# On the hypergeometric Curve expressed by the equation $y = 1 \cdot 2 \cdot 3 \cdots x$ *

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**1.** While here the letter $x$ denotes the abscissa and $y$ the ordinate, this equation immediately indicates the quantity only of those ordinates corresponding to integer numbers; for, if one had

$$\text{the abscissas } x \cdots \quad 0, \ 1, \ 2, \ 3, \ 4, \quad 5, \quad\ 6 \quad \text{etc.}$$

one will have

$$\text{the ordinates } y \cdots \quad 1, \ 1, \ 2, \ 6, \ 24, \ 120, \ 720 \ \text{ etc.}$$

so that, while the abscissas are taken according to the natural numbers, the ordinates proceed according to the Wallisian hypergeometric progression; therefore, it will be convenient to call also this curve hypergeometric. But even though through this equation just innumerable, but a discrete set of, points of this curve are assigned, nevertheless the nature of this curve is to be considered to be determined by this equation, so that to each abscissa a certain, and via this equation, well-defined ordinate corresponds. For, the nature of this equation requires that, if to a certain abscissa $x = p$ the ordinate $y = q$ corresponds, that then to the abscissa $x = p + 1$ the ordinate $y = q(p + 1)$

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**2.** But except these ordinates corresponding to abscissas expressed by integer numbers, those are especially noteworthy, which fall into the middle between them from the equation; and all are determined by the one I once showed to correspond to the abscissa $x = \frac{1}{2}$ and to be equal to $\frac{1}{2}\sqrt{\pi}$. Therefore, since

$$\sqrt{\pi} = 1.77245385090548,$$

all these ordinates so for the positive as for the negative abscissas will be as follows:

| $x$ | for the positive abscissas the ordinate $y$ is | | for the negative abscissas the ordinate $y$ is |
|---|---|---|---|
| 0 | 1 | 0 | $+1$ |
| $\frac{1}{2}$ | 0.8862269 | $-\frac{1}{2}$ | $+1.7724538$ |
| 1 | 1 | $-1$ | $\pm\infty$ |
| $1\frac{1}{2}$ | 1.3293404 | $-1\frac{1}{2}$ | $-3.5449077$ |
| 2 | 2 | $-2$ | $\mp\infty$ |
| $2\frac{1}{2}$ | 3.3233509 | $-2\frac{1}{2}$ | $+2.3632718$ |
| 3 | 6 | $-3$ | $\pm\infty$ |
| $3\frac{1}{2}$ | 11.6317284 | $-3\frac{1}{2}$ | $-0.9453087$ |
| 4 | 24 | $-4$ | $\mp\infty$ |
| $4\frac{1}{2}$ | 52.3427777 | $-4\frac{1}{2}$ | $+0.2700882$ |
| 5 | 120 | $-5$ | $\pm\infty$ |
| $5\frac{1}{2}$ | 287.8852775 | $-5\frac{1}{2}$ | $-0.0600196$ |
| 6 | 720 | $-6$ | $\mp\infty$ |
| $6\frac{1}{2}$ | 1871.2543038 | $-6\frac{1}{2}$ | $+0.0109126$ |
| 7 | 5040 | $-7$ | $\pm\infty$ |

From this I drew the curve seen in figure 1, which extends from the negative abscissa $x = -1$,

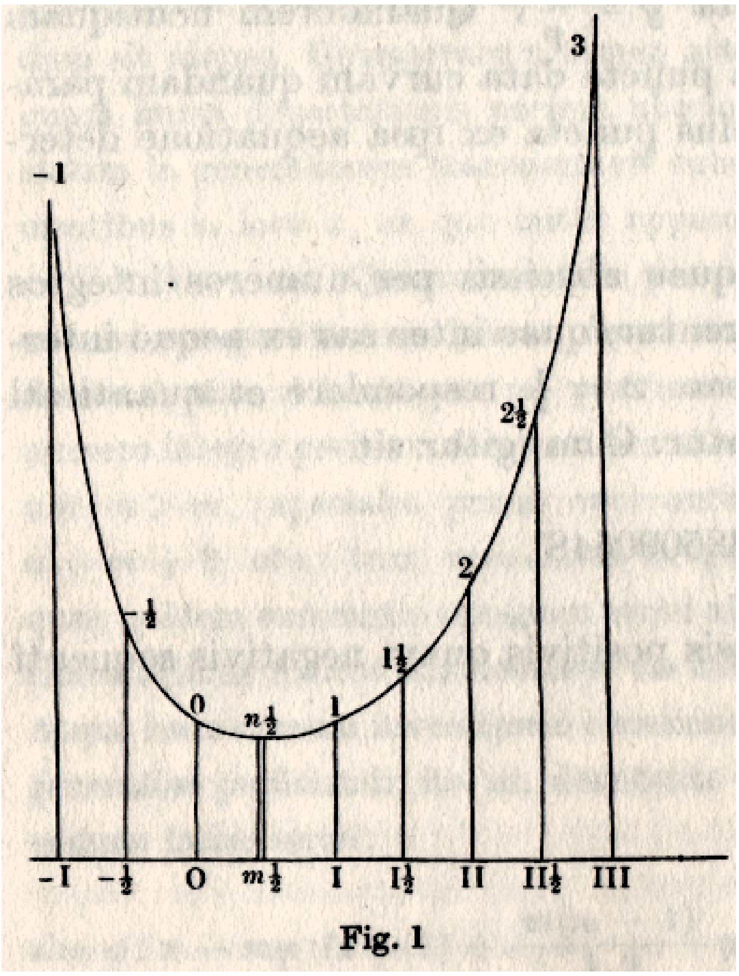


Figure 1 from E368. It shows the factorial function from $x = -1$ to $x = 3$. Euler uses Roman numbers to denote the abscissas.

The scan was taken from the Opera Onmia Version, i.e. p. 44 of volume 28 of the first series.

where the ordinate becomes infinite, to $x = 3$, where $y = 6$, and from this point on is to be understood to ascend to infinity; but the values on the left, where for each integer value of the abscissa the ordinates go over into asymptotes, I did not expresses beyond $x = -1$.

**3.** The consideration of this curve raises many rather curious questions, providing a reason to examine it more accurately; and their solution seem to be even more interesting, since the equation for our curve can not be expressed

in usual manner. Questions of this kind first concern the determination of the remaining points of a curve aside from those which are easily assigned. After this, in each point the tangents require an own investigation, so that the behaviour of the whole curve can be defined more easily. But the from the inspection of the figure it is perspicuous that between the abscissas $x = 0$ and $x = 1$ there must be a smallest ordinate somewhere: to assign so its abscissa as its value will be worth one's while.
Furthermore, between each two negative abscissas $-1$, $-2$, $-3$, $-4$, $-5$ etc., where the ordinates extend to infinity, there necessarily are smallest ordinates, which, the more we proceed to the left, become continuously smaller until eventually they vanish completely. Finally, also the question on the curvature radius in each point deserves our attention, and the point, where the curvature is the largest, seems to be especially remarkable, since it is manifest that in the elongation of the curve from the axis the branches come continuously closer to the straight line. Thus, I want to resolve those questions.

## FIRST QUESTION

*To find a continuous equation among the abscissa x and the ordinate y for the hypergeometric curve, which equally holds, no matter whether for x an integer or a fractional number is taken.*

**4.** Since the propounded equation $y = 1 \cdot 2 \cdot 3 \cdots x$ can only hold if $x$ is an integer number, it must be cast into another form which is not restricted by this condition; this can be achieved in multiple ways using expressions running to infinity, among which at first this one occurs:

$$y = \frac{1}{1+x}\left(\frac{2}{1}\right)^x \cdot \frac{2}{2+x}\left(\frac{3}{2}\right)^x \cdot \frac{3}{3+x}\left(\frac{4}{3}\right)^x \cdot \frac{4}{4+x}\left(\frac{5}{4}\right)^x \cdot \text{etc.}$$

which factors must be continued to infinity. The reason for the validity of this expression is obvious, since, the more factors are taken, the closer the true value is obtained, and, having taken infinitely many factors, the true value is obtained accurately: For, if the numbers of factors is $= n$, one has

$$y = \frac{1}{1+x} \cdot \frac{2}{2+x} \cdot \frac{3}{3+x} \cdots \frac{n}{n+x}(n+1)^x,$$

if whose numerator is represented this way:

$$1 \cdot 2 \cdot 3 \cdots x(x+1)(x+2)(x+3) \cdots n,$$

the denominator on the other hand this way

$$(1+x)(2+x)(3+x) \cdots n(n+1)(n+2) \cdots (n+x),$$

having cancelled the common factors, it results

$$y = \frac{1 \cdot 2 \cdot 3 \cdots x}{(n+1)(n+2)(n+3) \cdots (n+x)}(n+1)^x.$$

Hence, if $n$ is an infinite number, because of the $n+1$ single factors and the total amount of $x$ factors of the denominator, the whole denominator is cancelled by the factor $(n+1)^x$ and the propounded equation $y = 1 \cdot 2 \cdot 3 \cdots x$.

**5.** This formula can be generalised a bit; for, since the whole task reduces to this that the factor $(n+1)^x$ becomes equal to the last denominator

$$(n+1)(n+2)(n+3) \cdots (n+x),$$

in the case, in which $n$ is an infinite number, it is evident that this condition is also satisfied, if the factor is in general set $(n+a)^x$, while $a$ is an arbitrary finite number; but this formula is most appropriate for our task, if a certain mean value of 1 and $x$, e.g. $a = \frac{1+x}{2}$ or $a = \sqrt{x}$, is attributed to $a$. Now it is necessary that this factor $(n+a)^x$ is resolved into so many factors as $n$ contains units, which is conveniently achieved using this resolution:

$$(n+a)^x = a^x \cdot \left(\frac{a+1}{a}\right)^x \cdot \left(\frac{a+2}{a+1}\right)^x \cdot \left(\frac{a+3}{a+2}\right)^x \cdots \left(\frac{a+n}{a+n-1}\right)^x.$$

Therefore, for an arbitrary abscissa $x$ we will have the ordinate:

$$y = a^x \cdot \frac{1}{1+x}\left(\frac{a+1}{a}\right)^x \cdot \frac{2}{2+x}\left(\frac{a+2}{a+1}\right)^x \cdot \frac{3}{3+x}\left(\frac{a+3}{a+2}\right)^x \cdot \text{etc. to infinity,}$$

which expression is always true, whatever number is chosen for $a$, but leads to the truth most quickly, if one takes $a = \frac{1+x}{2}$, whence it will be:

$$y = \left(\frac{1+x}{2}\right)^x \cdot \frac{1}{1+x}\left(\frac{3+x}{1+x}\right)^x \cdot \frac{2}{2+x}\left(\frac{5+x}{3+x}\right)^x \cdot \frac{3}{3+x}\left(\frac{7+x}{5+x}\right)^x \cdot \text{etc.},$$

which expression consists of infinitely many factors of the form

$$\frac{m}{m+x}\left(\frac{a+m}{a-m-1}\right)^x$$

except the first $a^x$, and the more are multiplied by each other in a given case, the closer one will get to the truth. But the initial expression results, if one takes $a = 1$.

**6.** But this expression is the more useful, the faster the factors converge to one, which happens by taking $a = \frac{1+x}{2}$; indeed, then the calculation will become the easier the smaller numbers are substituted for $x$; but it always suffices to have investigated ordinates for abscissas $x$ between one and zero, since hence the ordinates corresponding to $x+1$, $x+2$, $x+3$, $x+4$ etc. are easily derived. Therefore, let $x = \frac{\alpha}{\beta}$, while $\alpha < \beta$, and it will be

$$y = \left(\frac{\alpha+\beta}{2\beta}\right)^{\frac{\alpha}{\beta}} \cdot \frac{\beta}{\alpha+\beta}\left(\frac{3\alpha+\beta}{\beta+\alpha}\right)^{\frac{\alpha}{\beta}} \cdot \frac{2\beta}{\alpha+2\beta}\left(\frac{5\beta+\alpha}{3\beta+\alpha}\right)^{\frac{\alpha}{\beta}} \cdot \frac{3\beta}{\alpha+3\beta}\left(\frac{7\beta+\alpha}{5\beta+\alpha}\right)^{\frac{\alpha}{\beta}} \cdot \text{etc.},$$

whence the power $y^\beta$ of the ordinate results expressed this way:

$$y^\beta = \left(\frac{\alpha+\beta}{2\beta}\right)^\alpha \cdot \frac{\beta^\beta(3\beta+\alpha)^\alpha}{(\beta+\alpha)^\beta(\beta+\alpha)^\alpha} \cdot \frac{(2\beta)^\beta(5\beta+\alpha)^\alpha}{(2\beta+\alpha)^\beta(3\beta+\alpha)^\alpha} \cdot \frac{(3\beta)^\beta(7\beta+\alpha)^\alpha}{(3\beta+\alpha)^\beta(5\beta+\alpha)^\alpha} \cdot \text{etc.}$$

But for the abscissa $x = -\frac{\alpha}{\beta}$ the ordinate is hence calculated to be

$$y^\beta = \left(\frac{2\beta}{\beta-\alpha}\right)^\alpha \cdot \frac{\beta^\beta(\beta-\alpha)^\alpha}{(\beta-\alpha)^\beta(3\beta-\alpha)^\alpha} \cdot \frac{(2\beta)^\beta(3\beta-\alpha)^\alpha}{(2\beta-\alpha)^\beta(5\beta-\alpha)^\alpha} \cdot \frac{(3\beta)^\beta(5\beta-\alpha)^\alpha}{(3\beta-\alpha)^\beta(7\beta-\alpha)^\alpha} \cdot \text{etc.}$$

For the sake of an example let us take $x = \frac{1}{2}$ and we will obtain:

$$y^2 = \frac{3}{4} \cdot \frac{2\cdot 2\cdot 7}{3\cdot 3\cdot 3} \cdot \frac{4\cdot 4\cdot 11}{5\cdot 5\cdot 7} \cdot \frac{6\cdot 6\cdot 15}{7\cdot 7\cdot 11} \cdot \frac{8\cdot 8\cdot 19}{9\cdot 9\cdot 15} \cdot \text{etc.},$$

since a general factor of which is

$$\frac{2n\cdot 2n(4n+3)}{(2n+1)(2n+1)(4n-1)} = \frac{16n^3+12nn}{16n^3+12nn-1} = 1+\frac{1}{(2n+1)^2(4n-1)},$$

hence it is seen in general, how quickly these factors converge to 1; therefore, it will be:

$$y^2 = \frac{3}{4}\left(1+\frac{1}{3^2\cdot 3}\right)\left(1+\frac{1}{5^2\cdot 7}\right)\left(1+\frac{1}{7^2\cdot 11}\right)\left(1+\frac{1}{9^2\cdot 15}\right)\left(1+\frac{1}{11^2\cdot 19}\right)\text{etc.},$$

where we know that $y^2 = \frac{\pi}{4}$. But if we put $x = -\frac{1}{2}$, to which $y = \sqrt{-\pi}$ corresponds, from the other expression it will be

$$\pi = 4\cdot\frac{2\cdot 2\cdot 1}{1\cdot 1\cdot 5}\cdot\frac{4\cdot 4\cdot 5}{3\cdot 3\cdot 9}\cdot\frac{6\cdot 6\cdot 9}{5\cdot 5\cdot 13}\cdot\frac{8\cdot 8\cdot 13}{7\cdot 7\cdot 17}\cdot\text{etc.}$$

or

$$\pi = 4\left(1-\frac{1}{1^2\cdot 5}\right)\left(1-\frac{1}{3^2\cdot 9}\right)\left(1-\frac{1}{5^2\cdot 13}\right)\left(1-\frac{1}{7^2\cdot 17}\right)\text{etc.},$$

hence

$$\pi = 3\left(1+\frac{1}{3^2\cdot 3}\right)\left(1+\frac{1}{5^2\cdot 7}\right)\left(1+\frac{1}{7^2\cdot 11}\right)\left(1+\frac{1}{9^2\cdot 15}\right)\text{etc.},$$

so that the one expression will get to the truth while increasing, the other while decreasing.

**7.** But the calculation is executed more conveniently, if our expression is terminated at each factor; for, then the following formulas coming continuously closer to the truth will result:

$$y = \frac{1}{1+x}\left(\frac{3+x}{2}\right)^x$$

$$y = \frac{1}{1+x}\cdot\frac{2}{2+x}\left(\frac{5+x}{2}\right)^x$$

$$y = \frac{1}{1+x}\cdot\frac{2}{2+x}\cdot\frac{3}{3+x}\left(\frac{7+x}{2}\right)^x$$

$$y = \frac{1}{1+x}\cdot\frac{2}{2+x}\cdot\frac{3}{3+x}\cdot\frac{4}{4+x}\left(\frac{9+x}{2}\right)^x$$

$$y = \frac{1}{1+x}\cdot\frac{2}{2+x}\cdot\frac{3}{3+x}\cdot\frac{4}{4+x}\cdot\frac{5}{5+x}\left(\frac{11+x}{2}\right)^x.$$

Since, if one writes $x-1$ instead of $x$, the ordinate $= \dfrac{y}{x}$ results, by similar formulas it will be

$$y = \left(\frac{2+x}{2}\right)^{x-1}$$

$$y = \frac{2}{1+x}\left(\frac{4+x}{2}\right)^{x-1}$$

$$y = \frac{2}{1+x}\cdot\frac{3}{2+x}\left(\frac{6+x}{2}\right)^{x-1}$$

$$y = \frac{2}{1+x}\cdot\frac{3}{2+x}\cdot\frac{4}{3+x}\left(\frac{8+x}{2}\right)^{x-1}$$

$$y = \frac{2}{1+x}\cdot\frac{3}{2+x}\cdot\frac{4}{3+x}\cdot\frac{5}{4+x}\left(\frac{10+x}{2}\right)^{x-1}$$

Hence, having put $x = \frac{1}{2}$, for the ordinate $y = \frac{1}{2}\sqrt{\pi}$ two series of formulas converging to it result:

| | |
|---|---|
| $\frac{1}{2}\sqrt{\pi} = \frac{2}{3}\sqrt{\frac{7}{4}}$ | $\frac{1}{2}\sqrt{\pi} = \sqrt{\frac{4}{5}}$ |
| $\frac{1}{2}\sqrt{\pi} = \frac{2\cdot 4}{3\cdot 5}\sqrt{\frac{11}{4}}$ | $\frac{1}{2}\sqrt{\pi} = \frac{4}{3}\sqrt{\frac{4}{9}}$ |
| $\frac{1}{2}\sqrt{\pi} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\sqrt{\frac{15}{4}}$ | $\frac{1}{2}\sqrt{\pi} = \frac{4\cdot 6}{3\cdot 5}\sqrt{\frac{4}{13}}$ |
| $\frac{1}{2}\sqrt{\pi} = \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}\sqrt{\frac{19}{4}}$ | $\frac{1}{2}\sqrt{\pi} = \frac{4\cdot 6\cdot 8}{3\cdot 5\cdot 7}\sqrt{\frac{4}{17}}$ |
| etc. | $\frac{1}{2}\sqrt{\pi} = \frac{4\cdot 6\cdot 8\cdot 10}{3\cdot 5\cdot 7\cdot 9}\sqrt{\frac{4}{21}}$ |
| | etc. |

**8.** But products of this kind are most conveniently expanded using logarithms; and first from the general formula involving the arbitrary number $a$ we obtain:

$$\log y = x\log a + x\log\frac{a+1}{a} + x\log\frac{a+2}{a+1} \;+ x\log\frac{a+3}{a+2} \;+ x\log\frac{a+4}{a+3} \;+ \text{etc.}$$
$$- \log(1+x) \;- \log\left(1+\frac{x}{2}\right) - \log\left(1+\frac{x}{3}\right) - \log\left(1+\frac{x}{4}\right) - \text{etc.}$$

and, having taken $a = \frac{1+x}{2}$, that this series is rendered most convergent:

$$\log y = x\log\frac{1+x}{2} + x\log\frac{x+3}{x+1} + x\log\frac{x+5}{x+3} + x\log\frac{x+7}{x+5} + x\log\frac{x+9}{x+7} + \text{etc.}$$
$$\log(1+x) \;-\; \log\left(1+\frac{x}{2}\right) \;-\; \log\left(1+\frac{x}{3}\right) \;-\; \log\left(1+\frac{x}{4}\right) \;-\; \text{etc.}$$

Therefore, having taken these logarithms, since in general:

$$x\log\frac{x+2m+1}{x+2m-1}=\frac{2x}{x+2m}+\frac{2x}{3(x+2m)^3}+\frac{2x}{5(x+2m)^5}+\frac{2x}{7(x+2m)^7}+\text{etc.}$$

$$\text{and}\log\left(1+\frac{x}{m}\right)=\frac{2x}{x+2m}+\frac{2x^3}{3(x+2m)^3}+\frac{2x^5}{5(x+2m)^5}+\frac{2x^7}{7(x+2m)^7}+\text{etc.}$$

we obtain the following formulas consisting of infinitely many series:

$$\begin{aligned}\log y=x\log\frac{1+x}{2}&+\frac{2}{3}x(1-xx)\left(\frac{1}{(x+2)^3}+\frac{1}{(x+4)^3}+\frac{1}{(x+6)^3}+\frac{1}{(x+8)^3}+\text{etc.}\right)\\&+\frac{2}{5}x(1-x^4)\left(\frac{1}{(x+2)^5}+\frac{1}{(x+4)^5}+\frac{1}{(x+6)^5}+\frac{1}{(x+8)^5}+\text{etc.}\right)\\&+\frac{2}{7}x(1-x^6)\left(\frac{1}{(x+2)^7}+\frac{1}{(x+4)^7}+\frac{1}{(x+6)^7}+\frac{1}{(x+8)^7}+\text{etc.}\right)\\&+\frac{2}{9}x(1-x^8)\left(\frac{1}{(x+2)^9}+\frac{1}{(x+4)^9}+\frac{1}{(x+6)^9}+\frac{1}{(x+8)^9}+\text{etc.}\right)\\&\qquad\qquad\qquad\text{etc.}\end{aligned}$$

**9.** Let us take a definite number of terms of the first series, which number we want to be $=n$, and since the upper part is reduced to the single term $x\log(a+n)$, it will be

$$\log y=x\log(a+x)-\log(1+x)-\log\left(1+\frac{1}{2}x\right)-\log\left(1+\frac{1}{3}x\right)-\cdots-\log\left(1+\frac{1}{n}x\right),$$

which expression comes the closer to the truth, the greater the number $n$ is taken. Therefore, let $n$ be very large, and first we will obviously have

$$\log(n+a)=\log n+\frac{a}{n}-\frac{aa}{2n^2}+\frac{a^3}{3n^3}-\text{etc.},$$

where it will be convenient to take $\frac{1+x}{2}$ for $a$; but then, for the sake of brevity, having put the fraction

$$0.5772156649015325 = \Delta,$$

we know the sum of the harmonic progression to be:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \Delta + \log n + \frac{1}{2n} - \frac{1}{12nn} + \frac{1}{120n^4} - \text{etc.},$$

whence, since:

$$\log(n+a) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \Delta - \frac{1}{2n} + \frac{1}{12nn} - \frac{1}{120n^4} + \text{etc.}$$
$$+ \frac{a}{n} - \frac{aa}{2nn} + \frac{a^3}{3n^3} - \text{etc.},$$

having taken $a = \frac{1+x}{2}$, we conclude

$$\log y = -\Delta x + x + \frac{1}{2}x + \frac{1}{3}x + \cdots + \frac{1}{n}x$$
$$- \log(1+x) - \log\left(1 + \frac{1}{2}x\right) - \log\left(1 + \frac{1}{3}x\right) - \cdots - \log\left(1 + \frac{1}{n}x\right)$$
$$+ \frac{xx}{2n} - \frac{x + 6xx + 3x^3}{24nn} + \text{etc.}$$

Therefore, by increasing the number $n$ to infinity, it will actually be:

$$\log y = -\Delta x + x + \frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x + \text{etc.}$$
$$- \log(1+x) - \log\left(1 + \frac{1}{2}x\right) - \log\left(1 + \frac{1}{3}x\right) - \log\left(1 + \frac{1}{4}x\right) - \text{etc.}$$

and, having expanded each logarithm into a series:

$$\begin{aligned}\log y = -\Delta x + \frac{1}{2}\, xx &\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{etc.}\right)\\ - \frac{1}{3}\, x^3 &\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.}\right)\\ + \frac{1}{4}\, x^4 &\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.}\right)\\ - \frac{1}{5}\, x^5 &\left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

**10.** But except those formulas, in which the ordinate $y$ corresponding to a certain abscissa $x$ is assigned, my method to sum progressions indefinitely provides us with an extraordinary expression accommodated to our purposes. For, since $\log y = \log 1 + \log 2 + \log 3 + \log 4 + \cdots + \log x$, this progression must be summed indefinitely; but introducing numerical values:

$$A = \frac{1}{6},\;\; B = \frac{1}{90},\;\; C = \frac{1}{945},\;\; D = \frac{1}{9450},\;\; E = \frac{1}{93555},\;\; F = \frac{691}{1 \cdot 3 \cdot 5 \cdots 15 \cdot 315}\;\; \text{etc.},$$

which progression is of such a nature that

$$5B = 2AA,\;\; 7C = 4AB,\;\; 9D = 4AC + 2BB,\;\; 11E = 4AD + 4BC\;\; \text{etc.},$$

I showed elsewhere that it will be:

$$\log y = \frac{1}{2}\log 2\pi + \left(x + \frac{1}{2}\right)\log x - x + \frac{A}{2x} - \frac{1 \cdot 2B}{2^3x^3} + \frac{1 \cdot 2 \cdot 3 \cdot 4C}{2^5x^5} - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6D}{2^7x^7} + \text{etc.},$$

which series, compared to the first, has the use that, the greater the abscissas $x$ are taken, the faster it exhibits the true value of the ordinate $y$. Therefore, since, if to the abscissa $x$ the ordinate $y$ corresponds, to the larger abscissa $x + n$ the following ordinate corresponds

$$y(x+1)(x+2)(x+3)\cdots(x+n),$$

we will have the following rapidly convergent series:

$$\log y = \frac{1}{2}\log 2\pi - \log(x+1) - \log(x+2) - \log(x+3) - \cdots - \log(x+n)$$
$$+\left(x+n+\frac{1}{2}\right)\log(x+n) - x - n$$
$$+\frac{A}{2(x+n)} - \frac{1\cdot 2B}{2^3(x+n)^3} + \frac{1\cdot 2\cdot 3\cdot 4C}{2^5(x+n)^5} - \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6D}{2^7(x+n)^7} + \text{etc.}$$

Therefore, if $e$ denotes the number whose natural logarithm is $= 1$, and for the sake of brevity one puts:

$$\frac{A}{2(x+n)} - \frac{1\cdot 2B}{2^3(x+n)^3} + \frac{1\cdot 2\cdot 3\cdot 4C}{2^5(x+n)^5} - \text{etc.} = s,$$

going back from logarithms to numbers we conclude:

$$y = \frac{\sqrt{2\pi(x+n)}}{(x+1)(x+2)(x+3)\cdots(x+n)}\left(\frac{x+n}{e}\right)^{x+n} e^s,$$

where the integer number $n$ is arbitrary; but the larger it is taken, the easier the true value of $s$ can be found.

**11.** Finally, the ordinate $y$ can even be exhibited by an integral formula; for, having put the abscissa $x = p$ and having introduced the new variable $u$, independent of the quantity $p$, the ordinate will be

$$y = \int du \left(\log\frac{1}{u}\right)^p,$$

if the integration is extended from the value $u = 0$ to the value $u = 1$. Or, if one prefers the exponential form, it will also be

$$y = \int e^{-v} v^p dv,$$

extending the integration from $v = 0$ to $v = \infty$. From those formulas, if the abscissa $p$ is an integer number, the integration indeed immediately yields

$$y = 1\cdot 2\cdot 3\cdots p,$$

but if $p$ was a fractional number, hence it is at the same time understood to which class of transcendental quantities the value of $y$ is to be referred. Indeed, I showed on another occasion, how the integral can then be expressed using quadratures of algebraic curves.

**12.** Therefore, lo and behold the many solutions of our first question, in which for an arbitrary abscissa $x$, even though it is expressed by a non-integer number, the value of the ordinate $y$ was in sought after; it will be helpful to have listed up the principal ones, that hence in each case the one which seems to be the most useful can be chosen:

$$\text{I.}\qquad y = \frac{1}{1+x}\left(\frac{2}{1}\right)^x \cdot \frac{2}{2+x}\left(\frac{3}{2}\right)^x \cdot \frac{3}{3+x}\left(\frac{4}{3}\right)^x \cdot \frac{4}{4+x}\left(\frac{5}{4}\right)^x \cdot \text{etc.}$$

$$\text{II.}\qquad y = \left(\frac{1+x}{2}\right)^x \cdot \frac{1}{1+x}\left(\frac{3+x}{2+x}\right)^x \cdot \frac{2}{2+x}\left(\frac{5+x}{3+x}\right)^x \cdot \frac{3}{3+x}\left(\frac{7+x}{5+x}\right)^x \cdot \text{etc.}$$

$$\text{III.}\quad \log y = x\log\frac{2}{1} + x\log\frac{3}{2} + x\log\frac{4}{3} + x\log\frac{5}{4} + \text{etc.}$$
$$- \log(1+x) - \log\left(1+\frac{1}{2}x\right) - \log\left(1+\frac{1}{3}x\right) - \log\left(1+\frac{1}{4}x\right) - \text{etc.}$$

$$\text{IV.}\quad \log y = x\log\frac{1+x}{2} + x\log\frac{x+3}{x+1} + x\log\frac{x+5}{x+3} + x\log\frac{x+7}{x+5} + x\log\frac{x+9}{x+7} + \text{etc.}$$
$$- \log(1+x) - \log\left(1+\frac{1}{2}x\right) - \log\left(1+\frac{1}{3}x\right) - \log\left(1+\frac{1}{4}x\right) - \text{etc.}$$

$$\text{V.}\quad \log y = -\Delta x + x + \frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x + \text{etc.}$$
$$- \log(1+x) - \log\left(1+\frac{1}{2}x\right) - \log\left(1+\frac{1}{3}x\right) - \log\left(1+\frac{1}{4}x\right) - \text{etc.}$$

$$\text{VI.}\quad \log y = \begin{cases} -\Delta x + \frac{1}{2}\,xx\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}\right) \\ \qquad -\frac{1}{3}\,x^3\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc.}\right) \\ \qquad +\frac{1}{4}\,x^4\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}\right) \\ \qquad -\frac{1}{5}\,x^5\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc.}\right) \\ \qquad + \quad \text{etc.} \end{cases}$$

$$\text{VII.}\quad \log y = \frac{1}{2}\log 2\pi + \left(x+\frac{1}{2}\right)\log x - x + \frac{A}{2x} - \frac{1\cdot 2B}{2^3x^3} + \frac{1\cdot 2\cdot 3\cdot 4C}{2^5x^5} - \frac{1\cdot 2\cdots 6D}{2^7x^7} + \text{etc.}$$

while $\Delta = 0.57721556649014225$ and

$$A = \frac{1}{6},\quad B = \frac{1}{90},\quad C = \frac{1}{945},\quad D = \frac{1}{9450},\quad E = \frac{1}{93555}\quad \text{etc.}$$

Then in the three last forms one has to use natural logarithms.

## SECOND QUESTION

*On the hypergeometric curve for each point to define the direction of its tangent.*

**13.** Therefore, here we assume that for the abscissa $x$ the value of the ordinate $y$ has already been found, and since the direction of the tangent is defined by the ratio of the differentials $\frac{dy}{dx}$, by which fraction the tangent of the angle, in which the tangent at that point is inclined to the axis, is usually expressed, it is just necessary that we differentiate one of the found formulas. To this end formula V seems especially suitable, from which we conclude:

$$\begin{aligned}\frac{dy}{ydx} = -\Delta + 1 \quad &+ \frac{1}{2} \quad + \frac{1}{3} \quad + \frac{1}{4} \quad + \text{etc.}\\ &- \frac{1}{1+x} - \frac{1}{2+x} - \frac{1}{3+x} - \frac{1}{4+x} - \text{etc.}\end{aligned}$$

which expression is contracted into this more convenient one:

$$\frac{dy}{ydx} = -\Delta + \frac{x}{1+x} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + \frac{x}{4(4+x)} + \text{etc.},$$

whence it is clear at the same time, if $x$ is a negative integer number, that not just the ordinate $y$ but also the formula $\frac{dy}{dx}$ becomes infinite, such that in this points the ordinates, since they are asymptotes, become the tangents. But let us in general put the angle the tangent constitutes with the axis $= \varphi$ that

$$\frac{dy}{dx} = \tan\varphi.$$

**14.** Therefore, first let us define the tangents for the abscissas $x$ which are expressed by positive numbers, since the ordinate $y$ are given.

I. Therefore, let $x = 0$ and, because of $y = 1$,

$$\frac{dy}{dx} = -\Delta = -0.5772156649 = \tan\varphi,$$

whence

$$\text{the angle } \varphi = -29°59'29'',$$

where the sign $-$ indicates that the tangent falls to the right of the axis and does not constitute an angle of 30° with it.

II. Let $x = 1$ and, because of $y = 1$,

$$\frac{dy}{dx} = 1 - \Delta = 0.422784335 = \tan\varphi,$$

and hence

$$\text{the angle } \varphi = 22°55'.$$

III. Let $x = 2$ and, because of $y = 2$,

$$\frac{dy}{dx} = 2\left(1 + \frac{1}{2} - \Delta\right) = 1.845568670 = \tan\varphi$$

and hence

$$\text{the angle } \varphi = 61°33'.$$

IV. Let $x = 3$ and, because of $y = 6$,

$$\frac{dy}{dx} = 6\left(1 + \frac{1}{2} + \frac{1}{3} - \Delta\right) = \tan\varphi$$

or

$$\tan\varphi = 7.536706010 \quad \text{and} \quad \varphi = 82°26'.$$

V. Let $x = 4$ and, because of $y = 24$,

$$\frac{dy}{dx} = 24\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \Delta\right)$$

and hence

$$\tan\varphi = 36.146824040 \quad \text{and} \quad \varphi = 88°25'.$$

Therefore, in general, if the abscissa $x$ is equal to an arbitrary integer number, because of $y = 1 \cdot 2 \cdots n$, it will be

$$\frac{dy}{dx} = \tan \varphi = 1 \cdot 2 \cdot 3 \cdots n \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \Delta\right).$$

**15.** Hence let us also define the tangents for the intermediate points, and first certainly for those corresponding to positive abscissas:

I. Let $x = \frac{1}{2}$, it will be $y = \frac{1}{2}\sqrt{\pi}$ and

$$\frac{dy}{ydx} = -\Delta + 1 - \frac{2}{3} + \frac{1}{2} - \frac{2}{5} + \frac{1}{3} - \frac{2}{7} + \text{etc.}$$

or

$$\frac{dy}{ydx} = -\Delta + 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \text{etc.}\right) = -\Delta + 2(1 - \log 2)$$

and hence

$$\frac{dy}{dx} = \tan \varphi = y(2(1 - \log 2) - \Delta) = 0.0364899739 \cdot y.$$

II. Let $x = \frac{3}{2}$, it will be $y = \frac{1 \cdot 3}{2 \cdot 2}\sqrt{\pi}$ and

$$\frac{dy}{ydx} = -\Delta + 2\left(1 + \frac{1}{3} - \log 2\right),$$

whence

$$\frac{dy}{dx} = \tan \varphi = y\left(2\left(1 + \frac{1}{3} - \log 2\right) - \Delta\right) = 0.7031566405 \cdot y.$$

III. Let $x = \frac{5}{2}$, it will be $y = \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2}\sqrt{\pi}$ and

$$\frac{dy}{ydx} = -\Delta + 2\left(1 + \frac{1}{3} + \frac{1}{5} - \log 2\right)$$

hence

$$\tan \varphi = y\left(2\left(1 + \frac{1}{3} + \frac{1}{5} - \log 2\right) - \Delta\right) = 1.1031566405 \cdot y.$$

Since now

$$\frac{1}{2}\sqrt{\pi \cdot (2(1-\log 2)-\Delta)} = 0.0323383973,$$

for these cases it will be:

$$\begin{aligned}
x &= \frac{1}{2}, & y &= 0.8862269, & \tan\varphi &= 0.0323384,\\
x &= \frac{3}{2}, & y &= 1.3293404, & \tan\varphi &= 0.9347345,\\
x &= \frac{5}{2}, & y &= 3.3233509, & \tan\varphi &= 3.6661767,\\
x &= \frac{7}{2}, & y &= 11.6317284, & \tan\varphi &= 16.1549694,\\
x &= \frac{9}{2}, & y &= 52.3427777, & \tan\varphi &= 84.3290907,
\end{aligned}$$

etc.

**16.** Before I proceed, I observe, if for any abscissa it was

$$x = p, \quad y = q, \quad \tan\varphi = r,$$

that then for the following abscissa it will be

$$x = p+1, \quad y = q(p+1) \quad \text{and} \quad \tan\varphi = r(p+1)+q,$$

but for the preceding one

$$x = p-1, \quad y = \frac{q}{p} \quad \text{and} \quad \tan\varphi = \frac{r}{p} - \frac{q}{pp},$$

whence we can easily continue the above values backwards:

$$
\begin{array}{llll}
x = \quad \frac{1}{2}, & y = \quad 0.8862269, & \tan\varphi = \quad 0.0323384, \\
x = -\frac{1}{2}, & y = \quad 1.7724538, & \tan\varphi = -3.4802308, \\
x = -\frac{3}{2}, & y = -3.5449077, & \tan\varphi = -0.1293538, \\
x = -\frac{5}{2}, & y = +2.3632718, & \tan\varphi = +1.6617504, \\
x = -\frac{7}{2}, & y = -0.9453087, & \tan\varphi = -1.0428236, \\
x = -\frac{9}{2}, & y = +0.2700882, & \tan\varphi = +0.3751176, \\
x = -\frac{11}{2}, & y = -0.0600196, & \tan\varphi = -0.0966971, \\
x = -\frac{13}{2}, & y = +0.0109126, & \tan\varphi = +0.0195654,
\end{array}
$$

etc.

**17.** The same differential equation serves for finding the point $\mu$ of the curve, where the ordinate is the smallest or the tangent is parallel to the axis. Therefore, having put $\frac{dy}{dx} = 0$, the corresponding abscissa $x$ must be found from this equation:

$$\Delta = \frac{x}{1+x} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + \frac{x}{4(4+x)} + \frac{x}{5(5+x)} + \text{etc.},$$

which is expanded into this one:

$$\begin{aligned}\Delta = &+ x\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}\right)\\ &- x^2\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc.}\right)\\ &+ x^3\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}\right)\\ &- x^4\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc.}\right)\\ &\quad\text{etc.}\end{aligned}$$

But having substituted the proximate sums of these series it will be

$$\begin{aligned}0 = &+ 0.5772156649 & &- 1.6449340668\, x\\ &+ 1.2020569032\, x^2 & &- 1.0823232337\, x^3\\ &+ 1.0369277551\, x^4 & &- 1.0173430620\, x^5\\ &+ 1.0083492774\, x^6 & &- 1.0040773562\, x^7\\ &+ 1.0020083928\, x^8 & &- 1.0009945751\, x^9\\ &+ 1.0004941886\, x^{10} & &- 1.0002460866\, x^{11}\\ &+ 1.0001227133\, x^{12} & &- 1.0000612481\, x^{13}\\ &+ 1.0000305882\, x^{14} & &- 1.0000152823\, x^{15}\\ & & \text{etc.}\end{aligned}$$

But if the first to fractions are kept, the following a lot more convergent series emerges

$$
\begin{aligned}
0 = \ & +0.5772156649 && -\tfrac{x}{1+x} - \tfrac{x}{2(2+x)} \\
& +0.0770569032x^2 && -0.3949340668x \\
& +0.0056777551x^4 && -0.0198232337x^3 \\
& +0.0005367774x^6 && -0.0017180620x^5 \\
& +0.0000552678x^8 && -0.0001711062x^7 \\
& +0.0000059074x^{10} && -0.0000180126x^9 \\
& +0.0000006430x^{12} && -0.0000019460x^{11} \\
& +0.0000000706x^{14} && -0.0000002130x^{13} \\
& +0.0000000078x^{16} && -0.0000000235x^{15} \\
& \quad \text{etc.}
\end{aligned}
$$

Hence one finds approximately $x = \frac{1}{2}$, but this minimal ordinate will be defined more easily by means of the following question.

## THIRD QUESTION

*For a given point of the hypergeometric curve to investigate the nature of an infinitesimal portion of this curve around this point.*

**18.** Therefore, for the given abscissa $x = p$ let the ordinate $y = q$ have been found; and now one has to find the ordinate, which corresponds to the abscissa $p + \omega$ differing from that one by just a small amount. Therefore, since according to formula V

$$
\begin{aligned}
\log q = -\Delta p + p \quad & + \frac{1}{2}p \quad + \frac{1}{3}p \quad + \frac{1}{4}p \quad + \text{etc.} \\
- \log(1+p) \quad & - \log\left(1 + \frac{1}{2}p\right) - \log\left(1 + \frac{1}{3}p\right) - \log\left(1 + \frac{1}{4}p\right) - \text{etc.},
\end{aligned}
$$

if one writes $p + \omega$ instead of $p$ here, instead of $\log q$ the value of $\log(q + \psi)$ will result, by which the question will be resolved. And if we put $\log q = P$, writing $p + \omega$ instead of $p$, it is known to result

$$\log(q+\psi) = P + \frac{\omega dP}{1dp} + \frac{\omega^2 ddP}{1\cdot 2dp^2} + \frac{\omega^3 d^3P}{1\cdot 2\cdot 3dp^3} + \frac{\omega^4 d^4P}{1\cdot 2\cdot 3\cdot 4dp^4} + \text{etc.}$$

But on the other hand, as we have seen:

$$\frac{dP}{dp} = -\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \frac{p}{4(4+p)} + \text{etc.}$$

and hence further:

$$\begin{aligned}
\frac{ddP}{1\cdot dp^2} &= \frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \frac{1}{(4+p)^2} + \text{etc.}\\
\frac{d^3P}{1\cdot 2dp^3} &= -\frac{1}{(1+p)^3} - \frac{1}{(2+p)^3} - \frac{1}{(3+p)^3} - \frac{1}{(4+p)^3} - \text{etc.}\\
\frac{d^4P}{1\cdot 2\cdot 3dp^4} &= \frac{1}{(1+p)^4} + \frac{1}{(2+p)^4} + \frac{1}{(3+p)^4} + \frac{1}{(4+p)^4} + \text{etc.}\\
&\text{etc.}
\end{aligned}$$

whence, because of $P = \log q$, we conclude:

$$\begin{aligned}
\log\left(1+\frac{\psi}{q}\right) = -\Delta\omega + {} & \omega\left(\frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \text{etc.}\right)\\
&+ \frac{1}{2}\omega^2\left(\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \text{etc.}\right)\\
&- \frac{1}{3}\omega^3\left(\frac{1}{(1+p)^3} + \frac{1}{(2+p)^3} + \frac{1}{(3+p)^3} + \text{etc.}\right)\\
&+ \frac{1}{4}\omega^4\left(\frac{1}{(1+p)^4} + \frac{1}{(2+p)^4} + \frac{1}{(3+p)^4} + \text{etc.}\right)\\
&- \frac{1}{5}\omega^2\left(\frac{1}{(1+p)^5} + \frac{1}{(2+p)^5} + \frac{1}{(3+p)^5} + \text{etc.}\right)\\
&\text{etc.}
\end{aligned}$$

**19.** Here now the coordinates $p$ and $q$ can be considered as constant, since the letters $\omega$ and $\psi$ denote two new coordinates taken from a given point of the curve and parallel to the first set; from their relation defined here the nature of the curve around that point is easily investigated. Thus, since we have already assigned innumerable points of the curve, hence the trace of each portion of the curve between two of those conjugated points can be defined approximately. First, from that differentiated equation, as before, the inclination $\varphi$ of the tangent to the axis is calculated and

$$\frac{d\psi}{d\omega} = \tan\varphi = q\left(-\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \text{etc.}\right).$$

Further, if for the differential equation, for the sake of brevity, we put

$$d\psi = Ad\omega + B\omega d\omega + C\omega^2 d\omega + \text{etc.},$$

the curvature radius at a given point of the curve will be

$$= \frac{(1+AA)^{\frac{3}{2}}}{B} = \frac{1}{B\cdot\cos^3\varphi}$$

because of $A = \tan\varphi$. But on the other hand

$$B = \tan\varphi\left(-\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \text{etc.}\right)$$
$$+q\left(\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \frac{1}{(4+p)^2} + \text{etc.}\right),$$

whence, if the curvature radius is put $= r$, it will be

$$\frac{1}{r} = \frac{\sin^2\varphi\cos\varphi}{q} + q\left(\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \text{etc.}\right).$$

**20.** But to be able to extend the investigation of the direction and the curvature from the principal point defined by the coordinates $p$ and $q$ to the points of the curve, for the sake of brevity, let us put

$$-\Delta + \frac{p}{1+p} + \frac{p}{2(2+p)} + \frac{p}{3(3+p)} + \frac{p}{4(4+p)} + \text{etc.} = P,$$

$$\frac{1}{(1+p)^2} + \frac{1}{(2+p)^2} + \frac{1}{(3+p)^2} + \frac{1}{(4+p)^2} + \text{etc.} = Q,$$

$$\frac{1}{(1+p)^3} + \frac{1}{(2+p)^3} + \frac{1}{(3+p)^3} + \frac{1}{(4+p)^3} + \text{etc.} = R,$$

$$\frac{1}{(1+p)^4} + \frac{1}{(2+p)^4} + \frac{1}{(3+p)^4} + \frac{1}{(4+p)^4} + \text{etc.} = S$$

etc.,

that

$$\log\left(1 + \frac{\psi}{q}\right) = P\omega + \frac{1}{2}Q\omega^2 - \frac{1}{3}R\omega^3 + \frac{1}{4}S\omega^4 - \frac{1}{5}T\omega^5 + \text{etc.}$$

Hence now differentiating we find:

$$\frac{d\psi}{d\omega} = (q+\psi)(P + Q\omega - R\omega^2 + S\omega^3 - T\omega^4 + \text{etc.})$$

and differentiating further

$$\frac{dd\psi}{d\omega^2} = (q+\psi)(P + Q\omega - R\omega^2 + S\omega^3 - T\omega^4 + \text{etc.})^2$$
$$+(q+\psi)(Q - 2R\omega + 3S\omega^2 - 4T\omega + \text{etc.})$$

$$\frac{d^3\psi}{d\omega^3} = 3(q+\psi)(Q - 2R\omega + 3S\omega^2 - 4T\omega^3 + \text{etc.})(P + Q\omega - R\omega^2 + S\omega^3 - \text{etc.})$$
$$+(q+\psi)(P + Q\omega - R\omega^2 + S\omega^3 - T\omega^4 + \text{etc.})^3$$
$$-(q+\psi)(2R - 6S\omega + 12T\omega^2 - \text{etc.}).$$

Having covered these calculations, for the point of the curve corresponding to the abscissa $x = p + \omega$ and $y = q + \psi$ the direction of the tangent will be

$$\tan\varphi = \frac{d\psi}{d\omega} = (q+\psi)(P + Q\omega - R\omega^2 + S\omega^3 - T\omega^4 + \text{etc.}).$$

But then, having put the curvature radius $= r$, we know that it will be:

$$r = \left(1 + \frac{d\psi^2}{d\omega^2}\right)^{\frac{3}{2}} : \frac{dd\psi}{d\omega^2} = 1 : \frac{dd\psi}{d\omega^2} \cos^3 \varphi$$

or

$$\frac{1}{r} = \frac{dd\psi}{d\omega^2} \cos^3 \varphi,$$

whence we find for the variability of the curvature:

$$-\frac{dr}{rrd\omega} = \frac{d^3\psi}{d\omega^3} \cos^3 \varphi - \frac{3dd\psi}{d\omega^2} \cdot \frac{d\varphi}{d\omega} \sin \varphi \cos^2 \varphi.$$

But on the other hand

$$\frac{d\varphi}{\cos^2 \varphi} = \frac{dd\psi}{d\omega},$$

whence:

$$-\frac{dr}{rrd\omega} = \frac{d^3\psi}{d\omega^3} \cos^3 \varphi - 3 \left(\frac{dd\psi}{d\omega^2}\right)^2 \sin \varphi \cos^4 \varphi.$$

## Fourth Question

*To investigate the nature of the hypergeometric curve around its lowest point $\mu$ where the ordinate is the smallest.*

**21.** Since this point is not far away from the point corresponding to the abscissa $= \frac{1}{2}$ and the ordinate $= \frac{1}{2}\sqrt{\pi}$, let us set $p = \frac{1}{2}$ that $q = \frac{1}{2}\sqrt{\pi}$, and hence first let us find the values of the letters $P$, $Q$, $R$, $S$ etc., which will result as:

$$
\begin{aligned}
P &= -\Delta + \frac{1}{3} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 7} + \text{etc.} = 2(1-\log 2) - \Delta = 0.03648997397857\\
Q &= \frac{4}{3^2} + \frac{4}{5^2} + \frac{4}{7^2} + \frac{4}{9^2} + \text{etc.} = 0.93480220054468\\
R &= \frac{8}{3^3} + \frac{8}{5^3} + \frac{8}{7^3} + \frac{8}{9^3} + \text{etc.} = 0.41439832211716\\
S &= \frac{16}{3^4} + \frac{16}{5^4} + \frac{16}{7^4} + \frac{16}{9^4} + \text{etc.} = 0.0.23484850566707\\
T &= \frac{32}{3^5} + \frac{32}{5^5} + \frac{32}{7^5} + \frac{32}{9^5} + \text{etc.} = 0.144760040831276\\
V &= \frac{64}{3^6} + \frac{64}{5^6} + \frac{64}{7^6} + \frac{64}{9^6} + \text{etc.} = 0.09261290502029\\
W &= \frac{128}{3^7} + \frac{128}{5^7} + \frac{128}{7^7} + \frac{128}{9^7} + \text{etc.} = 0.06035822809843
\end{aligned}
$$

Further,

$$q = \frac{1}{2}\sqrt{\pi} = 0.88622692545274.$$

**22.** Hence let us especially define the point $\mu$, where the ordinate is the smallest, since which simple approximations show to correspond to the abscissa $x = 0.4616$, having put

$$p + \omega = \frac{1}{2} + \omega = 0.4616,$$

one approximately finds

$$\omega = -0.0383,$$

which value must be investigated more accurately from the equation $\frac{d\psi}{d\omega} = 0$ or

$$P + Q\omega - R\omega^2 + S\omega^3 - T\omega^4 + \text{etc.} = 0.$$

Therefore, since approximately $\omega = -\frac{1}{26}$, set $\omega = -\frac{1}{26} - z$, and after the substitution it has to be

$$\begin{array}{rl}
 & + 0.03595393079018 + 0.934802200z \\
 & + 0.00061301526940 + 0.031876794z + 0.414398zz \\
 & + 0.00001336188585 + 0.001042227z + 0.027097zz \\
 & + 0.00000031677900 + 0.000032945z + 0.001285zz \\
 & + 0.00000000779479 + 0.000001013z + 0.000053zz \\
 & + 0.00000000019538 + 0.000000030z + 0.000002zz \\
 & + 0.00000000000496 + \qquad\qquad 2z + \\
 & + \qquad\qquad\qquad 13 + \\
\hline
 & 0.03658063271970 + 0.967755211z + 0.442835zz \\
 & 0.03648997397857 \\
\hline
0 = & 0.00009065874113 + 0.967755211z + 0.442835zz
\end{array}$$

whence one finds

$$z = -0.00009368323$$

and hence

$$\omega = -0.03836785523.$$

Therefore, the smallest ordinate $m\mu$ corresponds to the abscissa

$$Om = 0.46163214477.$$

For the ordinate $m\mu = q + \psi$ on the other hand one has to expand the equation

$$\log\left(1 + \frac{\psi}{q}\right) = P\omega + \frac{1}{2}Q\omega^2 - \frac{1}{3}R\omega^3 + \frac{1}{4}S\omega^4 - \frac{1}{5}T\omega^5 + \text{etc.},$$

from which one concludes

$$\log\left(1 + \frac{\psi}{q}\right) = -0.000704053$$

and further

$$1+\frac{\psi}{q}=1-0.000703805,$$

so that the smallest ordinate becomes

$$m\mu=q+\psi=0.8856031945.$$

**23.** Now let us in general differentiate define the value of $\psi$ from the logarithmic equation, and, having done the calculation, we will obtain:

$$\begin{aligned}\frac{\psi}{q}=&+0.0364899740\omega+0.468066860\omega^2\\&-0.121069221\omega^3+0.16321479\omega^4\\&-0.09360753\omega^5\quad+\text{etc.},\end{aligned}$$

which terms, if the value of $\omega$ is very small, suffice. But, for the sake of brevity, let us put

$$\frac{\psi}{q}=\mathfrak{P}\omega+\mathfrak{Q}\omega^2-\mathfrak{R}\omega^3+\mathfrak{S}\omega^4-\mathfrak{T}\omega^5,$$

that

$$\begin{aligned}&\mathfrak{P}=0.0364899740,\quad \mathfrak{Q}=0.468066860,\\&\mathfrak{R}=0.121069221,\quad\ \ \mathfrak{S}=0.16321479,\\&\mathfrak{T}=0.09360753,\end{aligned}$$

and hence we will have:

$$\begin{aligned}\frac{d\psi}{d\omega}&=q\left(\mathfrak{P}+2\mathfrak{Q}\omega-3\mathfrak{R}\omega^2+4\mathfrak{S}\omega^3-5\mathfrak{T}\omega^4\right),\\ \frac{dd\psi}{d\omega^2}&=q(2\mathfrak{Q}-6\mathfrak{R}\omega+12\mathfrak{S}\omega^2-20\mathfrak{T}\omega^3).\end{aligned}$$

If we now hence want to find the radius of curvature at the lowest point $\mu$, where

$$\omega = -0.03836785523,$$

since there $\frac{d\psi}{d\omega} = 0$, that radius of curvature will be $= \frac{d\omega^2}{dd\psi}$. Put the curvature radius in this point $= r$, and since

$$\frac{1}{r} = 2q(\mathfrak{Q} - 3\mathfrak{R}\omega + 6\mathfrak{S}\omega^2 - 10\mathfrak{T}\omega^3) = 0.9669949,$$

for the point $\mu$ the curvature radius results as

$$r = 1.166893.$$

**24.** I investigated these determinations of the lowest point $\mu$ of the curve with all eagerness, that it can not without any reason be conjectured, as this point has an extraordinary property, that so the numbers exhibiting its nature contain a certain elegance, and if they can not be expressed sufficiently simply in terms of a rational or irrational number, that they are at least to be referred to a certain simpler kind of transcendental quantities. But against the expectation it happened that such a criterion for elegance appears neither in the abscissa

$$Om = 0.46163214477$$

nor in the ordinate

$$m\mu = 0.8856031945$$

nor in the curvature radius at this point

$$= 1.166893;$$

for, no affinity to simpler rational numbers or irrational numbers or to the quadrature of the circle or to logarithmic or exponential numbers is detected. Since, if the abscissa $Om$ is considered as a logarithm, the number corresponding to it could seem to promise several things, I sought after this number and found

$$= 1.586616,$$

in which no affinity to any known quantities is recognized.

**25.** Before I end this speculation, it will helpful to have observed that the formula $1 \cdot 2 \cdot 3 \cdots x$ can also be expressed indefinitely in terms of the following series

$$x^x - x(x-1)^x + \frac{x(x-1)}{1 \cdot 2}(x-2)^x - \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}(x-3)^x + \text{etc.},$$

which, as often as $x$ is a positive integer number, immediately gives that product $1 \cdot 2 \cdot 3 \cdots x$. This is indeed also achieved by this further extending expression:

$$a^x - x(a-1)^x + \frac{x(x-1)}{1 \cdot 2}(a-2)^x - \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}(a-3)^x + \text{etc.},$$

for, if for $x$ one successively substitutes the numbers $1, 2, 3$ etc., it will be as follows:

$a^0 = 1$

$a^1 - (a-1)^1 = 1$

$a^2 - 2(a-1)^2 + (a-2)^2 = 1 \cdot 2$

$a^3 - 3(a-1)^3 + 3(a-2)^3 - (a-3)^3 = 1 \cdot 2 \cdot 3$

$a^4 - 4(a-1)^4 + 6(a-2)^4 - 4(a-3)^4 + (a-4)^4 = 1 \cdot 2 \cdot 3 \cdot 4$

$a^5 - 5(a-1)^5 + 10(a-2)^5 - 10(a-3)^5 + 5(a-4)^5 - (a-5)^5 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$

etc.

**26.** These are certainly obvious from the results demonstrated on the difference of each order of algebraic progressions, but nevertheless from the nature of these series the truth is not easily uncovered; thus, the following proof seems to be in order. Since for smaller exponents $x$ the matter is obvious, I reason as follows, i.e. that, having conceded the truth for the case $x = n$, I will show that it also follows for the case $x = n + 1$.
Therefore, let

$$\text{I.} \quad a^n - n(a-1)^n + \frac{n(n-1)}{1 \cdot 2}(a-2)^2 - \text{etc.} = N = 1 \cdot 2 \cdot 3 \cdots n,$$

and since the number $N$ does not depend on $a$, it will also be:

$$\text{II.}\quad (a-1)^n - n(a-2)^n + \frac{n(n-1)}{1\cdot 2}(a-3)^2 - \text{etc.} = N,$$

which subtracted from the first leaves:

$$\text{III.}\quad a^n - \frac{(n+1)}{1}(a-1)^n + \frac{(n+1)n}{1\cdot 2}(a-2)^n - \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}(a-3)^n + \text{etc.} = 0,$$

multiply this by $a$ so that it results

$$\text{IV.}\quad a^{n+1} - \frac{(n+1)}{1}a(a-1)^n + \frac{(n+1)n}{1\cdot 2}a(a-2)^n - \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}a(a-3)^n + \text{etc.} = 0,$$

to this one add equation II multiplied by $n+1$, i.e.:

$$\text{V.}\quad +(n+1)1(a-1)^n - \frac{(n+1)n}{1\cdot 2}2(a-2)^n + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}3(a-3)^n - \text{etc.} = (n+1)N$$

and the aggregate IV+V will give

$$\text{VI.}\ a^{n+1} - \frac{(n+1)}{1}(a-1)^{n+1} + \frac{(n+1)n}{1\cdot 2}(a-2)^{n+1} - \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}(a-3)^{n+1} + \text{etc.} = (n+1)N,$$

where, because of $N = 1\cdot 2\cdot 3\cdots N$, it will be $(n+1)N = 1\cdot 2\cdot 3\cdots(n+1)$. Therefore, it is proved that, if our proposition

$$a^x - x(a-1)^x + \frac{x(x-1)}{1\cdot 2}(a-2)^x - \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}(a-3)^x + \text{etc.} = 1\cdot 2\cdot 3\cdots x$$

was true in the case $x = n$, it will also be true in the case $x = n+1$. Therefore, since it is obviously true in the case $x = 1$, hence it follows that it is also true for all positive integer numbers assumed for $x$.

**27.** But although this expression is sufficiently elegant and worth one's complete attention, it is nevertheless less useful for our task, in which the hypergeometric curve it propounded, since for the cases, in which $x$ is a fractional number, this series not only runs to infinity but also, if the denominator is an even number, contains imaginary terms, such that its value can not even be calculated using approximations. So, having put $x = \frac{1}{2}$, this infinite series results:

$$\sqrt{a}-\frac{1}{2}\sqrt{a-1}-\frac{1\cdot 1}{2\cdot 4}\sqrt{a-2}-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\sqrt{a-3}-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\sqrt{a-4}-\text{etc.},$$

whose value can hardly be shown by anyone to be $=\frac{1}{2}\sqrt{\pi}$. In like manner, taking $x=-\frac{1}{2}$, from the above results we already know that

$$\sqrt{\pi}=\frac{1}{\sqrt{a}}+\frac{1}{2\sqrt{a-1}}+\frac{1\cdot 3}{2\cdot 4\sqrt{a-2}}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\sqrt{a-3}}+\text{etc.}$$

Nevertheless, a further investigation of this series is left to the mathematics, especially if it is extended and represented in this form:

$$s=x^n-m(x-1)^n+\frac{m(m-1)}{1\cdot 2}(x-2)^n-\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(x-3)^n+\text{etc.};$$

for, without much effort one soon detects extraordinary properties, whose expansion seems worth our complete attention. I will now present all extraordinary phenomena I was able to observe about it.

## OBSERVATIONS ON THE SERIES
## $s=x^n-m(x-1)^n+\frac{m(m-1)}{1\cdot 2}(x-2)^n-\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(x-3)^n+\text{ETC.}$

**I.** Therefore, in the preceding I already demonstrated, if the exponent $n$ was $=m$, that the sum of this series will be

$$s=1\cdot 2\cdot 3\cdots m,$$

so that in this case it does not depend on the number $x$. But hence I first conclude, if $n=m-1$, that then it will be $s=0$. For, since, having taken $n=m$,

$$♄\cdots 1\cdot 2\cdot 3\cdots m=x^m-m(x-1)^m+\frac{m(m-1)}{1\cdot 2}(x-2)^m-\text{etc.}$$

and, writing $x-1$ instead of $x$ and $m-1$ instead of $m$, in like manner:

$$♃\cdots 1\cdot 2\cdot 3\cdots(m-1)=(x-1)^{m-1}-(m-1)(x-2)^{m-1}+\frac{(m-1)(m-2)}{1\cdot 2}(x-3)^{m-1}-\text{etc.}$$

Now represent that equation [♄] this way:

$$♂ \cdots 1 \cdot 2 \cdot 3 \cdots m = x \cdot x^{m-1} - mx(x-1)^{m-1} + \frac{m(m-1)}{1 \cdot 2} x(x-2)^{m-1} - \text{etc.}$$
$$+ \quad (x-1)^{m-1} - \quad \frac{m(m-1)}{1}(x-2)^{m-1} + \text{etc.}$$

but the equation ♃ multiplied by $m$ gives:

$$\odot \cdots 1 \cdot 2 \cdot 3 \cdots m = m(x-1)^{m-1} - \frac{m(m-1)}{1}(x-2)^{m-1} + \frac{m(m-1)(m-2)}{1 \cdot 2}(x-3)^{m-1} - \text{etc.},$$

which subtracted from ♂ and divided by $x$ yields:

$$♀ \cdots 0 = x^{m-1} - \frac{m}{1}(x-1)^{m-1} + \frac{m(m-1)}{1 \cdot 2}(x-2)^{m-1} - \text{etc.},$$

which is the propounded equation for the case $n = m-1$, whose value thus is $= 0$.

**II.** In like manner it is shown that the sum $s$ of the propounded series also vanishes in the case $n = m-2$. For, represent the series ♀ this way:

$$☿ \cdots 0 = x \cdot x^{m-2} - \frac{m}{1} x(x-1)^{m-2} + \frac{m(m-1)}{1 \cdot 2} x(x-2)^{m-2} - \text{etc.}$$
$$+ \quad m(x-1)^{m-2} - \quad \frac{m(m-1)}{1 \cdot 2}(x-2)^{m-2} + \text{etc.}$$

and if in the same series ♀ one writes $x-1$ instead of $x$ and $m-1$ instead of $m$, but the whole series is multiplied by $m$, it becomes

$$☽ \cdots 0 = m(x-1)^{m-2} - \frac{m(m-1)}{1}(x-2)^{m-2} + \text{etc.}$$

Having subtracted this one from that one, divide the remainder by $x$ and it will result:

$$0 = x^{m-2} - \frac{m}{1}(x-1)^{m-2} + \frac{m(m-1)}{1 \cdot 2}(x-2)^{m-2} - \text{etc.}$$

And so the sum of the propounded series also vanishes in the case $n = m - 2$, and in like manner it can be shown that it also vanishes in the cases $n = m - 3$, $n = m - 4$ etc. and in general $n = m - i$, where $i$ is an arbitrary positive integer number. Therefore, keep in mind that the sum of the series $s$ is $= 1 \cdot 2 \cdot 3 \cdots m$ in the case $n = m$, but in the cases, in which the exponent $n$ is smaller than the number $m$, the sum vanishes, if the numbers $m$ and $n$ are integers or at least $n - m$ is a positive integer number, of course.

**III.** Therefore, to investigate the nature of the remaining cases, let us expand each term of our series and arrange them according to the powers of $x$, having done which we will obtain:

$$\begin{aligned}
s = x^n & \left(1 - m + \frac{m(m-1)}{1 \cdot 2} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} + \text{etc.}\right) \\
& + nx^{n-1}\left(m - \frac{2m(m-1)}{1 \cdot 2} + \frac{3m(m-1)(m-2)}{1 \cdot 2 \cdot 3} - \text{etc.}\right) \\
& - \frac{n(n-1)}{1 \cdot 2}x^{n-2}\left(m - \frac{4m(m-1)}{1 \cdot 2} + \frac{9m(m-1)(m-2)}{1 \cdot 2 \cdot 3} - \text{etc.}\right) \\
& + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}\left(m - \frac{8m(m-1)}{1 \cdot 2} + \frac{27m(m-1)(m-2)}{1 \cdot 2 \cdot 3} - \text{etc.}\right) \\
& \quad \text{etc.},
\end{aligned}$$

the sums of which series we will find as follows; first, exhibit them a bit more generally, and since its sum is known:

$$1 - mu + \frac{m(m-1)}{1 \cdot 2}u^2 - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}u^3 + \text{etc.} = (1-u)^m,$$

let us differentiate it continuously and always substitute $u$ for $du$ again, and, having changed the signs, it will be:

$$mu - \frac{2m(m-1)}{1\cdot 2}u^2 + \frac{3m(m-1)(m-2)}{1\cdot 2\cdot 3}u^3 - \text{etc.} = mu(1-u)^{m-1}$$

$$mu - \frac{2m(m-1)}{1\cdot 2}u^2 + \text{etc.} = mu(1-u)^{m-1} - m(m-1)u^2(1-u)^{m-2}$$

$$mu - \frac{2^3m(m-1)}{1\cdot 2}u^2 + \text{etc.} = mu(1-u)^{m-1} - 3m(m-1)u^2(1-u)^{m-2}$$

$$+m(m-1)(m-2)u^3(1-u)^{m-3}$$

$$mu - \frac{2^4m(m-1)}{1\cdot 2}u^2 + \text{etc.} = mu(1-u)^{m-1} - 7m(m-1)uu(1-u)^{m-2}$$

$$+6m(m-1)(m-2)u^3(1-u)^{m-3}$$

$$-m(m-1)(m-2)(m-3)u^4(1-u)^{m-4}$$

etc.

Therefore, here one now has to write $u = 1$, having done which all terms in each order vanish except those where the exponent of $1-u$ become $= 0$.

**IV.** Now successively attribute the values 1, 2, 3, 4, 5 etc. to $m$ and instead of the general coefficient

$$\frac{n(n-1)(n-2)\cdots(n-i)}{1\cdot 2\cdot 3\cdots(i+1)}$$

write $\left(\frac{n-i}{i+1}\right)$ for the sake of brevity, having done which we obtain the following values:

$$
\begin{array}{c|l}
\text{if} & \text{it will be} \\
m=1 & \dfrac{s}{1} = \left(\dfrac{n}{1}\right)x^{n-1} - \left(\dfrac{n-1}{2}\right)x^{n-2} + \left(\dfrac{n-2}{3}\right)x^{n-3} - \left(\dfrac{n-3}{4}\right)x^{n-4} + \left(\dfrac{n-4}{5}\right)x^{n-5} - \text{etc.} \\
m=2 & \dfrac{s}{1\cdot 2} = \left(\dfrac{n-1}{2}\right)x^{n-2} - 3\left(\dfrac{n-2}{3}\right)x^{n-3} + 7\left(\dfrac{n-3}{4}\right)x^{n-4} - 15\left(\dfrac{n-4}{5}\right)x^{n-5} + 31\left(\dfrac{n-5}{6}\right)x^{n-6} - \text{etc.} \\
m=3 & \dfrac{s}{1\cdot 2\cdot 3} = \left(\dfrac{n-2}{3}\right)x^{n-3} - 6\left(\dfrac{n-3}{4}\right)x^{n-4} + 25\left(\dfrac{n-4}{5}\right)x^{n-5} - 90\left(\dfrac{n-5}{6}\right)x^{n-6} + 301\left(\dfrac{n-6}{7}\right)x^{n-7} - \text{etc.} \\
m=4 & \dfrac{s}{1\cdot 2\cdots 4} = \left(\dfrac{n-3}{4}\right)x^{n-4} - 10\left(\dfrac{n-4}{5}\right)x^{n-5} + 65\left(\dfrac{n-5}{6}\right)x^{n-6} - 350\left(\dfrac{n-6}{7}\right)x^{n-7} + 1701\left(\dfrac{n-7}{8}\right)x^{n-8} - \text{etc.} \\
m=5 & \dfrac{s}{1\cdot 2\cdots 5} = \left(\dfrac{n-4}{5}\right)x^{n-5} - 15\left(\dfrac{n-5}{6}\right)x^{n-6} + 140\left(\dfrac{n-6}{7}\right)x^{n-7} - 1050\left(\dfrac{n-7}{8}\right)x^{n-8} + 6951\left(\dfrac{n-8}{9}\right)x^{n-9} - \text{etc.} \\
m=6 & \dfrac{s}{1\cdot 2\cdots 6} = \left(\dfrac{n-5}{6}\right)x^{n-6} - 21\left(\dfrac{n-6}{7}\right)x^{n-7} + 266\left(\dfrac{n-7}{8}\right)x^{n-8} - 2646\left(\dfrac{n-8}{9}\right)x^{n-9} + 22827\left(\dfrac{n-9}{10}\right)x^{n-10} - \text{etc.}
\end{array}
$$

where the formation of each numerical coefficient from the preceding is obvious; for, for the last sixth series:

$$21 = 6\cdot 1 + 15, \quad 266 = 6\cdot 21 + 140, \quad 2646 = 6\cdot 266 + 1050 \quad \text{etc.}$$

And hence it is immediately seen, if $m < n$, that the value of $s$ vanishes; for, in the last series, if $n < 6$ and hence either 5 or 4 or 3 etc., it will be

$$\left(\frac{n-5}{6}\right) = 0, \quad \left(\frac{n-6}{7}\right) = 0 \quad \text{etc.}$$

But then on the other hand, if $n = m$, it is also evident that

$$\frac{s}{1\cdot 2\cdots m} = 1,$$

for, in the lowest series:

$$\left(\frac{6-5}{6}\right) = 1, \quad \left(\frac{6-6}{7}\right) = 0, \quad \left(\frac{6-7}{8}\right) = 0, \quad \left(\frac{6-8}{9}\right) = 0 \quad \text{etc.}$$

## Expansion of the cases $n = m + 1$

**V.** Hence let us first expand the cases in which $n = m + 1$, and the last form yields

| if | these sums |
|---|---|
| $m=1,\quad n=2$ | $\dfrac{s}{1} = 2x - 1$ |
| $m=2,\quad n=3$ | $\dfrac{s}{1\cdot 2} = 3x - 3$ |
| $m=3,\quad n=4$ | $\dfrac{s}{1\cdot 2\cdot 3} = 4x - 6$ |
| $m=4,\quad n=5$ | $\dfrac{s}{1\cdot 2\cdot 3\cdot 4} = 5x - 10$ |
| $m=5,\quad n=6$ | $\dfrac{s}{1\cdot 2\cdots 6} = 6x - 15$ |
| | etc., |

where the first coefficients of $x$ are equal to $n$, but the absolute numbers are equal to the triangular number of $n$; we will have in general

| if | this equation |
|---|---|
| $n = m+1$ | $\dfrac{s}{1\cdot 2\cdots m} = (m+1)x - \dfrac{m(m+1)}{1\cdot 2} = (m+1)\left(x - \dfrac{m}{2}\right),$ |

such that

$$x^{m+1} - m(x-1)^{m+1} + \frac{m(m-1)}{1\cdot 2}(x-2)^{m+1} - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(x-3)^{m+1} + \text{etc.}$$
$$= 1\cdot 2\cdot 3\cdots(m+1)\left(x - \frac{m}{2}\right).$$

## Expansion of the Cases $n = m+2$

**VI.** Therefore, for these cases we will have:

$$
\begin{array}{c|cl}
\text{if it was} & \text{these equations} & \\
m=1,\quad n=3 & \dfrac{s}{1} & = \ \ 3x^2 - 3\cdot 1\ \ x + 1\cdot \quad 1 = \ \ 3\left(xx - \ \ x + \ \ \dfrac{2}{6}\right) \\
m=2,\quad n=4 & \dfrac{s}{1\cdot 2} & = \ \ 6x^2 - 4\cdot 3\ \ x + 1\cdot \quad 7 = \ \ 6\left(xx - 2x + \ \ \dfrac{7}{6}\right) \\
m=3,\quad n=5 & \dfrac{s}{1\cdot 2\cdot 3} & = 10x^2 - 5\cdot 6\ \ x + 1\cdot \ \ 25 = 10\left(xx - 3x + \dfrac{15}{6}\right) \\
m=4,\quad n=6 & \dfrac{s}{1\cdot 2\cdot 3\cdot 4} & = 15x^2 - 6\cdot 10x + 1\cdot \ \ 65 = 15\left(xx - 4x + \dfrac{26}{6}\right) \\
m=5,\quad n=7 & \dfrac{s}{1\cdot 2\cdots 5} & = 21x^2 - 7\cdot 15x + 1\cdot 140 = 21\left(xx - 5x + \dfrac{40}{6}\right) \\
m=6,\quad n=8 & \dfrac{s}{1\cdot 2\cdots 6} & = 28x^2 - 8\cdot 21x + 1\cdot 266 = 28\left(xx - 6x + \dfrac{57}{6}\right) \\
 & & \qquad\qquad \text{etc.,}
\end{array}
$$

which forms can be represented as follows:

| if it was | it will be |
|---|---|
| $m = 1, \quad n = 3$ | $\dfrac{s}{1} = \dfrac{2 \cdot 3}{1 \cdot 2}\left(xx - x + \dfrac{1 \cdot 4}{12}\right)$ |
| $m = 2, \quad n = 4$ | $\dfrac{s}{1 \cdot 2} = \dfrac{3 \cdot 4}{1 \cdot 2}\left(xx - 2x + \dfrac{2 \cdot 7}{12}\right)$ |
| $m = 3, \quad n = 5$ | $\dfrac{s}{1 \cdot 2 \cdot 3} = \dfrac{4 \cdot 5}{1 \cdot 2}\left(xx - 3x + \dfrac{3 \cdot 10}{12}\right)$ |
| $m = 4, \quad n = 6$ | $\dfrac{s}{1 \cdot 2 \cdot 3 \cdot 4} = \dfrac{5 \cdot 6}{1 \cdot 2}\left(xx - 4x + \dfrac{4 \cdot 13}{12}\right)$ |
| $m = 5, \quad n = 7$ | $\dfrac{s}{1 \cdot 2 \cdots 5} = \dfrac{6 \cdot 7}{1 \cdot 2}\left(xx - 5x + \dfrac{5 \cdot 16}{12}\right)$ |
| $m = 6, \quad n = 8$ | $\dfrac{s}{1 \cdot 2 \cdots 6} = \dfrac{7 \cdot 8}{1 \cdot 2}\left(xx - 6x + \dfrac{6 \cdot 19}{12}\right)$, |

whence it manifestly follows, if in general $n = m + 2$, that it will be

$$\frac{s}{1 \cdot 2 \cdots m} = \frac{m+1}{1} \cdot \frac{m+2}{2}\left(xx - mx + \frac{m(3m+1)}{12}\right)$$

or

$$\frac{s}{1 \cdot 2 \cdots m} = \frac{m+1}{1} \cdot \frac{m+2}{2}\left(\left(x - \frac{m}{2}\right)^2 + \frac{m}{12}\right).$$

Therefore, hence one obtains this summation

$$x^{m+2} - m(x-1)^{m+2} + \frac{m(m-1)}{1 \cdot 2}(x-2)^{m+2} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(x-3)^{m+2} + \text{etc.}$$
$$= 1 \cdot 2 \cdot 3 \cdots (m+2)\left(\frac{1}{2}\left(x - \frac{m}{2}\right)^2 + \frac{m}{24}\right).$$

## Expansion of the Cases $n = m + 3$

**VII.** For these cases we will have

| if it was | these equations |
|---|---|
| $m=1, \quad n=4$ | $\dfrac{s}{1} = 4x^3 - 6\cdot 1x^2 + 4\cdot 1x - 1\cdot 1$ |
| $m=2, \quad n=5$ | $\dfrac{s}{1\cdot 2} = 10x^3 - 10\cdot 3x^2 + 5\cdot 7x - 1\cdot 15$ |
| $m=3, \quad n=6$ | $\dfrac{s}{1\cdot 2\cdot 3} = 20x^3 - 15\cdot 6x^2 + 6\cdot 25x - 1\cdot 90$ |
| $m=4, \quad n=7$ | $\dfrac{s}{1\cdot 2\cdots 4} = 35x^3 - 21\cdot 10x^2 + 7\cdot 65x - 1\cdot 350$ |
| $m=5, \quad n=8$ | $\dfrac{s}{1\cdot 2\cdots 5} = 56x^3 - 28\cdot 15x^2 + 8\cdot 140x - 1\cdot 1050$ |

which can be represented this way:

$$\begin{aligned}
\frac{s}{1} &= \frac{2\cdot 3\cdot 4}{1\cdot 2\cdot 3}\left(x^3 - \frac{3}{2}x^2 + \frac{1\cdot 4}{4}x - \frac{1\cdot 1\cdot 2}{8}\right)\\
\frac{s}{1\cdot 2} &= \frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3}\left(x^3 - \frac{6}{2}x^2 + \frac{2\cdot 7}{4}x - \frac{2\cdot 2\cdot 3}{8}\right)\\
\frac{s}{1\cdot 2\cdot 3} &= \frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3}\left(x^3 - \frac{9}{2}x^2 + \frac{3\cdot 10}{4}x - \frac{3\cdot 3\cdot 4}{8}\right)\\
\frac{s}{1\cdot 2\cdots 4} &= \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}\left(x^3 - \frac{12}{2}x^2 + \frac{4\cdot 13}{4}x - \frac{4\cdot 4\cdot 5}{8}\right)\\
\frac{s}{1\cdot 2\cdots 5} &= \frac{6\cdot 7\cdot 8}{1\cdot 2\cdot 3}\left(x^3 - \frac{15}{2}x^2 + \frac{5\cdot 16}{4}x - \frac{5\cdot 5\cdot 6}{8}\right)
\end{aligned}$$

etc.,

whence in general for the case $n = m+3$ one concludes

$$\frac{s}{1\cdot 2\cdots m} = \frac{m+1}{1}\cdot\frac{m+2}{2}\cdot\frac{m+3}{3}\left(x^3 - \frac{3m}{2}x^2 + \frac{m(3m+1)}{4}x - \frac{mm(m+1)}{8}\right)$$
$$= \frac{m+1}{1}\cdot\frac{m+2}{2}\cdot\frac{m+3}{3}\left(\left(x-\frac{m}{2}\right)^2 + \frac{m}{4}\left(x-\frac{m}{2}\right)\right),$$

such that we obtain:

$$x^{m+3} - m(x-1)^{m+3} + \frac{m(m-1)}{1\cdot 2}(x-2)^{m+3} - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(x-3)^{m+3} + \text{etc.}$$
$$= 1\cdot 2\cdot 3\cdots(m+3)\left(\frac{1}{6}\left(x-\frac{m}{2}\right)^3 + \frac{m}{24}\left(x-\frac{m}{2}\right)\right).$$

## Preparation for the following Cases

**VIII.** Although in paragraph IV. we gave the formulas only up to the case $m = 6$, let us try to find a general formula for them. To this end let us set $n = m + \lambda$, and to abbreviate such an expression

$$\frac{k(k-1)(k-2)(k-3)\cdots(k-i+1)}{1\cdot 2\cdot 3\cdot 4\cdots i}$$

let us write

$$\left(\frac{k}{i}\right),$$

such that $k$ denotes the first factor of the numerator, $i$ on the other hand the last factor of the denominator. Therefore, let us put that for the case

$$\begin{array}{c|l}
m-1 & \frac{s}{1\cdot 2\cdots(m-1)} = \left(\frac{m+\lambda}{m-1}\right)x^{\lambda+1} - A\left(\frac{m+\lambda}{m}\right)x^{\lambda} + B\left(\frac{m+\lambda}{m+1}\right)x^{\lambda-1} - C\left(\frac{m+\lambda}{m+2}\right)x^{\lambda-2} + \text{etc.} \\
m & \frac{s}{1\cdot 2\cdot 3\cdots m} = \left(\frac{m+\lambda}{m}\right)x^{\lambda} - A^1\left(\frac{m+\lambda}{m+1}\right)x^{\lambda-1} + B^1\left(\frac{m+\lambda}{m+2}\right)x^{\lambda-2} - C^1\left(\frac{m+\lambda}{m+3}\right)x^{\lambda-3} + \text{etc.}
\end{array}$$

such that $A^1$, $B^1$, $C^1$, $D^1$ etc. are the coefficients which must be investigated. But from the law of these formulas we see that

$$A^1 = m\cdot 1 + A,\quad B^1 = mA^1 + B,\quad C^1 = mB^1 + C,\quad D^1 = mC^1 + D \quad \text{etc.},$$

where it is evident that

$$A = \frac{m(m-1)}{1 \cdot 2} \quad \text{and} \quad A^1 = \frac{(m+1)m}{1 \cdot 2}$$

or in our notation

$$A = \left(\frac{m}{2}\right) \quad \text{and} \quad A^1 = \left(\frac{m+1}{2}\right).$$

Now for the following operations I observe that:

$$\left(\frac{m+\mu+1}{\nu}\right) - \left(\frac{m+\mu}{\nu}\right) = \left(\frac{m+\mu}{\nu-1}\right),$$

which is clear, since by expanding:

$$\left(\frac{m+\mu+1}{\nu}\right) = \frac{(m+\mu+1)(m+\mu)(m+\mu-1)\cdots(m+\mu+2-\nu)}{1 \cdot 2 \cdot 3 \cdots \nu}$$

$$\left(\frac{m+\mu}{\nu}\right) = \frac{(m+\mu)(m+\mu-1)\cdots(m+\mu+2-\nu)(m+\mu+1-\nu)}{1 \cdot 2 \cdots (\nu-1)\nu},$$

whence it is seen that

$$\left(\frac{m+1}{2}\right) - \left(\frac{m}{2}\right) = \left(\frac{m}{1}\right) = m.$$

**IX.** Now for it to be

$$B^1 - B = mA^1 = \left(\frac{m+1}{2}\right) m = 3\left(\frac{m+1}{3}\right) + \left(\frac{m+1}{2}\right),$$

let us set

$$B = \alpha\left(\frac{m+1}{4}\right) + \beta\left(\frac{m+1}{3}\right)$$

and hence

$$B^1 = \alpha\left(\frac{m+2}{4}\right) + \beta\left(\frac{m+2}{3}\right)$$

and it will result

$$B^1 - B = \alpha \left(\frac{m+1}{3}\right) + \beta \left(\frac{m+1}{2}\right),$$

whence $\alpha = 3$ and $\beta = 1$, such that

$$B^1 = 3\left(\frac{m+2}{4}\right) + \left(\frac{m+2}{3}\right).$$

But for the following operations note that in general:

$$\left(\frac{m+\mu}{\nu}\right) m = (\nu+1)\left(\frac{m+\mu}{\nu+1}\right) + (\nu-\mu)\left(\frac{m+\mu}{\nu}\right),$$

which form results, if the value of $\left(\frac{m+\mu}{\nu}\right)$ expanded above is multiplied by

$$m = m+\mu-\nu-\nu-\mu = (\nu+1)\cdot\frac{m+\mu-\nu}{\nu+1} + (\nu-\mu).$$

**X.** Since, having observed these things, it must be $C^1 - C = mB^1$, because of

$$\left(\frac{m+2}{4}\right) m = 5\left(\frac{m+2}{5}\right) + 2\left(\frac{m+2}{4}\right)$$

and

$$\left(\frac{m+2}{3}\right) = 4\left(\frac{m+2}{4}\right) + 1\left(\frac{m+2}{3}\right),$$

it will be

$$mB^1 = 15\left(\frac{m+2}{5}\right) + 10\left(\frac{m+2}{4}\right) + 1\left(\frac{m+2}{3}\right);$$

therefore, set

$$C = 15\left(\frac{m+2}{6}\right) + 10\left(\frac{m+2}{5}\right) + 1\left(\frac{m+2}{4}\right)$$

hence

$$C^1 = 15\left(\frac{m+3}{6}\right) + 10\left(\frac{m+3}{5}\right) + 1\left(\frac{m+3}{4}\right).$$

**XI.** Since in like manner it has to be $D^1 - D = mC^1$, since

$$m\left(\frac{m+3}{6}\right) = 7\left(\frac{m+3}{7}\right) + 3\left(\frac{m+3}{6}\right)$$
$$m\left(\frac{m+3}{5}\right) = 6\left(\frac{m+3}{6}\right) + 2\left(\frac{m+3}{5}\right)$$
$$m\left(\frac{m+3}{4}\right) = 5\left(\frac{m+3}{5}\right) + 1\left(\frac{m+3}{4}\right),$$

it will be

$$mC^1 = 105\left(\frac{m+3}{7}\right) + 105\left(\frac{m+3}{6}\right) + 25\left(\frac{m+3}{5}\right) + \left(\frac{m+3}{4}\right),$$

whence we conclude:

$$D^1 = 105\left(\frac{m+4}{8}\right) + 105\left(\frac{m+4}{7}\right) + 25\left(\frac{m+4}{6}\right) + 1\left(\frac{m+4}{5}\right).$$

**XII.** Further, because of $E^1 - E = mD^1$, since

$$m\left(\frac{m+4}{8}\right) = 9\left(\frac{m+4}{9}\right) + 4\left(\frac{m+4}{8}\right)$$
$$m\left(\frac{m+4}{7}\right) = 8\left(\frac{m+4}{8}\right) + 3\left(\frac{m+4}{7}\right)$$
$$m\left(\frac{m+4}{6}\right) = 7\left(\frac{m+4}{7}\right) + 2\left(\frac{m+4}{6}\right)$$
$$m\left(\frac{m+4}{5}\right) = 6\left(\frac{m+4}{6}\right) + 1\left(\frac{m+4}{5}\right),$$

we conclude

$$mD^1 = 945\left(\frac{m+4}{9}\right) + 1260\left(\frac{m+4}{8}\right) + 490\left(\frac{m+4}{7}\right) + 56\left(\frac{m+4}{6}\right) + 1\left(\frac{m+4}{5}\right)$$

and hence

$$E^1 = 945\binom{m+5}{10} + 1260\binom{m+5}{9} + 490\binom{m+5}{8} + 56\binom{m+5}{7} + 1\binom{m+5}{6}$$

and, proceeding even further,

$$F^1 = 10395\binom{m+6}{12} + 17325\binom{m+6}{11} + 9450\binom{m+6}{10} + 1918\binom{m+6}{9} + 119\binom{m+6}{8} + \binom{m+6}{7}.$$

## Expansion of the case $n = m + \lambda$

**XIII.** Therefore, for our series in the case $n = m + \lambda$

$$s = x^{m+\lambda} - \frac{m}{1}(x-1)^{m+\lambda} + \frac{m(m-1)}{1\cdot 2}(x-2)^{m+\lambda} - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(x-3)^{m+\lambda} + \text{etc.},$$

if we divide the general equation exhibited above in paragraph VIII by

$$\binom{m+\lambda}{m} = \frac{(m+\lambda)(m+\lambda-1)(m+\lambda-2)\cdots(\lambda+1)}{1\cdot 2\cdot 3\cdots m},$$

we will get to this expression

$$\frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)} = x^{\lambda} - \frac{\lambda}{m+1}A^1x^{\lambda-1} + \frac{\lambda(\lambda-1)}{(m+1)(m+2)}B^1x^{\lambda-2} - \frac{\lambda(\lambda-1)(\lambda-2)}{(m+1)(m+2)(m+3)}C^1x^{\lambda-3} + \text{etc.}$$

where one has to substitute the following values for the letters $A^1$, $B^1$, $C^1$, $D^1$ etc.:

$$
\begin{aligned}
A^1 &= \binom{m+1}{2} = \frac{(m+1)m}{1\cdot 2} \\
B^1 &= 3\binom{m+2}{4} + \binom{m+2}{3} \\
C^1 &= 15\binom{m+3}{6} + 10\binom{m+3}{5} + \binom{m+3}{4} \\
D^1 &= 105\binom{m+4}{8} + 15\binom{m+4}{7} + 25\binom{m+3}{4} + \binom{m+4}{5} \\
E^1 &= 945\binom{m+5}{10} + 1260\binom{m+5}{9} + 490\binom{m+5}{8} + 56\binom{m+5}{7} + \binom{m+5}{6} \\
F^1 &= 10395\binom{m+6}{12} + 17325\binom{m+6}{11} + 9450\binom{m+6}{10} + 1918\binom{m+6}{9} + 119\binom{m+6}{7} + \binom{m+6}{7}
\end{aligned}
$$

where

$$
\begin{aligned}
10395 = 11\cdot 945, \quad 17325 &= 10\cdot 1260 + 5\cdot 945 \\
9450 &= 9\cdot 490 + 4\cdot 1260 \\
1918 &= 8\cdot 56 + 3\cdot 490 \\
119 &= 7\cdot 1 + 2\cdot 56 \\
1 &= 6\cdot 0 + 1\cdot 1
\end{aligned}
$$

hence, if for the following value one sets

$$
G^1 = \alpha\binom{m+7}{14} + \beta\binom{m+7}{13} + \gamma\binom{m+7}{12} + \delta\binom{m+7}{11} + \varepsilon\binom{m+7}{10} + \zeta\binom{m+7}{9} + \eta\binom{m+7}{8},
$$

these coefficients will be determined as follows:

$$
\begin{array}{l|l}
\alpha = 13\cdot 10395 & \varepsilon = 9\cdot 119 + 3\cdot 1918 \\
\beta = 12\cdot 17325 + 6\cdot 10395 & \zeta = 8\cdot 1 + 2\cdot 119 \\
\gamma = 11\cdot 9450 + 5\cdot 17325 & \eta = 7\cdot 0 + 1\cdot 1 \\
\delta = 10\cdot 1918 + 4\cdot 9450 &
\end{array}
$$

**XIV.** But the same values are expressed more conveniently this way:

$$A^1 = \left(\frac{m+1}{2}\right) \cdot 1$$

$$B^1 = \left(\frac{m+2}{3}\right)\left(1 + \quad 3 \cdot \frac{m-1}{4}\right)$$

$$C^1 = \left(\frac{m+3}{4}\right)\left(1 + \ 10 \cdot \frac{m-1}{5} + \quad 15 \cdot \frac{m-1}{5} \cdot \frac{m-2}{6}\right)$$

$$D^1 = \left(\frac{m+4}{5}\right)\left(1 + \ 25 \cdot \frac{m-1}{6} + \ 105 \cdot \frac{m-1}{6} \cdot \frac{m-2}{7} + 105 \cdot \frac{m-1}{6} \cdot \frac{m-2}{7} \cdot \frac{m-3}{8}\right)$$

$$E^1 = \left(\frac{m+5}{6}\right)\left(1 + \ 56 \cdot \frac{m-1}{7} + \ 490 \cdot \frac{m-1}{7} \cdot \frac{m-2}{8} + 1260 \cdot \frac{m-1}{7} \cdot \frac{m-2}{8} \cdot \frac{m-3}{9} \right.$$
$$\left. + 945 \cdot \frac{m-1}{7} \cdot \frac{m-2}{8} \cdot \frac{m-3}{9} \cdot \frac{m-4}{10}\right)$$

$$F^1 = \left(\frac{m+6}{7}\right)\left(1 + 119 \cdot \frac{m-1}{8} + 1918 \cdot \frac{m-1}{8} \cdot \frac{m-2}{9} + 9450 \cdot \frac{m-1}{8} \cdot \frac{m-2}{9} \cdot \frac{m-3}{10} \right.$$
$$+ 17325 \cdot \frac{m-1}{8} \cdot \frac{m-2}{9} \cdot \frac{m-3}{10} \cdot \frac{m-4}{11}$$
$$\left. + 10395 \cdot \frac{m-1}{8} \cdot \frac{m-2}{9} \cdot \frac{m-3}{10} \cdot \frac{m-4}{11} \cdot \frac{m-5}{12}\right)$$

to see the law of which progression more easily, let us in general put

$$M^1 = \left(\frac{m+\mu-1}{\mu}\right)\left(1 + \alpha \cdot \frac{m-1}{\mu+1} + \beta \cdot \frac{m-1}{\mu+1} \cdot \frac{m-2}{\mu+2} + \gamma \cdot \frac{m-1}{\mu+1} \cdot \frac{m-2}{\mu+2} \cdot \frac{m-3}{\mu+3} + \text{etc.}\right)$$

and the following one

$$N^1 = \left(\frac{m+\mu}{\mu+1}\right)\left(1 + \alpha^1 \cdot \frac{m-1}{\mu+2} + \beta^1 \frac{m-1}{\mu+2} \cdot \frac{m-2}{\mu+3} + \gamma^1 \cdot \frac{m-1}{\mu+2} \cdot \frac{m-2}{\mu+3} \cdot \frac{m-3}{\mu+4} + \text{etc.}\right),$$

and these coefficients are determined as follows by the preceding ones:

$$\begin{aligned}
\alpha^1 &= 2\alpha + \ \mu + 1 \\
\beta^1 &= 3\beta + (\mu+2)\alpha \\
\gamma^1 &= 4\gamma + (\mu+3)\beta \\
\delta^1 &= 5\delta + (\mu+4)\gamma \\
\varepsilon^1 &= 6\varepsilon + (\mu+5)\delta,
\end{aligned}$$

whence these formulas can easily be continued arbitrarily far.

**XV.** Let us now substitute these values, and for the sum $s$ of the propounded series, if $n = m + \lambda$, we will obtain the following expression:

$$\frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)}$$

$$= x^{\lambda} - \frac{\lambda m}{1\cdot 2}x^{\lambda-1} + \frac{\lambda(\lambda-1)m}{1\cdot 2\cdot 3}x^{\lambda-2}\left(1+\frac{3(m-1)}{4}\right)$$

$$- \frac{\lambda(\lambda-1)(\lambda-2)m}{1\cdot 2\cdot 3\cdot 4}x^{\lambda-3}\left(1+10\cdot\frac{m-1}{5}+15\cdot\frac{m-1}{5}\cdot\frac{m-2}{6}\right)$$

$$+ \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)m}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^{\lambda-4}\left(1+25\cdot\frac{m-1}{6}+105\cdot\frac{m-1}{6}\cdot\frac{m-2}{7}+105\cdot\frac{m-1}{6}\cdot\frac{m-2}{7}\cdot\frac{m-3}{8}\right)$$

$$- \frac{\lambda\cdots(\lambda-4)m}{1\cdots 5\cdot 6}x^{\lambda-5}\left(1+56\cdot\frac{m-1}{7}+490\cdot\frac{m-1}{7}\cdot\frac{m-2}{8}+1260\cdot\frac{m-1}{7}\cdots\frac{m-3}{9}\right.$$

$$\left.+\,945\cdot\frac{m-1}{7}\cdots\frac{m-4}{10}\right)$$

$$+ \frac{\lambda\cdots(\lambda-5)m}{1\cdots 6\cdot 7}x^{\lambda-6}\left(1+119\cdot\frac{m-1}{8}+1918\cdot\frac{m-1}{8}\cdot\frac{m-2}{9}+9450\cdot\frac{m-1}{8}\cdots\frac{m-3}{10}\right.$$

$$\left.+\,17325\cdot\frac{m-1}{8}\cdots\frac{m-4}{8}+10395\cdot\frac{m-1}{8}\cdots\frac{m-5}{12}\right)$$

etc.

hence first subtract the power

$$\left(x-\frac{m}{2}\right)^{\lambda} = x^{\lambda} - \frac{\lambda m}{2}x^{\lambda-1} + \frac{\lambda(\lambda-1)m^2}{1\cdot 2\cdot 4}x^{\lambda-2} - \frac{\lambda(\lambda-1)(\lambda-2)m^3}{1\cdot 2\cdot 3\cdot 8}x^{\lambda}$$
$$+ \frac{\lambda\cdots(\lambda-3)m^4}{1\cdots 4\cdot 16}x^{\lambda-4} - \frac{\lambda\cdots(\lambda-4)m^5}{1\cdots 5\cdot 32}x^{\lambda-5} + \frac{\lambda\cdots(\lambda-5)m^6}{1\cdots 6\cdot 64}x^{\lambda-6} - \text{etc.}$$

But here it conveniently happens that

$$\frac{15}{5\cdot 6} = \frac{4}{8}, \quad \frac{105}{6\cdot 7\cdot 8} = \frac{5}{16}, \quad \frac{945}{7\cdot 8\cdot 9\cdot 10} = \frac{6}{32}, \quad \frac{10395}{8\cdot 9\cdot 10\cdot 11\cdot 12} = \frac{7}{64},$$

the reason for which is obvious; therefore, the above expanded expression takes on the following form:

$$\left(x-\frac{m}{2}\right)^{\lambda}+\frac{\lambda(\lambda-1)m}{1\cdot 2\cdot 3}x^{\lambda-2}\cdot\frac{1}{4}-\frac{\lambda(\lambda-1)(\lambda-2)m}{1\cdot 2\cdot 3\cdot 4}x^{\lambda-3}\cdot\frac{m}{2}$$
$$+\frac{\lambda\cdots(\lambda-3)m}{1\cdots 4\cdot 5}x^{\lambda-4}\left(\frac{5}{8}m^2+\frac{5}{48}m-\frac{1}{24}\right)$$
$$-\frac{\lambda\cdots(\lambda-4)m}{1\cdots 5\cdot 6}x^{\lambda-5}\left(\frac{5}{8}m^3+\frac{5}{16}m^2-\frac{1}{8}m\right)$$
$$+\frac{\lambda\cdots(\lambda-5)m}{1\cdots 6\cdot 7}x^{\lambda-6}\left(\frac{35}{64}m^4+\frac{35}{64}m^3-\frac{91}{576}m^2-\frac{7}{96}+\frac{1}{30}\right)-\text{etc.}$$

**XVI.** The power of $x-\frac{m}{2}$, more precisely

$$\frac{\lambda(\lambda-1)m}{2\cdot 3\cdots 4}\left(x-\frac{m}{2}\right)^{\lambda-2}$$

is again detected to be contained in this expression; having separated this power, our expression will be:

$$\left(x-\frac{m}{2}\right)^{\lambda}+\frac{\lambda(\lambda-1)m}{2\cdot 3\cdot 4}\left(x-\frac{m}{2}\right)^{\lambda-2}+\frac{\lambda\cdots(\lambda-3)m}{1\cdots 4\cdot 5}x^{\lambda-4}\left(\frac{5}{48}m-\frac{1}{24}\right)$$
$$-\frac{\lambda\cdots(\lambda-4)m}{1\cdots 5\cdot 6}x^{\lambda-5}\left(\frac{5}{16}m^2-\frac{1}{8}m\right)$$
$$+\frac{\lambda\cdots(\lambda-5)m}{1\cdots 6\cdot 7}x^{\lambda-6}\left(\frac{35}{64}m^3-\frac{91}{576}m^2-\frac{7}{96}m+\frac{1}{36}\right)-\text{etc.},$$

which still contains

$$\frac{\lambda\cdots(\lambda-3)m}{1\cdots 4\cdot 5}\left(\frac{5}{48}m-\frac{1}{24}\right)\left(x-\frac{m}{2}\right)^{\lambda-4}$$

furthermore, there still is

$$\frac{\lambda\cdots(\lambda-5)m}{1\cdots 6\cdot 7}x^{\lambda-6}\left(\frac{35}{576}m^2-\frac{7}{96}m+\frac{1}{36}\right),$$

whence, without any doubt, this power additionally enters:

$$+\frac{\lambda\cdots(\lambda-5)m}{1\cdots 6\cdot 7}\cdot\frac{35m^2-42m+16}{576}\left(x-\frac{m}{2}\right)^{\lambda-6}.$$

Therefore, our expression will be of this nature:

$$\frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)} = \left(x-\frac{m}{2}\right)^{\lambda} + \frac{\lambda(\lambda-1)}{1\cdot 2}\cdot\frac{m}{12}\left(x-\frac{m}{2}\right)^{\lambda-2}$$
$$+\frac{\lambda(\lambda-1)\cdots(\lambda-3)}{1\cdot 2\cdots 4}\cdot\frac{m(5m-2)}{240}\left(x-\frac{m}{2}\right)^{\lambda-4} + \frac{\lambda(\lambda-1)\cdots(\lambda-5)}{1\cdot 2\cdots 6}\cdot\frac{m(35m^2-42m+16)}{4032}\left(x-\frac{m}{2}\right)^{\lambda-6} + \text{etc.}$$

**XVII.** Therefore, lo and behold the extraordinary transformation of our propounded general series:

$$s = x^{m+\lambda} - \frac{m}{1}(x-1)^{m+\lambda} + \frac{m(m-1)}{1\cdot 2}(x-2)^{m+\lambda} - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(x-3)^{m+\lambda} + \text{etc.}$$

which, since found in a so long-winded way and using such intricate operations, seems so weird, that a direct investigation will provide us with useful auxiliary tools for analysis: To investigate this transformation more easily, I will represent it this way that:

$$\frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)} = \left(x-\frac{m}{2}\right)^{\lambda} + \frac{\lambda(\lambda-1)}{1\cdot 2}P\left(x-\frac{m}{2}\right)^{\lambda-2}$$
$$+\frac{\lambda(\lambda-1)\cdots(\lambda-3)}{1\cdot 2\cdots 4}Q\left(x-\frac{m}{2}\right)^{\lambda-4} + \frac{\lambda(\lambda-1)\cdots(\lambda-6)}{1\cdot 2\cdots 6}R\left(x-\frac{m}{2}\right)^{\lambda-6}$$
$$+\frac{\lambda(\lambda-1)\cdots(\lambda-7)}{1\cdot 2\cdots 7}S\left(x-\frac{m}{2}\right)^{\lambda-8} + \frac{\lambda(\lambda-1)\cdots(\lambda-9)}{1\cdot 2\cdots 10}T\left(x-\frac{m}{2}\right)^{\lambda-10}$$
$$\text{etc.,}$$

for which expression up to this point I have found:

$$P = \frac{m}{3\cdot 4}$$
$$Q = \frac{m(5m-2)}{5\cdot 6\cdot 8}$$
$$R = \frac{m(35mm-42m+16)}{6\cdot 7\cdot 96}$$
$$S = \frac{m(175m^3-420m^2+404m-144)}{34560}$$

but a method to find the values of these letters more quickly is desired.

**XVIII.** But here it is especially helpful to have noted that our series is transformed into another one which is a power series in $x - \frac{m}{2}$, the exponents being $\lambda$, $\lambda - 2$, $\lambda - 4$ etc. continuously decreasing by two; but then the letters $P$, $Q$, $R$ etc. depend only on the number $m$, such that neither the exponent $\lambda$ nor he quantity $x$ enter it; furthermore, that the prefixed coefficients involve the number $\lambda$ and follow the law of progression resulting from the expansion of the binomial. Having studied this form carefully, it is manifest that the values of the letters $P$, $Q$, $R$, $S$ etc. can be found separately from the propounded series or its transformed counterpart in paragraph XV, whose law of progression likewise is known, if one sets $x = \frac{m}{2}$; for, if one takes $\lambda = 2$,

$$P = \frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)},$$

but, having put $\lambda = 4$,

$$Q = \frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)}$$

but, having put $\lambda = 6$,

$$R = \frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)} \quad \text{etc.}$$

**XIX.** Therefore, if here for

$$\frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)}$$

one substitutes the series found above in paragraph XV. and to this end, for the sake of brevity, one sets:

$$\mathfrak{A} = 1$$

$$\mathfrak{B} = 1 + 3 \cdot \frac{m-1}{4}$$

$$\mathfrak{C} = 1 + 10 \cdot \frac{m-1}{5} + 15 \cdot \frac{m-1}{5} \cdot \frac{m-2}{6}$$

$$\mathfrak{D} = 1 + 25 \cdot \frac{m-1}{6} + 105 \cdot \frac{m-1}{6} \cdot \frac{m-2}{7} + 105 \cdot \frac{m-1}{6} \cdot \frac{m-2}{7} \cdot \frac{m-3}{8}$$

$$\mathfrak{E} = 1 + 56 \cdot \frac{m-1}{7} + 490 \cdot \frac{m-1}{7} \cdot \frac{m-2}{8} + 1260 \cdot \frac{m-1}{7} \cdots \frac{m-3}{9} + 945 \cdot \frac{m-1}{7} \cdots \frac{m-4}{10}$$

$$\mathfrak{F} = 1 + 119 \cdot \frac{m-1}{8} + 1918 \cdot \frac{m-1}{8} \cdot \frac{m-2}{9} + 9450 \cdot \frac{m-1}{8} \cdots \frac{m-3}{10} + 17325 \cdot \frac{m-1}{8} \cdots \frac{m-4}{11}$$
$$+ 10395 \cdot \frac{m-1}{8} \cdots \frac{m-5}{12}$$

etc.

we obtain the following values

$$P = \frac{m^2}{2^2} - 2\mathfrak{A} \cdot \frac{m}{2} \cdot \frac{m}{2} + \mathfrak{B} \cdot \frac{m}{3}$$
$$Q = \frac{m^4}{2^4} - 4\mathfrak{A} \cdot \frac{m}{2} \cdot \frac{m^3}{2^3} + 6\mathfrak{B} \cdot \frac{m}{3} \cdot \frac{m^2}{2^2} - 4\mathfrak{C} \cdot \frac{m}{4} \cdot \frac{m}{2} + \mathfrak{D} \cdot \frac{m}{5}$$
$$R = \frac{m^6}{2^6} - 6\mathfrak{A} \cdot \frac{m}{2} \cdot \frac{m^5}{2^5} + 15\mathfrak{B} \cdot \frac{m}{3} \cdot \frac{m^4}{2^4} - 20\mathfrak{C} \cdot \frac{m}{4} \cdot \frac{m^3}{2^3} + \mathfrak{D} \cdot \frac{m}{5} \cdot \frac{m^2}{2^2} - 6\mathfrak{E} \cdot \frac{m}{6} \cdot \frac{m}{2} + \mathfrak{F} \cdot \frac{m}{7}$$

whence it will be convenient to expand the values of those letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., whence it results:

$$\mathfrak{A} = 1$$

$$\mathfrak{B} = \frac{3}{4}m + \frac{1}{4} = \frac{3}{4}\left(m + \frac{1}{3}\right)$$

$$\mathfrak{C} = \frac{1}{2}m^2 + \frac{1}{2}m = \frac{4}{8}(mm + m)$$

$$\mathfrak{D} = \frac{5}{16}m^3 + \frac{5}{8}m^2 + \frac{5}{48}m - \frac{1}{24} = \frac{5}{16}\left(m^3 + 2m^2 + \frac{1}{3}m - \frac{2}{15}\right)$$

$$\mathfrak{E} = \frac{3}{16}m^4 + \frac{5}{8}m^3 + \frac{5}{16}m^2 - \frac{1}{8}m = \frac{6}{32}\left(m^4 + \frac{10}{3}m^3 + \frac{5}{3}m^2 - \frac{2}{3}m\right)$$

$$\mathfrak{F} = \frac{7}{64}m^5 + \frac{35}{64}m^4 + \frac{35}{64}m^3 - \frac{91}{576}m^2 - \frac{7}{96}m + \frac{1}{36}$$

or

$$\mathfrak{F} = \frac{7}{64}\left(m^5 + 5m^4 + 5m^3 - \frac{13}{9}m^2 - \frac{2}{3}m + \frac{16}{63}\right);$$

but here, aside from the first terms, no structure is seen.

**XX.** But that the transformed series is a power series in $x - \frac{m}{2}$, was based only on induction, but it can be shown to happen necessarily this way. Since the propounded progression ends as it begins, such that the last two terms will be

$$\pm m(x - m + 1)^{m+\lambda} \mp (x - m)^{m+\lambda},$$

where the upper signs hold, if $m$ is an odd number, the lower on the other hand, if $m$ is even, let us assume that $m$ is an even number (for, the same conclusion follows, if it was odd) and put $x - \frac{m}{2} = y$, and it will be

$$2s = +\left(y+\frac{1}{2}m\right)^{m+\lambda} - \frac{m}{1}\left(y+\frac{1}{2}m-1\right)^{m+\lambda} + \frac{m(m-1)}{1\cdot 2}\left(y+\frac{1}{2}m-2\right)^{m+\lambda} - \text{etc.}$$
$$+\left(y-\frac{1}{2}m\right)^{m+\lambda} - \frac{m}{1}\left(y-\frac{1}{2}m-1\right)^{m+\lambda} + \frac{m(m-1)}{1\cdot 2}\left(y-\frac{1}{2}m-2\right)^{m+\lambda} - \text{etc.}$$

and after the expansion into powers of $y = x - \frac{m}{2}$ one finds:

$$s = y^{m+\lambda}\left(1 - \frac{m}{1} + \frac{m(m-1)}{1\cdot 2} - \text{etc.}\right)$$
$$+\left(\frac{m+\lambda}{2}\right) y^{m+\lambda-2}\left(\left(\frac{m}{2}\right)^2 - \frac{m}{1}\left(\frac{m}{2}-1\right)^2 + \frac{m(m-1)}{1\cdot 2}\left(\frac{m}{2}-2\right)^2 - \text{etc.}\right)$$
$$+\left(\frac{m+\lambda}{4}\right) y^{m+\lambda-4}\left(\left(\frac{m}{2}\right)^4 - \frac{m}{1}\left(\frac{m}{2}-1\right)^4 + \frac{m(m-1)}{1\cdot 2}\left(\frac{m}{2}-2\right)^4 - \text{etc.}\right)$$
$$\text{etc.}$$

But all these series vanish until one gets to the one in which the exponents are $m$, and we know its sum to be $= 1\cdot 2\cdot 3\cdots m$; therefore, having omitted those, whose sum becomes zero, we will obtain:

$$s = \left(\frac{m+\lambda}{m}\right) y^{\lambda}\quad\left(\left(\frac{m}{2}\right)^m - \frac{m}{1}\left(\frac{m}{2}-1\right)^m + \frac{m(m-1)}{1\cdot 2}\left(\frac{m}{2}-2\right)^m - \text{etc.}\right)$$
$$+\left(\frac{m+\lambda}{m+2}\right) y^{\lambda-2}\left(\left(\frac{m}{2}\right)^{m+2} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+2} + \frac{m(m-1)}{1\cdot 2}\left(\frac{m}{2}-2\right)^{m+2} - \text{etc.}\right)$$
$$\text{etc.}$$

and thus it is manifest, what I tried to demonstrate, that this series descends in the powers $y^{\lambda}$, $y^{\lambda-2}$, $y^{\lambda-4}$ etc.

**XXI.** Now let us attribute a form to the series similar to that we had in paragraph XVII., and it will be

$$\frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)} = \frac{y^{\lambda}}{1\cdot 2\cdots m}\left(\left(\frac{m}{2}\right)^{m} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m} + \text{etc.}\right)$$
$$+\frac{1\cdot 2y^{\lambda-2}}{1\cdot 2\cdots(m+2)}\cdot\frac{\lambda(\lambda-1)}{1\cdot 2}\left(\left(\frac{m}{2}\right)^{m+2} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+2} + \text{etc.}\right)$$
$$+\frac{1\cdot 2\cdot 3\cdot 4y^{\lambda-4}}{1\cdot 2\cdots(m+4)}\cdot\frac{\lambda(\lambda-1)\cdots(\lambda-3)}{1\cdot 2\cdot 3\cdot 4}\left(\left(\frac{m}{2}\right)^{m+4} - \text{etc.}\right)$$

etc.,

whence the values of the letters $P$, $Q$, $R$ etc. can be determined in a new way as follows:

$$P = \frac{1}{3\cdot 4\cdots(m+2)}\left(\left(\frac{m}{2}\right)^{m+2} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+2} + \text{etc.}\right)$$
$$Q = \frac{1}{5\cdot 6\cdots(m+4)}\left(\left(\frac{m}{2}\right)^{m+4} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+4} + \text{etc.}\right)$$
$$R = \frac{1}{7\cdot 8\cdots(m+6)}\left(\left(\frac{m}{2}\right)^{m+6} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+6} + \text{etc.}\right)$$
$$S = \frac{1}{9\cdot 10\cdots(m+8)}\left(\left(\frac{m}{2}\right)^{m+8} - \frac{m}{1}\left(\frac{m}{2}-1\right)^{m+8} + \text{etc.}\right)$$

etc.

Here certainly the summation of similar series is necessary; since these only involve he number $m$, our investigation is to be considered to be reduced to a simpler case. Furthermore, we realize just now that these letters depend only on the number $m$.

**XXII.** But if here we successively attribute the definite values 1, 2, 3, 4, 5, 6 etc. to the letter $m$, we will hence obtain as many values for the letters $P$, $Q$, $R$, $S$ etc., knowing which one can easily conclude their general forms. Thus, to find the letter $P$, we will have

$$\begin{array}{rcccccccccc}
\text{if } m = & 0, & & 1, & & 2, & & 3, & & 4, \text{ etc.} \\
3 \cdot 2^2 P = & 0, & & 1, & & 2, & & 3, & & 4, \text{ etc.} \\
\text{diff} & & 1, & & 1, & & 1, & & 1, &
\end{array}$$

such that hence $3 \cdot 2^2 P = m$ and $P = \frac{m}{2^2 \cdot 3}$ as before. Further, for the letter $Q$,

$$\begin{array}{rccccccccccccc}
\text{if } m = & 0, & & 1, & & 2, & & 3, & & 4, & & 5, & & 6 \\
2^4 \cdot 3 \cdot 5Q = & 0, & & 3, & & 16, & & 39, & & 72, & & 115, & & 168 \\
\text{diff. I.} & & 3, & & 13, & & 23, & & 33, & & 43, & & 53 & \\
\text{diff. II.} & & & 10, & & 10, & & 10, & & 10, & & 10, & &
\end{array}$$

therefore, it will be

$$2^4 \cdot 3 \cdot 5Q + 3m + 10\frac{m(m-1)}{1 \cdot 2} + m(5m-2)$$

and hence

$$Q = \frac{m(5m-2)}{2^4 \cdot 3 \cdot 5}.$$

In like manner for the letter $R$,

$$\begin{array}{rccccccccccccc}
\text{if } m = & 0, & & 1, & & 2, & & 3, & & 4, & & 5, & & 6 \\
2^6 \cdot 3 \cdot 7R = & 0, & & 3, & & 48, & & 205, & & 544, & & 1135, & & 2048 \\
\text{diff. I.} & & 3, & & 45, & & 157, & & 339, & & 591, & & 913 & \\
\text{diff. II.} & & & 42, & & 112, & & 182, & & 252, & & 322, & & \\
\text{diff. III.} & & & & 70, & & 70, & & 70, & & 70, & & &
\end{array}$$

whence one concludes

$$2^6 \cdot 3 \cdot 7R = 3m + 21m(m-1) + \frac{35}{3}m(m-1)(m-2)$$

and

$$R = \frac{m(35m^2 - 42m + 16)}{2^6 \cdot 3^2 \cdot 7},$$

which same values we obtained above already; therefore, let us apply the same operation to he following letters.

**XXIII.** Therefore, for the letter $S$ we will have:

$$\begin{array}{rccccccccccccc}
\text{if } m = & 0, & & 1, & & 2, & & 3, & & 4, & & 5, & & 6 \\
2^8 \cdot 5 \cdot 9S = & 0, & & 5, & & 256, & & 2013, & & 7936, & & 22085, & & 49920 \\
\text{diff. I.} & & 5, & & 251, & & 1757, & & 5923, & & 14149, & & 27835 & \\
\text{diff. II.} & & & 246, & & 1506, & & 4166, & & 8226, & & 13686, & & \\
\text{diff. III.} & & & & 1260, & & 2660, & & 4060, & & 5460, & & & \\
\text{diff. IV.} & & & & & 1400, & & 1400, & & 1400 & & & &
\end{array}$$

whence

$$2^8 \cdot 5 \cdot 9S = 5m + 123m(m-1) + 210m(m-1)(m-2) + \frac{175}{3}m(m-1)(m-2)(m-3)$$

Now further, for the letter $T$ we will have:

$$\begin{array}{rccccccccccccc}
\text{if } m = & 0, & & 1, & & 2, & & 3, & & 4, & & 5, & & 6 \\
2^{10} \cdot 3 \cdot 11T = & 0, & & 3, & & 512, & & 7665, & & 4680, & & 174255, & & 499968 \\
\text{diff. I.} & & 3, & & 509, & & 7153, & & 38415, & & 128175, & & 325713 & \\
\text{diff. II.} & & & 506, & & 6604, & & 31262, & & 89760, & & 197538, & & \\
\text{diff. III.} & & & & 6138, & & 24618, & & 58498, & & 107778, & & & \\
\text{diff. IV.} & & & & & 18480, & & 33880, & & 49280, & & & & \\
\text{diff. V.} & & & & & & 15400, & & 15400 & & & & &
\end{array}$$

whence

$$\begin{aligned}
2^{10} \cdot 3 \cdot 11T = {} & 3m + 253m(m-1) + 1023m(m-1)(m-2) \\
& + 770m(m-1)(m-2)(m-3) \\
& + \frac{385}{3}m(m-1)(m-2)(m-3)(m-4)
\end{aligned}$$

and

$$T = \frac{m(385m^4 - 1540m^3 + 2684m^2 - 2288m + 768)}{2^{10} \cdot 9 \cdot 11}.$$

**XXIV.** Now let us represent these values in such a way that the law of progression can be explored more easily:

$$\begin{aligned}
P &= \frac{1m}{12} \\
Q &= \frac{1 \cdot 3m}{12^2}\left(m - \frac{2}{5}\right) \\
R &= \frac{1 \cdot 3 \cdot 5m}{12^3}\left(m^2 - \frac{6}{5}m + \frac{16}{35}\right) \\
S &= \frac{1 \cdot 3 \cdot 5 \cdot 7m}{12^4}\left(m^3 - \frac{12}{5}m + \frac{404}{175}m - \frac{144}{175}\right) \\
T &= \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9m}{12^5}\left(m^4 - \frac{20}{5}m^3 + \frac{244}{35}m^2 - \frac{208}{35}m + \frac{768}{385}\right)
\end{aligned}$$

and here in the first and second terms the law of progression is so manifest, that the same can safely be assigned for all following letters, but in the remaining terms one can still not observe any law.

**XXV.** Therefore, to find the value of the letter $V$, let us set

$$V = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11m}{12^6}\left(m^5 - \frac{30}{5}m^4 + \alpha m^3 - \beta m^2 + \gamma m - \delta\right).$$

But from the general form

$$V = \frac{1}{13 \cdot 14 \cdots (m+12)}\left(\left(\frac{m}{2}\right)^{m+12} - \left(\frac{m}{2} - 1\right)^{m+12} + \frac{m(m-1)}{1 \cdot 2}\left(\frac{m}{2} - 2\right)^{m+12} - \text{etc.}\right)$$

we conclude that:

if it will be

$$
\begin{array}{llcl}
m = 1, & V = \dfrac{1}{2^{12}\cdot 13} & = & \dfrac{5\cdot 7\cdot 11}{2^{12}\cdot 3^3}\,(-\ \ 5 + \ \ \alpha - \ \ \beta + \ \gamma - \delta) \\[2ex]
m = 2, & V = \dfrac{1}{7\cdot 13} & = & \dfrac{5\cdot 7\cdot 11}{2^{11}\cdot 3^3}\,(-\ 64 + \ 8\alpha - \ 4\beta + 2\gamma - \delta) \\[2ex]
m = 3, & V = \dfrac{597871}{2^{12}\cdot 5\cdot 7\cdot 13} & = & \dfrac{5\cdot 7\cdot 11}{2^{12}\cdot 3^2}\,(- 243 + 27\alpha - \ 9\beta + 3\gamma - \delta) \\[2ex]
m = 4, & V = \dfrac{5461}{2^{2}\cdot 5\cdot 7\cdot 13} & = & \dfrac{5\cdot 7\cdot 11}{2^{10}\cdot 3^3}\,(- 512 + 64\alpha - 16\beta + 4\gamma - \delta) \\[2ex]
m = 5, & V = \dfrac{5838647}{2^{12}\cdot 7\cdot 13} & = & \dfrac{25\cdot 7\cdot 11}{2^{12}\cdot 3^3}\,(- 625 + 125\alpha - 25\beta + 5\gamma - \delta) \\[2ex]
m = 6, & V = \dfrac{63047}{2^{2}\cdot 3\cdot 7\cdot 13} & = & \dfrac{5\cdot 7\cdot 11}{2^{11}\cdot 3^2}\,(-\ \ 0 + 216\alpha - 26\beta + 6\gamma - \delta)
\end{array}
$$

Therefore, hence let us form the following equations:

$$
\begin{aligned}
\alpha - \beta + \gamma - \delta &= \frac{27}{5\cdot 7\cdot 11\cdot 13} + 5 \\
8\alpha - 4\beta + 2\gamma - \delta &= \frac{27\cdot 2048}{5\cdot 7^2\cdot 11\cdot 13} + 64 \\
27\alpha - 9\beta + 3\gamma - \delta &= \frac{9\cdot 597871}{5^2\cdot 7^2\cdot 11\cdot 13} + 243 \\
64\alpha - 16\beta + 4\gamma - \delta &= \frac{27\cdot 256\cdot 5461}{5^2\cdot 7^2\cdot 11\cdot 13} + 512 \\
125\alpha - 25\beta + 5\gamma - \delta &= \frac{27\cdot 5838647}{5^2\cdot 7^2\cdot 11\cdot 13} + 625 \\
216\alpha - 36\beta + 6\gamma - \delta &= \frac{3\cdot 512\cdot 63047}{5\cdot 7^2\cdot 11\cdot 13} + 0
\end{aligned}
$$

Now the first differences will look as follows:

$$
\begin{aligned}
7\alpha - \phantom{1}3\beta + \gamma &= \frac{27 \cdot 157}{5 \cdot 7^2 \cdot 11} + \phantom{1}59 \\
19\alpha - \phantom{1}5\beta + \gamma &= \frac{9 \cdot 43627}{5^2 \cdot 7^2 \cdot 11} + 179 \\
37\alpha - \phantom{1}7\beta + \gamma &= \frac{9 \cdot 276629}{5^2 \cdot 7^2 \cdot 11} + 269 \\
61\alpha - \phantom{1}9\beta + \gamma &= \frac{27 \cdot 341587}{5^2 \cdot 7^2 \cdot 11} + 113 \\
91\alpha - 11\beta + \gamma &= \frac{3 \cdot 8373269}{5^2 \cdot 7^2 \cdot 11} - 625
\end{aligned}
$$

the second differences, divided by 2, on the other hand give

$$
\begin{aligned}
6\alpha - \beta &= \frac{9 \cdot 268}{5^2 \cdot 7} + \phantom{3}60 \\
9\alpha - \beta &= \frac{9 \cdot 1513}{5^2 \cdot 7} + \phantom{3}45 \\
12\alpha - \beta &= \frac{9 \cdot 4858}{5^2 \cdot 7} - \phantom{3}78 \\
15\alpha - \beta &= \frac{3 \cdot 34409}{5^2 \cdot 7} - 369
\end{aligned}
$$

Finally, the third difference, divided by 3, yield

$$
\alpha = \frac{3 \cdot 249}{5 \cdot 7} - 5 = \frac{3 \cdot 669}{5 \cdot 7} - 41 = \frac{3967}{5 \cdot 7} - 97,
$$

which three equations give the same value

$$
\alpha = \frac{572}{5 \cdot 7} = \frac{4 \cdot 11 \cdot 13}{5 \cdot 7},
$$

from which value now the remaining ones are defined as follows:

$$\beta = \frac{6 \cdot 572}{5 \cdot 7} - \frac{9 \cdot 268}{5^2 \cdot 7} - 60 = \frac{12 \cdot 1229}{5^2 \cdot 7} - 60 = \frac{4248}{175} = \frac{8 \cdot 9 \cdot 59}{175}$$

$$\gamma = 3\beta - 7\alpha + \frac{27 \cdot 157}{5 \cdot 7^2 \cdot 11} + 59 = \frac{255968}{5^2 \cdot 7^2 \cdot 11}$$

$$\delta = \alpha - \beta + \gamma - \frac{27}{5 \cdot 7 \cdot 11 \cdot 13} - 5 = \frac{1061376}{5^2 \cdot 7^2 \cdot 11 \cdot 13}.$$

**XXVI.** Thus, let us list up the values of the letters $P$, $Q$, $R$ etc. found up to this point all at once

$$P = \frac{1m}{12}$$

$$Q = \frac{1 \cdot 3m}{12^2}\left(m - \frac{2}{5}\right)$$

$$R = \frac{1 \cdot 3 \cdot 5m}{12^3}\left(m^2 - \frac{6}{5}m + \frac{16}{35}\right)$$

$$S = \frac{1 \cdot 3 \cdot 5 \cdot 7m}{12^4}\left(m^3 - \frac{12}{5}m^2 + \frac{404}{175}m - \frac{144}{175}\right)$$

$$T = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9m}{12^5}\left(m^4 - \frac{20}{5}m^3 + \frac{244}{35}m^2 - \frac{208}{35}m + \frac{768}{385}\right)$$

$$V = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11m}{12^6}\left(m^5 - \frac{30}{5}m^4 + \frac{572}{35}m^3 - \frac{4248}{175}m^2 + \frac{255968}{13475}m - \frac{1061376}{175175}\right).$$

From the first terms I conclude that here powers occur, having separated which the structure seems that it can be seen more easily:

$$
\begin{aligned}
P &= \frac{1m}{12} \\
Q &= \frac{1\cdot 3m}{12^2}\left(m-\frac{2}{5}\right)^1 \\
R &= \frac{1\cdot 3\cdot 5m}{12^4}\left(\left(m-\frac{3}{5}\right)+\frac{17}{175}\right) \\
S &= \frac{1\cdot 3\cdot 5\cdot 7m}{12^4}\left(\left(m-\frac{4}{5}\right)^3+\frac{4\cdot 17}{175}m-\frac{16\cdot 17}{5\cdot 175}\right) \\
T &= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9m}{12^5}\left(\left(m-\frac{5}{5}\right)^4+\frac{2\cdot 17}{35}m^2-\frac{4\cdot 17}{35}m+\frac{383}{385}\right) \\
V &= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11m}{12^6}\left(\left(m-\frac{6}{5}\right)^5+\frac{4\cdot 17}{35}m^3-\frac{72\cdot 17}{175}m^2+\frac{581296}{5^3\cdot 7^2\cdot 11}m-\frac{78185568}{5^5\cdot 7^2\cdot 11\cdot 13}\right),
\end{aligned}
$$

yes, it even seems that the second terms can be approximately contracted this way that it results:

$$
\begin{aligned}
P &= \frac{1m}{12}\cdot 1 \\
Q &= \frac{1\cdot 3m}{12^2}\left(m-\frac{2}{5}\right) \\
R &= \frac{1\cdot 3\cdot 5m}{12^3}\left(\left(m-\frac{3}{5}\right)^2+\frac{17}{175}\right) \\
S &= \frac{1\cdot 3\cdot 5\cdot 7m}{12^4}\left(\left(m-\frac{4}{5}\right)^3+\frac{4\cdot 17}{175}\left(m-\frac{4}{5}\right)\right) \\
T &= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9m}{12^5}\left(\left(m-\frac{5}{5}\right)^4+\frac{10\cdot 17}{175}\left(m-\frac{5}{5}\right)^2+\frac{9}{385}\right) \\
V &= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11m}{12^6}\left(\left(m-\frac{6}{5}\right)^5+\frac{20\cdot 17}{175}\left(m-\frac{6}{5}\right)^3+\frac{15808}{5^3\cdot 7^2\cdot 11}m-\frac{4672128}{5^5\cdot 7^2\cdot 11\cdot 13}\right).
\end{aligned}
$$

If we had not found the last value, it would seem that all these expressions are reduced to powers of this kind, what we now have to admit not to happen. Therefore, one must investigate the law of these letters from another source.

**XXVIII.** Therefore, let us rather represent each term of these formulas this way:

$$P = \frac{m}{4 \cdot 3}$$

$$Q = \frac{mm}{16 \cdot 3} - \frac{m}{8 \cdot 3 \cdot 5}$$

$$R = \frac{5m^3}{64 \cdot 9} - \frac{mm}{32 \cdot 3} + \frac{m}{4 \cdot 9 \cdot 7}$$

$$S = \frac{5 \cdot 7m^4}{256 \cdot 27} - \frac{7m^3}{64 \cdot 9} + \frac{101m^2}{64 \cdot 27 \cdot 5} - \frac{m}{16 \cdot 3 \cdot 5}$$

$$T = \frac{5 \cdot 7m^5}{1024 \cdot 9} - \frac{5 \cdot 7m^4}{256 \cdot 9} + \frac{61m^3}{256 \cdot 9} - \frac{13m^2}{64 \cdot 9} + \frac{m}{4 \cdot 3 \cdot 1}$$

$$V = \frac{5 \cdot 7 \cdot 11m^6}{4096 \cdot 27} - \frac{5 \cdot 7 \cdot 11m^5}{2048 \cdot 9} + \frac{1573m^4}{1024 \cdot 27} - \frac{649m^3}{512 \cdot 3 \cdot 5} + \frac{7999m^2}{128 \cdot 27 \cdot 5 \cdot 7} - \frac{691m}{8 \cdot 9 \cdot 5 \cdot 7 \cdot 13},$$

where we have already noticed the structure in the first and second terms, but the last terms seemed to have no structure at all, until, having expanded the value of the letter $V$, the number 691 provided us with a criterion that the last terms contain the Bernoulli numbers.
Therefore, let us denote the Bernoulli numbers by the letters $\alpha$, $\beta$, $\gamma$, $\delta$ etc. such that

$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{6}, \quad \gamma = \frac{1}{6}, \quad \delta = \frac{3}{10}, \quad \varepsilon = \frac{5}{6}, \quad \zeta = \frac{691}{210} \quad \text{etc.}$$

and, concerning these, let us note this law of progression:

$$
\begin{aligned}
\alpha &= \frac{1}{2^1} \\
\beta &= \frac{5\cdot 4\alpha}{2^2\cdot 1\cdot 2\cdot 3} - \frac{2}{2^3} \\
\gamma &= \frac{7\cdot 6\beta}{2^2\cdot 1\cdot 2\cdot 3} - \frac{7\cdot 6\cdot 5\cdot 4\alpha}{2^4\cdot 1\cdot 2\cdots 5} + \frac{3}{2^5} \\
\delta &= \frac{9\cdot 8\gamma}{2^2\cdot 1\cdot 2\cdot 3} - \frac{9\cdot 8\cdot 7\cdot 6\cdot\beta}{2^4\cdot 1\cdot 2\cdots 5} + \frac{9\cdot 4\alpha}{2^6\cdot 1\cdots 7} - \frac{4}{2^7} \\
\varepsilon &= \frac{10\cdot 11\delta}{2^2\cdot 1\cdot 2\cdot 3} - \frac{11\cdots 8\gamma}{2^4\cdot 1\cdots 5} + \frac{11\cdots 6\beta}{2^6\cdot 1\cdots 7} - \frac{11\cdots 4\alpha}{2^8\cdot 1\cdots 9} + \frac{5}{2^9} \\
\zeta &= \frac{13\cdot 12\varepsilon}{2^2\cdot 1\cdot 2\cdot 3} - \frac{13\cdots 10\delta}{2^4\cdot 1\cdots 5} + \frac{13\cdots 8\gamma}{2^6\cdot 1\cdots 7} - \frac{13\cdots 6\beta}{2^8\cdot 1\cdots 9} + \frac{13\cdots 4\alpha}{2^{10}\cdot 1\cdot 11} - \frac{6}{2^{11}} \\
&\qquad\qquad \text{etc.}
\end{aligned}
$$

And the last terms of the letters $P$, $Q$, $R$, $S$ etc. can represented in short form as follows

$$\frac{\alpha m}{2\cdot 3}, \quad \frac{\beta m}{4\cdot 5}, \quad \frac{\gamma m}{6\cdot 7}, \quad \frac{\delta m}{8\cdot 9}, \quad \frac{\varepsilon m}{10\cdot 11}, \quad \frac{\zeta m}{12\cdot 13}.$$

**XXIX.** But to investigate how these letters $P$, $Q$, $R$, $S$ etc. proceed, let us subtract a multiple of it from the preceding one so that the first terms are cancelled, and since the letter $O = 1$ precedes those letters, we will have:

$$P - \frac{m}{12} O = 0$$

$$Q - \frac{3m}{12} P = -\frac{\beta m}{4 \cdot 5}$$

$$R - \frac{5m}{12} Q = -\frac{mm}{16 \cdot 9} + \frac{\gamma m}{6 \cdot 7} = -\frac{m}{12} P + \frac{\gamma m}{6 \cdot 7}$$

$$S - \frac{7m}{12} R = -\frac{7m^3}{128 \cdot 9} + \frac{3m^2}{16 \cdot 5} - \frac{\delta m}{8 \cdot 9}$$

$$T - \frac{9m}{12} S = -\frac{7m^4}{128 \cdot 9} + \frac{17m^3}{64 \cdot 3 \cdot 5} - \frac{7m^2}{8 \cdot 9 \cdot 5} + \frac{\varepsilon m}{10 \cdot 11}$$

$$V - \frac{11m}{2} T = -\frac{5 \cdot 7 \cdot 11 m^5}{2048 \cdot 27} + \frac{451 m^4}{512 \cdot 27} - \frac{121 m^3}{2048 \cdot 27 \cdot 5} + \frac{7159 m^2}{128 \cdot 27 \cdot 5 \cdot 7} - \frac{\zeta m}{12 \cdot 13}.$$

Therefore, if we now consider these forms with more attention and, for the sake of brevity, put:

$$\frac{\alpha m}{2 \cdot 3} = \alpha^1, \quad \frac{\beta m}{4 \cdot 5} = \beta^1, \quad \frac{\gamma m}{6 \cdot 7} = \gamma^1, \quad \frac{\delta m}{8 \cdot 9} = \delta^1 \quad \text{etc.,}$$

we will detect the following sufficiently simple law in our letters $P$, $Q$, $R$ etc:

$$
\begin{aligned}
&P - \frac{1}{1}\alpha^1 = 0\\
&Q - \frac{3}{1}\alpha^1 P + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}\beta^1 = 0\\
&R - \frac{5}{1}\alpha^1 Q + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}\beta^1 P - \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\gamma^1 = 0\\
&S - \frac{7}{1}\alpha^1 R + \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}\beta^1 Q - \frac{7\cdot 6\cdots 3}{1\cdot 2\cdots 5}\gamma^1 P + \frac{7\cdot 6\cdots 1}{1\cdot 2\cdots 7}\delta^1 = 0\\
&T - \frac{9}{1}\alpha^1 S + \frac{9\cdots 5}{1\cdots 3}\beta^1 R - \frac{9\cdots 5}{1\cdots 5}\gamma^1 Q + \frac{9\cdots 3}{1\cdots 7}\delta^1 P - \frac{9\cdots 1}{1\cdots 9}\varepsilon^1 = 0\\
&V - \frac{11}{1}\alpha^1 T + \frac{11\cdots 9}{1\cdots 3}\beta^1 S - \frac{11\cdots 7}{1\cdots 5}\gamma^1 R + \frac{11\cdots 5}{1\cdots 7}\delta^1 Q - \frac{11\cdots 3}{1\cdots 1}\varepsilon^1 P + \frac{11\cdots 1}{1\cdots 11}\zeta^1 = 0
\end{aligned}
$$

etc.

But these new letters $\alpha^1$, $\beta^1$, $\gamma^1$, $\delta^1$ etc. from the preceding ones follow this law:

$$
\begin{aligned}
&\alpha^1 - \frac{m}{2^2\cdot 3} = 0\\
&\beta^1 - \frac{3\cdot 2}{2^2\cdot 1\cdot 2\cdot 3}\alpha^1 + \frac{m}{2^4\cdot 5} = 0\\
&\gamma^1 - \frac{5\cdot 4}{2^2\cdot 1\cdot 2\cdot 3}\beta^1 + \frac{5\cdot 4\cdot 3\cdot 2}{2^4\cdot 1\cdots 5}\alpha^1 - \frac{m}{2^6\cdot 7} = 0\\
&\delta^1 - \frac{7\cdot 6}{2^2\cdot 1\cdots 3}\gamma^1 + \frac{7\cdots 4}{2^4\cdot 1\cdots 5}\beta^1 - \frac{7\cdots 2}{2^6\cdot 1\cdots 7}\alpha^1 + \frac{m}{2^8\cdot 9} = 0\\
&\varepsilon^1 - \frac{9\cdot 8}{2^2\cdot 1\cdots 3}\delta^1 + \frac{9\cdots 6}{2^4\cdot 1\cdots 5}\gamma^1 - \frac{9\cdots 4}{2^6\cdot 1\cdots 7}\beta^1 + \frac{9\cdots 2}{2^8 1\cdots 9}\alpha^1 - \frac{m}{2^{10}\cdot 11} = 0
\end{aligned}
$$

Therefore, I now have to believe to have answered the question on that extraordinary series, that I have contemplated, completely, whence I will now present the answer in short form here.

## PROBLEM

Having propounded this indefinite progression:

$$s = x^{m+\lambda} - \frac{m}{1}(x-1)^{m+\lambda} + \frac{m(m-1)}{1\cdot 2}(x-2)^{m+\lambda} - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}(x-3)^{m+\lambda} + \text{etc.}$$

to assign its sum, if $\lambda$ was an arbitrary positive integer.

## SOLUTION

Let the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. denote the Bernoulli numbers such that:

$$\begin{aligned}
\mathfrak{A} &= \frac{1}{2}, \quad \mathfrak{B} = \frac{1}{6}, \quad \mathfrak{C} = \frac{1}{6}, \quad \mathfrak{D} = \frac{3}{10}, \quad \mathfrak{E} = \frac{5}{6}, \\
\mathfrak{F} &= \frac{691}{210}, \quad \mathfrak{G} = \frac{35}{2}, \quad \mathfrak{H} = \frac{3617}{30}, \quad \mathfrak{I} = \frac{43867}{42}, \\
\mathfrak{K} &= \frac{1222277}{110}, \quad \mathfrak{L} = \frac{854513}{6}, \\
\mathfrak{M} &= \frac{1181820455}{546}, \quad \mathfrak{N} = \frac{76977927}{2}, \\
\mathfrak{O} &= \frac{23749461029}{30}, \quad \mathfrak{P} = \frac{8615841276005}{462}, \\
\mathfrak{Q} &= \frac{84802531453387}{170}, \quad \mathfrak{R} = \frac{90219075042845}{6} \\
& \text{etc.}
\end{aligned}$$

I observed that these numbers proceed in such a way that

$$
\begin{aligned}
\mathfrak{A} &= \frac{1}{2}\\
\mathfrak{B} &= \frac{4}{2}\cdot\frac{\mathfrak{A}^2}{3}\\
\mathfrak{C} &= \frac{6}{2}\cdot\frac{2\mathfrak{A}\mathfrak{B}}{3}\\
\mathfrak{D} &= \frac{8}{2}\cdot\frac{2\mathfrak{A}\mathfrak{C}}{3}+\frac{8\cdot7\cdot6}{2\cdot3\cdot4}\cdot\frac{\mathfrak{B}^2}{5}\\
\mathfrak{E} &= \frac{10}{2}\cdot\frac{2\mathfrak{A}\mathfrak{D}}{3}+\frac{10\cdot9\cdot8}{2\cdot3\cdot4}\cdot\frac{2\mathfrak{B}\mathfrak{C}}{5}\\
\mathfrak{F} &= \frac{12}{2}\cdot\frac{2\mathfrak{A}\mathfrak{E}}{3}+\frac{12\cdot11\cdot10}{2\cdot3\cdot4}\cdot\frac{2\mathfrak{B}\mathfrak{D}}{5}+\frac{12\cdot11\cdot10\cdot9\cdot8}{2\cdot3\cdot4\cdot5\cdot6}\cdot\frac{\mathfrak{C}\mathfrak{C}}{7}\\
\mathfrak{G} &= \frac{14}{2}\cdot\frac{2\mathfrak{A}\mathfrak{F}}{3}+\frac{14\cdot13\cdot12}{2\cdot3\cdot4}\cdot\frac{2\mathfrak{B}\mathfrak{E}}{5}+\frac{14\cdot13\cdot12\cdot11\cdot10}{2\cdot3\cdot4\cdot5\cdot6}\cdot\frac{2\mathfrak{C}\mathfrak{D}}{7}\\
&\qquad\text{etc.}
\end{aligned}
$$

Hence now find numbers $P$, $Q$, $R$, $S$ etc. that

$$
\begin{aligned}
P &= \frac{1\mathfrak{A}m}{1\cdot2\cdot3}\\
Q &= \frac{3\mathfrak{A}m}{1\cdot2\cdot3}P-\frac{3\cdot2\cdot1\mathfrak{B}m}{1\cdot2\cdot3\cdot4\cdot5}\\
R &= \frac{5\mathfrak{A}m}{1\cdot2\cdot3}Q-\frac{5\cdot4\cdot3\cdot\mathfrak{B}m}{1\cdot2\cdot3\cdot4\cdot5}P+\frac{5\cdot4\cdots1\mathfrak{C}m}{1\cdot2\cdots7}\\
S &= \frac{7\mathfrak{A}m}{1\cdot2\cdot3}R-\frac{7\cdot6\cdot5\cdot\mathfrak{B}m}{1\cdot2\cdots5}Q+\frac{7\cdots3\mathfrak{C}m}{1\cdot2\cdots7}P-\frac{7\cdots1\mathfrak{D}}{1\cdot2\cdots9}\\
&\qquad\text{etc.}
\end{aligned}
$$

where the law of progression also is perspicuous.
Having found this series, the sum $s$ in question will be expressed this way:

$$\frac{s}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)} = \left(x-\frac{m}{2}\right)^{\lambda} + \frac{\lambda(\lambda-1)}{1\cdot 2} P\left(x-\frac{m}{2}\right)^{\lambda-2}$$
$$+\frac{\lambda\cdots(\lambda-3)}{1\cdots 4} Q\left(x-\frac{m}{2}\right)^{\lambda-4}$$
$$+\frac{\lambda\cdots(\lambda-5)}{1\cdots 6} R\left(x-\frac{m}{2}\right)^{\lambda-6}$$
$$+\frac{\lambda\cdots(\lambda-7)}{1\cdots 8} S\left(x-\frac{m}{2}\right)^{\lambda-8}$$
$$\text{etc.}$$

where one should note, if the number $m$ is not an integer, that the value of the product

$$(\lambda+1)(\lambda+2)\cdots(\lambda+m)$$

can be defined through other artifices explained on another occasion.

## COROLLARY 1

If instead of the Bernoulli numbers we want to introduce the related ones, which I used for the sums of the powers of the reciprocals, and denote them by the letters $A$, $B$, $C$, $D$ etc. that $A=\frac{1}{6}$, $B=\frac{1}{90}$, $C=\frac{1}{945}$, $D=\frac{1}{9450}$, $E=\frac{1}{93555}$, since these numbers depend on the first in such a way that

$$\mathfrak{A}=\frac{1\cdot 2\cdot 3}{2^1}A, \quad \mathfrak{B}=\frac{1\cdots 5}{2^3}B, \quad \mathfrak{C}=\frac{1\cdots 7}{2^5}C \quad \text{etc.,}$$

but are connected to each other in such a way that:

$$5B=2A^2, \quad 7C=4AB, \quad 9D=4AC+2BB,$$
$$11E=4AD+4BC, \quad 13F=4AE+4BD+2CC \quad \text{etc.,}$$

then from these numbers the letters $P$, $Q$, $R$, $S$ etc. will be determined as follows:

$$P = \frac{1Am}{2}$$

$$Q = \frac{3Am}{2}P - \frac{3 \cdot 2 \cdot 1Bm}{2^3}$$

$$R = \frac{5Am}{2}Q - \frac{5 \cdot 4 \cdot 3Bm}{2^3}P + \frac{5 \cdot 4 \cdots 1Cm}{2^5}$$

$$S = \frac{7Am}{2}R - \frac{7 \cdot 6 \cdot 5Bm}{2^3}Q + \frac{7 \cdot 6 \cdots 3Cm}{2^5}P - \frac{7 \cdot 6 \cdots 1Dm}{2^7}$$

etc.

## COROLLARY 2

If for the various values of the number $\lambda$ we indicate the sum of the propounded progression by the sign $\int(\lambda)$ and now for $\lambda$ successively write the numbers 0, 1, 2, 3, 4 etc., for these cases the sums $\int(0)$, $\int(1)$, $\int(2)$, $\int(3)$ etc., for the sake of brevity having put $x - \frac{m}{2} = y$, will be expressed the following way:

$$\frac{\int(0)}{1 \cdot 2 \cdots m} = 1$$

$$\frac{\int(1)}{2 \cdot 3 \cdots (m+1)} = y$$

$$\frac{\int(2)}{3 \cdot 4 \cdots (m+2)} = y^2 + P$$

$$\frac{\int(3)}{4 \cdot 5 \cdots (m+3)} = y^3 + 3Py$$

$$\frac{\int(4)}{5 \cdot 6 \cdots (m+4)} = y^4 + 6Py^2 \quad + Q$$

$$\frac{\int(5)}{6 \cdot 7 \cdots (m+5)} = y^5 + 10Py^3 + 5Qy$$

$$\frac{\int(6)}{7 \cdot 8 \cdots (m+6)} = y^6 + 15Py^4 + 15Qy^2 + R$$

etc.

## COROLLARY 3

Therefore, hence these sums can be defined from the preceding ones as follows

$$\begin{aligned}
\int(1) &= \frac{m+1}{1} y \int(0) \\
\int(2) &= \frac{m+2}{2} y \int(1) + \frac{(m+2)(m+1)}{2\cdot 2} mA \int(0) \\
\int(3) &= \frac{m+3}{3} y \int(2) + \frac{(m+3)(m+2)}{2\cdot 3} mA \int(1) \\
\int(4) &= \frac{m+4}{4} y \int(3) + \frac{(m+4)(m+3)}{2\cdot 4} mA \int(2) - \frac{(m+4)\cdots(m+1)}{2^3\cdot 4} mB \int(0) \\
\int(5) &= \frac{m+5}{5} y \int(4) + \frac{(m+5)(m+4)}{2\cdot 5} mA \int(3) - \frac{(m+5)\cdots(m+2)}{2^3\cdot 5} mB \int(1) \\
\int(6) &= \frac{m+6}{6} y \int(5) + \frac{(m+6)(m+5)}{2\cdot 6} mA \int(4) - \frac{(m+6)\cdots(m+3)}{2^3\cdot 5} mB \int(2) + \frac{(m+6)\cdots(m+1)}{2^5\cdot 6} mC \int(0) \\
\int(7) &= \frac{m+7}{7} y \int(6) + \frac{(m+7)(m+6)}{2\cdot 7} mA \int(5) - \frac{(m+7)\cdots(m+4)}{2^3\cdot 7} mB \int(3) + \frac{(m+7)\cdots(m+2)}{2^5\cdot 7} mC \int(1) \\
\int(8) &= \frac{m+8}{8} y \int(7) + \frac{(m+8)(m+7)}{2\cdot 8} mA \int(6) - \frac{(m+8)\cdots(m+5)}{2^3\cdot 8} mB \int(4) + \frac{(m+8)\cdots(m+3)}{2^5\cdot 8} mC \int(2) \\
&\qquad - \frac{(m+8)\cdots(m+1)}{2^7\cdot 8} mD \int(0),
\end{aligned}$$

which law will become obvious soon to the attentive reader.

## Conclusion

Now there will be not much difficulty to generalize this task quite substantially, so that, if $\varphi : x$ denotes an arbitrary function of $x$, that we can assign the sum of this series

$$s = \varphi : x - m\varphi : (x-1) + \frac{m(m-1)}{1\cdot 2}\varphi : (x-2) - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\varphi : (x-3).$$

For, it is perspicuous that this form exhibits the difference of order $m$ of this progression

$$\varphi : x, \quad \varphi : (x-1), \quad \varphi : (x-2), \quad \varphi : (x-3) \quad \text{etc.}$$

For, from that, what I covered in *Institutiones Calculi Differentialis* pag. 343 [1], if we put $\varphi : x = y$, one concludes that the differences of the respective orders are:

[1] p. 264 in the Opera Onmia Version, i.e. Volume 10 of Series 1

$$\begin{aligned}
\Delta y &= \frac{dy}{dx} - \frac{ddy}{2dx^2} + \frac{d^3y}{2\cdot 3dx^3} - \frac{d^4y}{2\cdot 3\cdot 4dx^4} + \frac{d^5y}{2\cdots 5dx^5} - \text{etc.}\\
\Delta^2 y &= \frac{d^2y}{dx^2} - \frac{3d^3y}{3dx^3} + \frac{7d^4y}{3\cdot 4dx^4} - \frac{15d^5y}{3\cdot 4\cdot 5dx^5} + \frac{31d^6y}{3\cdots 6dx^6} - \text{etc.}\\
\Delta^3 y &= \frac{d^3y}{dx^3} - \frac{6d^4y}{4dx^4} + \frac{25d^5y}{4\cdot 5dx^5} - \frac{90d^6y}{4\cdot 5\cdot 6dx^6} + \frac{301d^7y}{4\cdots 7dx^7} - \text{etc.}\\
\Delta^4 y &= \frac{d^4y}{dx^4} - \frac{10d^5y}{5dx^5} + \frac{65d^6y}{5\cdot 6dx^6} - \frac{350d^7y}{5\cdot 6\cdot 7dx^7} + \frac{1701d^8y}{5\cdots 8dx^8} - \text{etc.}
\end{aligned}$$

etc.,

since which coefficients are those we had above in paragraph IV, in like manner we will understand that the difference of order $m$ or $\Delta^m y$, i.e. the sum of the propounded series, will be

$$s = \frac{d^m y}{dx^m} - \frac{A^1 d^{m+1}y}{(m+1)dx^{m+1}} + \frac{B^1 d^{m+2}y}{(m+1)(m+2)dx^{m+2}} - \frac{C^1 d^{m+3}y}{(m+1)\cdots(m+3)dx^{m+3}} + \text{etc.},$$

which coefficients $A^1$, $B^1$, $C^1$ etc. I determined above in paragraph XIII. Therefore, it will be

$$\begin{aligned}
&\frac{A^1}{m+1} = \frac{m}{2}\\
&\frac{B^1}{(m+1)(m+2)} = \frac{m}{1\cdot 2\cdot 3} + \frac{3m(m-1)}{1\cdot 2\cdot 3\cdot 4}\\
&\frac{C^1}{(m+1)\cdots(m+3)} = \frac{m}{1\cdot 2\cdot 3\cdot 4} + \frac{10m(m-1)}{1\cdot 2\cdots 5} + \frac{15m(m-1)(m-2)}{1\cdot 2\cdots 6}\\
&\frac{D^1}{(m+1)\cdots(m+4)} = \frac{m}{1\cdot 2\cdots 5} + \frac{25m(m-1)}{1\cdot 2\cdots 6} + \frac{105m(m-1)(m-2)}{1\cdot 2\cdots 7} + \frac{105m(m-1)(m-2)(m-3)}{1\cdot 2\cdots 8}
\end{aligned}$$

etc.

Therefore, if we now put $\varphi : \left(x - \frac{m}{2}\right) = v$, so that $v$ results from $v$, if one writes $x - \frac{m}{2}$ instead of $y$, it will obviously be

$$\frac{d^m v}{dx^m} = \frac{d^m y}{dx^m} - \frac{md^{m+1}y}{2dx^{m+1}} + \frac{m^2 d^{m+2}y}{2\cdot 4dx^{m+2}} - \text{etc.},$$

if which equation is subtracted from that, one will have to do exactly the same calculations as above. Hence introducing the same letters $P$, $Q$, $R$, $S$ etc., which we defined above, we will obtain the following value of the sum $s$:

$$s = \frac{d^m v}{dx^m} + \frac{Pd^{m+2}v}{1\cdot 2dx^{m+2}} + \frac{Qd^{m+4}v}{1\cdot 2\cdots 4dx^{m+4}} + \frac{Rd^{m+6}v}{1\cdot 2\cdots 6dx^{m+6}} + \frac{Sd^{m+8}v}{1\cdot 2\cdots 8dx^{m+8}} + \text{etc.}$$

and hence, if one takes

$$y = \varphi : x = x^{m+\lambda} \quad \text{and} \quad v = \left(x - \frac{m}{2}\right)^{m+\lambda},$$

manifestly the same summation we found before results, and thus the whole task reduces to the letters $P$, $Q$, $R$, $S$ etc., whose nature I derived from the Bernoulli numbers above.

Hence it follows immediately , what has been less obvious before, that, if in the function $y$ or $v$ the number of dimensions was smaller than the exponent $m$, which number certainly has to be a positive integer, then all differentials of order $m$ and higher vanish and the sum $s$ will be $= 0$.

Further, hence there is a more clear way to find the values of the letters $P$,$Q$, $R$, $S$ etc. For, since, having put

$$s = \frac{d^m y}{dx^m} - \frac{\alpha d^{m+1}y}{dx^{m+1}} + \frac{\beta d^{m+2}y}{dx^{m+2}} - \frac{\gamma d^{m+3}y}{dx^{m+3}} + \text{etc.},$$

we have

$$\alpha = \frac{m}{1\cdot 2}$$

$$\beta = \frac{m}{1\cdot 2\cdot 3} + \frac{3m(m-1)}{1\cdot 2\cdot 3\cdot 4}$$

$$\gamma = \frac{m}{1\cdots 4} + \frac{10m(m-1)}{1\cdots 5} + \frac{15m\cdots(m-2)}{1\cdots 6}$$

$$\delta = \frac{m}{1\cdots 5} + \frac{25m(m-1)}{1\cdots 6} + \frac{105m\cdots(m-2)}{1\cdots 7} + \frac{105m\cdots(m-3)}{1\cdots 8}$$

etc.,

but the function $y$ results from the function $v = \varphi : \left(x - \frac{m}{2}\right)$, if in it instead of $x$ one writes $x + \frac{m}{2}$, it will be in general

$$\frac{d^n y}{dx^n} = \frac{d^n v}{dx^n} + \frac{m}{2}\cdot\frac{d^{n+1}v}{dx^{n+1}} + \frac{m^2}{2\cdot 4}\cdot\frac{d^{n+2}v}{dx^{n+2}} + \frac{m^3}{2\cdot 4\cdot 6}\cdot\frac{d^{n+3}v}{dx^{n+3}} + \text{etc.},$$

whence, if one substitutes the differentials of $v$ for those of $y$, it will be

$$s = \frac{d^n v}{dx^n} + \left(\frac{m}{2} - \alpha\right)\frac{d^{n+1}v}{dx^{n+1}} + \left(\frac{m^2}{2\cdot 4} - \frac{m}{2}\alpha + \beta\right)\frac{d^{n+2}v}{dx^{n+2}} + \left(\frac{m^3}{2\cdot 4\cdot 6} - \frac{m^2}{2\cdot 4}\alpha + \frac{m}{2}\beta - \gamma\right)\frac{d^{n+3}v}{dx^{n+3}} + \text{etc.}$$

and so we will have:

$$
\begin{array}{l}
\frac{m}{2} - \alpha = 0 \\[2mm]
\frac{m^2}{2\cdot 4} - \frac{m}{2}\alpha + \beta = \frac{P}{1\cdot 2} \\[2mm]
\frac{m^3}{2\cdot 4\cdot 6} - \frac{m^2}{2\cdot 4}\alpha + \frac{m}{2}\beta - \gamma = 0 \\[2mm]
\frac{m^4}{2\cdot 4\cdot 6\cdot 8} - \frac{m^3}{2\cdot 4\cdot 6}\alpha + \frac{m^2}{2\cdot 4}\beta - \frac{m}{2}\gamma + \delta = \frac{Q}{1\cdot 2\cdot 3\cdot 4} \\[2mm]
\frac{m^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{m^4}{2\cdot 4\cdot 6\cdot 8}\alpha + \frac{m^3}{2\cdot 4\cdot 6}\beta - \frac{m^2}{2\cdot 4}\gamma + \frac{m}{2}\delta - \varepsilon = 0 \\[2mm]
\text{etc.;}
\end{array}
$$

for, one easily sees that these expressions must vanish alternately.