On Solids whose surface can be unfolded onto a plane *

Leonhard Euler

§1 Very well-known is the property of the cylinder and the cone according to which their surface can be unfolded onto a plane and this property is even extended to all cylindrical and conic bodies whose bases have any arbitrary shape; the sphere on the other hand, does not enjoy this property, since its surface by no means can be unfolded onto a plane and it cannot be covered by a plain surface; from this arises the question, as curious as noteworthy, whether except cones and cylinders other classes of solids exist whose surface can be onfolded onto the plane in the same way or not. Therefore, I decided to consider the following problem in this dissertation:

To find a general equation for all solids whose surface can be expanded onto a plane,

whose solution I will tackle in various ways.

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First Solution derived from mere analytical Principles

§2 Let (Fig. 1) Z be an arbitrary point on the surface of the solid we are looking for the location of which point in usual manner shall be expressed by three mutually orthogonal coordinates $AX = x$, $XY = y$ and $YZ = z$ such that an equation between these three coordinates is to be found by means of which the problem is satisfied.

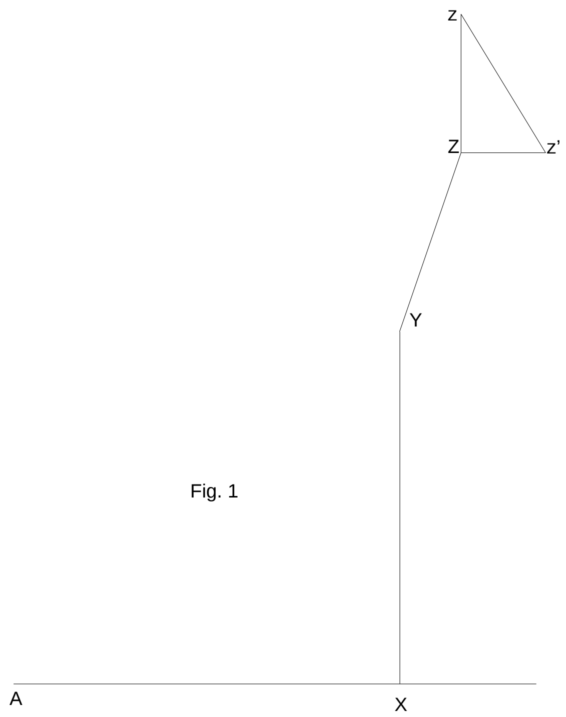

Fig. 1

Further, let us assume that the surface of a solid of such a kind is already unfolded onto the plane and it is represented in figure 2 in which the point Z falls on V whose location shall be defined by two orthogonal coordinates in such a way that it is $OT = t$ and $TV = u$, and it is manifest that the first three coordinates x, y and z have to depend on these two t and u in a certain way, and hence every single one of them can be considered a certain functions of t and u.

§3 In order to introduce this condition into the calculation in a more convenient way let us consider it in terms of differentials and, since x and y are functions of the two variables t and u, let us define their differentials by means of these formulas:

$$dx = ldt + \lambda du, \quad dy = mdt + \mu du \quad \text{and} \quad dz = ndt + \nu du,$$

where, since the letters l, m, n and λ, μ, ν in the same way denote certain functions of the two variables t and u, it is clear from the nature of functions of such a kind that is has to be:

$$\left(\frac{dl}{du}\right) = \left(\frac{d\lambda}{dt}\right), \quad \left(\frac{dm}{du}\right) = \left(\frac{d\mu}{dt}\right) \quad \text{and} \quad \left(\frac{dn}{du}\right) = \left(\frac{d\nu}{dt}\right).$$

§4 Now, let us in the unfolded surface (Fig.2)

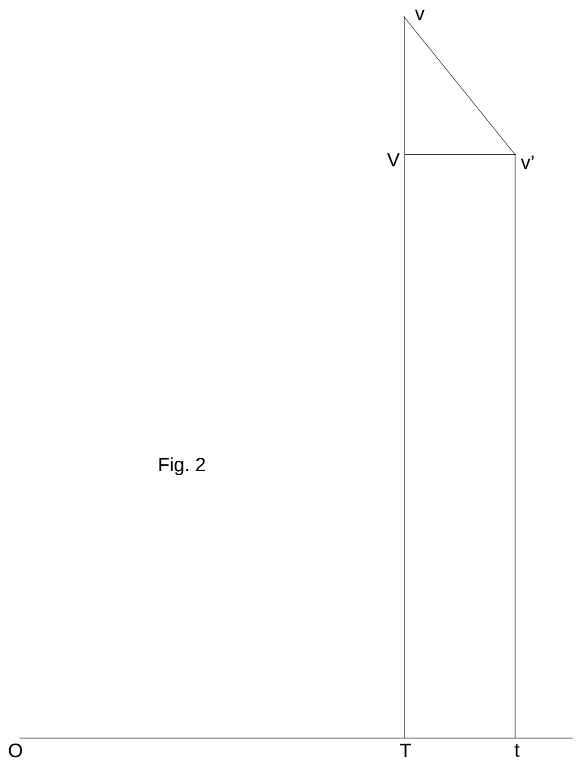

Fig. 2

except the point V contemplate two other infinitely close ones v and v' for the latter of which the coordinates shall be

$$OT = t \quad \text{and} \quad Tv = u + du,$$

for the first on the other hand:

$$Ot = t + dt \quad \text{and} \quad tv' = u,$$

such that the points V and v have the common abscissa $OT = t$, but the points V and v' have the common ordinate $= u$. Having drawn the infinitesimal short lines Vv' and vv' the sides of the elementary triangles Vvv' are determined in such a way that it is:

$$Vv = du, \quad Vv' = dt \quad \text{and} \quad vv' = \sqrt{du^2 + dt^2},$$

and now it is easily understood that the same triangle has to be found also in the surface of the solid we are looking for.

§5 Therefore, in the surface of the solid let z and z' be the points corresponding to the points v and v' and let us see, how the three coordinates behave for those points z and z'. But, the way how the point Z is defined by means of these three coordinates, the first $= x$, the second $= y$, the third $= z$, which all are functions of the two variables t and u, since for the point v the abscissa t remains the same, but the ordinate u on the other hand is augmented by its differential du, the three coordinates for the point z of the solid will behave as this:

$$\text{I.} \quad x + \lambda du, \quad \text{II.} \quad y + \mu du \quad \text{and} \quad \text{III.} \quad z + \nu du;$$

in similar way, because for the point v' the ordinate u remains the same, the abscissa t on the other hand is augmented by its differential dt, the three coordinates for the point z' will be:

$$\text{I.} \quad x + ldt, \quad \text{II.} \quad y + mdt \quad \text{and} \quad \text{III.} \quad z + ndt.$$

§6 But it is known, if for any arbitrary point in the surface of a solid the coordinates were x, y and z, but for another infinitely close point they were x', y' and z', that then the distance of the points will be:

$$= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2};$$

hence, we will have for the single sides of the triangle Zzz'

$$\begin{aligned} 1^\circ \quad Zz &= du\sqrt{\lambda^2+\mu^2+\nu^2}, \\ 2^\circ \quad Zz' &= dt\sqrt{l^2+m^2+n^2} \end{aligned}$$

and

$$3^\circ \quad zz' = \sqrt{(\lambda du - ldt)^2 + (\mu du - mdt)^2 + (\nu du - ndt)^2}$$

or

$$zz' = \sqrt{dt^2(ll+mm+nn)+du^2(\lambda\lambda+\mu\mu+\nu\nu)-2dtdu(l\lambda+m\mu+n\nu)}.$$

§7 Now, because the surface of the solid has to totally agree with the plain figure (Fig. 2), it is necessary that the triangles Zzz' and Vvv' are non only equal but also similar and hence the sides homologuously equal, namely:

$$\mathrm{I}^\circ. \quad Zz = Vv, \quad \mathrm{II}^\circ. \quad Zz' = Vv' \quad \text{and} \quad \mathrm{III}^\circ. \quad zz' = vv',$$

whence we obtain the following equations:

$$\begin{aligned} &\mathrm{I}^\circ. \quad \lambda^2+\mu^2+\nu^2 = 1 \\ &\mathrm{II}^\circ. \quad l^2+m^2+n^2 = 1 \\ &\mathrm{III}^\circ. \quad dt^2(l^2+m^2+n^2)+du^2(\lambda^2+\mu^2+\nu^2)-2dtdu(l\lambda+m\mu+n\nu) = dt^2+du^2, \end{aligned}$$

the third because of the first two is reduced to this one:

$$l\lambda + m\mu + n\nu = 0,$$

in which three equations the solution of our problem is contained, from which it is understood that it can be reduced to the following analytical problem:

Having propounded the two variables t and u to find six functions l, m, n and λ, μ, ν of them of such a nature that the following six conditions are satisfied:

$$\text{I}^\circ.\quad \left(\frac{dl}{du}\right) = \left(\frac{d\lambda}{dt}\right), \quad \text{II}^\circ.\quad \left(\frac{dm}{du}\right) = \left(\frac{d\mu}{dt}\right), \quad \text{III}^\circ,\quad \left(\frac{dn}{du}\right) = \left(\frac{d\nu}{dt}\right),$$

$$\text{IV}^\circ.\quad ll + mm + nn = 1, \quad \text{V}^\circ.\quad \lambda\lambda + \mu\mu + \nu\nu = 1,$$

$$\text{VI}^\circ\quad l\lambda + m\mu + n\nu = 0,$$

which problem considered for itself seems to be most difficult for a long time, of which nevertheless a sufficiently beautiful solution will be able to exhibited below.

Second Solution derived from geometrical Principles

§8 In order to derive this solution from first principles let us consider either prismatical or pyramidal bodies which having excepted the bases are understood to be covered by a chart, and on this chart rectilinear sharp bends will be detected either parallel to each other or converging to a certain point, the vertex of the pyramid, of course, which straight lines, whatsoever they were, shall be denoted by the letters Aa, Bb, Cc, Dd etc. If therefore the chart is unfolded onto the plane, in it the same straight lines Aa, Bb, Cc etc. will occur, and they will be either parallel to each other or converge to a certain point. Hence, vice versa, if on the plain chart such straight lines are drawn, according to which the chart can be folded, it will be apt for covering a certain either prismatic or pyramidal body.

§9 It will even be possible to draw the lines Aa, Bb, Cc, Dd etc., in the chart ad libitum, such that they are neither parallel to each other nor converge to a certain point, as long as they never cross each other, as figure 3 shows;

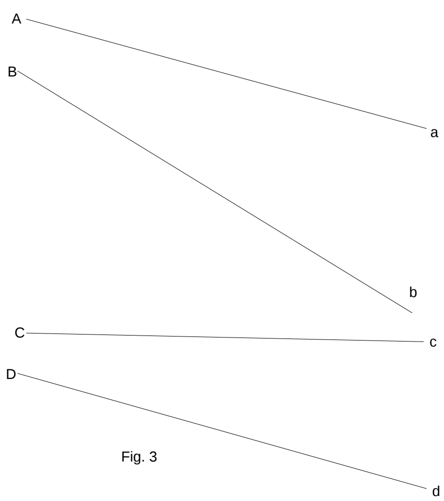

Fig. 3

for, no matter in which way this chart is folded according to these lines, it will always possible to conceive a solid of such a kind, to which this folded chart can be adapted. From this it is plain that except prismatic or pyramidal bodies also other classes of bodies are given which can be covered by a chart this way, and whose surface can therefore be unfolded onto a plain.

§10 Therefore, in the surface of these bodies any number of straight lines Aa, Bb, Cc, Dd etc. will be given which, even though they are neither parallel nor converge to a certain point, will nevertheless be of such a nature that two, very close to each other, such as Aa and Bb or Bb and Cc or Cc and Dd etc., if there are not parallel, at least intersect in a single point if the are drawn further; for, if this would not happen, the space between two infinitely close lines of such a kind in the surface of the bodies would not be intercepted and therefore it would not be possible to unfold the surface onto the plane, although in it are given arbitrary many straight lines Aa, Bb, Cc etc. From this we conclude that it does not suffice for bodies satisfying our scope that it is possible to draw an arbitrary number of lines Aa, Bb, Cc etc. on them, but that its is furthermore required that two lines infinitely close to each other live in the same plane and the space contained between them itself is plain.

§11 Now, let us continue the straight lines Aa, Bb, Cc etc. to infinity such that our body obtains a everywhere curved surface, as our problem postulates it because of the law of continuity. And now it is indeed immediately clear that a surface of such a kind has to be of such a nature that from any arbitrary point assumed in it at least one straight line can be drawn which completely lies in the surface itself; but this condition alone does not exhaust the total character of our problem, but additionally it is necessary that any two lines, infinitely close to each other, of this kind live in the same plane, this means that, if they are not parallel, they at least meet in one point, if continued. Hence, if those single lines are elongated to the point of intersection in this way, all these points of intersection will be found to lie on a certain curve, because which is not totally living in one plane, will have two curvatures and be of such a nature that its single elements, if they are elongated, exhibit those lines Aa, Bb, Cc etc. mentioned above themselves in the surface of the body.

§12 Therefore, as any body convenient for our problem leads to a certain curve with two curvatures, so vice versa having assumed a curve of this kind ad libitum we will be able to determine a body from it which satisfies our problem. But, at first project such a curve onto the plotting table plane, and let (Fig. 4) its projection be aUu,

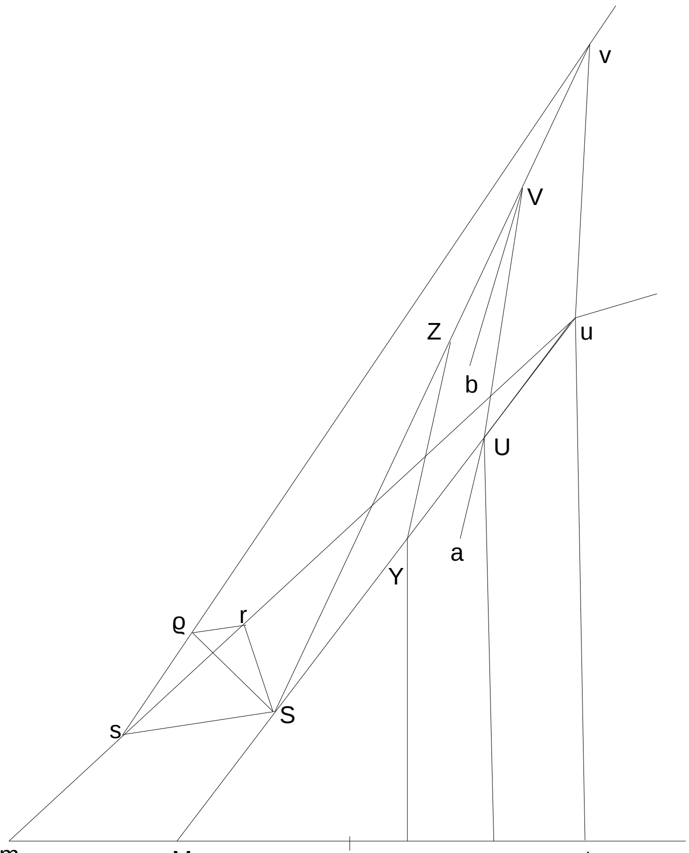

Fig. 4

for which we want to put the abscissa $AT = t$ and the ordinate $Tu = u$ such that an equation between t and u is considered as given, and let UM be the tangent of this curve at the point U, the line um on the other hand the tangent at the infinitely close point u; having put all this let bVv be the curve with the two curvatures itself, whose ordinate orthogonal to our plane shall be put $UV = v$, and let v be an infinitely close point on the same curve, and from both points V, v draw tangents of which the latter VS shall meet the line UM in the point S, the other vs on the other hand shall intersect the line rm in the point s. Here, we certainly could have drawn the infinitely close tangents in the points u and v, but, since it will be necessary in the following, it seemed advisable to indicate them in the figure here.

§13 Because therefore the nature of the curve bVv is expressed by two equations between the coordinates $AT = t$, $Tu = u$ and $UV = v$, so the letter u as v can be considered as a function to t, whence at the same time the position of both tangents UM and VS will be defined which is why we want to call the angles $TUM = \zeta$ and $UVS = \vartheta$, and having put the element $Tt = dt$ it will be

$$du = \frac{dt}{\tan \zeta}, \quad Uu = \frac{dt}{\sin \zeta},$$

then on the other hand

$$dv = \frac{dt}{\sin \zeta \tan \vartheta}$$

and finally the element of the curve

$$Vv = \frac{dt}{\sin \zeta \sin \vartheta}.$$

But, for the position of the tangents we will have

$$TM = u \tan \zeta, \quad UM = \frac{u}{\cos \zeta},$$

the line on the other hand

$$US = v \tan \vartheta \quad \text{and} \quad VS = v \sec \vartheta = \frac{v}{\cos \vartheta}.$$

§14 Since now the total line VS lies in the surface of the body we are looking for, let us on it take any indefinite point Z, whence having dropped the perpendicular ZY to the plotting table plane and having drawn the normal YX from point Y to the axis AT we will have for the surface we are looking for three coordinates itself we contemplated above, of course $AX = x$, $XY = y$ and $YZ = z$, between which the correct equation is therefore to be investigated, by means of which the nature of this surface is expressed.

§15 For this aim, let us call the indefinite interval $VZ = s$ which therefore is a variable quantity in no way depending on the point V, and hence is to be carefully distinguished from the variable t, of which not only the two ordinates $TU = u$ and $UV = v$ are functions, but also the two angles ζ and ϑ are. Hence, we obtain

$$ZY = z = v - s\cos\vartheta$$

and the interval

$$UY = s\sin\vartheta,$$

whence we further conclude

$$XY = y = u - s\sin\vartheta\cos\zeta$$

and

$$XT = s\sin\vartheta\sin\zeta,$$

and so we finally obtain the abscissa

$$AX = x = t - s\sin\vartheta\sin\zeta,$$

such that by means of the two variables t and s our three coordinates are succinctly determined this way:

$$\begin{aligned} &\text{I}^\circ. && x = t - s\sin\vartheta\sin\zeta, \\ &\text{II}^\circ. && y = u - s\sin\vartheta\cos\zeta, \\ &\text{III}^\circ. && z = v - s\cos\vartheta. \end{aligned}$$

§16 Therefore, against all expectations it happens here that we even found algebraic formulas for the three coordinates x, y, z, if for the quantities u and v algebraic functions of t are taken. For, these functions are completely subject to our desires, but having assumed them, the two angles ζ and ϑ are determined in such a way that it is $\tan\zeta = \frac{dt}{du}$ or

$$\sin\zeta = \frac{dt}{\sqrt{dt^2 + du^2}} \quad \text{and} \quad \cos\zeta = \frac{du}{\sqrt{dt^2 + du^2}},$$

then on the other hand

$$\tan\vartheta = \frac{dt}{dv\sin\zeta} = \frac{\sqrt{dt^2+du^2}}{dv}$$

and hence

$$\sin\vartheta = \frac{\sqrt{dt^2+du^2}}{\sqrt{dt^2+du^2+dv^2}} \quad \text{and} \quad \cos\vartheta = \frac{dv}{\sqrt{dt^2+du^2+dv^2}}.$$

But if therefore vice versa the two angles ζ and ϑ were given in terms of the variables t, the ordinates u and v themselves will be found expressed by the following integral formulas

$$u = \int \frac{dt}{\tan\zeta} \quad \text{and} \quad v = \int \frac{dt}{\sin\zeta\tan\vartheta}.$$

§17 In these formulas therefore completely all solids whose surface can be unfolded onto the plane are necessarily contained. Therefore, it will especially be worth the effort to show how the conic bodies are contained in them, since the cylindrical bodies are already contained in the conics, having removed the vertex to infinity. Therefore, let the point V be the vertex of the cone since which is fixed the constants t, u and v will have constant values. Since nothing impedes that this vertex is taken in the fixed point A itself, we will be able to put $t=0$, $u=0$ and $v=0$, but then because of

$$\tan\zeta = \frac{dt}{du} \quad \text{and} \quad \tan\vartheta = \frac{dt}{dv\sin\zeta} = \frac{\sqrt{dt^2+du^2}}{dv}$$

these angles ζ and ϑ arise as indefinite, nevertheless in such a way that the one can be considered as a certain function of the other, since all things extending to the position of the lines VS are to be referred to one single variable.

§18 Because it therefore is $t=0$, $u=0$ and $v=0$, we will have:

$$\begin{aligned} &\text{I}^\circ. && x = -s\sin\vartheta\sin\zeta,\\ &\text{II}^\circ. && y = -s\sin\vartheta\cos\zeta,\\ \text{and}\quad &\text{III}^\circ. && z = -s\cos\vartheta, \end{aligned}$$

whence it is

$$\frac{x}{y} = \tan\zeta \quad \text{and} \quad \frac{x}{z} = \tan\vartheta\sin\zeta,$$

from which it is concluded

$$\sin\zeta = \frac{x}{\sqrt{xx+yy}}$$

and hence from this

$$\tan\vartheta = \frac{\sqrt{xx+yy}}{z};$$

since therefore $\tan\vartheta$ becomes equal to an arbitrary function of $\tan\zeta$, we will have such an equation:

$$\frac{\sqrt{xx+yy}}{z} = \Phi : \left(\frac{x}{y}\right),$$

and so the quantity $\frac{\sqrt{xx+yy}}{z}$ will become equal to a homogeneous function of no dimension of x and y and hence further the quantity z itself will become equal to a homogeneous function of one dimension of x and y, or what reduces to the same, the equation between x, y and z will be of such a nature that in it the three variables x, y and z will fill the same number of dimensions everywhere. Therefore, if one of the coordinates x, y and z goes over to infinity, the equation for the solid will only contain the remaining two variables which is a criterion for cylindrical bodies.

§19 We do not spend more time on the expansion of other solids satisfying our problem here, because below, where we will offer a third method, we are able to cognize all species of bodies of this kind a lot easier. Meanwhile, while this second method provided us with such a simple solution, although by means of the first method hardly any solution could be hoped for, we will now also be able to expand the first solution further and even resolve those analytical formulas, on first sight exceedingly difficult, whence many light will be shed on the analysis. To do this it only will be necessary that we carefully reduce this second solution to the elements of the first.

APPLICATION OF THE SECOND METHOD TO THE FIRST SOLUTION

§20 Since in the second solution we already found formulas for the three coordinates x, y and z in which the nature of the solid is contained, we will have to elaborate on this that we also investigate formulas for the plain figure onto which the surface of the solid is unfolded. Here, it especially that curve bVv with the two curvature is to be studied more accurately, which by the unfolding of the surface is also reduced to the plane. But because this curve by means of inflections in infinitely many ways can be reduced to the plane and can even be extended into a straight line, it is especially to be inquired, according to which law this reduction to the plane has to happen. From the things mentioned above it is indeed manifest that this reduction has to happen in such a way that (Fig. 4) any two infinitesimally close tangents VS and vs conserve the same mutual position to each other or that the angle Svs enclosed between them remains the same. Of course, the curve bVv itself is to be reduced to the plane in such a way that any two infinitesimally close of its elements conserve the same inclination between each other.

§21 Therefore, the main task reduces to this that we find an infinitely small angle Svs for which aim it is to be started from the angle MUm. But because it is

$$\text{angle } TUM = \zeta \quad \text{and} \quad \text{angle } tum = \zeta + d\zeta,$$

it manifestly follows angle $Mum = d\zeta$, further, because we already found $US = v \tan \vartheta$ above, it will be from the nature of differentials:

$$us = v \tan \vartheta + d(v \tan \vartheta) = v \tan \vartheta + dv \tan \vartheta + \frac{vd\vartheta}{\cos^2 \vartheta},$$

where

$$dv = \frac{dt}{\sin \zeta \tan \vartheta};$$

because therefore it is

$$Uu = \frac{dt}{\sin \zeta},$$

it will be

$$Us = v \tan\vartheta + dv \tan\vartheta + \frac{vd\vartheta}{\cos^2\vartheta} - \frac{dt}{\sin\zeta} = v\tan\vartheta + \frac{vd\vartheta}{\cos^2\vartheta}.$$

Therefore, from S drop the perpendicular Sr to Us that one has

$$rs = \frac{vd\vartheta}{\cos^2\vartheta},$$

then it will indeed be

$$Sr = vd\zeta \tan\vartheta,$$

whence also the element Ss could be defined, if the would be of any necessity.

§22 Now, from the point r let us drop the perpendicular $r\rho$ to the tangent vs, that having drawn $S\rho$ becomes normal to vs, where it is to be noted that the triangle $Sr\rho$ will have an right angle at r, because Sr is normal to the plane sUV itself. Because

$$\text{angle } rs\rho = 90^\circ - \vartheta,$$

it will be

$$r\rho = sr \cdot \sin rs\rho = \frac{vd\vartheta}{\cos\vartheta},$$

whence it is calculated

$$S\rho = \sqrt{vv\zeta^2\tan^2\vartheta + \frac{vvd\vartheta^2}{\cos^2\vartheta}} = \frac{v}{\cos\vartheta}\sqrt{d\zeta^2\sin^2\zeta + d\vartheta^2}.$$

Because therefore it is $VS = \frac{v}{\cos\vartheta}$, hence it is concluded

$$\text{angle } SVs = \frac{S\rho}{VS} = \sqrt{d\zeta^2\sin^2\vartheta + d\vartheta^2}.$$

§23 So we therefore found the angle SVs in which the two infinitely close elements of the curve are inclined to each other, from which the radius of curvature of this curve in the point V can be defined very fast, which is of course

$$\frac{Vv}{SVs} = \frac{dt}{\sin\zeta\sin\vartheta\sqrt{d\zeta^2\sin^2\vartheta + d\vartheta^2}},$$

which task therefore is not impeded because of the two curvatures; it is enough to have remembered this in the transition. But because here the main issue lies in the determination of the elementary angle SVs, let us call the angle $SVs = d\omega$ such that it is

$$d\omega = \sqrt{d\zeta^2 \sin^2\vartheta + d\vartheta^2} \quad \text{or} \quad d\omega^2 - d\vartheta^2 = d\zeta^2 \sin^2\vartheta,$$

where, because the two angles ζ and ϑ are determined by the variable t, of which also the two ordinates u and v are functions, it is clear that also the angle ω has to be considered as a function of the same variable t.

§24 Now, according to the prescriptions given above (Fig. 5)

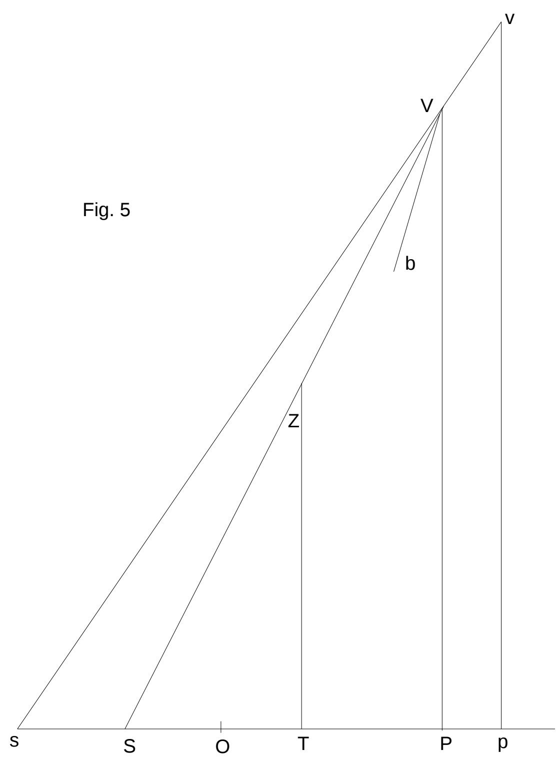

Fig. 5

let the curve bVv with the two curvatures be described in the plane, such that the angle SVs intercepted between to infinitely close tangents will be $= d\omega$, and having related this curve to the axis OP by means of the ordinate PV it is evident that the angle PVS will be $= \omega$. But let us put these coordinates $OP = p$ and $PV = q$, and we will have

$$\frac{dp}{dq} = \tan \omega,$$

and the element of the curve

$$Vv = \frac{dp}{\sin \omega},$$

but on the other hand by means of the preceding coordinates t, u and v with the angle ζ and ϑ the same element was

$$Vv = \frac{dt}{\sin \zeta \sin \vartheta},$$

whence we as a consequence obtain

$$dt \sin \omega = dp \sin \zeta \sin \vartheta,$$

which combined with the equation $\frac{dp}{dq} = \tan \omega$ will give the following integral values for the present coordinates p and q

$$p = \int \frac{dt \sin \omega}{\sin \zeta \sin \vartheta} \quad \text{and} \quad q = \int \frac{dt \cos \omega}{\sin \zeta \sin \vartheta};$$

having found these quantities p and q, which likewise are functions of the same variable t, take the interval $VZ = s$, which is the other variable to be introduced into the calculation, and having dropped the perpendicular ZT from the point Z to the axis, we find

$$OT = p - s \sin \omega \quad \text{and} \quad TZ = q - s \cos \omega.$$

§25 Since therefore for the point Z reduced to the plane we obtained the determination, let us put its coordinates $OT = T$ and $TZ = U$, which are defined by means of the two variables t and s that it is

$$T = p - s \sin \omega = \int \frac{dt \sin \omega}{\sin \zeta \sin \vartheta} - s \sin \omega,$$
$$U = q - s \cos \omega = \int \frac{dt \cos \omega}{\sin \zeta \sin \vartheta} - s \cos \omega,$$

where it is to be noted that the angle ω depends on the angles ζ and ϑ in such a way that it is

$$d\omega = \sqrt{d\zeta^2 \sin^2 \vartheta + d\vartheta^2}.$$

These coordinates T and U are indeed the same which we in the first solutions denoted by the letters t and u, whence having done the same change there the formulas found there for the solid reduce to these

$$dx = ldT + \lambda dU, \quad dy = mDT + \mu dU, \quad dz = ndT + \nu dU$$

while the conditions, we found there, remain, of course:

$$ll + mm + nn = 1, \quad \lambda\lambda + \mu\mu + \nu\nu = 1, \quad \text{and} \quad l\lambda + m\mu + n\nu = 0.$$

But here we found for the same coordinates x, y and z for the solid the following values:

$$x = t - s \sin\vartheta \sin\zeta, \quad y = u - s \sin\vartheta \cos\zeta \quad \text{and} \quad z = v - s\cos\vartheta,$$

which because of

$$du = \frac{dt}{\tan\zeta} \quad \text{and} \quad dv = \frac{dt}{\sin\zeta \tan\vartheta},$$

differentiated yield:

$$\begin{aligned}
dx &= dt - ds \sin\vartheta \sin\zeta - sd\zeta \sin\vartheta \cos\zeta - sd\vartheta \sin\zeta \cos\vartheta, \\
dy &= \frac{dt}{\tan\zeta} - ds \sin\vartheta \cos\zeta + sd\zeta \sin\zeta \sin\vartheta - sd\vartheta \cos\zeta \cos\vartheta, \\
dz &= \frac{dt}{\sin\zeta \tan\vartheta} - ds \cos\vartheta + sd\vartheta \sin\vartheta.
\end{aligned}$$

§27 Before we proceed further, it will not be a deviation to have noted the principal relations of these formulas, and at first by eliminating s we obtain these relations for the finite formulas themselves:

$$x\cos\zeta - y\sin\zeta = t\cos\zeta - u\sin\zeta,$$
$$x\sin\zeta + y\cos\zeta = t\sin\zeta + u\cos\zeta - s\sin\vartheta,$$
$$x\sin\zeta\cos\vartheta + y\cos\zeta\cos\vartheta - z\sin\vartheta = t\sin\zeta\cos\vartheta + u\cos\zeta\cos\vartheta - v\sin\vartheta.$$

Further, for the differentials the following:

$$\begin{aligned} &\mathrm{I}^\circ. && dx\cos\zeta - dy\sin\zeta = -sd\zeta\sin\vartheta,\\ &\mathrm{II}^\circ. && dx\sin\zeta + dy\cos\zeta = \frac{dt}{\sin\zeta} - ds\sin\vartheta - sd\vartheta\cos\vartheta\\ \text{and}\quad &\mathrm{III}^\circ. && dx\sin\zeta\cos\vartheta + dy\cos\zeta\cos\vartheta - dz\sin\vartheta = -sd\vartheta. \end{aligned}$$

§28 But because in this new calculation we reduced everything to the two variables t and s, while in the first calculation we used the two variables T and U, let us see, how these are expressed by those, and from the formulas found for T and U we indeed have

$$\begin{aligned} dT &= \frac{dt\sin\omega}{\sin\zeta\sin\vartheta} - ds\sin\omega - sd\omega\cos\omega\\ \text{and}\quad dU &= \frac{dt\cos\omega}{\sin\zeta\sin\vartheta} - ds\cos\omega + sd\omega\sin\omega, \end{aligned}$$

if we substitute which values in the formulas dx, dy and dz found before and carefully distinguish the two variables t and s, we will obtain the following expression:

$$\begin{aligned} dx &= dt\frac{l\sin\omega + \lambda\cos\omega}{\sin\zeta\sin\vartheta} - sd\omega(l\cos\omega - \lambda\sin\omega) - ds(l\sin\omega + \lambda\cos\omega),\\ dy &= dt\frac{m\sin\omega + \mu\cos\omega}{\sin\zeta\sin\vartheta} - sd\omega(m\cos\omega - \mu\sin\omega) - ds(m\sin\omega + \mu\cos\omega),\\ dz &= dt\frac{n\sin\omega + \nu\cos\omega}{\sin\zeta\sin\vartheta} - sd\omega(n\cos\omega - \nu\sin\omega) - ds(n\sin\omega + \nu\cos\omega), \end{aligned}$$

which we want to compare to those which arose in the last solution which are

$$
\begin{aligned}
dx &= dt && - sd\zeta \sin\vartheta \cos\zeta - sd\vartheta \sin\zeta \cos\vartheta - ds \sin\zeta \sin\vartheta,\\
dy &= \frac{dt}{\tan\zeta} && + sd\zeta \sin\zeta \sin\vartheta - sd\vartheta \cos\zeta \cos\vartheta - ds \cos\zeta \sin\vartheta,\\
dz &= \frac{dt}{\sin\zeta \tan\vartheta} && \qquad\qquad\qquad\qquad + sd\vartheta \sin\vartheta \qquad\quad - ds \cos\vartheta;
\end{aligned}
$$

and first, the terms affected by ds have to be equal on both sides whence we obtain these equations:

$$
\begin{aligned}
&\mathrm{I}^\circ. && l \sin\omega + \lambda \cos\omega = \sin\zeta \sin\vartheta,\\
&\mathrm{II}^\circ. && m \sin\omega + \mu \cos\omega = \cos\zeta \sin\vartheta,\\
&\mathrm{III}^\circ. && n \sin\omega + \nu \cos\omega = \cos\vartheta.
\end{aligned}
$$

§29 If therefore now these values are substituted in the first terms, which involve the differential dt and those depending on it, namely $d\zeta$, $d\vartheta$ and $d\omega$, we will obtain the following equations:

$$
\begin{aligned}
l \cos\omega - \lambda \sin\omega &= \frac{d\zeta \cos\zeta \sin\vartheta + d\vartheta \sin\zeta \cos\vartheta}{d\omega} = \frac{d(\sin\zeta \sin\vartheta)}{d\omega},\\
m \cos\omega - \mu \sin\omega &= \frac{-d\zeta \sin\zeta \sin\vartheta + d\vartheta \cos\zeta \cos\vartheta}{d\omega} = \frac{d(\cos\zeta \sin\vartheta)}{d\omega},\\
n \cos\omega - \nu \sin\omega &= -\frac{d\vartheta \sin\vartheta}{d\omega} = \frac{d \cos\vartheta}{d\omega}.
\end{aligned}
$$

Here it is especially noteworthy that from these found formulas the one variable s went out completely such that now the quantities l, λ, m, μ, n, ν are determined by the single variable t and do not involve the other s at all, whereas the quantities T and U implicate both variables t and s.

§30 Now, we found these two equations for defining the functions l and λ:

$$
\begin{aligned}
l \cos\omega + \lambda \cos\omega &= \sin\zeta \sin\vartheta,\\
l \cos\omega - \lambda \sin\omega &= \frac{d(\sin\zeta \sin\vartheta)}{d\omega}.
\end{aligned}
$$

Hence, the first multiplied by $\sin\omega +$ the second multiplied by $\cos\omega$:

$$l = \sin\zeta \sin\vartheta \sin\omega + \cos\omega \frac{d(\sin\zeta\sin\vartheta)}{d\omega},$$

but I. $\cos\omega -$II. $\sin\omega$ gives:

$$\lambda = \sin\zeta \sin\vartheta \cos\omega - \sin\omega \frac{d(\sin\zeta\sin\vartheta)}{d\omega}.$$

In similar way the remaining letters will be found as follows:

$$\begin{aligned} m &= \cos\zeta\sin\vartheta\sin + \cos\omega\frac{d(\cos\zeta\sin\vartheta)}{d\omega}, \\ \mu &= \cos\zeta\sin\vartheta\cos\omega - \sin\omega\frac{d(\cos\zeta\sin\vartheta)}{d\omega}, \\ n &= \cos\vartheta\sin\omega + \frac{\cos\omega\ d\cos\vartheta}{d\omega}, \\ \nu &= \cos\vartheta\cos\omega - \frac{\sin\omega\ d\cos\vartheta}{d\omega}. \end{aligned}$$

Glow and behold these suitable values for the letters l, λ, m, μ and n, ν which are of such a nature that those three formulas $ldT + \lambda dU$, $mdT + \mu dT$ and $ndT + \nu dU$ become integrable and even the integrals themselves can easily be exhibited which are of course

$$x = t - s\sin\vartheta\sin\zeta, \quad y = u - s\sin\vartheta\cos\zeta, \quad z = v - s\cos\vartheta.$$

§31 Since our two solutions have to agree completely with each other, there is no doubt that the remaining conditions mentioned above are also satisfied, it will certainly be:

$$ll + mm + nn = 1, \quad \lambda\lambda + \mu\mu + \nu\nu = 1, \quad l\lambda + m\mu + n\nu = 0.$$

To show this for the sake of brevity let us put

$$\sin\zeta\sin\vartheta = p, \quad \cos\zeta\sin\vartheta = q \quad \text{and} \quad \cos\vartheta = r,$$

such that it is

$$pp + qq + rr = 1 \quad \text{and hence} \quad pdp + qdq + rdr = 0,$$

now, because we have

$$\begin{aligned}
l \ &= p \sin\omega + \frac{dp}{d\omega}\cos\omega, \\
m &= q \sin\omega \ + \frac{dq}{d\omega}\cos\omega, \\
n \ &= r \sin\omega \ + \frac{dq}{d\omega}\cos\omega, \\
\lambda \ &= p \cos\omega - \frac{dp}{d\omega}\sin\omega, \\
\mu \ &= q \cos\omega \ - \frac{dq}{d\omega}\sin\omega, \\
\nu \ &= r \cos\omega \ - \frac{dr}{d\omega}\sin\omega,
\end{aligned}$$

having done the calculation we will hence find:

$$1^\circ \quad ll + mm + nn = (pp + qq + rr)\sin^2\omega + \frac{2\sin\omega\cos\omega}{d\omega}(pdp + qdq + rdr)$$

$$+\frac{\cos^2\omega}{d\omega^2}(dp^2 + dq^2 + dr^2)$$

or

$$ll + mm + nn = \sin^2\omega + \frac{\cos^2\omega}{d\omega^2}(dp^2 + dq^2 + dr^2),$$

and so the whole question now is on the investigation of the value $dp^2 + dq^2 + dr^2$. But because it is

$$\begin{aligned}
& dp = +d\zeta\cos\zeta\sin\vartheta + d\vartheta\sin\zeta\cos\vartheta, \\
& dq = -d\zeta\sin\zeta\sin\vartheta + d\vartheta\cos\zeta\cos\vartheta \\
\text{and} \quad & dr = -d\vartheta\sin\vartheta,
\end{aligned}$$

we conclude

$$dp^2 + dq^2 + dr^2 = d\zeta^2\sin^2\vartheta + d\vartheta^2 = d\omega^2,$$

so that it is certain that it is

$$\frac{dp^2 + dq^2 + dr^2}{d\omega^2} = 1,$$

whence it is manifest that it will be:

$$ll + mm + nn = \sin^2\omega + \cos^2\omega = 1.$$

§32 In similar manner we will find for the Greek letters:

$$\lambda\lambda + \mu\mu + \nu\nu = (pp + qq + rr)\cos^2\omega - \frac{2\sin\omega\cos\omega}{d\omega}(pdp + qdq + rdr)$$

$$+\frac{\sin^2\omega}{d\omega^2}(dp^2 + dq^2 + dr^2),$$

which manifestly yields as before

$$\lambda\lambda + \mu\mu + \nu\nu = \cos^2\omega + \sin^2\omega = 1.$$

Therefore, it remains that we examine the third property, for which we obtain:

$$\begin{aligned}
&l\lambda \;\; pp\sin\omega\cos\omega - \frac{pdp}{d\omega}\sin^2\omega + \frac{pdp}{d\omega}\cos^2\omega - \frac{dp^2}{d\omega^2}\sin\omega\cos\omega,\\
&m\mu qq\sin\omega\cos\omega - \frac{qdq}{d\omega}\sin^2\omega + \frac{qdq}{d\omega}\cos^2\omega - \frac{dq^2}{d\omega^2}\sin\omega\cos\omega,\\
&n\nu \;\; rr\sin\omega\cos\omega - \frac{rdr}{d\omega}\sin^2\omega + \frac{rdr}{d\omega}\cos^2\omega - \frac{dr^2}{d\omega^2}\sin\omega\cos\omega,
\end{aligned}$$

having collected which into one sum, it will be

$$l\lambda + m\mu + n\nu = \sin\omega\cos\omega - \sin\omega\cos\omega = 0.$$

And in this way we gave the solution of that analytical Problem mentioned above (§ 7) which solution in short is as follows.

Analytical Problem

§33 *Having propounded the two variables T and U to find six functions l, m, n and λ, μ, ν of them of such a nature that the following six conditions are satisfied:*

$$\text{I}^\circ.\ \left(\frac{dl}{dU}\right)=\left(\frac{d\lambda}{dT}\right),\quad \text{II}^\circ.\ \left(\frac{dm}{dU}\right)=\left(\frac{d\mu}{dT}\right),\quad \text{III}^\circ.\ \left(\frac{dn}{dU}\right)=\left(\frac{d\nu}{dT}\right),$$
$$\text{IV}^\circ\ \ ll+mm+nn=1,\quad \text{V}^\circ.\ \lambda\lambda+\mu\mu+\nu\nu=1,$$
$$\text{VI}^\circ.\ l\lambda+m\mu+n\nu=0.$$

Solution

Having introduced the two new variables s and t into the calculation, imagine any two functions ζ and ϑ of the latter t, which functions are considered as angles, of course, from which let the new angle ω be formed such that it is

$$d\omega=\sqrt{d\zeta^2\sin^2\vartheta+d\vartheta^2},$$

then from this the two variables T and U are indeed determined in such a way that it is

$$T=\int\frac{dt\sin\omega}{\sin\zeta\sin\vartheta}-s\sin\omega,$$
$$U=\int\frac{dt\cos\omega}{\sin\zeta\sin\vartheta}-s\cos\omega,$$

having done which the six functions we are looking for will behave as this

$$\begin{aligned}
l &= \sin\zeta\sin\vartheta\sin\omega+\frac{\cos\omega}{d\omega}d(\sin\zeta\sin\vartheta),\\
\lambda &= \sin\zeta\sin\vartheta\cos\omega-\frac{\sin\omega}{d\omega}d(\sin\zeta\sin\vartheta),\\
m &= \cos\zeta\sin\vartheta\sin\omega+\frac{\cos\omega}{d\omega}d(\cos\zeta\sin\vartheta),\\
\mu &= \cos\zeta\sin\vartheta\cos\omega-\frac{\sin\omega}{d\omega}d(\cos\zeta\sin\vartheta),\\
n &= \cos\vartheta\sin\omega+\frac{\cos\omega}{d\omega}d\cos\vartheta,\\
\nu &= \cos\vartheta\cos\omega-\frac{\sin\omega}{d\omega}d\cos\vartheta.
\end{aligned}$$

But, by means of these three values the following three differential formulas:

$$\text{I}^\circ.\ \ ldT + \lambda dU, \quad \text{II}^\circ.\ \ mdT + \mu dU, \quad \text{III}^\circ.\ \ ndT + \nu dU,$$

in which the first three conditions are contained, of course, are not only made integrable, but also the integrals themselves will be expressed in the following manner:

$$\begin{aligned}
&\text{I}^\circ. && \int (ldT \ \ + \lambda dU) = t - s \sin\vartheta \sin\zeta, \\
&\text{II.}^\circ. && \int (mdT + \mu dU) = \int \frac{dt}{\tan\zeta} - s \sin\vartheta \cos\zeta, \\
&\text{III.}^\circ. && \int (ndT \ \ + \nu dU) = \int \frac{dt}{\sin\zeta \tan\vartheta} - s \cos\vartheta,
\end{aligned}$$

which solution is therefore to be considered as complete because it contains two arbitrary functions.

§34 This expansion without any doubt is of greatest importance and especially deserves it that we with all eagerness inquire its single elements. And at first, because having introduced the letters p, q and r in such a ways that it is

$$pp + qq + rr = 1, \quad \text{and} \quad dp^2 + dq^2 + dr^2 = d\omega^2,$$

we found

$$l \sin\omega + \lambda \cos\omega = p \quad \text{and} \quad l \cos\omega - \lambda \sin\omega = \frac{dp}{d\omega},$$

if we differentiate, we will have

$$dl \sin\omega + d\lambda \cos\omega + ld\omega \cos\omega - \lambda d\omega \sin\omega = dp$$

and hence

$$dl \sin\omega + d\lambda \cos\omega = 0,$$

such that it is

$$\frac{d\lambda}{dl} = -\tan\omega.$$

In similar manner we will indeed also find

$$\frac{d\mu}{dm} = -\tan\omega \quad \text{and} \quad \frac{d\nu}{dn} = -\tan\omega.$$

Therefore, glow and behold this most beautiful property which intercedes between our six functions l, m, n and λ, μ, ν which can also represented in this way that it is

$$dl : d\lambda = dm : d\mu = dn : d\nu = -\cos\omega : \sin\omega.$$

§35 If we therefore carefully consider those things we will discover certain traces following which we will be able to find a direct solution of this most difficult problem. Of course, having constituted these equations:

$$dx = ldT + \lambda dU, \quad dy = mdT + \mu dU, \quad dz = ndT + \nu dU$$

it is convenient to observe at first that the quantities l, m, n and λ, μ, ν have to be functions of one single new variable which nevertheless has a certain relation to the two principal variables T and U. Therefore, let ω be this new variable, of which our six quantities shall be certain functions. And we already saw, if the letters p, q and r are such functions of ω that it is

$$pp + qq + rr = 1 \quad \text{and} \quad dp^2 + dq^2 + dr^2 = d\omega^2,$$

then by putting:

$$\begin{aligned}
l &= p\sin\omega + \frac{dp}{d\omega}\cos\omega, \\
m &= q\sin\omega + \frac{dq}{d\omega}\cos\omega, \\
n &= r\sin\omega + \frac{dr}{d\omega}\cos\omega, \\
\lambda &= p\cos\omega - \frac{dp}{d\omega}\sin\omega, \\
\mu &= q\cos\omega - \frac{dq}{d\omega}\sin\omega, \\
\nu &= r\cos\omega - \frac{dr}{d\omega}\sin\omega,
\end{aligned}$$

now these three conditions have to be satisfied, of course:

$$ll + mm + nn = 1, \quad \lambda\lambda + \mu\mu + \nu\nu = 1 \quad \text{and} \quad l\lambda + m\mu + n\nu = 0,$$

we furthermore already deduced from this the extraordinary property that it is

$$d\lambda = -dl \tan\omega, \quad d\mu = -dm \tan\omega \quad \text{and} \quad -dn \tan\omega,$$

which will be of immense use for us to satisfy the remaining conditions, as it will become clear soon.

§36 These three conditions certainly demand that those differential formulas exhibited for dx, dy and dz are made integrable, for which aim one has to find the relation which must intercede between the two variables T and U and between ω. To achieve this by means of an integration convert these differential equations into the following forms:

$$\begin{aligned} x &= lT \ + \lambda U - \int (Tdl \ + Ud\lambda), \\ y &= mT + \mu U - \int (Tdm + Ud\mu), \\ z &= nT \ + \nu U - \int (Tdn \ + Ud\nu); \end{aligned}$$

now, these three new integral formulas indeed will take the following forms:

$$\begin{aligned} x &= lT \ + \lambda U - \int dl \ (T - U \tan\omega), \\ y &= mT + \mu U - \int dm (T - U \tan\omega), \\ z &= nT \ + \nu U - \int dn \ (T - U \tan\omega). \end{aligned}$$

Since l,m, n are functions of the same variable ω, it is manifest that these three formulas are indeed made integrable, if only the expression $T - U \tan\omega$ was any function of the new variable ω; hence, if such a function is indicated by the letter Ω, we will have

$$T - U \tan\omega = \Omega,$$

by means of which equations the equation we are looking for interceding between the variables T, U and ω is determined.

§37 Hence, if for Ω ad libitum an arbitrary function of ω is taken, of which also, as we saw, the letters p, q and r are certain functions by means of which we already defined the letters l, m, n and λ, μ, ν, the two variables T and U must be of such a nature that $T = \Omega + U\tan\omega$, of course we only want to keep the two variables U and ω in the calculation and therefore let us introduce this value instead of T, then our three integral formulas can be represented this way:

$$\begin{aligned}
x &= l\Omega + lU\tan\omega + \lambda U - \int \Omega dl,\\
y &= m\Omega + mU\tan\omega + \mu U - \int \Omega dm,\\
z &= n\Omega + nU\tan\omega + \nu U - \int \Omega dn,
\end{aligned}$$

which expressions are easily transformed into the following

$$\begin{aligned}
x &= U(l\tan\omega + \lambda) + \int ld\Omega = \frac{Up}{\cos\omega} + \int p\sin\omega d\Omega + \int \frac{dpd\Omega}{d\omega}\cos\omega,\\
y &= U(m\tan\omega + \mu) + \int md\Omega = \frac{Uq}{\cos\omega} + \int q\sin\omega d\Omega + \int \frac{dqd\Omega}{d\omega}\cos\omega,\\
z &= U(n\tan\omega + \nu) + \int nd\Omega = \frac{Ur}{\cos\omega} + \int r\sin\omega d\Omega + \int \frac{drd\Omega}{d\omega}\cos\omega.
\end{aligned}$$

Third solution of the principal problem derived from the theory of light and shadow

§38 What is usually treated in Optics on light and shadow, mostly is restricted to the highly special case in with both the glowing and the opaque body from which the shadow is projected have a spherical shape whence either or cylindrical or a conic or a convergent or divergent shadow arises, depending on whether the opaque body was either equal to or smaller or greater than the glowing body. But whenever the shape of either the shining or the opaque or of both recedes from that of a sphere, we hardly find anything which could make us content in the books which were written on this subject; if we wanted to treat this subject in general, attributing to both bodies, the shining and the opaque, any shapes, a most difficult question arises and this questions is to be counted to that part of the analysis of the Infinite on functions of two or more variables which was begun to be constructed not so long ago.

§39 But what especially from this theory extends to our undertaking is that the shapes of the shadows are always of such a nature that their surface can be unfolded onto a plane, whence it is vice versa understood, if we were able to determine the shape of the shadow for any figure of both the shining and the opaque body, that then at the same time also our problem will be perfectly solved.

§40 That the shape of the shadow is indeed always subjected to our problem can be easily shown this way. Since the shadow is terminated by the extreme rays of the shining body which at the same time touch upon the opaque body, first it is plain that in the surface of a certain shadow infinitely many straight lines are given since the single rays proceed in straight lines; furthermore, all these rays will touch upon both the shining and the opaque body, whence, if any plane is imagined, which those two bodies touch at the same time and the point of contact on the shining body is denoted by the letter M, on the opaque on the other hand by the letter m, it is perspicuous that the straight line Mm if elongated exhibits the ray of the light by which the shadow is terminated, which is also to be understood about the other infinitely close rays which a emitted from the point M on the same tangent plane which can be considered also as the tangents of the opaque body, from which the palmary properties of our problem arises that any two infinitely close lines to be drawn in the surface at the same time are found in the same plane.

§41 But this theory of light and shadow extends too far to permit the space to discuss it here in more detail; therefore, we will only take from it what suffices to finish our present undertaking. Having put aside the shape of both the shining and the opaque body, let is consider only the shape of a conic shadow, for which aim we want to contemplate two parallel sections distant from a each by a given interval, because it is possible to attribute any arbitrary shape to which, it is manifest that this consideration contains completely all shapes of the shadows.

§42 Therefore, let (Fig. 6) these two sections be normal to the table plane and be footed perpendicular on the line Aa,

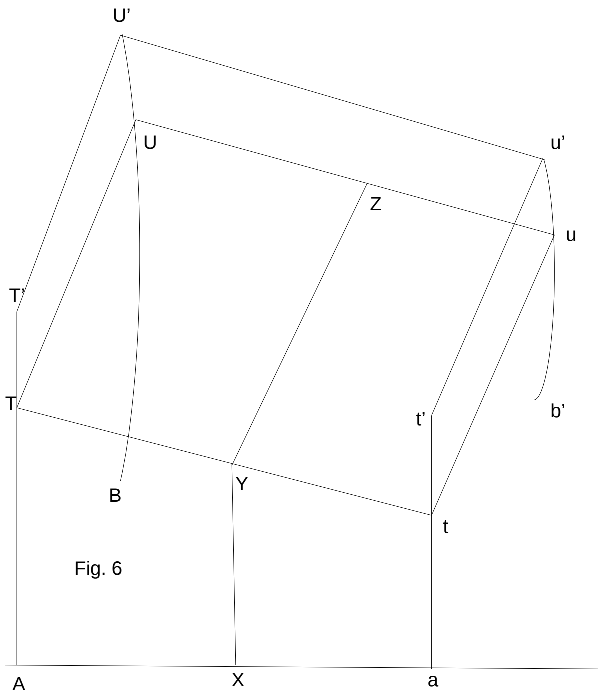

Fig. 6

and at first let BUU' be the curve, whose nature shall be expressed by an equation between the coordinates $AT = T$ and $TU = U$; in similar way, let buu' be another curve somehow different from the first, for which an equation between the coordinates $at = t$, $tu = u$ let be given, but the interval between these sections shall be put $Aa = a$; here, it will certainly be possible to consider the one section BUU' as shining plain disk, and while the other buu' refers to a opaque plain disk, the shadow-cone which we contemplate will arise from the light rays.

§43 But let the points U and u be taken in such a way that the line Uu if elongated refers to the ray terminating the shadow, because which has to lie in the plane touching both the disks, it is necessary that both elements UU' and uu' lie in the same plane with the line Uu, from what it is perspicuous that these two elements are parallel to each other whence it follows that between the differentials the same ratio has to hold such that $dT : dU = dt : du$, which is why, if it one puts $dU = \varphi dT$, it will also be $du = \varphi dt$.

§44 Therefore, consider this quantity φ as the main variable, by which all the remaining are determined in the following way. For the first curve BU let T be any function of φ, whose nature defines the properties of the curve BUU', but then it will be

$$dU = \varphi dT \quad \text{and} \quad U = \int \vartheta dT,$$

it is evident that in this way any any arbitrary curve can be expressed by means of the variable φ. In similar way, for the other curve buu' the abscissa t will certainly become equal to a function of φ and then one will equally have

$$du = \varphi dt \quad \text{and} \quad u = \int \varphi dt,$$

whence, because the two curves are completely subject to our desire, it is possible to assume any functions of φ for the letters T and t, having constituted which at the same time the two ordinates U and u are determined.

§45 Now, let us take an arbitrary point Z on the line Uu, because which point lies on the surface we investigate, from these let us drop the perpendicular ZY intersecting the line Tt to the plotting table plane and from Y let us draw the normal YX to our axis Aa that for the indefinite point Z we obtain three coordinates which we want to call:

$$AX = x, \quad XY = y \quad \text{and} \quad YZ = z,$$

and now it will be easy to find an equation between these three coordinates by means of which the nature of the surface in question is expressed.

§46 The principles of geometry immediately give us these analogies

$$T-t : a = T-y : x, \quad \text{or} \quad Tx - tx = aT - ay,$$
$$U-u : a = T-z : x, \quad \text{or} \quad Ux - ux = aU - az,$$

whence by means of the two variables φ and x it will be possible to define the two coordinates y and z, since we will have:

$$y = T - \frac{x(T-t)}{a} \quad \text{and} \quad z = U - \frac{x(U-u)}{a};$$

for, if from these two equations the variable φ together with the one depending on it T, t and U, u are eliminated, an equation expressing the nature of our surface will result.

§47 But we by no means to not want to seek for such an elimination, since the nature of the surface can be seen much clearer from the two found equations, which per se are already so simple that it would be an injustice to desire a more convenient solution, meanwhile it will nevertheless not be without use to manipulate the forms of these equations a little bit. In a more general way let us represent the values for y and z as this

$$y = P + Qx \quad \text{and} \quad z = R + Sx,$$

where the letters P, Q, R, S now denote functions of the other variable φ, and now the question is about of which kind this functions have to be that the two exhibited equations define an surface unfoldable onto a plane.

§48 Therefore, let us compare these assumed forms to those we found before and we will have, of course:

$$P = T \quad \text{and} \quad R = U, \quad Q = \frac{t-T}{a}, \quad S = \frac{u-U}{a},$$

because if T and t are arbitrary functions of φ, it is evident that the functions P and Q can be taken ad libitum, and since U and u depend on T and t, the functions R and S will also have to depend on the first two P and Q in a certain way. But because it is

$$T = P, \quad t = P + aQ, \quad U = R \quad \text{and} \quad u = R + aS,$$

let us substitute these values in the fundamental formulas

$$dU = \varphi dT \quad \text{and} \quad du = \varphi dt$$

and we will obtain

$$dR = \varphi dP \quad \text{and} \quad dR + adS = \varphi dP + a\varphi dQ$$

or $dS = \varphi dQ$.

§49 Therefore, we will also be able to eliminate the quantity φ from the calculation, because it is either $\varphi = \frac{dR}{dP}$ or $\varphi = \frac{dS}{dQ}$, such that instead of it one of the letters R and S are subject to our desires, if hence P, Q ad R were any arbitrary functions of the same variable, then S has be such a function of the same variable that it is:

$$dS = \frac{dQdR}{dP} \quad \text{or} \quad \frac{dS}{dR} = \frac{dQ}{dP};$$

this solution can even be made more beautiful in such a way that we say that for the letters P, Q, R, S one has to assume functions of certain variable of such a kind that it is $\frac{dS}{dR} = \frac{dQ}{dP}$ or even $\frac{dS}{dQ} = \frac{dR}{dP}$ if which was done these two equations

$$y = P + Qx \quad \text{and} \quad z = R + Sx$$

will express the nature of the solid we are looking for.

§50 It does not matter by means of which letter the variable, of which P, Q, R and S are functions, is indicated, one can even take one of these P, Q, R, S for it, as functions of which the remaining are then to be understood. Hence, as long as one of them retains a constant value, the remaining ones will also be constant, and then from the variability of x all straight lines will arise which can be drawn on the surface.

§51 The prescribed condition $\frac{dS}{dQ} = \frac{dR}{dP}$ will manifestly be satisfied by taking the quantities P and R as constants; hence, a particular solution of our problems follows. For, let us put that it is $P = A$ and $R = B$, such that now S is to be considered as a function of Q. But it is always possible to vary the coordinates in such a way that $A = 0$ and $B = 0$ having done which because of $Q = \frac{y}{x}$ $\frac{z}{x} = S$ will be a homogeneous function of no dimension of x and y, or z will become equal to a homogeneous function of one dimension of x and y which is the criterion for a conic surface.

§52 The condition is also satisfied by taking $Q = 0$ and $S = 0$ such that R remains an function of P, in which case for z a function of y will arise, because which involves only two variables,y and z, it will be for a cylindrical solid; the same happens, if we put either $P = 0$ and $Q = 0$ or $R = 0$ and $S = 0$; for, in the first case one has $y = 0$, in the second on the other hand $z = 0$, in both cases it is the equation for a plane.

§53 But to cognize also other species of solids of this kind let us assume for the simpler ones:

$$P = a\varphi^{\alpha}, \quad Q = b\varphi^{\beta}, \quad R = c\varphi^{\gamma}, \quad S = d\varphi^{\delta},$$

and to satisfy the prescribed condition it is necessary that it is

$$\frac{b\beta}{a\alpha}\varphi^{\beta-\alpha} = \frac{d\delta}{c\gamma}\varphi^{\delta-\gamma},$$

whence a double determination arises, the first of the exponents

$$\beta - \alpha = \delta - \gamma,$$

the other for the coefficients:

$$\frac{b\beta}{a\alpha} = \frac{d\delta}{c\gamma},$$

both of which are satisfied by taking the values as follows:

$$a = \frac{fg}{\varkappa + \lambda}, \quad b = \frac{fh}{\varkappa + \mu}, \quad c = \frac{gk}{\lambda + \nu}, \quad d = \frac{hk}{\mu + \nu},$$

$$\alpha = \vartheta + \lambda, \quad \beta = \varkappa + \mu, \quad \gamma = \lambda + \nu, \quad \delta = \mu + \nu,$$

then the equations will be:

$$y = a\varphi^{\alpha} + b\varphi^{\beta}x, \quad z = c\varphi^{\gamma} + d\varphi^{\delta}x.$$

§54 Therefore, in numbers let us consider this case:

$$y = 2\varphi + 3\varphi^2 x \quad \text{and} \quad z = \varphi^2 + 2\varphi^3 x,$$

whence after the elimination of the letter φ the following equation is found:

$$-4xy^3 - y^2 + 18xyz + 27x^2z^2 + 4z = 0,$$

which is therefore for a solid whose surface can be unfolded onto the plane.