} = \frac{P + \sqrt{PP - QQ}}{Q},$$

for the sake of brevity having set

$$\begin{aligned} P &= A\cos\eta + B\cos\zeta + D\cos\zeta\cos\eta + E\sin\zeta\sin\eta, \\ Q &= A\cos\zeta + B\cos\eta + D, \end{aligned}$$

I reduce it to this form:

$$\frac{d\zeta\sin\eta + d\eta\sin\zeta}{d\zeta\sin\eta - d\eta\sin\zeta} = \frac{P + Q + \sqrt{PP - QQ}}{P - Q + \sqrt{PP - QQ}} = \frac{\sqrt{P+Q}}{\sqrt{P-Q}}.$$

Then having put $\tan\frac{1}{2}\zeta = p$ and $\tan\frac{1}{2}\eta = q$, since hence

$$\frac{d\zeta}{\sin\zeta} = \frac{dp}{p} \quad \text{and} \quad \frac{d\eta}{\sin\eta} = \frac{dq}{q},$$

our equation which is to be solved will be

$$\frac{qdp + pdq}{qdp - pdq} = \sqrt{\frac{P+Q}{P-Q}}.$$

§37 But having put $\tan\frac{1}{2}\zeta = p$ and $\tan\frac{1}{2}\eta = q$, it will be

$$\sin\zeta = \frac{2p}{1+pp}, \quad \cos\zeta = \frac{1-pp}{1+pp} \quad \sin\eta = \frac{2q}{1+qq}, \quad \cos\eta = \frac{1-qq}{1+qq}.$$

Since hence

$$P + Q = (A+B)(\cos\zeta + \cos\eta) + D(1 + \cos\zeta\cos\eta) + E\sin\zeta\sin\eta,$$

it will be

$$P + Q = \frac{2(A+B)(1-ppqq) + 2D(1+ppqq) + 4Epq}{(1+pp)(1+qq)}.$$

Further, since

$$P - Q = (A-B)(\cos\eta - \cos\zeta) - D(1 - \cos\zeta\cos\eta) + E\sin\zeta\sin\eta,$$

it will be

$$P - Q = \frac{2(A-B)(pp-qq) - 2D(pp+qq) + 4Epq}{(1+pp)(1+qq)}.$$

Therefore, having introduced these values, our equation which is to be solved will be

$$\frac{qdp + pdq}{qdp - pdq} = \sqrt{\frac{(A+B)(1-ppqq) + D(1+ppqq) + 2Epq}{(A-B)(pp-qq) - D(pp+qq) + 2Epq}}$$

which is easily plain to be reducible to the separation of variables, since the numerator of the second there is a function of pq, but in the denominator the quantities p and q add up to two dimension everywhere.

§38 To this end, let us set

$$pq = r \quad \text{and} \quad \frac{p}{q} = s,$$

that

$$p = \sqrt{rs} \quad \text{and} \quad q = \sqrt{\frac{r}{s}},$$

whence, because of $pdq + qdp = dr$ and $qdp - pdq$, it will be

$$\frac{dr}{qqds} = \sqrt{\frac{(A+B)(1-rr) + D(1+rr) + 2Er}{qq((A+B)(ss-1) - D(ss+1) + 2Es}}$$

or

$$\frac{dr}{ds} = \sqrt{\frac{r((A+B)(1-rr) + D(1+rr) + 2Er)}{s((A-B)(ss-1) - D(ss+1) + 2Es}},$$

from which form the separation of the variables r and s is obvious; for, it will be

$$\frac{dr}{\sqrt{r(A+B+D+2Er-(A+B-D)rr)}} = \frac{ds}{\sqrt{s(-A+B-D+2Es+(A-B-D)ss}}.$$

Or, if we set $r = xx$ and $s = yy$, one will have

$$\frac{dx}{\sqrt{A+B+D+2Exx-(A+B-D)x^4}} = \frac{dy}{\sqrt{-A+B-D+2Eyy+(A-B-D)y^4}}.$$

But since the values r and s could have negative values, this transformation could cause inconveniences.

§39 But even though this way we got to a separated equation, nevertheless the integration of each of both sides is extremely difficult, since it can be done neither by the the quadrature of the circle nor by logarithms; but the construction by arcs of conic sections would hardly lead to any understanding here. And this difficulty is not lowered, even if we set $B = 0$, in which case the solution is nevertheless known from elsewhere; yes, even the case $A = 0$ and $B = 0$, in which the curve described by the body M certainly is a line, is also not any easier. Therefore, it is necessary that in the cases both transcendental quantities arising from the respective integrations have a relation of such a kind that they are contained in an algebraic equation between x and y. From this a new field of inquiring algebraic equations, which are maybe contained in differential equations of this kind, is opened. And in this task there is no other method known than the one I explained several years ago, and by means of which I have compared infinitely many arcs, so of ellipses as of hyperbolas, to each other, which at that time seemed that it will have an immense use sometime in the future.

§40 But before I escape to this method, it will be in order to indicate the origin of the error made above which is now manifest. For, since we got to a separated differential equation between r and s, it is evident that it is satisfied, if either a constant value α of such a kind is attributed to r that

$$A+B+D+2Er-(A+B-D)rr=0,$$

or a constant value β of such a kind to s that

$$-A+B-D+2Es+(A-B-D)ss=0,$$

both of which can be done in two ways. And from the first assumption certainly $r=\alpha$, it follows

$$pq=\tan\frac{1}{2}\zeta\tan\frac{1}{2}\eta=\alpha,$$

which is an equation for an ellipse, but from the other assumption $s=\beta$ $\frac{p}{q}=\beta$ results or

$$\tan\frac{1}{2}\zeta=\beta\tan\frac{1}{2}\eta,$$

which is the equation for the hyperbola. And indeed, these curves only solve the problem, if the same values are contained in the integrated equation. Therefore, it is evident that we will have made a similar error, if we would have had assumed the motion of the body M to happen on a hyperbola.

§41 Now let us consider the integral equation which we found in the case $B=0$ above in paragraph 22 from our differential equation, which will be

$$\frac{\sin\zeta\cos\eta}{\sin\eta}=\frac{A+D\cos\zeta}{G}+\sin\zeta\sqrt{\frac{2E}{G}+\frac{AA-DD}{GG}-1}$$

and this, having put $\tan\frac{1}{2}\zeta=p$ and $\tan\frac{1}{2}\eta=q$ and further $pq=r$ and $\frac{p}{q}=s$, goes over into this form:

$$\left.\begin{aligned} GG(r+s)^2+2G(A+D)(r-s)-8EGrs+(A+D)^2 \\ +2G(A-D)rs(r-s)-2(AA-DD)rs+(A-D)^2 rrss \end{aligned}\right\} = 0,$$

which therefore is the complete integral equation corresponding to this differential equation

$$\frac{dr}{\sqrt{r(A+D+2Er-(A-D)rr)}} = \frac{ds}{\sqrt{s(-A-D+2Es+(A-D)ss)}},$$

since in it the new constant G is contained.

§42 To inquire such an integral equation in general, for the sake of brevity, let us put

$$\frac{A+B}{2E} = m, \quad \frac{A-B}{2E} = n \quad \text{and} \quad \frac{D}{2E} = k,$$

that the equation which is to be integrated, if it can be done, of course, is

$$\frac{dr}{\sqrt{r(m+k+r-(m-k)rr)}} = \frac{ds}{\sqrt{s(-n-k+s+(n-k)ss)}},$$

the integral of which we want to assume to be contained in this form:

$$\mathfrak{A} + 2\mathfrak{B}r + 2\beta s + \mathfrak{C}rr + \gamma ss + 2\mathfrak{D}rs + 2\mathfrak{E}rrs + 2\varepsilon rss + \mathfrak{F}rrss = 0,$$

whence we deduce:

$$rr = \frac{-2\mathfrak{B}r - 2\mathfrak{D}rs - 2\varepsilon rss - \mathfrak{A} - 2\beta s - \gamma ss}{\mathfrak{C} + 2\mathfrak{E}s + \mathfrak{F}ss}$$

and

$$ss = \frac{-2\beta s - 2\mathfrak{D}rs - 2\mathfrak{E}rrs - \mathfrak{A} - 2\mathfrak{B}r - \mathfrak{C}rr}{\gamma + 2\varepsilon r + \mathfrak{F}rr}.$$

But then by differentiation

$$\begin{aligned} &dr(\mathfrak{B} + \mathfrak{C}r + \mathfrak{D}s + 2\mathfrak{E}rs + \varepsilon ss + \mathfrak{F}rss) \\ &+ds(\beta + \gamma s + \mathfrak{D}r + \mathfrak{E}rr + 2\varepsilon rs + \mathfrak{F}rrs) = 0. \end{aligned}$$

§43 But from those equations, extracting the root, we obtain

$$\begin{aligned}&r(\mathfrak{C}+2\mathfrak{E}s+\mathfrak{F}ss)+\mathfrak{B}+\mathfrak{D}s+\varepsilon ss=S\\&=\sqrt{((\mathfrak{B}+\mathfrak{D}s+\varepsilon ss)^2-(\mathfrak{A}+2\beta s+\gamma ss)(\mathfrak{C}+2\mathfrak{E}s+\mathfrak{F}ss)}\end{aligned}$$

and

$$\begin{aligned}&s(\gamma+2\varepsilon r+\mathfrak{F}rr)+\beta+\mathfrak{D}r+\mathfrak{E}rr=-R\\&=-\sqrt{(\beta+\mathfrak{D}r+\mathfrak{E}rr)^2-(\mathfrak{A}+2\mathfrak{B}r+\mathfrak{C}rr)(\gamma+2\varepsilon r+\mathfrak{F}rr)}\end{aligned}$$

which values in the applied differential yield

$$Sdr-Rds=0 \quad \text{or} \quad \frac{dr}{R}=\frac{ds}{S}.$$

Therefore, it just remains that the irrational formulas R and S are made equal to those our equation to be solved contains, or that

$$R=\sqrt{(m+k)r+rr-(m-k)r^3}$$

and

$$S=\sqrt{-(n+k)s+ss+(n-k)s^3}.$$

§44 Therefore, since here on both sides so the first terms are constants as the last containing r^4 and s^4 are zero, it must be

$$\mathfrak{B}\mathfrak{B}-\mathfrak{A}\mathfrak{C}=0,\quad \varepsilon\varepsilon-\gamma\mathfrak{F}=0,\quad \beta\beta-\mathfrak{A}\gamma=0,\quad \mathfrak{E}\mathfrak{E}-\mathfrak{C}\mathfrak{F}=0.$$

Therefore, it will be

$$\mathfrak{C}=\frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}=\frac{\mathfrak{E}\mathfrak{E}}{\mathfrak{F}} \quad \text{and} \quad \gamma=\frac{\varepsilon\varepsilon}{\mathfrak{F}}=\frac{\beta\beta}{\mathfrak{A}}$$

and hence

$$\frac{\mathfrak{A}}{\mathfrak{F}}=\frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{E}\mathfrak{E}}=\frac{\beta\beta}{\varepsilon\varepsilon}.$$

Further, because of the terms rr and ss it must be

$$\mathfrak{D}\mathfrak{D} + 2\mathfrak{B}\varepsilon \ - \mathfrak{A}\mathfrak{F} - 4\beta\mathfrak{E} - \gamma\mathfrak{C} = 1,$$
$$\mathfrak{D}\mathfrak{D} + 2\mathfrak{B}\mathfrak{E} - \mathfrak{A}\mathfrak{F} - 4\beta\varepsilon \ - \gamma\mathfrak{C} = 1,$$

whence

$$6\mathfrak{B}\varepsilon = 6\beta\mathfrak{E} \quad \text{or} \quad \frac{\mathfrak{B}}{\mathfrak{E}} = \frac{\beta}{\varepsilon},$$

but then

$$\mathfrak{D}\mathfrak{D} - 2\beta\mathfrak{E} - \mathfrak{A}\mathfrak{F} - \gamma\mathfrak{C} = 1,$$

or

$$\mathfrak{D}\mathfrak{D} - 2\beta\mathfrak{E} - \mathfrak{A}\mathfrak{F} - \frac{\beta\beta\mathfrak{E}\mathfrak{E}}{\mathfrak{A}\mathfrak{F}} = 1,$$

and hence

$$\mathfrak{A}\mathfrak{F}(\mathfrak{D}\mathfrak{D} - 1) = (\beta\mathfrak{E} + \mathfrak{A}\mathfrak{F})^2$$

or

$$\beta\mathfrak{E} = -\mathfrak{A}\mathfrak{F} + \sqrt{\mathfrak{A}\mathfrak{F}(\mathfrak{D}\mathfrak{D} - 1)} = \mathfrak{B}\varepsilon$$

or since

$$\mathfrak{F} = \frac{\mathfrak{A}\mathfrak{E}\mathfrak{E}}{\mathfrak{B}\mathfrak{B}},$$

it will be

$$\mathfrak{D}\mathfrak{D} = 2\beta\mathfrak{E} + \frac{\mathfrak{A}\mathfrak{A}\mathfrak{E}\mathfrak{E}}{\mathfrak{B}\mathfrak{B}} + \frac{\mathfrak{B}\mathfrak{B}\beta\beta}{\mathfrak{A}\mathfrak{A}} + 1 = \left(\frac{\mathfrak{A}\mathfrak{E}}{\mathfrak{B}} + \frac{\mathfrak{B}\beta}{\mathfrak{A}}\right)^2 + 1.$$

§45 The remaining terms give

$$2\beta\mathfrak{D} \ - 2\mathfrak{A}\varepsilon \ - 2\mathfrak{B}\gamma = m + k,$$
$$2\mathfrak{D}\mathfrak{E} \ - 2\mathfrak{C}\varepsilon \ - 2\mathfrak{B}\mathfrak{F} = -m + k,$$
$$2\mathfrak{B}\mathfrak{D} - 2\mathfrak{A}\mathfrak{E} - 2\beta\mathfrak{C} \ = -n - k,$$
$$2\mathfrak{D}\varepsilon \ - 2\gamma\mathfrak{E} - 2\beta\mathfrak{F} \ = n - k,$$

the sum of which yields this equality:

$$\mathfrak{D}(\beta+\mathfrak{B}+\varepsilon+\mathfrak{E})-\mathfrak{A}(\varepsilon+\mathfrak{E})-\mathfrak{F}(\beta+\mathfrak{B})-\mathfrak{C}\varepsilon-\gamma\mathfrak{E}-\mathfrak{B}\gamma-\beta\mathfrak{C}=0.$$

Since we already found $\frac{\mathfrak{B}}{\mathfrak{E}}=\frac{\beta}{\varepsilon}$, let us put $\mathfrak{B}=\lambda\beta$ and $\mathfrak{E}=\lambda\varepsilon$, it will be

$$\mathfrak{C}=\frac{\lambda\lambda\beta\beta}{\mathfrak{A}},\quad \gamma=\frac{\beta\beta}{\mathfrak{A}}\quad\text{and}\quad \mathfrak{F}=\frac{\varepsilon\varepsilon}{\beta\beta}\mathfrak{A},$$

whence further put $\mathfrak{A}=\mu\beta\beta$ and $\mathfrak{F}=\mu\varepsilon\varepsilon$ that

$$\mathfrak{C}=\frac{\lambda\lambda}{\mu}\quad\text{and}\quad \gamma=\frac{1}{\mu},$$

and hence

$$\mathfrak{D}\mathfrak{D}=1+\left(\mu\beta\varepsilon+\frac{\lambda}{\mu}\right)^2,$$

having substituted which values, except the last, there we will obtain

$$\mathfrak{D}(\lambda+1)(\beta+\varepsilon)-\mu\beta\varepsilon(\lambda+1)(\beta+\varepsilon)-\frac{\lambda(\lambda+1)(\beta+\varepsilon)}{\mu}=0,$$

or

$$(\lambda+1)(\beta+\varepsilon)\left(\mathfrak{D}-\mu\beta\varepsilon-\frac{\lambda}{\mu}\right)=0,$$

the three factors of which equation yield as many solutions.

§46 *Resolution I.* Let $\lambda=-1$, it will be $\mathfrak{B}=-\beta$, $\mathfrak{E}=-\varepsilon$, $\mathfrak{C}=\frac{1}{\mu}$, $\gamma=\frac{1}{\mu}$, $\mathfrak{A}=\mu\beta\beta$, $\mathfrak{F}=\mu\varepsilon\varepsilon$, and hence

$$\mathfrak{D}\mathfrak{D}=\left(\mu\beta\varepsilon-\frac{1}{\mu}\right)^2+1,$$

whence the conditions which are to be satisfied will be:

$$
\begin{aligned}
k &= \mathfrak{D}(\beta-\varepsilon) + \frac{\beta-\varepsilon}{\mu} - \mu\beta\varepsilon(\beta-\varepsilon) = (\beta-\varepsilon)\left(\mathfrak{D} + \frac{1}{\mu} - \mu\beta\varepsilon\right),\\
m &= \mathfrak{D}(\beta+\varepsilon) + \frac{\beta+\varepsilon}{\mu} - \mu\beta\varepsilon(\beta+\varepsilon) = (\beta-\varepsilon)\left(\mathfrak{D} + \frac{1}{\mu} - \mu\beta\varepsilon\right),\\
n &= \mathfrak{D}(\beta+\varepsilon) + \frac{\beta+\varepsilon}{\mu} - \mu\beta\varepsilon(\beta+\varepsilon) = (\beta-\varepsilon)\left(\mathfrak{D} + \frac{1}{\mu} - \mu\beta\varepsilon\right),
\end{aligned}
$$

Therefore, hence it will be $m = n$ and $B = 0$ such that this resolution can only be accommodated to the case $B = 0$. Therefore, since in this case

$$\frac{m}{k} = \frac{\beta+\varepsilon}{\beta-\varepsilon},$$

put $\beta+\varepsilon = m$ and $\beta-\varepsilon = k$ that

$$\beta = \frac{k+m}{2} \quad \text{and} \quad \varepsilon = \frac{m-k}{3};$$

and it must be $\mathfrak{D} + \frac{1}{\mu} - \mu\beta\varepsilon = 1$, whence it results

$$1 + 2\left(\mu\beta\varepsilon - \frac{1}{\mu}\right) + \left(\mu\beta\varepsilon - \frac{1}{\mu}\right)^2 = 1 + \left(\mu\beta\varepsilon - \frac{1}{\mu}\right)^2$$

and hence

$$\mu\mu = \frac{1}{\beta\varepsilon} = \frac{4}{mm-kk} \quad \text{and} \quad \mu = \frac{2}{\sqrt{mm-kk}}.$$

hence we conclude $\mathfrak{D} = 1$, and for the case $m = n$ the integral equation will be

$$\mathfrak{A} + 2\mathfrak{B}r + 2\beta s + \mathfrak{C}rr + \gamma ss + 2\mathfrak{D}rs + 2\mathfrak{E}rrs + 2\varepsilon rss + \mathfrak{F}rrss = 0.$$

§47 But this integral equation, since it contains no new constant, is not the complete one; the reason for this is that the quantities $\beta-\varepsilon$ and $\beta+\varepsilon$ are not equal to the numbers k and m, but must be only proportional to them. Therefore, let

$$\beta-\varepsilon = \frac{k}{\nu}, \quad \beta+\varepsilon = \frac{m}{\nu}, \quad \text{it will be} \quad \beta = \frac{m+k}{2\nu}, \quad \varepsilon = \frac{m-k}{2\nu}$$

and

$$\mathfrak{D} = \nu + \mu\beta\varepsilon - \frac{1}{\mu} = \sqrt{1 + \left(\mu\beta\varepsilon - \frac{1}{\mu}\right)^2},$$

whence

$$\mu\beta\varepsilon - \frac{1}{\mu} = \frac{1-\nu\nu}{2\nu} \quad \text{and} \quad \mathfrak{D} = \frac{1+\nu\nu}{2\nu},$$

where ν is an arbitrary constant in terms of which the number μ must be defined. Therefore, since β and ε are given in terms of m and k and ν, the integral will be

$$0 = \mu\beta\beta + 2\beta(s-r) + \frac{1}{\mu}(rr+ss) + \frac{1+\nu\nu}{\nu}rs + 2\varepsilon rs(s-r) + \mu\varepsilon\varepsilon rrss,$$

which, having substituted the values for β and ε, multiplying by μ goes over into this form:

$$0 = \frac{\mu\mu(m+k)^2}{4\nu\nu} + \frac{\mu}{\nu}(m+k)(s-r) + rr + ss + \frac{\mu\mu}{4\nu\nu}(m-k)^2 rrss$$
$$+ \frac{\mu}{\nu}(m-k)rs(s-r) + \frac{\mu(1+\nu\nu)}{\nu}rs.$$

Let $\frac{\mu}{\nu} = 2f$ that f is an arbitrary constant, and having taken

$$f(mm-kk) = \frac{1}{f} + 1 - \nu\nu,$$

the complete integral equation will be

$$0 = ff(m+k)^2 + 2f(m+k)(s-r) + rr + ss + ff(m-k)^2 rrss$$
$$+2f(m-k)rs(s-r) + 2f(1\nu\nu)rs.$$

§48 But since

$$\nu\nu = 1 + \frac{1}{f} - f(mm - kk),$$

it will be

$$2f(1 + \nu\nu) = 4f + 2 - 2ff(mm - kk);$$

and hence the complete integral equation in expanded form will look as follows:

$$\begin{aligned} 0 = ff(m+k)^2 + 2f(m+k)(s-r) + (r+s)^2 + ff(m-k)^2 rrss \\ +2f(m-k)rs(s-r) + 4frs - 2ff(mm-kk)rs, \end{aligned}$$

which having extracted the square root takes this form:

$$s - r + f(m+k) + f(m-k)rs = 2\sqrt{rs(ff(mm-kk) - f - 1)}$$

and after having substituted $s = \frac{p}{q}$, $r = pq$ again

$$\frac{p(1-qq)}{q} + f(m+k) + f(m-k)pp = 2p\sqrt{ff(mm-kk) - f - 1},$$

which agrees with the complete integral exhibited above. But it is to be noted carefully that this integral extends only to the case $B = 0$.

§49 *Resolution II.* Let us now put $\varepsilon = -\beta$, and first we will have

$$\mathfrak{B} = \lambda\beta, \quad \mathfrak{E} = -\lambda\beta, \quad \mathfrak{C} = \frac{\lambda\lambda}{\mu}, \quad \gamma = \frac{1}{\mu}, \quad \mathfrak{A} = \mu\beta\beta, \quad \mathfrak{F} = \mu\beta\beta$$

and

$$\mathfrak{D}\mathfrak{D} = 1 + \left(\frac{\lambda}{\mu} - \mu\beta\beta\right)^2.$$

But then hence we conclude:

$$k = \beta(1-\lambda)\left(\mathfrak{D} - \frac{\lambda}{\mu} + \mu\beta\beta\right),$$

$$m = \beta(1+\lambda)\left(\mathfrak{D} - \frac{\lambda}{\mu} + \mu\beta\beta\right) = -n,$$

such that this resolution only holds if $m+n=0$ and hence $A=0$. But for this case it will further be

$$\frac{k}{m} = \frac{1-\lambda}{1+\lambda} \quad \text{and hence} \quad \lambda = \frac{m-k}{m+k},$$

whence it follows

$$k = \frac{2\beta k}{m+k}\left(\mathfrak{D} - \frac{\lambda}{\mu} + \mu\beta\beta\right) \quad \text{or} \quad \mathfrak{D} = \frac{\lambda}{\mu} - \mu\beta\beta + \frac{m+k}{2\beta};$$

therefore,

$$1 = \frac{m+k}{\beta}\left(\frac{\lambda}{\mu} - \mu\beta\beta\right) + \frac{(m+k)^2}{4\beta\beta}$$

and

$$\frac{\lambda}{\mu} - \mu\beta\beta = \frac{\beta}{m+k} - \frac{(m+k)}{4\beta},$$

and hence

$$\mathfrak{D} = \frac{\beta}{m+k} + \frac{m+k}{4\beta}.$$

The letter β remains undefined and μ is defined by this equation:

$$\frac{m-k}{\mu(m+k)} - \mu\beta\beta = \frac{\beta}{m+k} - \frac{(m+k)}{4\beta}.$$

§50 Having substituted these values this complete integral equation for the case $A=0$ results:

$$0 = \mu\beta\beta + 2\lambda\beta r + 2\beta s + \frac{\lambda\lambda}{\mu}rr + \frac{1}{\mu}ss + \frac{2\beta rs}{m+k} + \frac{(m+k)}{2\beta}rs$$
$$-2\lambda\beta rrs - 2\beta rss + \mu\beta\beta rrss.$$

Let us set $\mu\beta = f$, it will be

$$\frac{m-k}{m+k} - ff = \frac{f}{m+k} - \frac{\mu\mu(m+k)}{4f}$$

and that equation multiplied by μ will be

$$0 = ff + 2\lambda fr + 2fs + \lambda\lambda rr + ss + \frac{2frs}{m+k} + \frac{\mu\mu(m+k)}{2f}rs$$
$$-2\lambda frrs - 2frss + ffrrss,$$

which, because of

$$\frac{\mu\mu(m+k)}{f} = \frac{2f}{m+k} + 2ff - 2\lambda,$$

takes this form:

$$0 = ff(1+rs)^2 + (\lambda r - s)^2 + \frac{4frs}{m+k} + 2f(\lambda r + s) - 2frs(\lambda r + s)$$

or this one:

$$0 = ff(1-rs)^2 + (\lambda r + s)^2 + 2f(1-rs)(\lambda r + s) + \frac{4frs}{m+k} + 4ffrs - 4\lambda rs,$$

which having extracted the square root yields

$$f(1-rs) + \lambda r + s = 2\sqrt{rs\left(\frac{m-k}{m+k} - ff - \frac{f}{m+k}\right)}.$$

And this solution is completely similar to preceding one, while that one was restricted to the case $B = 0$, this one on the other hand is restricted to the case $A = 0$.

§51 The third factor

$$\mathfrak{D} - \mu\beta\varepsilon - \frac{\lambda}{\mu}$$

shows nothing, since its annihilation can not be consistent with the equation

$$\mathfrak{D}\mathfrak{D} = 1 + \left(\mu\beta\varepsilon + \frac{\lambda}{\mu}\right)^2,$$

and so we have just two cases admitting an algebraic solution, namely when either $B = 0$ or $A = 0$. Furthermore, even the third case expanded above, in which it was $A = B$ and $D = 0$, $E = 0$, can be covered here without further effort; for, then the equation of paragraph 38 goes over into this one:

$$\frac{dr}{\sqrt{(A+B)r(1-rr)}} = \frac{ds}{\sqrt{s \cdot 0}},$$

which can only hold, if $ds = 0$, and hence

$$s = \frac{p}{q} = \frac{\tan\frac{1}{2}\zeta}{\tan\frac{1}{2}\eta} = \text{Const.},$$

by which equation a hyperbola is defined. In the remaining cases the construction of the equation

$$\frac{dr}{\sqrt{r(A+B+D+2Er-(A+B-D)rr)}} = \frac{ds}{\sqrt{s(B-A-D+2Es-(B-A+D)ss)}}$$

is to be called for help. For, that this equation does not admit an algebraic integral is plain even from the case $D = A + B$, in which the left-hand side depends of the quadrature of conic section, the right-hand side on the other hand required higher quadratures.

§52 But having found the relation between r and s whence at the same time the ratio of the angles ζ and η becomes known, the cognition of the motion is concluded from the nature of time. For, since

$$vvuud\zeta d\eta = 2gadt^2(A\cos\zeta + B\cos\eta + D),$$

because of

$$v = \frac{a\sin\eta}{\sin(\zeta+\eta)} \quad \text{and} \quad u = \frac{a\sin\zeta}{\sin(\zeta+\eta)}$$

and

$$\tan\frac{1}{2}\zeta = p \quad \text{and} \quad \tan\frac{1}{2}\eta = q,$$

it will be

$$d\zeta = \frac{dp\sin\zeta}{p} \quad \text{and} \quad d\eta = \frac{dq\sin\eta}{q},$$

or

$$d\zeta = \frac{2dp}{1+pp} \quad \text{and} \quad d\eta = \frac{2dq}{1+qq},$$

and hence further

$$v = \frac{aq(1+pp)}{(p+q)(1-pq)} \quad \text{and} \quad u = \frac{ap(1+qq)}{(p+q)(1-pq)},$$

whence

$$\frac{4a^3ppqq(1+pp)(1+qq)dpdq}{(p+q)^4(1-pq)^4} = 2gdt^2\left(\frac{A(1-pp)}{1+pp} + \frac{B(1-qq)}{1+qq} + D\right).$$

Further, let $pq = r$, $\frac{p}{q} = s$ or $pp = rs$ and $qq = \frac{r}{s}$, it will be

$$2pdp = rds + sdr \quad \text{and} \quad 2qdq = \frac{sdr - rds}{ss},$$

therefore,

$$4pqdpdq = \frac{ssdr^2 - rrds^2}{ss},$$

and hence finally:

$$\frac{a^3(r+s)(1+rs)(ssdr^2 - rrds^2)}{rs(1+s)^4(1-r)^4} = 2gdt^2\left(\frac{A(1-rs)}{1+rs} + \frac{B(s-r)}{r+s} + D\right).$$

§53 Now, for the sake of brevity, let us set

$$RR = r(A + B + D + 2Er - (A + B - D)rr),$$
$$SS = s(B - A - D + 2Es - (B - A + D)ss),$$

that $\frac{dr}{R} = \frac{ds}{S}$, and let us set

$$\frac{dr}{R} = \frac{ds}{S} = dV,$$

that $dr = RdV$ and $ds = SdV$ and it will be

$$2gdt^2 = \frac{a^3(r+s)^2(1+rs)^2(RRss - SSrr)dV^2}{rs(1+s)^4(1-r)^4(A(r+s)(1-rs) + B(s-r)(1+rs) + D(s+r)(1+rs))}.$$

But on the other hand

$$RRss - SSrr = rs(A(r+s)(1-rs) + B(s-r)(1+rs) + D(r+s)(1+rs)),$$

having substituted which value it results

$$2gdt^2 = \frac{a^3(r+s)^2(1+rs)^2 dV^2}{(1+s)^3(1-r)^4},$$

and hence

$$dt\sqrt{\frac{2g}{a}} = \frac{a(r+s)(1+rs)dV}{(1+s)^2(1-r)^2} = adV\left(\frac{r}{(1-r)^2} + \frac{s}{(1+s)^2}\right),$$

such that

$$\frac{dt\sqrt{2g}}{a\sqrt{a}} = \frac{dr\sqrt{r}}{(1-r)^2\sqrt{A+B+D+2Er-(A+B-D)rr}} + \frac{ds\sqrt{s}}{(1+s)^2\sqrt{B-A-D+2Es-(B-A+D)s^2}},$$

and so even the determination of time is reduced to the integration of simple formulas.

§54 Therefore, since this problem which on first sight seemed to be hardly easier than that in which all three bodies are assumed as mobile, could be resolved perfectly, we now have greater hope that sometime it will be that even that problem which is to be considered as the foundation of whole Astronomy is resolved. I personally think that I hence still do not see a way to arrive at this goal, but believe that for this still many and maybe cumbersome attempts are needed. Furthermore, I add an observation, probably interesting for the Geometers, on the problem I treated here, namely that aside from the cases expanded here there are innumerable others, in which the curve describes by the body M is algebraic, the investigation of which seems to lead to increments of analysis not to be contemned.

§55 But although we reduced the solution of this problem to quadratures of curves, it would nevertheless be bothersome to define the curve described by the body M and a lot more bothersome to assign the position of the body at a given time. But if cases of this kind would actually exist, it would be worth one's while to expand this solution more accurately, since this way it seems to happen the most convenient way. Of course, for the case, after by many attempts the constants A, B, D, E had become known approximately and were then to be corrected, a table listing the values of r corresponding to the values of s for each pair of values will have to be constructed, to which thereafter a table exhibiting the time t must be added, from which further vice versa for a given time t the values of the letters r and s and hence the angles ζ and η could be concluded. If this determination would agree less with observations, this would be an indication that the constants would not be assumed correctly, and so eventually, having constructed many tables, the truth would easily be found from this.

§56 But since it is especially convenient to know the location of the body M, where its distance from one of the fixed points A and B is maximal or minimal, let us see how this must be defined. Since the distance AM is

$$v = \frac{a \sin \eta}{\sin(\zeta + \eta)},$$

its differential

$$\frac{dv}{a} = \frac{d\eta \cos \eta \sin(\zeta + \eta) - (d\zeta + d\eta) \sin \eta \cos(\zeta + \eta)}{\sin^2(\zeta + \eta)} = \frac{d\eta \sin \zeta - d\zeta \sin \eta \cos(\zeta + \eta)}{\sin^2(\zeta + \eta)}$$

set equal to zero or

$$\frac{d\eta}{\sin \eta} = \frac{d\zeta}{\sin \zeta} \cos(\zeta + \eta)$$

will indicate the locations at which the distance AM is maximal or minimal. Therefore, having put $\tan \frac{1}{2}\zeta = p$ and $\tan \frac{1}{2}\eta = q$, because of

$$\cos(\zeta + \eta) = \frac{(1-pp)(1-qq) - 4pq}{(1+pp)(1+qq)},$$

we will have

$$\frac{dq}{q} = \frac{dp}{p} \cdot \frac{(1-pq)^2 - (p+q)^2}{(1+pp)(1+qq)}.$$

And having further set $pq = r$, $\frac{p}{q} = s$, or $p = \sqrt{rs}$, $q = \sqrt{\frac{r}{s}}$, it will be

$$\left(\frac{dr}{r} - \frac{ds}{s}\right)(s(1+rr) + r(1+ss)) = \left(\frac{dr}{r} + \frac{ds}{s}\right)(s(1-r)^2 - r(1+s)^2)$$

or

$$dr(1+s)^2 = ds(1-r)^2.$$

Therefore, where

$$\frac{dr}{(1-r)^2} = \frac{ds}{(1+s)^2},$$

there the distance $AM = v$ is either maximal or minimal.

§57 Therefore, since we found $\frac{dr}{R} = \frac{ds}{S}$ above, for these locations we have

$$\frac{R}{(1-r)^2} = \frac{S}{(1+s)^2},$$

whence the relation among the finite quantities r and s is found which is:

$$r(1+s)^4(A+B+D+2Er-(A+B-D)rr)$$
$$= s(1-r)^4(B-A-D+2Es-(B-A+D)ss),$$

whence, for the sake of brevity having set

$$\frac{A+B+D}{2E} = m, \qquad \frac{B-A-D}{2E} = n,$$
$$\frac{A+B-D}{2E} = \mu, \qquad \frac{B-A+D}{2E} = \nu,$$

that $\mu + \nu = m + n$, this equation results

$$\left.\begin{array}{llllll} +mr+rr & -\mu r^3 & +4(n-\mu)r^4s & -nr^4s & -r^4ss & \\ -ns\ +4(m+n)rs & +(4-6n)rrs & +4(m-\nu)rs^3 & +(4-6\mu)r^3ss & -4(\mu+\nu)r^3s^3 & +\nu r^4s^3 \\ \quad -ss & +(4+6m)rss & & +(4+6\nu)rrs^3 & +rrs^4 & -\mu r^3s^4 \\ & +\nu s^3 & & +mrs^4 & & \end{array}\right\} = 0$$

which equation in general does not seem to have any factors. But the equation among p and q will be

$$(p+q)^4(m+pq-\mu ppqq) = (1-pq)^4(nqq+pq-\nu pp).$$

§58 But having substituted the angles ζ and η again, the equation between them for the case in which the distance is maximal or minimal will look as follows:

$$\sin^4\left(\frac{\zeta+\eta}{2}\right)(D(1+\cos\zeta\cos\eta)+(A+B)(\cos\zeta+\cos\eta)+E\sin\zeta\sin\eta)$$

$$=\cos^4\left(\frac{\zeta+\eta}{2}\right)(D(\cos\zeta\cos\eta-1)+(B-A)(\cos\zeta-\cos\eta)+E\sin\zeta\sin\eta),$$

where

$$\sin^4\left(\frac{\zeta+\eta}{2}\right)=\frac{1}{4}(1-\cos(\zeta+\eta))^2 \quad\text{and}\quad \cos^4\left(\frac{\zeta+\eta}{2}\right)=\frac{1}{4}(1+\cos(\zeta+\eta))^2,$$

whence we conclude:

$$(1+\cos^2(\zeta+\eta))(A\cos\zeta+B\cos\eta+D)$$
$$=2\cos(\zeta+\eta)(A\cos\eta+B\cos\zeta+D\cos\zeta\cos\eta+E\sin\zeta\sin\eta),$$

which equation likewise hardly admits a resolution. Furthermore, since having permuted the angles ζ, η and the masses A and B the equation is not changed, it indicates the same position where the distance $BM = u$ is maximal or minimal. But having extracted the square root both cases are separated from each other.