On the sums of series of reciprocals*

Leonhard Euler

§1 The series of the reciprocals of the powers of natural numbers have already been treated and investigated that often that it seems hardly probable that anything new can be found about them. For almost everyone, who meditated about the sums of series, also tried to find the sums of series of this kind and has nevertheless not been able to express them by any method in a suitable way. Even I, after I had given various summation methods, persecuted these series diligently and nevertheless have obtained nothing other than that I defined their true sums either approximately or reduced them to quadratures of highly transcendental curves; I did the first in my dissertation red last, but the latter in the preceding ones. But here I talk about the series of fractions, whose numerators are 1, but the denominators on the other hand are either squares or cubes or other powers of the natural numbers; of this kind are

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.},$$

likewise

$$1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\text{etc.}$$

and similar ones of the higher powers, whose general terms are contained in this form $\frac{1}{x^n}$.

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§2 But I was recently unexpectedly led to an elegant expression for the sum of this series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.},$$

which depends on the quadrature of the circle, such that, if one would have the true sum of this series, hence at the same time the quadrature of the circle would follow. For, I found that six times the sum of this series is equal to the square of the circumference of the circle, whose diameter is 1, or having put the sum of this series $= s$, $\sqrt{6s}$ to 1 will have the ratio of the circumference to the diameter. But I recently showed that the sum of this series approximately is

$$1.6449340668482264364;$$

if of the sextuple of this series the square root is taken, indeed the following number will arise

$$3.141592653589793238$$

expressing the circumference of this circle whose diameter is 1. Further, following the same path, on which I was led to this sum, I also detected the sum of this series

$$1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625} + \text{etc.}$$

to depend on the quadrature of the circle. For, its sum multiplied by 90 gives the bisquare of the circumference of the circle, whose diameter is 1. And in similar manner I was also able to determine the sums of the following series, in which the exponents of the powers are even numbers.

§3 Therefore, to show in the most convenient way, how I obtained all this, I want to explain everything I used in order. In the circle (Fig. 1) $AMBNA$ described about the center C with the radius BC or $BC = 1$ I contemplated an arbitrary arc AM, whose sine is MP, but its cosine CP. Now having put the arc $AM = s$, the sine $PM = y$ and the cosine $CP = x$ by means of a sufficiently well-known method one is able to define so the sine y as the cosine x from the given arc s by means of series; for, it is, as one can see in various sources,

$$y = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}$$

and

$$x = 1 - \frac{s^2}{1 \cdot 2} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

From the consideration of these equations I got to the sum of the series of reciprocals mentioned above; both of these equations lead almost to the same scope, and therefore it will be sufficient to have treated only the one in the way, which I will explain here.

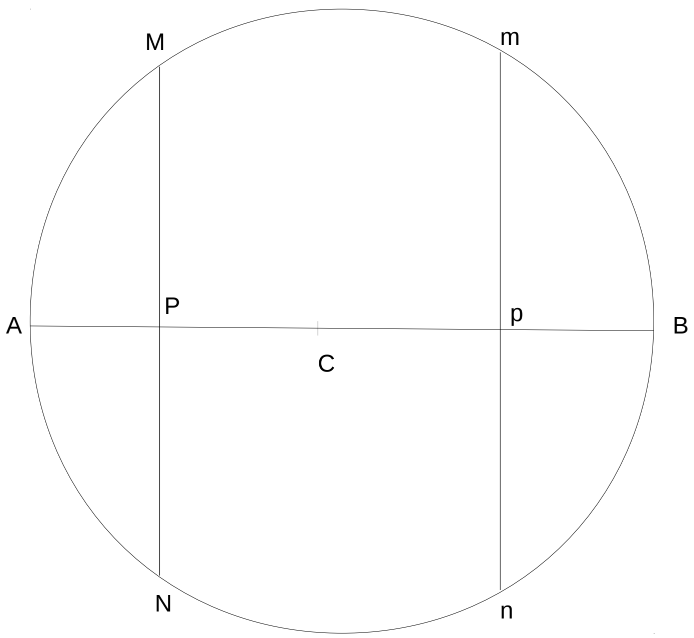

Fig. 1

§4 Therefore, the first equation

$$y = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}$$

expresses the relation between the arc and the sine. Hence so from the given arc its sine as from the given sine its arc can be determined. But I consider the sine y as given and investigate, how the arc s must be found from y. Here, it is especially to be considered that to the same sine y innumerable arcs correspond, which innumerable arcs therefore the propounded equation will have to yield. If in this equation s is considered as an unknown, it has infinite dimensions and hence it is not surprising, if this equation contains innumerable simple factors, of which each set equal to zero must give a suitable value for s.

§5 But how, if all factors of this equation would be known, also all its roots or all values of s would become known, so vice versa, if all values of s can be assigned, then one will have all factors of it. But that I adjudicate better so on the roots as on the factors, I transform the propounded equation into this form

$$0 = 1 - \frac{s}{y} + \frac{s^3}{1 \cdot 2 \cdot 3y} - \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5y} + \text{etc.}$$

If now all roots of this equation or all arcs, whose sine is the same y, were A, B, C, D, E etc., then also the factors will be all these quantities

$$1 - \frac{s}{A}, \quad 1 - \frac{s}{B}, \quad 1 - \frac{s}{C}, \quad 1 - \frac{s}{D} \quad \text{etc.}$$

Therefore, it will be

$$1 - \frac{s}{y} + \frac{s^3}{1 \cdot 2 \cdot 3y^3} - \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5y} + \text{etc.} = \left(1 - \frac{s}{A}\right)\left(1 - \frac{s}{B}\right)\left(1 - \frac{s}{C}\right)\left(1 - \frac{s}{D}\right) \text{etc.}$$

§6 But from the nature and the resolution of equations it is known that the coefficient of the term, in which s is contained, or $\frac{1}{y}$ is equal to the sum of all coefficients of s in the factors or

$$\frac{1}{y} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} + \text{etc.}$$

Further, the coefficient of s^2, which is $= 0$ because of the missing term in the equation, is equal to the aggregate of products of two terms of the series $\frac{1}{A}$, $\frac{1}{B}$, $\frac{1}{C}$, $\frac{1}{D}$ etc. Further, $-\frac{1}{1 \cdot 2 \cdot 3y}$ will be equal to the aggregate of products of three factors of the same series $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{1}{D}$ etc. And in similar manner $0 =$ the aggregate of the products of four terms of the series and $+\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5y} =$ the aggregate of products of five terms of this series, and so forth.

§7 But having put the smallest arc $AM = A$, whose sine is $PM = y$, and having put the half of the circumference of the circle $= p$

$$A, \quad p+A, \quad 2p-A, \quad 3p+A, \quad 4p-A, \quad 5p+A, \quad 6p+A \quad \text{etc}$$

likewise

$$-p-A, \quad -2p+A, \quad -3p-A, \quad -4p+A, \quad -5p-A \quad \text{etc.}$$

will be all the arcs, whose sine is the same y. Therefore, the series we assumed before

$$\frac{1}{A}, \quad \frac{1}{B}, \quad \frac{1}{C}, \quad \frac{1}{D} \quad \text{etc.}$$

is transformed into this one

$$\frac{1}{A}, \frac{1}{p-A}, \frac{1}{-p-A}, \frac{1}{2p+A}, \frac{1}{-2p+A}, \frac{1}{3p-A}, \frac{1}{-3p-A}, \frac{1}{4p+A}, \frac{1}{-4p+A} \text{ etc.}$$

Therefore, the sum of all these terms is $= \frac{1}{y}$, but the sum of the products of two terms of this series is equal to 0, the sum of the products of three is $= \frac{-1}{1\cdot 2\cdot 3y}$, the sum of the products of four is $= 0$, the sum of products of five is $= \frac{+1}{1\cdot 2\cdot 3\cdot 4\cdot 5y}$, the sum of products of six is $= 0$. And so forth.

§8 But if one has an arbitrary series

$$a+b+c+d+e+f+\text{etc.},$$

whose sum shall be α, the sum of products of two terms $= \beta$, the sum of products of three $= \gamma$, the sum of factors of four $= \delta$ etc., the sum of the squares of the single terms will be

$$a^2+b^2+c^2+d^2+\text{etc.} = \alpha^2 - 2\beta,$$

the sum of the cubes on the other hand

$$a^3+b^3+c^3+d^4+\text{etc.} = \alpha^3 - 3\alpha\beta + 3\gamma,$$

the sum of the bisquares

$$= \alpha^4 - 4\alpha^2\beta + 4\alpha\gamma + 2\beta^2 - 4\delta.$$

But that it becomes more clear, how these formulas proceed, let us put that the sum of the terms a, b, c, d etc. is $= P$, the sum of the squares is $= Q$, the sum of the cubes is $= R$, the sum of the bisquares $= S$, the sum of the fifth powers $= T$, the sum of the sixth $= V$ etc. Having put these it will be

$$P = \alpha, \quad Q = P\alpha - 2\beta, \quad R = Q\alpha - P\beta + 3\gamma, \quad S = R\alpha - Q\beta + P\gamma - 4\delta,$$

$$T = S\alpha - R\beta + Q\gamma - P\delta + 5\varepsilon \quad \text{etc.}$$

§9 Therefore, since in our case the sum of all terms of the series

$$\frac{1}{A}, \quad \frac{1}{p-A}, \quad \frac{1}{-p-A}, \quad \frac{1}{2p+A}, \quad \frac{1}{-2p+A}, \quad \frac{1}{3p-A}, \quad \frac{1}{-3p-A} \quad \text{etc.}$$

is α or $= \frac{1}{y}$, the sum of the products of two or $\beta = 0$ and further

$$\gamma = \frac{-1}{1 \cdot 2 \cdot 3y}, \quad \delta = 0, \quad \varepsilon = \frac{+1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5y}, \quad \zeta = 0 \quad \text{etc.,}$$

the sum of those terms itself will be

$$P = \frac{1}{y},$$

the sum of the squares of those terms

$$Q = \frac{P}{y} = \frac{1}{y^2},$$

the sum of the cubes of those terms

$$R = \frac{Q}{y} - \frac{1}{1 \cdot 2y},$$

the sum of the bisquares

$$S = \frac{R}{y} - \frac{P}{1 \cdot 2 \cdot 3y}$$

and further

$$
\begin{aligned}
T &= \frac{S}{y} - \frac{Q}{1\cdot 2\cdot 3y} + \frac{1}{1\cdot 2\cdot 3\cdot 4y},\\
V &= \frac{T}{y} - \frac{R}{1\cdot 2\cdot 3y} + \frac{P}{1\cdot 2\cdot 3\cdot 4\cdot 5y},\\
W &= \frac{V}{y} - \frac{S}{1\cdot 2\cdot 3y} + \frac{Q}{1\cdot 2\cdot 3\cdot 4\cdot 5y} - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}.
\end{aligned}
$$

From this law the sums of the remaining higher powers are easily determined.

§10 Now let us put the sine $PM = y$ equal to the radius, that $y = 1$; the smallest arc A, whose sine is 1, will be the fourth part of the circumference $= \frac{1}{2}p$, or while q denotes the fourth part of the circumference it will be $A = q$ and $p = 2q$. Therefore, the superior series will go over into this one

$$\frac{1}{q}, \quad \frac{1}{q}, \quad -\frac{1}{3q}, \quad -\frac{1}{3q}, \quad +\frac{1}{5q}, \quad +\frac{1}{5q}, \quad -\frac{1}{7q}, \quad -\frac{1}{7q}, \quad +\frac{1}{9q}, \quad +\frac{1}{9q} \quad \text{etc.}$$

while each two terms are equal. Therefore, the sum of these terms, which is

$$\frac{2}{q}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}\right),$$

is equal to $P = 1$. Therefore, hence it arises

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.} = \frac{q}{2} = \frac{p}{4}.$$

Therefore, the quadruple of this series becomes equal to the half of the circumference of the circle, whose radius is 1, or to the circumference of the while circle, whose diameter is 1. And this is the series found first by Leibniz some time ago, by which he defined the quadrature of the circle. From this the general foundation of this method, if to anyone it maybe did not seem to be sufficiently clear, will become evident, such that the remaining ones, which will be derived from this method, cannot be in any doubt.

§11 Now let us take the squares of the found terms for the case, in which it is $y = 1$, and this series will arise

$$+\frac{1}{q^2} + \frac{1}{q^2} + \frac{1}{9q^2} + \frac{1}{9q^2} + \frac{1}{25q^2} + \frac{1}{25q^2} + \text{etc.},$$

whose sum is

$$\frac{2}{q^2}\left(\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.}\right),$$

which therefore must be equal to $Q = P = 1$. Hence it follows that the sum of this series

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.}$$

is $= \frac{q^2}{2} = \frac{p^2}{8}$, while p denotes the whole circumference of the circle, whose diameter is $= 1$. But the sum of this series

$$1 + \frac{1}{9} + \frac{1}{25} + \text{etc.}$$

depends on the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.},$$

since this diminished by its further part gives the first series. Therefore, the sum of this series is equal to the sum of that series together with is third part. Therefore, it will be

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \text{etc.} = \frac{p^2}{6},$$

and hence the sum of this series multiplied by 6 is equal to the square of the circumference of the circle, whose diameter is 1; this is the proposition itself I mentioned at the beginning.

§12 Since therefore in the case, in which it is $y = 1$, it is $P = 1$ and $Q = 1$, the values of remaining letters R, S, T, V etc. will be as follows

$$R = \frac{1}{2}, \quad S = \frac{1}{3}, \quad T = \frac{5}{24}, \quad V = \frac{2}{15}, \quad W = \frac{61}{720}, \quad X = \frac{17}{315} \quad \text{etc.}$$

But since the sum of the cubes is equal to $R = \frac{1}{2}$, it will be

$$\frac{2}{q^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.}\right) = \frac{1}{2}.$$

Hence it will be

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} = \frac{q^3}{4} = \frac{p^3}{32}.$$

Therefore, the sum of this series multiplied by 32 gives the cube of the circumference of the circle, whose diameter is 1. In similar manner the sum of the bisquares, which is

$$\frac{2}{q^4}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.}\right)$$

must be equal to $\frac{1}{3}$ and hence it will be

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = \frac{q^4}{6} = \frac{p^4}{96}.$$

But on the other hand this series multiplied by $\frac{16}{15}$ is equal to this one

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.};$$

hence this series is equal to $\frac{p^4}{90}$, or the sum of this series

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.}$$

multiplied by 90 gives the bisquare of the circumference of the circle, whose diameter is 1.

§13 In similar manner the sum of the higher powers will be found; but it will arise as follows

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.} = \frac{5q^5}{48} = \frac{5p^5}{1536}$$

and

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} = \frac{q^6}{15} = \frac{p^6}{960}.$$

But having found the sum of this series at the same time the sum of this series will be known

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.},$$

which will be

$$=\frac{p^6}{945}.$$

Further, for the seventh powers it will be

$$1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\text{etc.}=\frac{61q^7}{1440}=\frac{61p^7}{184320}$$

and for the eighth

$$1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{9^8}+\text{etc.}=\frac{17q^8}{630}=\frac{17p^8}{161280},$$

whence one deduces

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\frac{1}{6^8}+\text{etc.}=\frac{p^8}{9450}.$$

But it is observed about these series that in the powers of odd exponents the signs of the terms alternate, for the even powers on the other hand they are equal; and this is the reason that the sum of this general series

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.}$$

can only be exhibited in the cases, in which n is an even number. Furthermore, it is also to be noted, if the general term of the series 1, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{5}{24}$, $\frac{2}{15}$, $\frac{61}{720}$, $\frac{17}{315}$ etc., which values we found for the letters P, Q, R, S etc., could be assigned, then then by it the quadrature of the circle would be exhibited.

§14 In these we put the sine PM equal to the radius; therefore, let us see, what kind of series arises, if other values are attributed to y. Therefore, let $y=\frac{1}{\sqrt{2}}$, the smallest arc to which sine is $\frac{1}{4}p$. Therefore, having put $A=\frac{1}{4}p$ the sum of the simple terms or the first power will be this one

$$\frac{4}{p}+\frac{4}{3p}-\frac{4}{5p}-\frac{4}{7p}+\frac{4}{9p}+\frac{4}{11p}-\text{etc.},$$

the sum P of which series is $\frac{1}{y} = \sqrt{2}$. Therefore, one will have

$$\frac{p}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.},$$

which series only with respect to the sings differs from Leibniz's series and was given by Newton a long time ago. But the sum of the squares of those terms, namely

$$\frac{16}{p^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.}\right),$$

is equal to $Q = 2$. Therefore, it will be

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.} = \frac{p^2}{8},$$

as it was found before.

§15 If it is $y = \frac{\sqrt{3}}{2}$, the smallest arc corresponding to this sine will be 60° and hence $A = \frac{1}{3}p$. Therefore, in this case the following series of terms arises

$$\frac{3}{p} + \frac{3}{2p} - \frac{3}{4p} - \frac{3}{5p} + \frac{3}{7p} + \frac{3}{8p} - \text{etc.},$$

the sum of which terms is equal to $\frac{1}{y} = \frac{2}{\sqrt{3}}$. Therefore, one will have

$$\frac{2p}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \text{etc.}$$

But the sum of the squares of those terms is $= \frac{1}{y^2} = \frac{4}{3}$; hence it follows that it will be

$$\frac{4p^2}{27} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{25} + \frac{1}{49} + \frac{1}{64} + \text{etc.},$$

in which series the terms contain three are missing. But this series also depends on this one

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \quad \text{etc.},$$

whose sum had been found $= \frac{p^2}{6}$; for, if this series is diminished by its ninth part, the superior series arises, whose sum therefore must be $= \frac{p^2}{6}\left(1 - \frac{1}{9}\right) =$

$\frac{4pp}{27}$. In similar manner, if other sines are assumed, other series will arise, so of the simple terms as of the squares and the higher powers of the terms, whose sum involve the quadrature of the circle.

§16 But if one puts $y = 0$, series of this kind cannot further be assigned because of the y put in the denominator or the initial equation divided by y. But series can hence be deduced in another way; since they are the series

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.},$$

themselves, if n is an even number, I will deduce how the sum of these series are to be found separately from this case, in which it is $y = 0$. But having put $y = 0$ the fundamental equation goes over into this one

$$0 = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.},$$

the roots of which equation give all arcs, whose sine is $= 0$. But one and the smallest root is $s = 0$, whence the equation divided by s will exhibit all remaining arcs, whose sine is $= 0$, which arcs will equally be the roots of this equation

$$0 = 1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.}$$

But the arcs, whose sine is $= 0$, are

$$p, \quad -p, \quad +2p, \quad -2p, \quad 3p, \quad -3p \quad \text{etc.},$$

the one of which each two is the negative of the other, what also the equation itself because of the only even dimensions of s indicates. Hence the divisors of that equation will be

$$1 - \frac{s}{p}, \quad 1 + \frac{s}{p}, \quad 1 - \frac{s}{2p}, \quad 1 + \frac{s}{2p} \quad \text{etc.}$$

and by combing each two of this terms it will be

$$\begin{aligned} &1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{s^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \text{etc.} \\ &= \left(1 - \frac{s^2}{p^2}\right)\left(1 - \frac{s^2}{4p^2}\right)\left(1 - \frac{s^2}{9p^2}\right)\left(1 - \frac{s^2}{16p^2}\right) \text{etc.} \end{aligned}$$

§17 It is now manifest form the nature of equations that the coefficient of ss or $\frac{1}{1\cdot 2\cdot 3}$ will be equal to

$$\frac{1}{p^2}+\frac{1}{4p^2}+\frac{1}{9p^2}+\frac{1}{16p^2}+\text{etc.}$$

The sum of the products of two terms of this series will be $=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}$ and the sum of the products of three will be $=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$ etc. Therefore, according to § 8 it will be

$$\alpha=\frac{1}{1\cdot 2\cdot 3}, \quad \beta=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}, \quad \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} \quad \text{etc.}$$

and having put the sum of the terms

$$\frac{1}{p^2}+\frac{1}{4p^2}+\frac{1}{9p^2}+\frac{1}{16p^2}+\text{etc.}=P$$

and the sum of the squares of the same terms $=Q$, the sum of the cubes $=R$, the sum of the bisquares $=S$ etc. it will be by means of § 8

$$\begin{aligned}
P &= \alpha=\frac{1}{1\cdot 2\cdot 3}=\frac{1}{6},\\
Q &= P\alpha-2\beta=\frac{1}{90},\\
R &= Q\alpha-P\beta+3\gamma=\frac{1}{945},\\
S &= R\alpha-Q\beta+P\gamma-4\delta=\frac{1}{9450},\\
T &= S\alpha-r\beta+Q\gamma-P\delta+5\varepsilon=\frac{1}{93555}\\
V &= T\alpha-S\beta+R\gamma-Q\gamma+P\varepsilon-6\zeta=\frac{691}{6825\cdot 93555}\text{etc.}
\end{aligned}$$

§18 Therefore, from these the following sums are derived

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = \frac{p^2}{6} = P',$$
$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = \frac{p^4}{90} = Q',$$
$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = \frac{p^6}{945} = R',$$
$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = \frac{p^8}{9450} = S',$$
$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \text{etc.} = \frac{p^{10}}{93555} = T',$$
$$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} + \text{etc.} = \frac{691 p^{12}}{6825 \cdot 93555} = V',$$
$$\text{etc.},$$

which series from the given law with a lot of work can be continued to the higher powers. But by dividing the single series by the preceding ones the following equations will arise

$$p^2 = 6P' = \frac{15Q'}{P'} = \frac{21R'}{2Q'} = \frac{10S'}{R'} = \frac{99T'}{10S'} = \frac{6825V'}{691T'} \quad \text{etc.},$$

to which single expressions the square of the circumference of the circle, whose diameter is 1, becomes equal.

§19 But since the sum of these series, even though they can be exhibited approximately easily, nevertheless do not have much use to express the circumference of this circle approximately because of the square root, which would have to be extracted, from the first series we will find expressions, which are equal to the circumference p itself. But it will arise as follows

$$
\begin{aligned}
p &= 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.}\right)\\
p &= 2\cdot\frac{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\text{etc.}}{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.}}\\
p &= 4\cdot\frac{1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}}{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\text{etc.}}\\
p &= 3\cdot\frac{1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{11^4}+\text{etc.}}{1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}}\\
p &= \frac{16}{5}\cdot\frac{1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\frac{1}{11^5}+\text{etc.}}{1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{11^4}+\text{etc.}}\\
p &= \frac{25}{8}\cdot\frac{1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\frac{1}{11^6}+\text{etc.}}{1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\frac{1}{11^5}+\text{etc.}}\\
p &= \frac{192}{61}\cdot\frac{1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\frac{1}{11^7}+\text{etc.}}{1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\frac{1}{11^6}+\text{etc.}}
\end{aligned}
$$