A more general Discussion of the Formulas serving for the comparison of curves*

Leonhard Euler

§1 What is known about the comparison of circular arcs from the elements and what Fagnano found on the similar comparison of the arcs of the lemniscate with extraordinary ingenuity, as I have already shown several times, can be formulated more generally in such a way that, if the arc of a certain curved line is expressed indefinitely by this integral formula

$$\int \frac{\mathfrak{A}dz}{\sqrt{A+Czz+Ez^4}},$$

then, having taken an arbitrary arc on the same curve, starting from a certain point one can geometrically separate an arc equal to the first one. And hence having propounded an arbitrary arc, starting from a certain point, one will also be able to separate an arc, which is the double or triple or in general has any rational ration to the given one. Hence it follows that one can compare the arcs of all curves, whose rectification is contained in this form, to each other as it is possible for circular arcs.

§2 Further, what has been found on the comparison of parabolic arcs a long time ago and what was in like manner by Fagnano accomplished in the case of elliptic and hyperbolic arcs with greatest ingenuity, I demonstrated later to

*Original title: „Evolutio generalior formularum comparationi curvarum inservientium", first published in „*Novi Commentarii academiae scientiarum Petropolitanae* 12, 1768, pp. 42-86", reprinted in „*Opera Omnia*: Series 1, Volume 20, pp. 318 - 356 ", Eneström-Number E347, translated by: Alexander Aycock for the project „Euler-Kreis Mainz"

extend so far that with the same success they can be extended to all curves, whose arcs can be expressed indefinitely by this integral formula

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}zz + \mathfrak{C}z^4 + \mathfrak{D}z^6 + \text{etc.})}{\sqrt{A + Czz + Dz^4}},$$

Having taken an arbitrary arc beginning at a certain point in such a curve, one will be able to separate another arc, which differs from that arc by a geometrically assignable quantity. But then one will also be able to separate arcs of such a kind, which differ from the double, triple or any arbitrary multiple by a geometrically assignable quantity. Yes, one will even be able to take that point, whence the arc must be separated, in such a way that this difference vanishes completely.

§3 Therefore, whatever has already been accomplished some time ago on parabolic arcs, the same can with the same success also be done in the case of all curves, whose rectification can be reduced to this integral formula. But whereas Fagnano got to this remarkable comparisons by such cumbersome substitutions and which were not clear how to find them, I explained a clear method, which quasi directly leads to the same comparisons. And this method even solves this much more general problem, which contains all comparisons in most general manner; for, it is equivalent to to complete integration, which involves an arbitrary constant at that same time, whereas those substitutions are to be considered to yield only particular integrations, which is why it was possible for me to proceed a lot further by means of this method, as it is clear from the several specimens I once gave.

§4 But as in these formulas, that I considered, this surdic expression $\sqrt{A + Czz + Ez^4}$ is contained, which already leads to very hard to solve cases, so I observed that it can be extended to this more complicated surdic expression

$$\sqrt{A + 2Bz + Czz + 2Dz^3 + Ez^4};$$

this opens at lot broader field to make similar comparisons in many other curved lines. But this investigation is indeed not only extremely useful in the case of curved lines, but also seems to lead to huge progress in Analysis and Integral Calculus; to pave the way to generalize these results even further, I will explain the expansions concerning this more general formula more diligently. For this aim, let the following equation be propounded expressing the relation among the variables x and y.

THE CANONICAL EQUATION TO BE CONSIDERED

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy$$

§5 This equation, except for the two variables x and y, contains six constant quantities, which, since only their ratio is considered, are reduced to five, so that we have only five arbitrary determinations. Further, even though this equation, considering the variables, rises to four dimensions, nevertheless both of them separately never rises higher than to two dimensions, so that the value of each of them can be exhibited by the resolution of a quadratic equation, which is absolutely necessary for the present investigation. Finally, both of the variables x and y enter this equation equally, and even though the are permuted, induce no change, that each of them are expressed by the other formula in the completely same way. And therefore, the terms x^3+y^3, x^4+y^4 and $xy(xx+yy)$ and the higher dimensions had to be omitted.

§6 If we now extract the value of x and of y from this equation, we will find

$$x = \frac{-\beta-\delta y-\varepsilon yy \pm \sqrt{(\beta+\delta y+\varepsilon yy)^2-(\alpha+2\beta y+\gamma yy)(\gamma+2\varepsilon y+\zeta yy)}}{\gamma+2\varepsilon y+\zeta yy},$$

$$y = \frac{-\beta-\delta x-\varepsilon xx \pm \sqrt{(\beta+\delta x+\varepsilon xx)^2-(\alpha+2\beta x+\gamma xx)(\gamma+2\varepsilon y+\zeta xx)}}{\gamma+2\varepsilon x+\zeta x}.$$

For the sake of brevity, let us put

$$\pm\sqrt{(\beta+\delta y+\varepsilon yy)^2-(\alpha+2\beta y+\gamma yy)(\gamma+2\varepsilon y+\zeta yy)} = Y,$$

$$\pm\sqrt{(\beta+\delta x+\varepsilon xx)^2-(\alpha+2\beta x+\gamma xx)(\gamma+2\varepsilon x+\zeta xx)} = X,$$

that we have

$$x = \frac{-\beta-\delta y-\varepsilon yy+Y}{\gamma+2\varepsilon y+\zeta yy} \quad \text{and} \quad y = \frac{-\beta-\delta x-\varepsilon xx+X}{\gamma+2\varepsilon x+\zeta xx}$$

and hence

$$Y = \beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy),$$
$$X = \beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx).$$

Let us differentiate this canonical equation and, having divided by two, the following differential equation will result

$$0 = +\beta dx + \gamma xdx + \delta ydx + 2\varepsilon xydx + \varepsilon yydx + \zeta xyydx$$
$$+\beta dy + \gamma ydy + \delta xdy + 2\varepsilon xydy + \varepsilon xxdy + \zeta xxydy;$$

since this is reduced to this form

$$0 = +dx(\beta + \delta y + \varepsilon yy) + xdx(\gamma + 2\varepsilon y + \zeta yy)$$
$$+dy(\beta + \delta y + \varepsilon xx) + ydy(\gamma + 2\varepsilon x + \zeta xx),$$

since the coefficients of dx and dy are the quantities, we just exhibited for the formulas X and Y, the differential equation will be

$$0 = Ydx + Xdy \quad \text{and} \quad \frac{dx}{X} + \frac{dy}{Y} = 0;$$

since in this equation the variables x and y are separated, if we substitute those surdic values for X and Y, by means of integration we will hence obtain this finite equation

$$\int \frac{dx}{X} + \int \frac{dy}{Y} = \text{Const.}$$

§8 Therefore, since this integral equation expresses a certain relation among the variables x and y, it can not be different from the relation contained in the equation and so the canonical equation will contain this integral equation. Therefore, even though in the differential equation $\frac{dx}{X} + \frac{dy}{Y} = 0$ none of both parts is integrable and hence cannot be expressed by the quadrature of the circle or by logarithms, the integration nevertheless yields an algebraic relation among both variables x and y, since this equation, having integrated it, agrees with the canonical equation. Yes, I even say that the canonical equation does not only yields a particular case of the integral, cases of which kind often satisfy highly complicated equations, but it even exhibits the complete integral.

§9 To show this, in which without any doubt the high power of this method of integration must be acknowledged, it suffices to have noted that in the canonical equation there is one constant more than in this differential equation. For, we have seen that the canonical equation involves five arbitrary constants; hence let us see, how many arbitrary constant of this constants the differential equation contains. But it is obvious that it has a form of this kind

$$\frac{dx}{\sqrt{A+2Bx+Cxx+2Dx^3+Ex^4}}+\frac{dy}{\sqrt{A+2By+Cyy+2Dy^3+Ey^4}}=0,$$

in which seemingly the five arbitrary constants A, B, C, D, E are contained; but it is evident that each one of them can be cancelled by division so that there actually are only four constants. Hence, although the integral equation contains five constants, one is arbitrary, which is an obvious sign of an complete integral.

§10 But of whatever nature these five coefficients A, B, C, D, E are, one can always determine the coefficients of the canonical equation conform to them is such away that one remains undetermined. For, let us divide the differential equation by the indefinite quantity p, which is to be seen to have been removed, that it actually was

$$X=\sqrt{Ap+2Bpx+Cpxx+2Dpx^3+Epx^4}.$$

But now let us also expand the primitive values of X into a power series in x, which will be

$$X=\sqrt{\begin{matrix}\beta\beta\\ -\alpha\gamma\end{matrix}\;\left.\begin{matrix}+2\beta\delta\\ -2\alpha\varepsilon\\ -2\beta\gamma\end{matrix}\right\}x\;\left.\begin{matrix}+2\beta\varepsilon\\ +\ \delta\delta\\ -\ \alpha\zeta\\ -4\beta\varepsilon\\ -\ \gamma\gamma\end{matrix}\right\}x^2\;\left.\begin{matrix}+2\delta\varepsilon\\ -2\beta\zeta\\ -2\gamma\varepsilon\end{matrix}\right\}x^3\;\left.\begin{matrix}\beta\beta\\ -\alpha\gamma\end{matrix}\right\}x^4},$$

and these letters α, β, γ, δ, ε, ζ are determined in such a way that this form agrees with the first; for, so it will be plain that one determination is still arbitrary.

§11 Therefore, the following five equations must be satisfied

$$\begin{aligned} \text{I.}\quad & \beta\beta - \alpha\gamma = Ap \\ \text{II.}\quad & \beta\delta - \alpha\varepsilon - \beta\gamma = Bp \\ \text{III.}\quad & \delta\delta - \alpha\zeta - 2\beta\varepsilon - \gamma\gamma = Cp \\ \text{IV.}\quad & \delta\varepsilon - \beta\zeta - \gamma\varepsilon = Dp \\ \text{V.}\quad & \varepsilon\varepsilon - \gamma\zeta = Ep. \end{aligned}$$

For the sake of brevity let us put $\delta - \gamma = \lambda$ or $\delta = \gamma + \lambda$ and let us start from II and IV

$$\text{II}\quad \beta\lambda - \alpha\varepsilon = BP \quad \text{and} \quad \text{IV}\quad \varepsilon\lambda - \beta\zeta = Dp,$$

whence we will define β and ε in such way that

$$\beta = \frac{D\alpha + B\lambda}{\lambda\lambda - \alpha\zeta}p \quad \text{and} \quad \varepsilon = \frac{B\zeta + D\lambda}{\lambda\lambda - \alpha\zeta}p.$$

But I and IV combined give

$$\beta\beta\zeta - \alpha\varepsilon\varepsilon = Ap\zeta - Ep\alpha = \frac{BB\zeta - DD\alpha}{\lambda\lambda - \alpha\zeta}pp,$$

whence one finds

$$p = \frac{(\lambda\lambda - \alpha\zeta)(A\zeta - E\alpha)}{BB\zeta - DD\alpha},$$

which value substituted in one of the two remaining ones yields

$$\gamma = \frac{(A\zeta - E\alpha)((ADD - BEE)\lambda\lambda + 2BD(A\zeta - E\alpha)\lambda + ABB\zeta\zeta - DDE\alpha\alpha)}{(BB\zeta - DD\alpha)^2}.$$

§12 Therefore, equation III remains, which, because of $\delta = \gamma + \lambda$ goes over into

$$2\gamma\lambda + \lambda\lambda - \alpha\zeta - 2\beta\varepsilon = Cp.$$

Since now, having substituted the values of p it is

$$\beta = \frac{(A\zeta - E\alpha)(D\alpha + B\lambda)}{BB\zeta - DD\alpha} \quad \text{and} \quad \varepsilon = \frac{(A\zeta - E\alpha)(B\zeta + D\lambda)}{BB\zeta - DD\alpha},$$

if these values are substituted for γ, β, ε and p, the whole equation can be divided by $\lambda\lambda - \alpha\zeta$, having done which one will find

$$\lambda = \frac{C(A\zeta - E\alpha)(BB\zeta - DD\alpha) - 2BD(A\zeta - E\alpha)^2 - (BB\zeta - DD\alpha)^2}{2(A\zeta - E\alpha)(ADD - BBE)}.$$

Therefore, since now all conditions are satisfied, the two coefficients α and ζ or their mutual relation are still arbitrary. Hence it is manifest that in the integral equation order in the canonical equation there is one arbitrary constant not depending on the differential equation.

ANOTHER RESOLUTION OF THE SAME FORMULAS

§13 Since the application of these values is only possible in the cases in which

$$ADD - BBE = 0,$$

I will give another resolution not obstructed by this inconvenience. Put having put $\delta = \gamma + \lambda$ I further set

$$\lambda\lambda - \alpha\zeta = \mu \quad \text{or} \quad \lambda\lambda = \mu + \alpha\zeta$$

and as before from the equations II and IV we will have

$$\beta = \frac{p}{\mu}(D\alpha + B\lambda), \quad \varepsilon = \frac{p}{\mu}(B\zeta + D\lambda).$$

But then, since I and V combined give

$$A\zeta - E\alpha = (BB\zeta - DD\alpha)\frac{p}{\mu},$$

hence I define the relation among α and ζ, or since the one of them can be assumed arbitrarily, I define each of them in such a way that

$$\alpha = \mu A - BBp \quad \text{and} \quad \zeta = \mu E - DDp$$

and hence

$$\lambda\lambda = \mu + (\mu A - BBp)(\mu E - DDp).$$

But the other of the two equations I and V having substituted the values up to this point will yield

$$\gamma = \frac{pp}{\mu\mu}(2BD\lambda + (ADD + BBE)\mu) - \frac{2BBDDp^3}{\mu\mu} - \frac{p}{\mu}.$$

§14 Hence if these values are substituted in equation III, it is reduced to a very long equation; but the problem will be solved more conveniently, if the values found for α and ζ are substituted in the last formula of the preceding resolution; for, then it will result

$$\lambda = \frac{\mu\mu}{2p} + BDp - \frac{1}{2}C\mu,$$

whose square equated to the above value of $\lambda\lambda$ yields

$$\mu(\mu - Cp)^2 + 4(BD - AE)pp\mu + 4(ADD - BCD + BBE)p^3 = 4pp;$$

to resolve it let us put $\mu = pM$