Observations on the integrals of the form $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ having put $x=1$ after the integration*

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§1 In this paper I will consider the integrals of this form

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$$

or expressed this way

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$$

and I assume the exponents n, p and q to be positive integer numbers, since, if they were not such numbers, they could easily be reduced to this form. Further, I decided not to consider the integral of this formula in general, but only the value it obtains, if after the integration one sets $x=1$ and the integration was done in such a way, of course, that the integral vanishes for $x=0$. For, first there is no doubt that in this case $x=1$ the integral is expressed a lot simpler; and furthermore, as often as in Analysis one gets to formulas of this kind, in most cases not so the indefinite integral for an arbitrary value of x as for the definite value $x=1$ is especially in question.

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§2 But it is known that in the case, in which after the integration one puts $x = 1$, the integral $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ is expressed in terms of an infinite product this way that it is

$$\frac{p+q}{pq} \cdot \frac{n(p+q+n)}{(p+n)(q+n)} \cdot \frac{2n(p+q+2n)}{(p+2n)(q+2n)} \cdot \frac{3n(p+q+3n)}{(p+3n)(q+3n)} \cdot \text{etc.};$$

the first factor $\frac{p+q}{pq}$ of this product does certainly not fit into the structure of the following. But it is nevertheless perspicuous that the exponents p and q are interchangeable, so that it is

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1} = \int x^{q-1}dx(1-x^n)^{\frac{p}{n}-1},$$

which equality is also easily shown directly. But this infinite product will lead us to other much greater ones illustrating these integrals a lot better.

§3 But for the sake of brevity let us introduce the following notation to indicate $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$: for each exponent n I will write

$$\left(\frac{p}{q}\right),$$

so that $\left(\frac{p}{q}\right)$ denotes the value of the integral formula $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ in the case, in which after the integration one puts $x = 1$. And since we saw that in this case it is

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1} = \int x^{q-1}dx(1-x^n)^{\frac{p}{n}-1},$$

it is manifest that it will be

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right),$$

so that for each value of the exponent n these expressions $\left(\frac{p}{q}\right)$ and $\left(\frac{q}{p}\right)$ signify the same quantity. So if, for the sake of an example, it was $n = 4$, it will be

$$\left(\frac{3}{2}\right) = \left(\frac{2}{3}\right) = \int \frac{x^2dx}{\sqrt[4]{(1-x^4)^2}} = \frac{xdx}{\sqrt[4]{1-x^4}}.$$

But by means of the infinite product one will have

$$\left(\frac{3}{2}\right) = \left(\frac{2}{3}\right) = \frac{5}{2\cdot 3}\cdot\frac{4\cdot 9}{6\cdot 7}\cdot\frac{8\cdot 13}{10\cdot 11}\cdot\frac{12\cdot 17}{14\cdot 15}\cdot \text{etc.}$$

§4 First, I now observe, if the exponents p and q were larger than the exponent n, that the integral formula can always be reduced to another one, in which these exponents are smaller than n. For, because since it is

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \frac{p-n}{p+q-n}\int \frac{x^{p-n-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}},$$

in our notation it will be

$$\left(\frac{p}{q}\right) = \frac{p-n}{p+q-n}\left(\frac{p-n}{q}\right);$$

by this reduction, if it was $p > n$, the formula is reduced to another one, in which the exponent p is smaller than n, which is to be understood also for the other exponent q because of their interchangeability. Therefore, for us going to examine these formulas, it will be sufficient to assume the exponents p and q smaller than that exponent n; since all cases, in which they would have larger values, can be reduced to the cases of smaller n.

§5 But it is immediately plain that the cases, in which it is either $p = n$ or $q = n$, are absolutely or algebraically integrable. For, if it was $q = n$, because of

$$\left(\frac{p}{n}\right) = \int x^{p-1}dx = \frac{x^p}{p}$$

having put $x = 1$ it will be $\left(\frac{p}{n}\right) = \frac{1}{n}$ and in like manner $\left(\frac{n}{q}\right) = \frac{1}{q}$. And these are the only cases, in which the integral of our formula can be exhibited absolutely, if p and q do not exceed the exponent n, of course. In all remaining cases the integration will involve either the quadrature of the circle or even higher quadratures, which we intend to consider more accurately here. Therefore, after the formulas $\left(\frac{p}{n}\right)$ or $\left(\frac{n}{p}\right)$, whose absolute value is $= \frac{1}{p}$, those come, whose values are expressed only by terms of the quadrature of the circle; but then these cases will follow, which require a certain higher quadrature, and I will try to reduce these higher quadratures so to the simplest form as to the smallest number.

§6 Since the numbers p and q are assumed to be smaller than the exponent n, those formulas $\left(\frac{p}{q}\right)$ become integrable by means of the quadrature of the circle in the cases, in which it is $p+q=n$. For, let it be $q=n-p$ and our formula

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^p}}=\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^q}}$$

will be expressed by means of this infinite product

$$\frac{n}{p(n-p)}\cdot\frac{n\cdot 2n}{(n+p)(2n-p)}\cdot\frac{2n\cdot 3n}{(2n+p)(3n-p)}\cdot\frac{3n\cdot 4n}{(3n+p)(4n-p)}\cdot\text{etc.},$$

which represented this way

$$\frac{1}{p}\cdot\frac{nn}{nn-pp}\cdot\frac{4nn}{4nn-pp}\cdot\frac{9nn}{9nn-pp}\cdot\text{etc.}$$

is identical to the product expressing the sine of a certain angle. Hence, if π is taken for the half of the circumference of the circle, whose radius is $=1$, and at the same time exhibits the measure of two right angles, it will be

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin\frac{p\pi}{n}}=\frac{\pi}{n\sin\frac{q\pi}{n}}.$$

§7 In the remaining cases, in which it is neither $p=n$ nor $q=n$ and also not $p+q=n$, the integral can neither be exhibited absolutely nor by the quadrature of the circle, but contains other certain higher quadratures. But on the other not every single formula requires a peculiar quadrature of this kind, but many reductions are given, making it possible to compare the different formulas to each other. But these reductions are derived from the infinite product exhibited above; for, because it is

$$\left(\frac{p}{q}\right)=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\text{etc.},$$

it will in like manner be

$$\left(\frac{p+q}{r}\right)=\frac{p+q+r}{(p+q)r}\cdot\frac{n(p+q+r+n)}{(p+q+n)(r+n)}\cdot\frac{2n(p+q+r+2n)}{(p+q+2n)(r+2n)}\cdot\text{etc.};$$

having multiplied the two expressions by each other one obtains

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn(p+q+r+n)}{(p+n)(q+n)(r+n)}\cdot\frac{4nn(p+q+r+2n)}{(p+2n)(q+2n)(r+2n)}\cdot\text{etc.},$$

where the three quantities p, q, r are interchangeable.

§8 Therefore, hence by permuting these quantities p, q, r we obtain the following reductions

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),$$

whence from a few of the given formulas many others can be determined. As if it is $q+r=n$ or $r=n-q$, because of

$$\left(\frac{q+r}{p}\right)=\frac{1}{p}\quad\text{and}\quad\left(\frac{q}{r}\right)=\frac{\pi}{n\sin\frac{q\pi}{n}}$$

it will be

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{n-q}\right)=\frac{\pi}{np\sin\frac{q\pi}{n}}$$

and also

$$\left(\frac{p}{n-q}\right)\left(\frac{n+p-q}{q}\right)=\frac{\pi}{np\sin\frac{q\pi}{n}}.$$

Further, if it is $p+q+r=n$ or $r=n-p-q$, it will be

$$\frac{\pi}{n\sin\frac{r\pi}{n}}=\frac{\pi}{n\sin\frac{q\pi}{n}}\left(\frac{p}{r}\right)=\frac{\pi}{n\sin\frac{p\pi}{n}}\left(\frac{q}{r}\right),$$

whence extraordinary reductions of the ones to the others result, by means of which the amount of quadratures necessary for our goal is vehemently decreased.

§9 But furthermore, by assuming determined numbers for p, q, r we obtain the following equalities of the products of two formulas

$$\left(\frac{1}{1}\right)\left(\frac{2}{2}\right)=\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)$$
$$\left(\frac{1}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)$$
$$\left(\frac{2}{1}\right)\left(\frac{3}{3}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{2}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{1}\right)$$
$$\left(\frac{2}{2}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)$$
$$\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{1}\right)$$
$$\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{2}\right)$$
$$\left(\frac{2}{2}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)$$
$$\left(\frac{3}{1}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{1}\right)$$
$$\left(\frac{2}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{5}{1}\right)\left(\frac{6}{2}\right)=\left(\frac{5}{2}\right)\left(\frac{7}{1}\right)$$
$$\left(\frac{1}{1}\right)\left(\frac{6}{2}\right)=\left(\frac{6}{1}\right)\left(\frac{7}{1}\right)$$
etc.,

where certainly many occur, which are already contained in the remaining ones.

§10 Having mentioned these principles in advance I will divide the general formula $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$, in which I assume the numbers p and q not to exceed the exponent n, into classes according to the exponent n, so that the values $n=1$, $n=2$, $n=3$, $n=4$ etc. will yield the first, second, third etc. class.

And the first class, in which it is $n=1$, contains the single formula $\left(\frac{1}{1}\right)$, whose value is $=1$. But the second class, in which it is $n=2$, contains these formulas $\left(\frac{1}{1}\right)$, $\left(\frac{2}{1}\right)$ and $\left(\frac{2}{2}\right)$, whose expansion is manifest per se. The third class, in which it is $n=3$, has these

$$\binom{1}{1}, \quad \binom{2}{1}, \quad \binom{3}{1}, \quad \binom{2}{2}, \quad \binom{3}{2}, \quad \binom{3}{3}.$$

But the fourth class, in which it is $n = 4$, on the other hand contains these

$$\binom{1}{1}, \quad \binom{2}{1}, \quad \binom{3}{1}, \quad \binom{4}{1}, \binom{2}{2}, \quad \binom{3}{2}, \quad \binom{4}{2}, \quad \binom{3}{3}, \quad \binom{4}{3}, \quad \binom{4}{4};$$

and so the number of the formulas increases according to the triangular numbers in the following classes. Therefore, let us go through these classes in order.

$$2. \textit{ Class of the form } \int \frac{x^{p-1}dx}{\sqrt[2]{(1-x^2)^{2-q}}} = \binom{p}{q}$$

Here it is certainly perspicuous that these formulas are expressed either absolutely or by means of the quadrature of the circle; for, these $\binom{2}{1}$ and $\binom{2}{2}$ are given absolutely and the remaining one $\binom{1}{1}$ because of $1+1=2$ is $\frac{\pi}{2\sin\frac{\pi}{2}} = \frac{\pi}{2}$; therefore, if for the sake of brevity we set $\frac{\pi}{2} = \alpha$, as we will certainly do it in the following cases, all formulas of this class are defined this way:

$$\binom{2}{1} = 1, \quad \binom{2}{2} = \frac{1}{2},$$
$$\binom{1}{1} = \alpha.$$

$$3. \textit{ Class of the form } \int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}} = \binom{p}{q}$$

Since here it is $n = 3$, the formula involving the quadrature of the circle is

$$\binom{2}{1} = \frac{\pi}{3\sin\frac{\pi}{3}};$$

therefore, let us put $\binom{2}{1} = \alpha$; but the remaining formulas, which are not given absolutely, involve a higher quadrature and the formula $\binom{1}{1}$, which we will indicate by the letter A; having conceded this we will be able to assign the values of all formulas of this class:

$$\binom{3}{1} = 1, \quad \binom{3}{2} = 1\frac{1}{2}, \quad \binom{3}{3} = \frac{1}{3},$$
$$\binom{2}{1} = \alpha, \quad \binom{2}{2} = \frac{\alpha}{A};$$
$$\binom{1}{1} = A.$$

4. class of the form $\int \frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}} = \binom{p}{q}$

Since here it is $n = 4$, we have two formulas depending on the quadrature of the circle, whose values, since they are known, we want to indicate this way:

$$\binom{3}{1} = \frac{\pi}{4\sin\frac{\pi}{4}} = \alpha \quad \text{and} \quad \binom{2}{2} = \frac{\pi}{4\sin\frac{2\pi}{4}} = \beta.$$

Furthermore, one single formula involving a higher quadrature is necessary; having conceded this formula we will know all the remaining ones. For, let us put $\binom{2}{1} = A$ and all formulas of this class will be determined this way:

$$\binom{4}{1} = 1, \quad \binom{4}{2} = \frac{1}{2}, \quad \binom{4}{3} = \frac{1}{3}, \quad \binom{4}{4} = \frac{1}{4};$$
$$\binom{3}{1} = \alpha, \quad \binom{3}{2} = \frac{\beta}{A}, \quad \binom{3}{3} = \frac{\alpha}{2A};$$
$$\binom{2}{1} = A, \quad \binom{2}{2} = \beta;$$
$$\binom{1}{1} = \frac{\alpha A}{\beta}.$$

5. Class of the form $\int \frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}} = \binom{p}{q}$

Since here it is $n = 5$, let us immediately note the formulas depending on the quadrature of the circle

$$\binom{4}{1} = \frac{\pi}{5\sin\frac{\pi}{5}} = \alpha, \quad \binom{3}{2} = \frac{\pi}{5\sin\frac{2\pi}{5}} = \beta.$$

But additionally two new quadratures are necessary for this class; we will denote them this way

$$\left(\frac{3}{1}\right) = A \quad \text{and} \quad \left(\frac{2}{2}\right) = B,$$

from which all remaining ones will be determined as this:

$$\begin{array}{lllll}
\left(\frac{5}{1}\right) = 1, & \left(\frac{5}{2}\right) = \frac{1}{2}, & \left(\frac{5}{3}\right) = \frac{1}{3}, & \left(\frac{5}{4}\right) = \frac{1}{4}, & \left(\frac{5}{5}\right) = 1, \\
\left(\frac{4}{1}\right) = \alpha, & \left(\frac{4}{2}\right) = \frac{\beta}{A}, & \left(\frac{4}{3}\right) = \frac{\beta}{2B}, & \left(\frac{4}{4}\right) = \frac{\alpha}{3A}; & \\
\left(\frac{3}{1}\right) = A, & \left(\frac{3}{2}\right) = \beta, & \left(\frac{3}{3}\right) = \frac{\beta\beta}{\alpha B}; & & \\
\left(\frac{2}{1}\right) = \frac{\alpha B}{\beta}, & \left(\frac{2}{2}\right) = B; & & & \\
\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}. & & & &
\end{array}$$

6. Class of the form $\int \frac{x^{p-1}dx}{\sqrt[6]{(1-x^6)^{6-q}}} = \left(\frac{p}{q}\right)$

Here it is $n = 6$ and the formulas involving the quadrature of the circle are

$$\left(\frac{5}{1}\right) = \frac{\pi}{6\sin\frac{\pi}{6}} = \alpha, \quad \left(\frac{4}{2}\right) = \frac{\pi}{6\sin\frac{2\pi}{6}} = \beta, \quad \left(\frac{3}{3}\right) = \frac{\pi}{6\sin\frac{3\pi}{6}} = \gamma.$$

But the values of all remaining ones additionally depend on these two quadratures

$$\left(\frac{4}{1}\right) = A \quad \text{and} \quad \left(\frac{3}{2}\right) = B$$

and they are detected to be:

$$\binom{6}{1} = 1, \quad \binom{6}{2} = \frac{1}{2}, \quad \binom{6}{3} = \frac{1}{3}, \quad \binom{6}{4} = \frac{1}{4}, \quad \binom{6}{5} = \frac{1}{5}, \quad \binom{6}{6} = \frac{1}{6};$$
$$\binom{5}{1} = \alpha, \quad \binom{5}{2} = \frac{\beta}{A}, \quad \binom{5}{3} = \frac{\gamma}{2B}, \quad \binom{5}{4} = \frac{\beta}{3B}, \quad \binom{5}{5} = \frac{\alpha}{4A},$$
$$\binom{4}{1} = A, \quad \binom{4}{2} = \beta, \quad \binom{4}{3} = \frac{\beta\gamma}{\alpha B}, \quad \binom{4}{4} = \frac{\beta\gamma A}{2\alpha BB};$$
$$\binom{3}{1} = \frac{\alpha B}{\beta}, \quad \binom{3}{2} = B, \quad \binom{3}{3} = \gamma;$$
$$\binom{2}{1} = \frac{\alpha B}{\gamma}, \quad \binom{2}{2} = \frac{\alpha BB}{\gamma A};$$
$$\binom{1}{1} = \frac{\alpha A}{\beta}.$$

7. Class of the form $\int \frac{x^{p-1}dx}{\sqrt[7]{(1-x^7)^{7-q}}} = \binom{p}{q}$

Since it is $n = 7$, denote the formulas depending on the quadrature of the circle this way

$$\binom{6}{1} = \frac{\pi}{7\sin\frac{\pi}{7}} = \alpha, \quad \binom{5}{2} = \frac{\pi}{7\sin\frac{2\pi}{7}} = \beta, \quad \binom{4}{3} = \frac{\pi}{7\sin\frac{3\pi}{7}} = \gamma,$$

furthermore, introduce these quadratures

$$\binom{5}{1} = A, \quad \binom{4}{2} = B, \quad \binom{3}{3} = C;$$

having given those quadratures the formulas will be determined this way:

$$\binom{7}{1} = 1, \quad \binom{7}{2} = \frac{1}{2}, \quad \binom{7}{3} = \frac{1}{3}, \quad \binom{7}{4} = \frac{1}{4}, \quad \binom{7}{4} = \frac{1}{5}, \quad \binom{7}{6} = \frac{1}{6}, \quad \binom{7}{7} = \frac{1}{7};$$
$$\binom{6}{1} = \alpha, \quad \binom{6}{2} = \frac{\beta}{A}, \quad \binom{6}{3} = \frac{\gamma}{2B}, \quad \binom{6}{4} = \frac{\gamma}{3C}, \quad \binom{6}{5} = \frac{\gamma}{3C}, \quad \binom{6}{6} = \frac{\alpha}{5A};$$
$$\binom{5}{1} = A, \quad \binom{5}{2} = \beta, \quad \binom{5}{3} = \frac{\beta\gamma}{\alpha B}, \quad \binom{5}{4} = \frac{\gamma\gamma A}{2\alpha BC}, \quad \binom{5}{5} = \frac{\beta\gamma A}{3\alpha BC};$$
$$\binom{3}{1} = \frac{\alpha C}{\gamma}, \quad \binom{3}{2} = \frac{\alpha BC}{\gamma A}, \quad \binom{3}{3} = C;$$
$$\binom{2}{1} = \frac{\alpha B}{\gamma}, \quad \binom{2}{2} = \frac{\alpha\beta BC}{\gamma\gamma A};$$
$$\binom{1}{1} = \frac{\alpha A}{\beta}.$$

8. Class of the form $\int \frac{x^{p-1}dx}{\sqrt[8]{(1-x^8)^{8-q}}} = \binom{p}{q}$

Since here it is $n = 8$, the formulas involving the quadrature of the circle will be

$$\binom{7}{1} = \frac{\pi}{8\sin\frac{\pi}{8}} = \alpha, \quad \binom{6}{2} = \frac{\pi}{8\sin\frac{2\pi}{8}} = \beta,$$
$$\binom{5}{3} = \frac{\pi}{8\sin\frac{3\pi}{8}} = \gamma, \quad \binom{4}{4} = \frac{\pi}{8\sin\frac{4\pi}{8}} = \delta.$$

Now, on the other hand consider these three formulas as known

$$\binom{6}{1} = A, \quad \binom{5}{2} = B \quad \text{and} \quad \binom{4}{3} = C$$

and using these the formulas of this class will be determined this way:

$$\binom{8}{1} = 1, \quad \binom{8}{2} = \frac{1}{2}, \quad \binom{8}{3} = \frac{1}{3}, \quad \binom{8}{4} = \frac{1}{4}, \quad \binom{8}{5} = \frac{1}{5}, \quad \binom{8}{6} = \frac{1}{6}, \quad \binom{8}{7} = \frac{1}{7}, \quad \binom{8}{8} = \frac{1}{8};$$
$$\binom{7}{1} = \alpha, \quad \binom{7}{2} = \frac{\beta}{A}, \quad \binom{7}{3} = \frac{\gamma}{2B}, \quad \binom{7}{4} = \frac{\delta}{3C}, \quad \binom{7}{5} = \frac{\gamma}{4C}, \quad \binom{7}{6} = \frac{\beta}{5B}, \quad \binom{7}{7} = \frac{\alpha}{6A};$$
$$\binom{6}{1} = A, \quad \binom{6}{2} = \beta, \quad \binom{6}{3} = \frac{\beta\gamma}{\alpha B}, \quad \binom{6}{4} = \frac{\gamma\delta A}{2\alpha BC}, \quad \binom{6}{5} = \frac{\gamma\delta A}{3\alpha CC}, \quad \binom{6}{6} = \frac{\beta\gamma A}{4\alpha BC};$$
$$\binom{5}{1} = \frac{\alpha B}{\beta}, \quad \binom{5}{2} = B, \quad \binom{5}{3} = \gamma, \quad \binom{5}{4} = \frac{\gamma\delta}{\alpha C}, \quad \binom{5}{5} = \frac{\gamma\gamma\delta A}{2\alpha\beta CC};$$
$$\binom{4}{1} = \frac{\alpha C}{\gamma}, \quad \binom{4}{2} = \frac{\alpha BC}{\gamma A}, \quad \binom{4}{3} = C, \quad \binom{4}{4} = \delta;$$
$$\binom{3}{1} = \frac{\alpha C}{\delta}, \quad \binom{3}{2} = \frac{\alpha\beta CC}{\gamma\delta A}, \quad \binom{3}{3} = \frac{\alpha CC}{\delta A};$$
$$\binom{2}{1} = \frac{\alpha B}{\gamma}, \quad \binom{2}{2} = \frac{\alpha\beta BC}{\gamma\delta A};$$
$$\binom{1}{1} = \frac{\alpha A}{\beta}.$$

Hence it is possible to continue these reductions to the following classes arbitrarily far. Therefore, let us explain, how the integrals of the single formulas will be calculated in general.

$$\textit{Expansion of the general form} \int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \binom{p}{q}$$

Therefore, these formulas are absolutely integrable

$$\binom{n}{1} = 1, \quad \binom{n}{2} = \frac{1}{2}, \quad \binom{n}{3} = \frac{1}{3}, \quad \binom{n}{4} = \frac{1}{4} \quad \text{etc.,}$$

further, the formulas depending on the quadrature of the circle are

$$\binom{n-1}{1} = \alpha, \quad \binom{n-2}{2} = \beta, \quad \binom{n-3}{3} = \gamma, \quad \binom{n-4}{4} = \delta \quad \text{etc.,}$$

the progression of which formulas repeats, since it also is

$$\binom{4}{n-4} = \delta, \quad \binom{3}{n-3} = \gamma, \quad \binom{2}{n-2} = \beta, \quad \binom{1}{n-1} = \alpha.$$

Furthermore, also higher quadratures must be used, which are represented this way

$$\left(\frac{n-2}{1}\right) = A, \quad \left(\frac{n-3}{2}\right) = B, \quad \left(\frac{n-4}{3}\right) = C, \quad \left(\frac{n-5}{4}\right) = D \quad \text{etc.},$$

whose number is immediately determined in each case, since these formulas eventually repeat.

But having admitted these formulas, one will be able to determine completely all formulas extending to the same class. But by going backwards from the formula $\left(\frac{n-1}{1}\right) = \alpha$, as we ordered the formulas above, we will have

$$\left(\frac{n-1}{1}\right) = \alpha, \quad \left(\frac{n-2}{1}\right) = A, \quad \left(\frac{n-3}{1}\right) = \frac{\alpha B}{\beta}, \quad \left(\frac{n-4}{1}\right) = \frac{\alpha C}{\gamma},$$
$$\left(\frac{n-5}{1}\right) = \frac{\alpha D}{\delta}, \quad \left(\frac{n-6}{1}\right) = \frac{\alpha E}{\varepsilon} \quad \text{etc.},$$

which values taken backwards will be as follows

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}, \quad \left(\frac{2}{1}\right) = \frac{\alpha B}{\gamma}, \quad \left(\frac{3}{1}\right) = \frac{\alpha C}{\delta} \quad \text{etc.}$$

But then by proceeding horizontally from the same formula $\left(\frac{n-1}{1}\right) = \alpha$ these formulas are defined

$$\left(\frac{n-1}{1}\right) = \alpha, \quad \left(\frac{n-1}{2}\right) = \frac{\beta}{A}, \quad \left(\frac{n-1}{3}\right) = \frac{\gamma}{2B}, \quad \left(\frac{n-1}{4}\right) = \frac{\delta}{3C} \quad \text{etc.};$$

the last of them will be

$$\left(\frac{n-1}{n-1}\right) = \frac{\alpha}{(n-2)A},$$

the penultimate

$$\left(\frac{n-1}{n-2}\right) = \frac{\beta}{(n-3)B},$$

the second-to-last

$$\left(\frac{n-1}{n-3}\right) = \frac{\gamma}{(n-4)C}$$

etc.

Similarly, so by descending downwards as proceeding horizontally from the formula $\binom{n-2}{2} = \beta$ we will obtain the values of the others; by descending, of course, we will obtain

$$\binom{n-2}{2} = \beta, \quad \binom{n-3}{2} = B, \quad \binom{n-4}{2} = \frac{\alpha BC}{\gamma A}, \quad \binom{n-5}{2} = \frac{\alpha\beta CD}{\gamma\delta A},$$
$$\binom{n-6}{2} = \frac{\alpha\beta DE}{\delta\varepsilon A}, \quad \binom{n-7}{2} = \frac{\alpha\beta EF}{\varepsilon\zeta A} \quad \text{etc.},$$

where the last will be

$$\binom{2}{2} = \frac{\alpha\beta BC}{\gamma\delta A},$$

the penultimate

$$\binom{3}{2} = \frac{\alpha\beta CD}{\delta\varepsilon A}$$

etc.;

but by proceeding horizontally

$$\binom{n-2}{2} = \beta, \quad \binom{n-2}{3} = \frac{\beta\gamma}{\alpha B}, \quad \binom{n-2}{4} = \frac{\gamma\delta A}{2\alpha BC}, \quad \binom{n-2}{5} = \frac{\delta\varepsilon A}{3\alpha CD},$$
$$\binom{n-2}{6} = \frac{\varepsilon\zeta A}{4\alpha DE}, \quad \binom{n-2}{7} = \frac{\zeta\eta A}{5\alpha EF} \quad \text{etc.},$$

the last of which will be

$$\binom{n-2}{n-2} = \frac{\beta\gamma A}{(n-4)\alpha BC},$$

the penultimate

$$\binom{n-2}{n-3} = \frac{\gamma\delta A}{(n-5)\alpha CD}$$

etc.

Further, by descending from the formula $\left(\frac{n-3}{n-3}\right) = \gamma$ we get to these formulas

$$\left(\frac{n-3}{3}\right) = \gamma, \quad \left(\frac{n-4}{4}\right) = C, \quad \left(\frac{n-5}{3}\right) = \frac{\alpha CD}{\delta A}, \quad \left(\frac{n-6}{3}\right) = \frac{\alpha\beta CDE}{\delta\varepsilon AB},$$
$$\left(\frac{n-7}{3}\right) = \frac{\alpha\beta\gamma DEF}{\delta\varepsilon\zeta AB}, \quad \left(\frac{n-8}{3}\right) = \frac{\alpha\beta\gamma EFG}{\varepsilon\zeta\eta AB} \quad \text{etc.}$$

and by proceeding horizontally

$$\left(\frac{n-3}{3}\right) = \gamma, \quad \left(\frac{n-3}{4}\right) = \frac{\delta\gamma}{\alpha C}, \quad \left(\frac{n-3}{5}\right) = \frac{\gamma\delta\varepsilon A}{2\alpha\beta CD}, \quad \left(\frac{n-3}{6}\right) = \frac{\delta\varepsilon\zeta AB}{3\alpha\beta CDE},$$
$$\left(\frac{n-3}{7}\right) = \frac{\varepsilon\zeta\eta AB}{4\alpha\beta DEF}, \quad \left(\frac{\zeta\eta\theta AB}{5\alpha\beta EFG}\right) \quad \text{etc.}$$

In the same way by descending from the formula $\left(\frac{n-4}{4}\right) = \delta$ we obtain

$$\left(\frac{n-4}{4}\right) = \delta, \quad \left(\frac{n-5}{4}\right) = D, \quad \left(\frac{n-6}{4}\right) = \frac{\alpha DE}{\varepsilon A}, \quad \left(\frac{n-7}{4}\right) = \frac{\alpha\beta DEF}{\varepsilon\zeta AB},$$
$$\left(\frac{n-8}{4}\right) = \frac{\alpha\beta\gamma DEFG}{\varepsilon\zeta\eta ABC}, \quad \left(\frac{n-9}{4}\right) = \frac{\alpha\beta\gamma\delta EFGH}{\varepsilon\zeta\eta\theta ABC} \quad \text{etc.}$$

and by proceeding horizontally

$$\left(\frac{n-4}{4}\right) = \delta, \quad \left(\frac{n-4}{5}\right) = \frac{\delta\varepsilon}{\alpha D}, \quad \left(\frac{n-4}{6}\right) = \frac{\delta\varepsilon\zeta A}{2\alpha\beta DE}, \quad \left(\frac{n-4}{7}\right) = \frac{\delta\varepsilon\zeta\eta AB}{3\alpha\beta\gamma DEF},$$
$$\left(\frac{n-4}{8}\right) = \frac{\varepsilon\zeta\eta\theta ABC}{4\alpha\beta\gamma DEFG}, \quad \left(\frac{n-4}{9}\right) = \frac{\zeta\eta\theta\iota ABC}{5\alpha\beta\gamma EFGH} \quad \text{etc.}$$

And this way finally the values of all formulas are found.

Let us apply these general reductions to the

$$\textit{9. Class of form} \int \frac{x^{p-1}dx}{\sqrt[9]{(1-x^9)^{9-q}}} = \left(\frac{p}{q}\right)$$

There because of $n = 9$ the formulas involving the quadrature of the circle will be

$$\binom{8}{1} = \alpha, \quad \binom{7}{2} = \beta, \quad \binom{6}{3} = \gamma, \quad \binom{5}{4} = \delta;$$

hence $\varepsilon = \delta$, $\zeta = \gamma$, $\eta = \beta$, $\theta = \alpha$.

Further, put the new quadratures required for this

$$\binom{7}{1} = A, \quad \binom{6}{2} = B, \quad \binom{5}{3} = C, \quad \binom{4}{4} = D$$

and so it will be

$$E = C, \quad F = B \quad \text{and} \quad G = A;$$

and having conceded these four values one will be able to assign the values of all formulas of the ninth class, which we want to represent in the same order as we did before:

$$\binom{9}{1} = 1, \quad \binom{9}{2} = \frac{1}{2}, \quad \binom{9}{3} = \frac{1}{3}, \quad \binom{9}{4} = \frac{1}{4}, \quad \binom{9}{5} = \frac{1}{5},$$
$$\binom{9}{6} = \frac{1}{6}, \quad \binom{9}{7} = \frac{1}{7}, \quad \binom{9}{8} = \frac{1}{8}, \quad \binom{9}{9} = \frac{1}{9};$$

$$\binom{8}{1} = \alpha, \quad \binom{8}{2} = \frac{\beta}{A}, \quad \binom{8}{3} = \frac{\gamma}{2B}, \quad \binom{8}{4} = \frac{\delta}{3C}, \quad \binom{8}{5} = \frac{\delta}{4D},$$
$$\binom{8}{6} = \frac{\gamma}{5C}, \quad \binom{8}{7} = \frac{\beta}{6B}, \quad \binom{8}{8} = \frac{\alpha}{7A};$$

$$\binom{7}{1} = A, \quad \binom{7}{2} = \beta, \quad \binom{7}{3} = \frac{\beta\gamma}{\alpha B}, \quad \binom{7}{4} = \frac{\gamma\delta A}{2\alpha BC}, \quad \binom{7}{5} = \frac{\delta\delta A}{3\alpha CD},$$
$$\binom{7}{6} = \frac{\gamma\delta A}{4\alpha CD}, \quad \binom{7}{7} = \frac{\beta\gamma A}{5\alpha BC};$$

$$\binom{6}{1} = \frac{\alpha B}{\beta}, \quad \binom{6}{2} = B, \quad \binom{6}{3} = \gamma, \quad \binom{6}{4} = \frac{\gamma\delta}{\alpha C}, \quad \binom{6}{5} = \frac{\gamma\delta\delta A}{2\alpha\beta CD},$$

$$V66 = \frac{\gamma\delta\delta AB}{3\alpha\beta CCD};$$

$$\binom{5}{1} = \frac{\alpha C}{\gamma}, \quad \binom{5}{2} = \frac{\alpha BC}{\gamma A}, \quad \binom{5}{3} = C, \quad \binom{5}{4} = \delta, \quad \binom{5}{5} = \frac{\delta\delta}{\alpha D};$$

$$\binom{4}{1} = \frac{\alpha D}{\delta}, \quad \binom{4}{2} = \frac{\alpha\beta CD}{\gamma\delta A}, \quad \binom{4}{3} = \frac{\alpha CD}{\delta A}, \quad \binom{4}{4} = D;$$

$$\binom{3}{1} = \frac{\alpha C}{\delta}, \quad \binom{3}{2} 0 \frac{\alpha\beta CD}{\delta\delta A}, \quad \binom{3}{3} = \frac{\alpha\beta CCD}{\delta\delta AB};$$

$$\binom{2}{1} = \frac{\alpha B}{\gamma}, \quad \binom{2}{2} = \frac{\alpha\beta BC}{\gamma\delta A};$$

$$\binom{1}{1} = \frac{\alpha A}{\beta}.$$

The structure of these formula deserves it to be noted even while proceeding diagonally from the left to the right, where certainly two species of progressions occur, depending on whether we start from the first vertical series or from the topmost horizontal series. This way, first by beginning from the vertical series:

$$
\begin{array}{llll}
\binom{n-1}{1} = \alpha, & \binom{n-2}{2} = \dfrac{\beta}{\alpha} \times \binom{n-1}{1}, & \binom{n-3}{3} = \dfrac{\gamma}{\beta} \times \binom{n-2}{2}, & \binom{n-4}{4} = \dfrac{\delta}{\gamma} \times \binom{n-3}{3} \\
\binom{n-2}{1} = A, & \binom{n-3}{2} = \dfrac{B}{A} \times \binom{n-2}{1}, & \binom{n-4}{3} = \dfrac{C}{B} \times \binom{n-3}{2}, & \binom{n-5}{4} = \dfrac{D}{C} \times \binom{n-4}{3} \\
\binom{n-3}{1} = \dfrac{\alpha B}{\beta}, & \binom{n-4}{2} = \dfrac{\beta C}{\gamma A} \times \binom{n-3}{1}, & \binom{n-5}{3} = \dfrac{\gamma D}{\delta B} \times \binom{n-4}{2}, & \binom{n-6}{4} = \dfrac{\delta E}{\varepsilon C} \times \binom{n-5}{3} \\
\binom{n-4}{1} = \dfrac{\alpha C}{\gamma}, & \binom{n-5}{2} = \dfrac{\beta D}{\delta A} \times \binom{n-4}{1}, & \binom{n-6}{3} = \dfrac{\gamma E}{\varepsilon B} \times \binom{n-5}{2}, & \binom{n-7}{4} = \dfrac{\delta F}{\zeta C} \times \binom{n-6}{3} \\
\binom{n-5}{1} = \dfrac{\alpha D}{\delta}, & \binom{n-6}{2} = \dfrac{\beta E}{\varepsilon A} \times \binom{n-5}{1}, & \binom{n-7}{3} = \dfrac{\gamma F}{\zeta B} \times \binom{n-6}{2}, & \binom{n-8}{4} = \dfrac{\delta G}{\eta C} \times \binom{n-7}{3} \\
\binom{n-6}{1} = \dfrac{\alpha E}{\varepsilon}, & \binom{n-7}{2} = \dfrac{\beta F}{\zeta A} \times \binom{n-6}{1}, & \binom{n-8}{3} = \dfrac{\gamma G}{\eta B} \times \binom{n-7}{2}, & \binom{n-9}{4} = \dfrac{\delta H}{\theta C} \times \binom{n-6}{3}
\end{array}
$$

etc.

further, by starting from the topmost horizontal one:

$$
\begin{array}{llll}
\binom{n}{1} = 1, & \binom{n-1}{2} = \dfrac{\beta}{A} \times \binom{n}{1}, & \binom{n-2}{3} = \dfrac{\gamma A}{\alpha B} \times \binom{n-1}{2}, & \binom{n-3}{4} = \dfrac{\delta B}{\beta C} \times \binom{n-2}{3} \\
\binom{n}{2} = \dfrac{1}{2}, & \binom{n-1}{3} = \dfrac{\gamma}{B} \times \binom{n}{2}, & \binom{n-2}{4} = \dfrac{\delta A}{\alpha C} \times \binom{n-1}{3}, & \binom{n-3}{5} = \dfrac{\varepsilon B}{\beta D} \times \binom{n-2}{4} \\
\binom{n}{3} = \dfrac{1}{3}, & \binom{n-1}{4} = \dfrac{\delta}{C} \times \binom{n}{3}, & \binom{n-2}{5} = \dfrac{\varepsilon A}{\alpha D} \times \binom{n-1}{4}, & \binom{n-3}{6} = \dfrac{\zeta B}{\beta E} \times \binom{n-2}{5} \\
\binom{n}{4} = \dfrac{1}{4}, & \binom{n-1}{5} = \dfrac{\varepsilon}{D} \times \binom{n}{4}, & \binom{n-2}{6} = \dfrac{\zeta A}{\alpha E} \times \binom{n-1}{5}, & \binom{n-3}{7} = \dfrac{\eta B}{\beta F} \times \binom{n-2}{6} \\
\binom{n}{5} = \dfrac{1}{5}, & \binom{n-1}{6} = \dfrac{\zeta}{E} \times \binom{n}{5}, & \binom{n-2}{7} = \dfrac{\eta A}{\alpha F} \times \binom{n-1}{6}, & \binom{n-3}{8} = \dfrac{\theta B}{\beta G} \times \binom{n-2}{7}
\end{array}
$$

etc.

Here the rule, how these formulas depend on each other, is plain, if we only note that in each of the two series of letters α, β, γ, δ etc. and A, B, C, D etc. the terms preceding the first are equal to each other.

Conclusion

Therefore, whereas we can exhibit the formula of the second class having conceded only the quadrature of the circle, the formulas of the third class additionally require a quadrature contained either in this formula

$$\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A \quad \text{or in this one} \quad \int \frac{xdx}{\sqrt[3]{(1-x^3)}} = \frac{\alpha}{A},$$

since, having given one, at the same time the other is given. If we express these formulas in terms of an infinite product, their value is found to be

$$\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2}{1} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{9 \cdot 11}{10 \cdot 10} \cdot \frac{12 \cdot 14}{13 \cdot 13} \cdot \text{etc.},$$

whence its quantity can conveniently be calculated approximately; in the same way it is

$$\int \frac{xdx}{\sqrt[3]{(1-x^3)}} = 1 \cdot \frac{3 \cdot 7}{5 \cdot 5} \cdot \frac{6 \cdot 10}{8 \cdot 8} \cdot \frac{9 \cdot 13}{11 \cdot 11} \cdot \frac{12 \cdot 16}{14 \cdot 14} \cdot \text{etc.}$$

Further, we will be able to integrate all formulas of the fourth class, if only except for the quadrature of the circle one of these four formulas was known $\left(\frac{2}{1}\right)$, $\left(\frac{1}{1}\right)$, $\left(\frac{3}{2}\right)$, $\left(\frac{3}{3}\right)$, which yield these forms

$$\int \frac{xdx}{\sqrt[4]{(1-x^4)^3}} = \frac{1}{2}\int \frac{dx}{\sqrt[4]{(1-xx)^3}} = \int \frac{dx}{\sqrt{(1-x^4)}} = A,$$

$$\int \frac{dx}{\sqrt[4]{(1-x^4)^3}} = \frac{\alpha A}{\beta}, \quad \int \frac{xxdx}{\sqrt[4]{(1-x^4)}} = \frac{\alpha}{2A},$$

$$\int \frac{xxdx}{\sqrt{(1-x^4)}} = \int \frac{xdx}{\sqrt[4]{(1-x^4)}} = \frac{1}{2}\int \frac{dx}{\sqrt[4]{(1-xx)}} = \frac{\beta}{A};$$

but by means of an infinite product it will be

$$A = \frac{3}{1 \cdot 2} \cdot \frac{4 \cdot 7}{5 \cdot 6} \cdot \frac{8 \cdot 11}{9 \cdot 10} \cdot \frac{12 \cdot 15}{13 \cdot 14} \cdot \frac{16 \cdot 19}{17 \cdot 18} \cdot \text{etc.}$$

The fifth class requires the two higher quadratures $\left(\frac{3}{1}\right) = A$ and $\left(\frac{2}{2}\right) = B$; instead of those two others depending on them could be assumed, which might seem simpler, even though because of the prime number 5 the ones can hardly be considered as simpler as the others.

For the sixth class also these two formulas are required $\left(\frac{4}{1}\right) = A$ and $\left(\frac{3}{2}\right) = B$. But here instead of the one the other, which was necessary in the third class, can be assumed, that only a single new one is to be used. For, because it is

$$\left(\frac{2}{2}\right) = \int \frac{xdx}{\sqrt[6]{(1-x^6)^4}} = \frac{1}{2}\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = \frac{\alpha BB}{\gamma A},$$

it will be

$$\frac{2\alpha BB}{\gamma A} = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}},$$

which is the formula required for the third class. Therefore, if this one was given and if this formula is known

$$\left(\frac{3}{2}\right) = \int \frac{xdx}{\sqrt{1-x^6}} = \frac{1}{2}\int \frac{dx}{\sqrt{(1-x^3)}} = B$$

or even this one

$$\left(\frac{4}{3}\right) = \int \frac{xxdx}{\sqrt[3]{(1-x^6)}} = \frac{1}{3}\int \frac{dx}{\sqrt[3]{(1-xx)}} = \frac{\beta\gamma}{\alpha B},$$

which are the simplest of this kind, all remaining ones can be defined using only these. But having combined these it is plain that it will be

$$\int \frac{dx}{\sqrt{(1-x^3)}} \cdot \int \frac{dx}{\sqrt[3]{(1-xx)}} = \frac{6\beta\gamma}{\alpha} = \frac{\pi}{\sqrt{3}}.$$

In like manner, from the formulas of the first class one concludes

$$\int \frac{dx}{\sqrt{1-x^4}} \cdot \int \frac{dx}{\sqrt[4]{(1-x^2)}} = \frac{\pi}{2};$$

a lot of theorems of this kind can be deduced, of which this is especially notable

$$\int \frac{dx}{\sqrt[m]{(1-x^n)}} \cdot \int \frac{dx}{\sqrt[n]{1-x^n}} = \frac{\pi \sin \frac{(m-n)}{mn}\pi}{(m-n)\sin\frac{\pi}{m}\cdot\sin\frac{\pi}{n}},$$

which, if m and n are fractional numbers, is transmuted into this form

$$\int \frac{x^{q-1}dx}{\sqrt[r]{(1-x^p)^s}} \cdot \int \frac{x^{s-1}dx}{\sqrt[p]{(1-x^r)^q}} = \frac{\pi \sin\left(\frac{s}{r}-\frac{q}{p}\right)\pi}{(ps-qr)\sin\frac{q\pi}{p}\cdot\sin\frac{s\pi}{r}}.$$

On the other hand it is in general

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right) = \frac{\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right)}{(q-p)\left(\frac{n-q+p}{q-p}\right)},$$

which yields this form

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^q}} \cdot \int \frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^p}} = \frac{\pi \sin \frac{(q-p)\pi}{n}}{n(q-p)\sin\frac{\pi\pi}{n} \cdot \sin\frac{q\pi}{n}},$$

whence not only the preceding theorems, but also many others are easily derived. For, having put $n = \frac{pq}{m}$ we will have

$$\int \frac{x^{m-1}dx}{\sqrt[n]{(1-x^q)^m}} \cdot \int \frac{x^{m-1}dx}{\sqrt[q]{(1-x^p)^m}} = \frac{\pi \sin\left(\frac{m}{p} - \frac{m}{q}\right)\pi}{m(q-p)\sin\frac{m\pi}{q} \cdot \sin\frac{m\pi}{p}},$$

which can be extend further this way

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^m)^q}} \cdot \int \frac{x^{q-1}dx}{\sqrt[m]{(1-x^n)^p}} = \frac{\pi \sin\left(\frac{q}{n} - \frac{p}{m}\right)\pi}{(mq-np)\sin\frac{p\pi}{m} \cdot \sin\frac{q\pi}{n}};$$

if in this equation one puts $n = 2q$, it will be

$$\int \frac{x^{p-1}dx}{\sqrt{(1-x^m)}} \cdot \int \frac{x^{q-1}dx}{\sqrt[m]{(1-x^{2q})^p}} = \frac{\pi \cos\frac{p\pi}{m}}{q(m-2p)\sin\frac{p\pi}{m}}.$$

But if in the last integral formula one puts $x^{2q} = 1 - y^m$, it will be

$$\int \frac{x^{q-1}dx}{\sqrt[m]{(1-x^{2q})^p}} = \frac{m}{2q} \int \frac{y^{m-p-1}dy}{\sqrt{(1-y^m)}},$$

whence having written x for y

$$\int \frac{x^{p-1}dx}{\sqrt{1-x^m}} \cdot \int \frac{x^{m-p-1}dx}{\sqrt{(1-x^m)}} = \frac{2\pi \cos\frac{p\pi}{m}}{m(m-2p)\sin\frac{ppi}{m}}.$$

If in like manner in general for the other integral formula one puts $1 - x^n = y^m$, it will be

$$\int \frac{x^{q-1}dx}{\sqrt[m]{(1-x^n)^p}} = \frac{m}{n} \int \frac{y^{m-p-1}dy}{\sqrt[n]{(1-y^m)^{n-q}}},$$

whence having again written x for y one obtains

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^m)^q}} \cdot \int \frac{x^{m-p-1}dx}{\sqrt[n]{(1-x^m)^{n-q}}} = \frac{n\pi \sin\left(\frac{q}{n} - \frac{p}{m}\right)\pi}{m(mq-np)\sin\frac{p\pi}{m} \cdot \sin\frac{q\pi}{n}},$$

which value is reduced to $\frac{n\pi}{m(mq-np)}\left(\cot\frac{p\pi}{m} - \cot\frac{q\pi}{n}\right)$. And hence this more convenient form results

$$\int \frac{x^{\frac{m-r}{2}-1}dx}{\sqrt[n]{(1-x^m)^{\frac{n-s}{2}}}} \cdot \int \frac{x^{\frac{m+r}{2}-1}dx}{\sqrt[n]{(1-x^m)^{\frac{n+s}{2}}}} = \frac{2n\pi\left(\tan\frac{r\pi}{2m} - \tan\frac{s\pi}{2n}\right)}{m(nr-ms)}.$$

Since the foundation of these reductions lies in this equality

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right) = \frac{\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right)}{(q-p)\left(\frac{n-q+p}{q-p}\right)},$$

which is reduced to this form

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right) = \left(\frac{n}{q-p}\right)\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right),$$

its truth can shown directly this way.

Having taken these three numbers $n-q$, $q-p$, q for those three numbers p, q, r in the reduction given in § 8 we will have

$$\left(\frac{n-q}{q-p}\right)\left(\frac{n-p}{q}\right) = \left(\frac{n-q}{q}\right)\left(\frac{n}{q-p}\right);$$

but then having taken $n-q$, $q-p$, p instead of them it will be

$$\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right) = \left(\frac{n-q}{q-p}\right)\left(\frac{n-p}{p}\right);$$

having multiplied these equations by each other and having got rid of the formula $\left(\frac{n-1}{q-p}\right)$ common to both sides by division it will be

$$\left(\frac{n-p}{q}\right)\left(\frac{n-q}{p}\right)\left(\frac{n-q+p}{q-p}\right) = \left(\frac{n}{q-p}\right)\left(\frac{n-p}{p}\right)\left(\frac{n-q}{q}\right).$$

Yes, even an equality not depending on the exponent n of three formulas of this kind can be exhibited, of course

$$\begin{aligned}\left(\frac{s}{p}\right)\left(\frac{r+s}{q}\right)\left(\frac{p+s}{r}\right) &= \left(\frac{r+s}{p}\right)\left(\frac{s}{q}\right)\left(\frac{q+s}{r}\right)\\ = \left(\frac{r}{p}\right)\left(\frac{r+s}{q}\right)\left(\frac{p+r}{s}\right) &= \left(\frac{r+s}{p}\right)\left(\frac{r}{q}\right)\left(\frac{q+r}{s}\right),\end{aligned}$$

which involves four letters not depending on n and is similar to the equality of the products of two formulas

$$\binom{r}{p}\binom{p+r}{q}=\binom{q+r}{p}\binom{q}{r}=\binom{q}{p}\binom{p+q}{r}.$$

But one also has this equality of the products of three formulas

$$\binom{p}{q}\binom{r}{s}\binom{p+q}{r+s}=\binom{q}{r}\binom{s}{p}\binom{q+r}{p+s}=\binom{p}{r}\binom{q}{s}\binom{p+r}{q+s}$$
$$=\binom{p}{q}\binom{p+q}{r}\binom{p+q+r}{s}=\binom{p}{q}\binom{p+q}{s}\binom{p+q+s}{r}\quad \text{etc.}$$

For, in these the letters p, q, r, s can be arbitrarily permuted.