Analytical Observations *

Leonhard Euler

Considering the powers which result from the expansion of the trinomial formula

$$1 + x + xx,$$

the middle terms are found to have the largest numerical coefficients; although this is rather obvious, it seems worth of one's complete attention, since speculations of this kind are often quite fruitful for analysis. Therefore, first I will present the powers for small exponent in a clear way:

Exponent of the power	Power in expanded Form
0	1
1	$1 + x + xx$
2	$1 + 2x + 3x^2 + 2x^3 + x^4$
3	$1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$
4	$1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$
6	$1 + 5x + 15x^2 + 30x^2 + 45x^4 + 51x^5 + 45x^6 + 30x^7 + 15x^8 + 5x^9 + x^{10}$
	etc;

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if the middle terms from each power are written in order, this progression results

$$1, \quad 1x, \quad 3x^2, \quad 7x^3, \quad 19x^4, \quad 51x^5, \quad 141x^6 \quad \text{etc.};$$

it seems worth of one's while to investigate according to which law these numbers progress such that not only the general term, i.e. the coefficient corresponding to the power x^n, is found but also the extraordinary properties of this series are explored. To this end, I want to propound the following problems, the solution of which will later on lead to other quite interesting speculations.

PROBLEM 1

§1 *Having expanded the indefinite power $(1+x+xx)^n$, to find the coefficient of the middle term, i.e. the coefficient of the power x^n.*

SOLUTION

Represent the power given in this way as a binomial $(x(1+x)+1)^n$; if this is expanded in the usual way, it gives

$$x^n(1+x)^n + \frac{n}{1}x^{n-1}(1+x)^{n-1} + \frac{n(n-1)}{1\cdot 2}x^{n-2}(1+x)^{n-2}$$
$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}(1+x)^{n-3} + \text{etc.};$$

if it is expanded further, from that expansion each term of the form x^n must be found. The first member obviously gives

$$x^n,$$

whereas all remaining powers of x that result from its expansion will be higher. But from the second member this term results for the power x^n:

$$\frac{n}{1}x^{n-1}\cdot\frac{n-1}{1}x = \frac{n(n-1)}{1\cdot 1}x^n;$$

in like manner, from the third member we obtain

$$\frac{n(n-1)}{1\cdot 2}x^{n-2}\cdot\frac{(n-2)(n-3)}{1\cdot 2}x^2 = \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 1\cdot 2}x^n$$

etc.

if all these parts are collected into one sum, one obtains the coefficient in question of the power x^n

$$1+\frac{n(n-1)}{1\cdot 1}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 1\cdot 2}+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}+\text{etc.}$$

COROLLARY 1

§**2** Therefore, this series, which terminates for each integer number n, gives the coefficient of the power x^n for the propounded series

$$1+x+3x^2+7x^3+19x^4+\text{etc.}$$

and in this way, by means of it, every term, no matter how far away from the initial term, can be found without the preceding ones.

§**3** If we substitute the numbers 1, 2, 3 etc. for n successively, the following values are found:

n	coefficient of x^n
0	1
1	1
2	$1 + 2 = 3$
3	$1 + 6 = 7$
4	$1 + 12 + 6 = 19$
5	$1 + 20 + 30 = 51$
6	$1 + 30 + 90 + 20 = 141$
7	$1 + 42 + 210 + 140 = 393$
8	$1 + 56 + 420 + 560 + 70 = 1107$
9	$1 + 72 + 756 + 1680 + 630 = 3139$
10	$1 + 90 + 1260 + 4200 + 3150 + 252 = 8953$
11	$1 + 110 + 1980 + 9240 + 11550 + 2772 = 25653$
12	$1 + 132 + 2970 + 18480 + 34650 + 16632 + 924 = 73789$
etc.	etc.

COROLLARY 3

§4 The series of these numbers is of such a nature that it seems that every term can conveniently be compared with the triple of the precursor, from which comparison the following differences result:

1,	1,	3,	7,	19,	51,	141,	393,	1107,	3139	etc.
	3,	3,	9,	21,	57,	153,	423,	1179,	3321	etc.
	2,	0,	2,	2,	6,	12,	30,	72,	182	etc.

SCHOLIUM 1

§**5** If we consider these differences more accurately, it seems to be the case that they are the pronic numbers, i.e. doubled triangular numbers of the form $mm+$; and if we consider the indices of these pronic numbers, which constitute this series

$$1, \quad 0, \quad 1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13 \quad \text{etc.},$$

it is obviously a recurring series, each term of which is the sum of two preceding ones. Since this structure is detected up to the tenth term, who would doubt that it extends to the whole series? Certainly, less certain inductions have turned out to be correct. Therefore, it will be worth of one's while to consider this example more accurately; since the number 13 of the series corresponds to the terms x^9, to the general power x^n this number will correspond:

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2};$$

the corresponding pronic number is

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2} + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-4} + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{2n-4}$$
$$-\frac{2}{5}(-1)^{n-2}.$$

If in the propounded series two contiguous terms are in general exhibited in this way:

$$1 + x + 3x^2 + 7x^3 + 10x^4 + \cdots + Px^n + Qx^{n+1} + \text{etc.},$$

it will be

$$3P - Q = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-2} + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-2}$$
$$-\frac{2}{5}(-1)^{n-1},$$

whence we conclude

$$P = \frac{3^n + (-1)^n}{10} + \frac{1}{5}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{3-\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

such that even the propounded series itself would be a recurring series with the scale of relation

$$6, \quad -8, \quad -8, \quad 14, \quad 4, \quad -3,$$

according to which it will be

$$3139 = 6 \cdot 1107 - 8 \cdot 393 - 8 \cdot 141 + 14 \cdot 51 + 4 \cdot 19 - 3 \cdot 7.$$

SCHOLIUM 2

§6 But no matter how probable this law of progression might seem, while it holds for the first ten terms, it is nevertheless found to be false, while it fails in the eleventh term 8953; for, having subtracted this from the triple of the preceding term, 9417, the remainder 464 is not even a pronic number, even less does it have the pronic index $21 = 13 + 8$; for, $21^2 + 21 = 462$, which number differs from 464, which would have to result according to the observed law, by two. For this reason, I will now investigate the true law of progression of this series such that it becomes clear how each term is actually defined via several preceding terms.

PROBLEM 2

§7 *Given the series*

$$1, \quad x, \quad 3x^2, \quad 7x^3, \quad 19x^4, \quad 51x^5 \quad etc.$$

to investigate the law, according to which each term is determined by several preceding ones.

SOLUTION

Consider several subsequent terms of this series in general

$$1, \quad x, \quad 3x^2, \quad 7x^3 \quad \cdots \quad Px^n, \quad Qx^{n+1}, \quad Rx^{n+2} \quad \text{etc.},$$

and since in the preceding problem we saw that

$$P = 1 + \frac{n(n-1)}{1\cdot 1} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3} + \text{etc.},$$

in like manner, it will be

$$Q = 1 + \frac{(n+1)n}{1\cdot 1} + \frac{(n+1)n(n-1)(n-2)}{1\cdot 1\cdot 2\cdot 2} + \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3} + \text{etc.},$$

$$R = 1 + \frac{(n+2)(n+1)}{1\cdot 1} + \frac{(n+2)(n+1)n(n-1)}{1\cdot 1\cdot 2\cdot 2} + \frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3} + \text{etc.},$$

whence, subtracting each term from the respective following term, we calculate

$$Q - P = \frac{2n}{1} + \frac{2n(n-1)(n-2)}{1\cdot 1\cdot 2} + \frac{2n(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2\cdot 2\cdot 3} + \text{etc.},$$

$$R - Q = \frac{2(n+1)}{1} + \frac{2(n+1)n(n-1)}{1\cdot 1\cdot 2} + \frac{2(n+1)n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2\cdot 3} + \text{etc.};$$

hence let us take this form

$$\frac{n+2}{n+1}(R-Q) = \frac{2(n+2)}{1} + \frac{2(n+2)n(n-1)}{1\cdot 1\cdot 2} + \frac{2(n+2)n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2\cdot 3} + \text{etc.};$$

if $Q - P$ is subtracted from this one such that

$$\frac{n+2}{n+1}(R-Q) - (Q-P) = 4 + \frac{4n(n-1)}{1\cdot 1} + \frac{4n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2} + \text{etc.};$$

since this series is $= 4P$, we will have

$$R = Q + \frac{(n+1)(Q-P)}{n+2} + \frac{4(n+1)P}{n+2}$$

or

$$R = \frac{(2n+3)Q + 3(n+1)P}{n+2}.$$

COROLLARY 1

§8 Therefore, lo and behold the law by which each term of this series is determined by the two preceding ones and which is

$$R = Q + \frac{n+1}{n+2}(Q+3P),$$

whence from the two following terms Q and R the preceding one P is defined in this way

$$P = \frac{(n+2)R - (2n+3)Q}{3(n+1)}.$$

COROLLARY 2

§9 That it becomes clear how this law holds in the propounded series, let us illustrate this in some cases:

$$\begin{aligned}
&\text{if} \quad n=0, \quad & 3 &= 1 + \frac{1}{2}(1 + 3\cdot 1);\\
&\text{if} \quad n=1, \quad & 7 &= 3 + \frac{2}{3}(3 + 3\cdot 1);\\
&\text{if} \quad n=2, \quad & 19 &= 7 + \frac{3}{4}(7 + 3\cdot 3);\\
&\text{if} \quad n=3, \quad & 51 &= 19 + \frac{4}{5}(19 + 3\cdot 7);\\
&\text{if} \quad n=4, \quad & 141 &= 51 + \frac{5}{6}(51 + 3\cdot 19)
\end{aligned}$$

etc.

Corollary 3

§10 Since the exponent n enters into the relation, which holds between three contiguous terms, from this it is easily concluded that this series is not a recurring series.

Corollary 4

§11 But a relation not involving the letter n between the four contiguous letters P, Q, R, S can be exhibited; since from the preceding three

$$n = \frac{2R - 3Q - 3P}{3P + 2Q - R},$$

in like manner, it will be

$$n + 1 = \frac{2S - 3R - 3Q}{3Q + 2R - S},$$

whence we conclude that

$$S = R + Q + \frac{3P(Q + R) + 2QR}{6P + 3Q - R},$$

which is a constant relation by which the a term is determined by the preceding three contiguous terms.

Scholium 1

§12 After having found the law by which each term of our series depends on the two preceding ones, it is now a lot easier to continue this progression arbitrarily far. Since the powers x^{11} and x^{12} have the coefficients 25653 and 73789, respectively, because of $n = 11$ the coefficient of the following power x^{13} reads as

$$73789 + \frac{12}{13}(73789 + 3 \cdot 25653) = 212941$$

and the coefficient of the power as

$$212941 + \frac{13}{14}(212941 + 3 \cdot 73789) = 616227,$$

whence our progression continued to the twentieth power will look as follows:

1	
$1x$	$25653x^{11}$
$3x^2$	$73789x^{12}$
$7x^3$	$212941x^{13}$
$19x^4$	$616227x^{14}$
$51x^5$	$1787607x^{15}$
$141x^6$	$5196627x^{16}$
$393x^7$	$15134931x^{17}$
$1107x^8$	$44152809x^{18}$
$3139x^9$	$128996853x^{19}$
$8953x^{10}$	$377379369x^{20}$

I observe about these numbers that none of them is divisible by 5, but the coefficients of the powers $x^{3\alpha+2}$ are divisible by 3, and those of the powers $x^{7\alpha+3}$ by 7; and from this nothing about the nature of these numbers can be concluded. But using the law of progression we found here we will be able to define its infinite sum, to which the following problem is devoted.

SCHOLIUM 2

§**13** If each term of our progression is subtracted from the triple of the preceding term, the differences constitute this progression

$$1\cdot 2,\quad 2\cdot 1,\quad 3\cdot 2,\quad 4\cdot 3,\quad 5\cdot 6,\quad 6\cdot 12,\quad 7\cdot 26,\quad 8\cdot 58,\quad 9\cdot 134,\quad 10\cdot 317,$$

$$11\cdot 766,\quad 12\cdot 1883,\quad 13\cdot 4698,\quad 14\cdot 11871,\quad 15\cdot 30330,\quad 16\cdot 78249,\quad 17\cdot 203662,$$
$$18\cdot 533955\quad \text{etc.},$$

for which in general we want to set

$$mp,\quad (m+1)q,\quad (m+2)r,$$

where it is remarkable that the first factors of these terms progress in the series of the natural numbers, the second factors on the other hand are of such a nature that they are composed from the two preceding ones in this way

$$r = \frac{3mp + 2(m+1)q}{m+4}.$$

PROBLEM 3

§14 *If in our series*

$$1 + x + 3x^2 + 7x^3 + 19x^4 + \text{etc.}$$

is continued to infinity, to investigate its sum.

SOLUTION

Since the relation of each term to the two preceding ones was defined, let us set

$$s = 1 + x + 3x^2 + \cdots + Px^n + Qx^{n+1} + Rx^{n+2} + \text{etc.},$$

where one has to note that

$$(n+2)R - (2n+3)Q - 3(n+1)P = 0;$$

to satisfy this condition, let us take the differential

$$\frac{ds}{dx} = 1 + 6x + \cdots + nPx^{n-1} + (n+1)Qx^n + (n+2)Rx^{n+1} + \text{etc.},$$

which multiplied by $1 - 2x - 3xx$ gives

$$\begin{array}{lllllll} \frac{ds}{dx}(1-2x-3xx) = 1 & + \; 6x & + \; 21xx & + \cdots & + \; nPx^{n-1} & + \; (n+1)Qx^n & + \; (n+2)Rx^{n+1} + \text{etc.,} \\ & - \; 2 & - \; 12 & + & & - \; 2Pn & - \; (2n+2)Q \\ & & - \; 3 & & & & - \; 3nP \end{array}$$

which is reduced to this one

$$1 + 4x + 6xx + \cdots + (Q + 3P)x^{n+1} + \text{etc.}$$

But the propounded series multiplied by $1 + 3x$ gives

$$s(1 + 3x) = 1 + 4x + 6xx + \cdots + (Q + 3P)x^{n+1} + \text{etc.},$$

whence it is manifest that

$$\frac{ds}{dx}(1 - 2x - 3xx) = s(1 + 3x)$$

and hence

$$\frac{ds}{s} = \frac{dx(1 + 3x)}{1 - 2x - 3xx},$$

the integral of which gives

$$s = \frac{1}{\sqrt{1 - 2x - 3xx}} = \frac{1}{\sqrt{(1 + x)(1 - 3x)}},$$

which is the sum of the propounded series, if it is continued to infinity.

Corollary 1

§15 Therefore, it is clear that the sum of this series is imaginary, if one does not take $x < \frac{1}{3}$, but becomes infinite in the case $x = \frac{1}{3}$. Attributing negative values to x, say $x = -y$, the sum becomes finite for $y < 1$, but in the case $y > 1$ it becomes imaginary. For example, setting $x = -\frac{1}{2}$

$$\frac{2}{\sqrt{5}} = 1 - \frac{1}{2} + \frac{3}{4} - \frac{7}{8} + \frac{19}{16} - \frac{51}{32} + \frac{141}{64} - \text{etc.}$$

Corollary 2

§16 Therefore, we know now that our series results, if the irrational formula

$$(1 - 2x - 3xx)^{-\frac{1}{2}}$$

is expanded into a series in the usual manner; since this formula can be represented in this way

$$s = ((1-x)^2 - 4xx)^{-\frac{1}{2}},$$

it results

$$s = \frac{1}{1-x} + \frac{2}{1} \cdot \frac{xx}{(1-x)^3} + \frac{2 \cdot 6}{1 \cdot 2} \cdot \frac{x^4}{(1-x)^5} + \frac{2 \cdot 6 \cdot 10}{1 \cdot 2 \cdot 3} \cdot \frac{x^6}{(1-x)^7} + \text{etc.};$$

from its further expansion this results

$$\begin{array}{llllllllll}
s = 1 + & x^2 + & x^3 + & x^4 + & x^5 + & x^6 + & x^7 + & x^8 + & x^9 + & x^{10} \\
 & + 2 \cdot 1 & + 2 \cdot 3 & + 2 \cdot 6 & + 2 \cdot 10 & + 2 \cdot 15 & + 2 \cdot 21 & + 2 \cdot 28 & + 2 \cdot 36 & + 2 \cdot 45 \\
 & & & 6 \cdot 1 & + 6 \cdot 5 & + 6 \cdot 15 & + 6 \cdot 35 & + 6 \cdot 70 & + 6 \cdot 126 & + 6 \cdot 210 \\
 & & & & & + 20 \cdot 1 & + 20 \cdot 7 & + 20 \cdot 28 & + 20 \cdot 84 & + 20 \cdot 210 \\
 & & & & & & & + 70 \cdot 1 & + 70 \cdot 9 & + 70 \cdot 45 \\
 & & & & & & & & & + 252 \cdot 1
\end{array}$$

etc.

Corollary 3

§17 From this we conclude that in general the numerical coefficient of the power x^n results expressed in this way

$$1 + \frac{2}{1} \cdot \frac{n(n-1)}{1 \cdot 2} + \frac{2 \cdot 6}{1 \cdot 2} \cdot \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.},$$

which form does not differ from that one which we found in the first problem.

Scholium

§18 If we consider the form of this sum more accurately, we will easily find a method which extends a lot further and by means of which even the more general power $(a + bx + cxx)^n$ can be treated such that not only the middle term of each power but even the terms equally far away from the middle term can be assigned. Therefore, I will present this method in the following problem.

Problem

§19 *If each power of the trinomial $a+bx+xx$ is expanded and both the middle terms and the terms equally far away from the middle term are arranged into series, to investigate the nature and the sum of these series.*

Solution

Consider this formula

$$\frac{1}{1-y(a+bx+cxx)},$$

which, in expanded form, gives

$$1+y(a+bx+cxx)+yy(a+bx+xx)^2+y^3(a+bx+cxx)^3+\text{etc.};$$

since each power of the propounded trinomial occurs here, having expanded them, it will result

$$\begin{array}{llllllll}
 & & & 1 & & & & \\
 & & & y(a+bx+cxx) & & & & \\
 & y^2(a^2 & +2abx & +2acx^2 & +2bcx^3 & +ccx^4) & & \\
 & & & +\ bb & & & & \\
y^3(a^3 & +3a^2bx & +3a^2cx^2 & +6abcx^3 & +3bbcx^4 & +3bccx^5 & +c^3x^6) & \\
 & & +3ab^2 & +\ b^3 & +3aac & & & \\
 & & & \text{etc.}; & & & &
\end{array}$$

if first the middle terms, then the terms equally far away from the middle terms are taken, the following series will result:

$$1 + bxy + (2ac + bb)xxyy + (6abc + b^3)x^3y^3 + \text{etc.},$$

$$y(a + cxx)(1 + 2bxy + (3ac + 3bb)xxyy + \text{etc.}),$$

$$y^2(a^2 + c^2x^4)(1 + 3bxy + \text{etc.}),$$

$$y^3(a^3 + c^3x^6)(1 + 4bxy + \text{etc.}),$$

$$y^4(a^4 + c^4x^8)(1 + 5bxy + \text{etc.})$$

etc.

Therefore, having omitted these factors, since powers of xy occur in the series, let us set $xy = z$ and indicate these series this way:

$$\begin{aligned} 1 + \quad bz + (2ac + \quad bb)zz + (6abc + b^3)z^3 + \text{etc.} &= P, \\ 1 + 2bz + (3ac + 3bb)zz + \text{etc.} &= Q, \\ 1 + 3bz + \text{etc.} &= R, \\ 1 + 4bz + \text{etc.} &= S, \end{aligned}$$

etc.

such that, because of $y = \frac{z}{x}$, we have

$$\frac{1}{1 - bz - z\left(\frac{a}{x} + cx\right)}$$

$$= P + z\left(\frac{a}{x} + cx\right) + zz\left(\frac{aa}{xx} + ccxx\right)R + z^3\left(\frac{a^3}{x^3} + c^3x^3\right)S + \text{etc};$$

multiply both sides by

$$1 - bz - z\left(\frac{a}{x} + cx\right),$$

and since the quantities P, Q, R etc. depend only on z, arrange all terms according to both positive and negative powers of z; having done this we will obtain

$$
\begin{aligned}
1 = P(1-bz) &+ Qz(1-bz)cx + Rzz(1-bz)c^2x^2 + Sz^3(1-bz)c^3x^3 + \text{etc.} \\
&- Pzcx \qquad\quad - Qzzccx^2 \qquad\quad - Rz^3c^3x^3 \\
-2Qaczz &- Rz^3accx \qquad - Sz^4ac^3x^2 \qquad - Tz^5ac^4x^3 \\
&+Qz(1-bz)\frac{a}{x} + Rzz(1-bz)\frac{a^2}{x^2} + Sz^3(1-bz)\frac{a^3}{x^3} + \text{etc.}, \\
&-Pz\cdot\frac{a}{x} \qquad - Qzz\cdot\frac{aa}{xx} \qquad - Rz^3\cdot\frac{a^3}{x^3} \\
&-Rz^3ac\cdot\frac{a}{x} \qquad - Sz^4ac\cdot\frac{aa}{xx} \qquad - Tz^5ac\cdot\frac{a^3}{x^3}
\end{aligned}
$$

where it is evident that the negative powers of z are reduced to zero by the same conditions as the positive powers. Therefore, we obtain the following equations

$$
\begin{aligned}
Q &= \frac{P(1-bz)-1}{2aczz}, \\
R &= \frac{Q(1-bz)-P}{aczz}, \\
S &= \frac{T(1-bz)-Q}{aczz}, \\
T &= \frac{S(1-bz)-R}{aaczz}
\end{aligned}
$$

etc.

Therefore, we see that the quantities P, Q, R, S etc. progress as recurring series with the scale of relation

$$\frac{1-bz}{aczz}, \quad -\frac{1}{aczz};$$

if indices are attributed these quantities, i.e.

$$
\begin{array}{cccccc}
0 & 1 & 2 & 3 & & n \\
P, & Q, & R, & S, & \cdots & Z
\end{array}
$$

such that Z is the one which corresponds to the index n, from the nature of recurring series

$$Z = A\left(\frac{1-bz-\sqrt{(1-bz)^2-4aczz}}{2aczz}\right)^n + B\left(\frac{1-bz+\sqrt{(1-bz)^2-4aczz}}{2aczz}\right)^n;$$

since we know that the quantity Z is expressed by a series of such a kind that

$$Z = 1 + (n+1)bz + \cdots zz + \cdots z^3 + \cdots z^4 + \text{etc.},$$

it is obvious that B must be $= 0$, since otherwise the terms from the preceding term would result affected by negative powers of z. Therefore, having set $B = 0$, it will be

$$Z = A\left(\frac{1-bz-\sqrt{(1-bz)^2-4aczz}}{2aczz}\right)^n.$$

Now let $n = 0$ and it has to be

$$A = P,$$

but for $n = 1$ it has to be

$$A \cdot \frac{1-bz-\sqrt{1-2bz+(bb-4ac)zz}}{2aczz} = Q.$$

Since $A = P$ and $2aczzQ + 1 = P(1-bz)$, it follows that

$$P(1-bz) - P\sqrt{1-2bz+(bb-4ac)zz} = P(1-bz) - 1$$

and hence

$$P = \frac{1}{\sqrt{1-2bz+(bb-4ac)^2}}.$$

Therefore, the general term of our series $P, Q, R, S, \cdots Z$ is

$$Z = \frac{1}{\sqrt{1-2bz+(bb-4ac)^2zz}}\left(\frac{1-bz-\sqrt{(1-bz)^2-4aczz}}{2aczz}\right)^n.$$

For $y = 1$ such that $x = z$, if all powers of the trinomial $a + bz + czz$ are expanded, the series of intermediate terms $1 + bz + (2ac + bb)zz +$ etc. will be $= P$, but the sum of the terms removed from the middle terms by n places towards the preceding ones is $= a^n Z$, but the sum of the terms removed by as many placed towards the following terms is $= c^n z^n Z$. But the sum of all these series together is

$$= \frac{1}{1 - a - bz - czz}.$$

Corollary 1

§20 Therefore, the quantities P, Q, R, S etc, constitute a geometric progression, the first term of which is

$$P = \frac{1}{\sqrt{1 - 2bz + (bb - 4ac)zz}}$$

and the denominator of the progression is

$$\frac{1 - bz + \sqrt{1 - 2bz + (bb - 4ac)zz}}{2aczz}.$$

Corollary 2

§21 If we take $a = 1$, $b = 1$ and $c = 1$, the case discussed before results, in which we considered the powers of the trinomial $1 + z + zz$; their middle terms constitute a series with a sum $= \frac{1}{\sqrt{1-2z-3zz}}$, as we saw above.

Problem

§22 *To convert the formula that we found in the preceding problem, i.e.*

$$\frac{1}{\sqrt{1 - 2bz + (bb - 4ac)zz}} \left(\frac{1 - bz - \sqrt{1 - 2bz + (bb - 4ac)zz}}{2aczz} \right)^n,$$

into a series, the terms of which progress according to the powers of z.

SOLUTION

For the sake of brevity, set $bb - 4ac = e$ and put

$$s = \frac{1}{\sqrt{1-2bz+ezz}}\left(\frac{1-bz-\sqrt{1-2bz+ezz}}{2aczz}\right)^n,$$

which relation among z and s must be liberated from irrational quantities by differentiation. To this end, set

$$\frac{1-bz-\sqrt{1-2bz+ezz}}{2aczz} = v$$

such that

$$acvvzz - (1-bz)v + 1 = 0,$$

whence by differentiation

$$dv(2acvzz - 1 + bz) + vdz(2acvz + b) = 0$$

or

$$dv\sqrt{1-2bz+ezz} = \frac{vdz}{z}(1-\sqrt{1-2bz+ezz})$$

and hence

$$\frac{dv}{v} = \frac{dz}{z\sqrt{1-2bz+ezz}} - \frac{dz}{z}.$$

After logarithmic differentiation of this equation we have

$$\frac{ds}{s} = \frac{dz(b-ez)}{1-2bz+ezz} - \frac{ndz}{z} + \frac{ndz}{z\sqrt{1-2bz+ezz}}.$$

Let us set

$$\frac{dt}{t} = \frac{ds}{s} + \frac{ndz}{z} - \frac{dz(b-ez)}{1-2bz+ezz}$$

such that

$$\frac{dt}{t} = \frac{ndz}{z\sqrt{1-2bz+ezz}},$$

whence by squaring the equation we conclude

$$zzdt^2(1-2bz+ezz) = nnt^2dz^2,$$

which equation differentiated again, where the element dz is assumed to be constant, gives

$$zzddt(1-2bz+ezz) + zdtdz(1-3bz+2ezz) = nntdz^2$$

or

$$\frac{ddt}{t} + \frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)} \cdot \frac{dt}{t} - \frac{nndz^2}{zz(1-2bz+ezz)} = 0.$$

Since

$$\frac{ddt}{t} = d.\frac{dt}{t} + \frac{dt^2}{tt},$$

it will be

$$\frac{ddt}{t} = \frac{dds}{s} - \frac{ds^2}{ss} - \frac{ndz^2}{zz} + \frac{dz^2(e-2bb-2bez-eezz)}{(1-2bz+ezz)^2} + \frac{nndz^2}{zz} - \frac{2ndz^2(b-ez)}{z(1-2bz+ezz)}$$
$$+\frac{ds^2}{ss} + \frac{2ndzds}{sz} - \frac{2dzds(b-ez)}{s(1-2bz+ezz)} + \frac{dz^2(bb-2bez+eezz)}{(1-2bz+ezz)^2}.$$

Therefore, after the substitution the above equation goes over into this form

$$\frac{dds}{s} + \frac{2ndz}{z} \cdot \frac{ds}{s} - \frac{2dz(b-ez)}{1-2bz+ezz} \cdot \frac{ds}{s} + \frac{n(n-1)dz^2}{zz} - \frac{2ndz^2(be-z)}{z(1-2bz+ezz)}$$
$$+\frac{dz^2(e-bb)}{(1-2bz+ezz)^2} + \frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)} \cdot \frac{ds}{s} + \frac{ndz^2(1-3bz+2ezz)}{zz(1-2bz+ezz)}$$
$$-\frac{dz^2(b-(e+3bb)z+5bezz-2eez^3)}{z(1-2bz+ezz)^2} - \frac{nndz^2}{zz(1-2bz+ezz)} = 0,$$

where, if the terms divided by $(1-2bz+ezz)^2$ are collected into one sum, the numerator and denominator of that fraction can be divided by $1-2bz+ezz$; thus, after the reduction we obtain

$$\frac{dds}{s}+\frac{2ndz}{z}\cdot\frac{ds}{s}+\frac{dz(1-5bz+4ezz)}{z(1-2bz+ezz)}\cdot\frac{ds}{s}-\frac{nndz^2(2b-ez)}{z(1-2bz+ezz)}-\frac{3ndz^2(b-ez)}{z(1-2bz+ezz)}$$
$$-\frac{dz^2(b-2ez)}{z(1-2bz+ezz)}=0,$$
which, properly arranged, gives

$$\begin{gathered}zdds(1-2bz+ezz)+dzds(2n+1-(4n+5)bz+2(n+2)ezz)\\ -sdz^2((n+1)(2n+1)b-(n+1)(n+2)ez)=0.\end{gathered}$$

Since $z=1$ for $s=1$, let us assume this series

$$s=1+Az+Bzz+Cz^3+Dz^4+Ez^5+\text{etc.};$$

after the substitution of this series the following form must be reduced to zero

$$\begin{array}{rcrcrcrcrcr}
 & + & 2Bz & + & 6Czz & + & 12Dz^3 & + & 20Ez^5 \\
 & & & - & 4Bb & - & 12Cb & - & 24Db \\
 & & & & & + & 2Be & + & 6Ce \\
(2n+1)A & + & 2(2b+1)B & + & 3(2n+1)C & + & 4(2n+1)D & + & 5(2n+1)E \\
 & - & (4n+5)Ab & - & 2(4n+5)Bb & - & 3(4n+5)Cb & - & 4(4n+5)Db \\
 & & & + & 2(n+2)Ae & + & 4(n+2)Be & + & 6(n+2)Ce \\
-(n+1)(2n+1)b & - & (n+1)(2n+1)Ab & - & (n+1)(2n+1)Bb & - & (n+1)(2n+1)Cb & - & (n+1)(2n+1)Db \\
 & + & (n+1)(n+2)e & + & (n+1)(n+2)Ae & + & (n+1)(n+2)Be & + & (n+1)(n+2)Ce \\
 & & & & \text{etc.,} & & & &
\end{array}$$

whence we derive these equation:

$$
\begin{aligned}
A &= (n+1)b, \\
B &= \frac{(n+2)((2n+3)Ab-(n+1)e}{2(2n+2)}, \\
C &= \frac{(n+3)((2n+5)Bb-(n+2)Ae}{3(2n+3)}, \\
D &= \frac{(n+4)((2n+7)Cb-(n+3)Be}{4(2n+4)}, \\
E &= \frac{(n+5)((2n+9)Db-(n+4)Ce}{5(2n+5)} \\
& \qquad \text{etc.,}
\end{aligned}
$$

where one should note that $e = bb - 4ac$. And so each term of the series in question is determined by two preceding ones.