Expansion of the integral formula $\int x^{f-1}dx(\log x)^{\frac{m}{n}}$ having extended the integration from the value $x = 0$ to $x = 1$ *

Leonhard Euler

Theorem 1

§1 *If n denotes any positive integer and the integration of the formula*

$$\int x^{f-1}dx(1-x^g)^n$$

is extended from the value $x = 0$ to $x = 1$, its value will be

$$= \frac{g^n}{f} \cdot \frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}.$$

Demonstration

It is known, that in general the integration of the formula $\int x^{f-1}dx(1-x^g)^m$ can be reduced to the integration of this one $\int x^{f-1}dx(1-x^g)^{m-1}$, since it is possible to define constant quantities A and B in such a way, that it is

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$$\int x^{f-1}dx(1-x^g)^m = A\int x^{f-1}dx(1-x^g)^{m-1} + Bx^f(1-x^g)^m;$$

for having taken differentials this equation arises

$$x^{f-1}dx(1-x^g)^m$$
$$= Ax^{f-1}dx(1-x^g)^{m-1} + Bfx^{f-1}dx(1-x^g)^m - Bmgx^{f+g-1}dx(1-x^g)^{m-1},$$

which divided by $x^{f-1}dx(1-x^g)^{m-1}$ gives

$$1-x^g = A + Bf(1-x^g) - Bmgx^g$$

or

$$1-x^g = A - Bmg + B(f+mg)(1-x^g);$$

that this equation can hold, it is necessary, that it is

$$1 = B(f+mg) \quad \text{and} \quad A = Bmg,$$

whence we conclude

$$B = \frac{1}{f+mg} \quad \text{and} \quad A = \frac{mg}{f+mg}.$$

Therefore we will have the following general reduction

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1} + \frac{1}{f+mg}x^f(1-x^g)^m;$$

because it vanishes for $x=0$, if $f>0$ of course, the addition of a constant is not necessary. Hence having extended both integrals to $x=1$ the last integral part vanishes by itself and for the case $x=1$ it will be

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1}.$$

Since for $m=1$ it is

$$\int x^{f-1}dx(1-x^g)^0 = \frac{1}{f}x^f = \frac{1}{f},$$

having put $x=1$, we obtain the following values for the same case $x=1$

$$\int x^{f-1}dx(1-x^g)^1 = \frac{g}{f} \cdot \frac{1}{f+g},$$
$$\int x^{f-1}dx(1-x^g)^2 = \frac{g^2}{f} \cdot \frac{1}{f+g} \cdot \frac{2}{f+2g},$$
$$\int x^{f-1}dx(1-x^g)^3 = \frac{g^3}{f} \cdot \frac{1}{f+g} \cdot \frac{2}{f+2g} \cdot \frac{3}{f+3g}$$

and hence for any positive integer n we conclude, that it will be

$$\int x^{f-1}dx(1-x^g)^n = \frac{g^n}{f} \cdot \frac{1}{f+g} \cdot \frac{2}{f+2g} \cdot \frac{3}{f+3g} \cdots \frac{n}{f+ng},$$

if only the number f and g are positive.

COROLLARY 1

§2 Hence vice versa the value of a product of this kind, formed from an arbitrary amount of factors, can be expressed by an integral formula, so that it is

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} = \frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$$

having extended this integral from the value $x=0$ to $x=1$.

COROLLARY 2

§3 So if one therefore has a progression of this kind

$$\frac{1}{f+g}, \quad \frac{1 \cdot 2}{(f+g)(f+2g)}, \quad \frac{1 \cdot 2 \cdot 3}{(f+g)(f+2g)(f+3g)}, \quad \frac{1 \cdot 2 \cdot 3 \cdot 4}{(f+g)(f+2g)(f+3g)(f+4g)} \quad \text{etc.},$$

its general term, that corresponds to the indefinite index n, is conveniently represented by this integral formula $\frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$, by whose means the progression and its terms, corresponding to fractional indices, can be exhibited.

COROLLARY 3

§4 If instead of n we write $n-1$, we will have

$$\frac{1\cdot 2\cdot 3\cdots(n-1)}{(f+g)(f+2g)(f+3g)\cdots(f+(n-1)g)}=\frac{f}{g^{n-1}}\int x^{f-1}dx(1-x^g)^{n-1},$$

which multiplied by $\frac{n}{f+ng}$ yields

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}=\frac{f\cdot ng}{g^n(f+ng)}\int x^{f-1}dx(1-x^g)^{n-1}.$$

REMARK 1

It would have been possible to derive this last formula immediately from the preceding one, since we just proved, that

$$\int x^{f-1}dx(1-x^g)^n=\frac{ng}{f+ng}\int x^{f-1}dx(1-x^g)^{n-1},$$

if both integrals are extended from the value $x=0$ to $x=1$; this is to be kept in mind about the determination of the integrals in everything that follows. Furthermore it is also always to be noted, that the quantities f and g are positive, which condition the given proof demands, of course. Concerning the number n, if by it one denotes the index of a certain term of the progression (§3), there is nothing to impede, that either any positive or negative numbers are denoted by it, because all terms, also corresponding to negative indices, of its progression are considered to be exhibited by the given integral formula. Nevertheless it is carefully to be noted, that this reduction

$$\int x^{f-1}dx(1-x^g)^m=\frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1}$$

is only true, if $m>0$, because otherwise the algebraic part $\frac{1}{f+mg}x^f(1-x^g)^m$ would non vanish for $x=1$.

REMARK 2

§6 Series of this kind, which can be called transcendental, because the terms, corresponding to fractional indices, are transcendental quantities, I already once studied in COMMENT. ACAD. SC. PETROP., BOOK 5 in more detail;

hence in this place I did not scrutinise those progressions as the remarkable comparisons of the integral formula, which are derived from there, more diligently. After I had shown that the value of the indefinite product $1 \cdot 2 \cdot 3 \cdots n$ is expressed by the integral formula $\int dx\left(\log \frac{1}{x}\right)^n$ extended from $x = 0$ to $x = 1$, which, if n is a postive integer, by integration itself is manifest, I examined the cases, in which a fractional number is taken for n; there it is indeed anything but clear from the integral formula itself, to what kind of transcendental quantities these terms are to be referred. But by a singular artifice I reduced the same terms to better-known quadratures, what therefore seems most worthy to be considered in more detail.

PROBLEM 1

§7 *Since it was demonstrated, that*

$$\frac{1 \cdot 2 \cdot 3 \cdot n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} = \frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$$

having extended the integral from $x = 0$ to $x = 1$, to assign the value of the same product in the case $g = 0$ by means of an integral formula.

SOLUTION

Having put $g = 0$ in the integral formula the member $(1-x^g)^n$ vanishes, but at the same time also the denominator g^n vanishes, whence the question reduces to that, that the value of the fraction $\frac{(1-x^g)^n}{g^n}$ is defined in the case $g = 0$, in which so the numerator as the denominator vanishes. For this aim let us consider g as an infinitely small quantity, and because $x^g = e^{g \log x}$, it will be $x^g = 1 + g\log x$ and hence $(1-x^g)^n = g^n(-\log x)^n = g^n\left(\log \frac{1}{x}\right)^n$; from there for this case our integral formula changes into $f\int x^{f-1}dx\left(\log \frac{1}{x}\right)^n$, so that one now has

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{f^n} = f\int x^{f-1}dx\left(\log \frac{1}{x}\right)^n$$

or

$$1 \cdot 2 \cdot 3 \cdots n = f^{n+1}\int x^{f+1}dx\left(\log \frac{1}{x}\right)^n.$$

COROLLARY 1

§8 As often as n is a positive integer, the integration of the formula $\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n$ succeds and having extended it from $x=0$ to $x=1$ indeed the product arises, that we found to be equal to this formula. But if fractional numbers are taken for n, the same formula will serve for interpolating this hypergeometric progression

$$1, \quad 1\cdot 2, \quad 1\cdot 2\cdot 3, \quad 1\cdot 2\cdot 3\cdot 4, \quad 1\cdot 2\cdot 3\cdot 4\cdot 5 \quad \text{etc.}$$

or

$$1, \quad 2, \quad 6, \quad 24, \quad 24, \quad 120, \quad 720, \quad 5040 \quad \text{etc.}$$

COROLLARY 2

§9 If the expression just found is divided by the principal one, a product will arise, whose factors proceed in an arithmetic progression,

$$(f+g)(f+2g)(f+3g)\cdots(f+ng)=f^n g^n \frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n}{\int x^{f-1}dx\,(1-x^g)^n},$$

whose values can also, if n is a fractional number, can be assigned from there.

COROLLARY 3

§10 Because it is

$$\int x^{f-1}dx(1-x^g)^n=\frac{ng}{f+ng}\int x^{f-1}dx(1-x^g)^{n-1},$$

it will also in the same way be for the case $g=0$

$$\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n=\frac{n}{f}\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}$$

and hence by those other integral formulas

$$1\cdot 2\cdot 3\cdots n=nf^n\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}$$

and

$$(f+g)(f+2g)\cdots(f+ng) = f^{n-1}g^{n-1}(f+ng)\frac{\int x^{f-1}dx\,\left(\log\frac{1}{x}\right)^{n-1}}{\int x^{f-1}dx(1-x^g)^{n-1}}.$$

REMARK

§11 Because we found, that

$$1\cdot 2\cdot 3\cdots n = f^{n+1}\int x^{f-1}dx\left(\log\frac{1}{x}\right)^n,$$

it is clear, that this integral formula does not depend on the value of the quantity f, what is also easily seen by putting $x^f = y$, whence it becomes

$$fx^{f-1}dx = dy \quad \text{and} \quad \log\frac{1}{x} = -\log x = -\frac{1}{f}\log y = \frac{1}{f}\log\frac{1}{y}$$

and therefore

$$f^n\left(\log\frac{1}{x}\right)^n = \left(\log\frac{1}{y}\right)^n,$$

so that it is

$$1\cdot 2\cdot 3\cdots n = \int dy\left(\log\frac{1}{y}\right)^n,$$

which formula arises from the first by putting $f=1$. For an interpolation of this kind the whole task is therefore reduced to that, that the values of the integral formula $\int dx\,\left(\log\frac{1}{x}\right)^n$ are defined, whenever the exponent n is a fractional number. As for example when $n=\frac{1}{2}$, one has to assign the value of the formual $\int dx\sqrt{\log\frac{1}{x}}$, which value I already once showed to be $=\frac{1}{2}\sqrt{\pi}$, while π denotes the circumference of the circle, whose diameter is $=1$; but for other fractional numbers I taught to reduce its value to quadratures of algebraic curves of higher order. Because this reduction is by no means obvious and is only valid, when the integration of the formula $\int dx\,\left(\log\frac{1}{x}\right)^n$ is extended from the value $x=0$ to $x=1$, it seems to be worth of special attention. But even though I already treated this subject once, I nevertheless, because I was led there on a quite non straightforward way, decided to resume the same here an expand it in more detail.

THEOREM 2

§12 *If the integral formulas are extended from the value $x = 0$ to $x = 1$ and n denotes a positive integer, it will be*

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)(n+2)\cdots 2n} = \frac{1}{2}ng\int x^{f+ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}},$$

no matter which positive numbers are taken for f and g.

DEMONSTRATION

Because above (§4) we showed, that

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)\cdots(f+ng)} = \frac{f \cdot ng}{g^n(f+ng)}\int x^{f-1}dx(1-x^g)^{n-1},$$

we will, if we write $2n$ instead of n, have

$$\frac{1 \cdot 2 \cdot 3 \cdots 2n}{(f+g)(f+2g)\cdots(f+2ng)} = \frac{f \cdot 2ng}{g^{2n}(f+2ng)}\int x^{f-1}dx(1-x^g)^{2n-1}.$$

Now divide the first equation by the second and this third one will arise

$$\frac{(f+(n+1)g)(f+(n+2)g)\cdots(f+2ng)}{(n+1)(n+2)\cdots 2n} = \frac{g^n(f+2ng)}{2(f+ng)} \cdot \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

But if in the first equation instead of f one writes $f + ng$, this fourth equation will arise

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+(n+1)g)(f+(n+2)g)\cdots(f+2ng)} = \frac{(f+ng)ng}{g^n(f+2ng)}\int x^{f+ng-1}dx(1-x^g)^{n-1}.$$

Multiply this fourth equation by that third one and one will find the equation to be demonstrated itself

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)(n+3)\cdots 2n} = \frac{1}{2}ng\int x^{f+ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

COROLLARY 1

§13 If in the first equation one sets $f = n$ and $g = 1$, the same product will arise

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots 2n} = \frac{1}{2}n \int x^{n-1}dx(1-x^g)^{n-1},$$

having compared which eqaution to that one we obtain

$$\frac{\int x^{n-1}dx(1-x)^{n-1}}{g\int x^{f+ng-1}dx(1-x^g)^{n-1}} = \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

COROLLARY 2

§14 If we write x^g instead of x in that equation, it will be

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots 2n} = \frac{1}{2}ng \int x^{ng-1}dx(1-x^g)^{n-1},$$

so that we reach this comparison between the following integral formulas

$$\int x^{ng-1}dx(1-x^g)^{n-1} = \int x^{f+ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

COROLLARY 3

§15 If in the equation of the theorem we put $g = 0$, because of $(1-x^g)^m = g^m\left(\log\frac{1}{x}\right)^m$ the powers of g will cancel each other and this equation will arise

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots 2n} = \frac{1}{2}n \int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1} \cdot \frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}},$$

whence we conclude

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}} = g\int x^{ng-1}dx(1-x^g)^{n-1}$$

or because of

$$\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}=\frac{f}{n}\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n}$$

this one

$$\frac{2f}{n}\cdot\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n}\right)^{2}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n}}=g\int x^{ng-1}dx(1-x^{g})^{n-1}.$$

COROLLAR 4

§16 Let us put $f=1$, $g=2$ and $n=\frac{m}{2}$ here, that m is a positive integer, and because of

$$\int dx\left(\log\frac{1}{x}\right)^{m}=1\cdot 2\cdot 3\cdots m$$

it will be

$$\frac{4}{m}\cdot\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{\frac{m}{2}}\right)^{2}}{1\cdot 2\cdot 3\cdots m}=2\int x^{m-1}dx(1-x^{2})^{\frac{m}{2}-1}$$

and hence

$$\int dx\left(\log\frac{1}{x}\right)^{\frac{m}{2}}=\sqrt{1\cdot 2\cdot 3\cdots m\cdot\frac{m}{2}\int x^{m-1}dx(1-x^{2})^{\frac{m}{2}-1}}$$

and by taking $m=1$ because of

$$\int\frac{dx}{\sqrt{1-xx}}=\frac{\pi}{2}$$

one will have

$$\int dx\sqrt{\log\frac{1}{x}}=\sqrt{\frac{1}{2}\int\frac{dx}{\sqrt{1-xx}}}=\frac{1}{2}\sqrt{\pi}.$$

REMARK

§17 So behold this succinct proof of the theorem once propounded by me, that $\int dx\sqrt{\log\frac{1}{x}}=\frac{1}{2}\sqrt{\pi}$, and behold that I did not use an argument involving interpolation, which I had used then. Here it was of course deduced from this theorem, in which I found, that

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}}=g\int x^{ng-1}dx(1-x^g)^{n-1}.$$

But the principal theorem, whence this one is deduced, behaves as follows

$$g\frac{\int x^{f-1}dx(1-x^g)^{n-1}\cdot\int x^{f+ng-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}=\int x^{n-1}dx(1-x)^{n-1};$$

for, each member by integration extended from $x=0$ to $x=1$ is expanded in this numerical product

$$\frac{1\cdot 2\cdot 3\cdots(n-1)}{(n+1)(n+2)\cdots(2n-1)}.$$

And if we want to assign a class extending further to the other member, the theorem can be propounded so, that it is

$$g\frac{\int x^{f-1}dx(1-x^g)^{n-1}\cdot\int x^{f+ng-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}=k\int x^{nk-1}dx(1-x^k)^{n-1},$$

and if here one takes $g=0$, it is

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}}=k\int x^{nk-1}dx(1-x^k)^{n-1}.$$

Therefore it is especially to be noted, that that equality holds, no matther what numbers are taken for f and g; in the case $f=g$ it is indeed clear, because

$$\int x^{g-1}dx(1-x^g)^{n-1}=\frac{1-(1-x^g)^n}{ng}=\frac{1}{ng};$$

for it will be

$$2g\int x^{ng+g-1}dx(1-x^g)^{n-1} = k\int x^{nk-1}dx(1-x^k)^{n-1},$$

and because

$$\int x^{ng+g-1}dx(1-x^g)^{n-1} = \frac{1}{2}\int x^{ng-1}dx(1-x^g)^{n-1},$$

the equality is perspicuous, because k can be taken ad libitum. But in the same way, on which we got to this theorem, it is possible to get to other similar ones.

Theorem 3

§18 *If the following integral formulas are extended from the value $x=0$ to $x=1$ and n denotes any positive integer, it will be*

$$\frac{1\cdot 2\cdot 3\cdots n}{(2n+1)(2n+2)\cdots 3n} = \frac{2}{3}ng\int x^{f+2ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}},$$

no matter which positive numbers are taken for f and g.

Demonstration

In the preceding theorem we already saw, that

$$\frac{1\cdot 2\cdot 3\cdots 2n}{(f+g)(f+2g)\cdots(f+2ng)} = \frac{f\cdot 2ng}{g^{2n}(f+2ng)}\int x^{f-1}dx(1-x^g)^{2n-1};$$

if in the same way we write $3n$ instead of n in the principal formula, we will have

$$\frac{1\cdot 2\cdot 3\cdots 3n}{(f+g)(f+2g)\cdots(f+3ng)} = \frac{f\cdot 3ng}{g^{3n}(f+3ng)}\int x^{f-1}dx(1-x^g)^{3n-1},$$

from where that equation divided by this one produces

$$\frac{(f+(2n+1)g)(f+(2n+2)g)\cdots(f+3ng)}{(2n+1)(2n+2)\cdots 3n} = \frac{2g^n(f+3ng)}{3(f+2ng)}\cdot\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}}.$$

But if we write $f+2gn$ instead of f in the principal equation (§4), we obtain this equation

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+(2n+1)g)(f+(2n+2)g)\cdots(f+3ng)}=\frac{(f+2ng)ng}{g^n(f+3ng)}\int x^{f+2ng-1}dx(1-x^g)^{n-1}.$$

Now multiply this equation by the preceding and the equation itself, which is to be proved, will arise

$$\frac{1\cdot 2\cdot 3\cdots n}{(2n+1)(2n+2)\cdot 3n}=\frac{2}{3}ng\int x^{f+2ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}}.$$

COROLLARY 1

§19 We obtain the same value from the principal equation by putting $f=2n$ and $g=1$, so that

$$\frac{1\cdot 2\cdot 3\cdots n}{(2n+1)(2n+2)\cdots 3n}=\frac{2}{3}n\int x^{2n-1}dx(1-x)^{n-1},$$

which integral formula, by writing x^k instead of x, is transformed into this one

$$\frac{2}{3}nk\int x^{2nk-1}dx(1-x^k)^{n-1},$$

so that

$$g\int x^{f+2ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{3n-1}}=k\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

COROLLARY 2

§20 If we set $g=0$ here, because of $1-x^g=g\log\frac{1}{x}$ we will have this equation

$$\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\cdot\frac{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{3n-1}}=k\int x^{2nk-1}dx(1-x^k)^{n-1};$$

because we had found before

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{2n-1}} = k\int x^{nk-1}dx(1-x^k)^{n-1},$$

by multiplying them by each other we will have these equations

$$\frac{\left(\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{n-1}\right)^2}{\int x^{f-1}dx\left(\log\frac{1}{x}\right)^{3n-1}} = k^2\int x^{nk-1}dx(1-x^k)^{n-1}\cdot\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

COROLLARY 3

§21 Without any restriction one can put $f=1$ here; then for $n=\frac{1}{3}$ and $k=3$ it will therefore be

$$\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{-\frac{2}{3}}\right)^3}{\int dx\left(\log\frac{1}{x}\right)^0} = 9\int dx(1-x^3)^{-\frac{2}{3}}\cdot\int xdx(1-x^3)^{-\frac{2}{3}}$$

and because of

$$\int dx\left(\log\frac{1}{x}\right)^{-\frac{2}{3}} = 3\int dx\left(\log\frac{1}{x}\right)^{\frac{1}{3}} \quad\text{and}\quad \int dx\left(\log\frac{1}{x}\right)^0 = 1$$

$$\left(\int dx\left(\log\frac{1}{x}\right)^{\frac{1}{3}}\right)^3 = \frac{1}{3}\int dx(1-x^3)^{-\frac{2}{3}}\cdot\int xdx(1-x^3)^{-\frac{2}{3}};$$

but then for $n=\frac{2}{3}$ and $k=3$ it will be

$$\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{-\frac{1}{3}}\right)^3}{\int dx\log\frac{1}{x}} = 9\int xdx(1-x^3)^{-\frac{1}{3}}\cdot\int x^3dx(1-x^3)^{-\frac{1}{3}}$$

or

$$\left(\int dx\left(\log\frac{1}{x}\right)^{\frac{2}{3}}\right)^3 = \frac{4}{3}\int xdx(1-x^3)^{-\frac{1}{3}}\cdot\int x^3dx(1-x^3)^{-\frac{1}{3}}.$$

General Theorem

§22 *If the following integral formulas are extended from the value $x = 0$ to $x = 1$ and n denotes any positive integer, it will be*

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(\lambda n+1)(\lambda n+2)\cdots(\lambda+1)n} = \frac{\lambda}{\lambda+1} ng \int x^{f+\lambda ng-1} dx (1-x^g)^{n-1} \cdot \frac{\int x^{f-1} dx (1-x^g)^{\lambda n-1}}{\int x^{f-1} dx (1-x^g)^{(\lambda+1)n-1}},$$

no matter which positive numbers are taken for the letters f and g.

Demonstration

Because, as we showed above, it is

$$\frac{1 \cdot 2 \cdots n}{(f+g)(f+2g)\cdots(f+ng)} = \frac{f \cdot ng}{g^n(f+ng)} \int x^{f-1} dx (1-x^g)^{n-1},$$

if we write λn instead of n here at first, but then $(\lambda+1)n$, we will obtain these two equations

$$\frac{1 \cdot 2 \cdots \lambda n}{(f+g)(f+2g)\cdots(f+\lambda ng)} = \frac{f \cdot \lambda ng}{g^{\lambda n}(f+\lambda ng)} \int x^{f-1} dx (1-x^g)^{\lambda n-1},$$

$$\frac{1 \cdot 2 \cdots (\lambda+1)n}{(f+g)(f+2g)\cdots(f+(\lambda+1)ng)} = \frac{f \cdot (\lambda+1)ng}{g^{(\lambda+1)n}(f+(\lambda+1)ng)} \int x^{f-1} dx (1-x^g)^{(\lambda+1)n-1},$$

of which that one divided by this one yields

$$\frac{(f+\lambda ng+g)(f+\lambda ng+2g)\cdots(f+\lambda ng+ng)}{(\lambda n+1)(\lambda n+2)\cdots(\lambda n+n)} = g^n \frac{\lambda(f+\lambda ng+ng)}{(\lambda+1)(f+\lambda ng)} \cdot \frac{\int x^{f-1} dx (1-x^g)^{\lambda n-1}}{\int x^{f-1} dx (1-x^g)^{(\lambda+1)n-1}}.$$

But if we in the first equation write $f + \lambda ng$ instead of f, we will obtain

$$\frac{1 \cdot 2 \cdots n}{(f+\lambda ng+g)(f+\lambda ng+2g)\cdots(f+\lambda ng+ng)} = \frac{(f+\lambda ng)ng}{g^n(f+\lambda ng+ng)} \int x^{f+\lambda ng-1} dx (1-x^g)^{n-1},$$

which two equations multiplied by each other produce the equation to be demonstrated itself

$$\frac{1 \cdot 2 \cdots n}{(\lambda n+1)(\lambda n+2)\cdots(\lambda n+n)} = \frac{\lambda ng}{\lambda+1} \int x^{f+\lambda ng-1} dx (1-x^g)^{n-1} \cdot \frac{\int x^{f-1} dx (1-x^g)^{\lambda n-1}}{\int x^{f-1} dx (1-x^g)^{(\lambda+1)n-1}}.$$

COROLLARY 1

§23 If we put $f = \lambda n$ and $g = 1$ in the principal equation, we will also find

$$\frac{1 \cdot 2 \cdots n}{(\lambda n + 1)(\lambda n + 2) \cdots (\lambda n + n)} = \frac{\lambda n}{\lambda + 1} \int x^{\lambda n - 1} dx (1 - x)^{n-1},$$

which form, by writing x^k instead of x changes into this one

$$\frac{\lambda n k}{\lambda + 1} \int x^{\lambda n k - 1} dx (1 - x^k)^{n-1},$$

so that we have this very far-extending theorem

$$g \int x^{f + \lambda n g - 1} dx (1 - x^g)^{n-1} \cdot \frac{\int x^{f-1} dx (1 - x^g)^{\lambda n - 1}}{\int x^{f-1} dx (1 - x^g)^{\lambda n + n - 1}} = k \int x^{\lambda n k - 1} dx (1 - x^k)^{n-1}.$$

COROLLARY 2

§24 This theorem now holds, even if n is not an integer; let us even, because the number λ can be taken ad libitum, write m instead of λn and we will reach this theorem

$$\frac{\int x^{f-1} dx (1 - x^g)^{m-1}}{\int x^{f-1} dx (1 - x^g)^{m+n-1}} = \frac{k \int x^{mk-1} dx (1 - x^k)^{n-1}}{g \int x^{f + mg - 1} dx (1 - x^g)^{n-1}}.$$

COROLLARY 3

§25 If we put $g = 0$, because of $1 - x^g = g \log \frac{1}{x}$ this theorem will take this form

$$\frac{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{m-1}}{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{m+n-1}} = \frac{k \int x^{mk-1} dx (1 - x^k)^{n-1}}{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{n-1}},$$

which is more conveniently represented as follows

$$\frac{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{n-1} \cdot \int x^{f-1} dx \left(\log \frac{1}{x}\right)^{m-1}}{\int x^{f-1} dx \left(\log \frac{1}{x}\right)^{m+n-1}} = k \int x^{mk-1} dx (1 - x^k)^{n-1},$$

where its evident, that the numbers m and n can be permutated.

Remark

§26 So we found two sources, whence it is possible to scoop innumerable comparisions of integral formulas; the one source, opened up in § 24, contains integral formulas of this kind

$$\int x^{p-1}dx(1-x^g)^{q-1},$$

which I already treated some time ago in the observations on the integral formulas

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1},$$

extended from the value $x=0$ to $x=1$, where I showed at first, that the letters p and q can be commuted, that it is

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1} = \int x^{q-1}dx(1-x^n)^{\frac{p}{n}-1},$$

but then that

$$\int \frac{x^{p-1}dx}{(1-x^n)^{\frac{p}{n}}} = \frac{\pi}{n\sin\frac{p\pi}{n}};$$

But especially I demonstrated that

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}\cdot\int \frac{x^{p+q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}} = \int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int \frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}},$$

in which equation the comparison found in § 24 is already contained, so that from this nothing new, I did not already expand, can be deduced. Therefore I here mainly attempt to investigate the other source indicated in § 25; since without any restriction one can take $f=1$, our primary question will be

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{n-1}\cdot\int dx\left(\log\frac{1}{x}\right)^{m-1}}{\int dx\left(\log\frac{1}{x}\right)^{m+n-1}} = k\int x^{mk-1}dx(1-x^k)^{n-1},$$

by means of which the values of the integral formula $\int dx\left(\log\frac{1}{x}\right)^{\lambda}$, whenever λ is not an integer, can be reduced to quadratures of algebraic curves; since, as often as λ is an integer, one can carry out the integration, because

$$\int dx \left(\log \frac{1}{x}\right)^{\lambda} = 1 \cdot 2 \cdot 3 \cdots \lambda.$$

But the question of greatest importance is about the cases, in which λ is a fractional number. Therefore I will define these for the kinds of the denomination successively here.

PROBLEM 2

§27 *While i denotes a positive integer, to define the value of the integral formula $\int dx \left(\log \frac{1}{x}\right)^{\frac{i}{2}}$, having extended the integration from $x = 0$ to $x = 1$.*

SOLUTION

In our general equation let us put $m = n$ and it will be

$$\frac{\left(\int dx \left(\log \frac{1}{x}\right)^{n-1}\right)^2}{\int dx \left(\log \frac{1}{x}\right)^{2n-1}} = k \int x^{nk-1} dx (1 - x^k)^{n-1}.$$

Now let be $n - 1 = \frac{i}{2}$ and because of $2n - 1 = i + 1$ it will be

$$\int dx \left(\log \frac{1}{x}\right)^{2n-1} = 1 \cdot 2 \cdot 3 \cdots (i+1);$$

now further take $k = 2$, that $nk - 1 = i + 1$, and it will be

$$\frac{\left(\int dx \sqrt{\left(\log \frac{1}{x}\right)^i}\right)^2}{1 \cdot 2 \cdot 3 \cdots (i+1)} = 2 \int x^{i+1} dx (1 - x^2)^{\frac{i}{2}}$$

and hence

$$\frac{\int dx \sqrt{\left(\log \frac{1}{x}\right)^i}}{\sqrt{1 \cdot 2 \cdot 3 \cdots (i+1)}} = \sqrt{2 \int x^{i+1} dx (1 - x^2)^{\frac{i}{2}}},$$

where it is evident, that it is convenient to take only odd numbers for i, because for the even ones the expansion is manifest per se.

COROLLARY 1

§28 But all cases are easily reduced to $i = 1$ or even to $i = -1$; for, as long as $i + 1$ is not a negative number, the reduction found holds. For this case it will therefore be

$$\int \frac{dx}{\sqrt{\log \frac{1}{x}}} = \sqrt{2 \int \frac{dx}{\sqrt{1 - xx}}} = \sqrt{\pi}$$

because of $\int \frac{dx}{\sqrt{1-xx}} = \frac{\pi}{2}$.

COROLLARY 2

§29 But having settled this principal cases because of

$$\int dx \left(\log \frac{1}{x}\right)^n = n \int dx \left(\log \frac{1}{x}\right)^{n-1}$$

we will have

$$\int dx \sqrt{\log \frac{1}{x}} = \frac{1}{2}\sqrt{\pi}, \quad \int dx \left(\log \frac{1}{x}\right)^{\frac{3}{2}} = \frac{1 \cdot 3}{2 \cdot 2}\sqrt{\pi},$$

and in general

$$\int dx \left(\log \frac{1}{x}\right)^{\frac{2n+1}{2}} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdots \frac{2n+1}{2}\sqrt{\pi}.$$

PROBLEM 3

§30 *While i denotes a positive integer, to define the value of the integral formula* $\int dx \left(\log \frac{1}{x}\right)^{\frac{i}{3}-1}$, *having extended the integration from* $x = 0$ *to* $x = 1$.

SOLUTION

Let us start from the equation of the preceding problem

$$\frac{\left(\int dx \left(\log \frac{1}{x}\right)^{n-1}\right)^2}{\int dx \left(\log \frac{1}{x}\right)^{2n-1}} = k \int x^{nk-1} dx (1 - x^k)^{n-1}$$

and let us put $m = 2n$ in the general formula, that one has

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{n-1}\cdot\int dx\left(\log\frac{1}{x}\right)^{2n-1}}{\int dx\left(\log\frac{1}{x}\right)^{3n-1}} = k\int x^{2nk-1}dx(1-x^k)^{n-1},$$

and by multiplying these two equations we obtain

$$\frac{\left(\int dx\left(\log\frac{1}{x}\right)^{n-1}\right)^3}{\int dx\left(\log\frac{1}{x}\right)^{3n-1}} = kk\int x^{nk-1}dx(1-x^k)^{n-1}\cdot\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

Now just put $n = \frac{i}{3}$ here, that

$$\int dx\left(\log\frac{1}{x}\right)^{i-1} = 1\cdot 2\cdot 3\cdots(i-1),$$

and take $k = 3$ and it will arise

$$\frac{\left(\int dx\sqrt[3]{\left(\log\frac{1}{x}\right)^{i-3}}\right)^3}{1\cdot 2\cdot 3\cdots(i-1)} = 9\int x^{i-1}dx\sqrt[3]{(1-x^3)^{i-3}}\cdot\int x^{2i-1}dx\sqrt[3]{(1-x^3)^{i-3}},$$

whence we conclude

$$\frac{\int dx\sqrt[3]{\left(\log\frac{1}{x}\right)^{i-3}}}{\sqrt{1\cdot 2\cdot 3\cdots(i-1)}} = \sqrt[3]{9\int\frac{x^{i-1}dx}{\sqrt[3]{(1-x^3)^{3-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[3]{(1-x^3)^{3-i}}}}$$

COROLLARY 1

§31 Here two principal cases occur, on which all remaining depend, of course by putting either $i = 1$ or $i = 2$, which are

$$\text{I.}\quad \int\frac{dx}{\sqrt[3]{\left(\log\frac{1}{x}\right)^2}} = \sqrt[3]{9\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}},$$

$$\text{II.}\quad \int\frac{dx}{\sqrt[3]{\log\frac{1}{x}}} = \sqrt[3]{9\int\frac{dx}{\sqrt[3]{1-x^3}}\cdot\int\frac{x^3dx}{\sqrt[3]{1-x^3}}},$$

which last formula because of

$$\int \frac{x^3 dx}{\sqrt[3]{1-x^3}} = \frac{1}{3} \int \frac{dx}{\sqrt[3]{1-x^3}}$$

changes into

$$\int \frac{dx}{\sqrt[3]{\log \frac{1}{x}}} = \sqrt[3]{\int \frac{dx}{\sqrt[3]{1-x^3}} \cdot \int \frac{xdx}{\sqrt[3]{1-x^3}}}$$

COROLLARY 2

§32 If we as in my observations mentioned before for the sake of brevity put

$$\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right)$$

and as there for this class

$$\left(\frac{2}{1}\right) = \frac{\pi}{3\sin\frac{\pi}{3}} = \alpha,$$

but then

$$\left(\frac{1}{1}\right) = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A,$$

it will be

$$\text{I.} \quad \int \frac{dx}{\sqrt[3]{\left(\log\frac{1}{x}\right)^2}} = \sqrt[3]{9\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)} = \sqrt[3]{9\alpha A},$$

$$\text{II.} \quad \int \frac{dx}{\sqrt[3]{\left(\log\frac{1}{x}\right)^1}} = \sqrt[3]{3\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)} = \sqrt[3]{\frac{3\alpha\alpha}{A}}.$$

COROLLARY 3

§33 For the first case we will therefore have

$$\int dx \sqrt[3]{\left(\log\frac{1}{x}\right)^{-2}} = \sqrt[3]{9\alpha A}, \quad \int dx \sqrt[3]{\log\frac{1}{x}} = \frac{1}{3}\sqrt[3]{9\alpha A}$$

and

$$\int dx \sqrt[3]{\left(\log\frac{1}{x}\right)^{3n+1}} = \frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdots\frac{3n+1}{3}\sqrt[3]{9\alpha A},$$

but for the other case on the other hand

$$\int dx \sqrt[3]{\left(\log\frac{1}{x}\right)^{-1}} = \sqrt[3]{\frac{3\alpha\alpha}{A}}, \quad \int dx \sqrt[3]{\left(\log\frac{1}{x}\right)^{2}} = \frac{2}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}$$

and

$$\int dx \sqrt[3]{\left(\log\frac{1}{x}\right)^{3n-1}} = \frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\cdots\frac{3n-1}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}.$$

PROBLEM 4

While i denotes a positive integer, to define the value of the integral formula $\int dx \left(\log\frac{1}{x}\right)^{\frac{i}{4}-1}$, *having extended the integration from* $x = 0$ *to* $x = 1$.

SOLUTION

In the solution of the preceding problem we were led to this equation

$$\frac{\left(\int dx \left(\log\frac{1}{x}\right)^{n-1}\right)^3}{\int dx \left(\log\frac{1}{x}\right)^{3n-1}} = kk\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}};$$

but the general formula by taking $m = 3n$ yields

$$\frac{\int dx \left(\log\frac{1}{x}\right)^{n-1}\cdot\int dx \left(\log\frac{1}{x}\right)^{3n-1}}{\int dx \left(\log\frac{1}{x}\right)^{4n-1}} = k\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}},$$

by combining which formulas we obtain

$$\frac{\left(\int dx \left(\log \frac{1}{x}\right)^{n-1}\right)^4}{\int dx \left(\log \frac{1}{x}\right)^{4n-1}} = k^3 \int \frac{x^{nk-1}dx}{(1-x^k)^{1-n}} \cdot \int \frac{x^{2nk-1}dx}{(1-x^k)^{1-n}} \cdot \int \frac{x^{3nk-1}dx}{(1-x^k)^{1-n}}.$$

Let $n = \frac{i}{4}$ and take $k = 4$ and it will be

$$\frac{\int dx \left(\log \frac{1}{x}\right)^{\frac{i}{4}-1}}{\sqrt[4]{1 \cdot 2 \cdot 3 \cdots (i-1)}} = \sqrt[4]{4^3 \int \frac{x^{i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}} \cdot \int \frac{x^{2i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}} \cdot \int \frac{x^{3i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}}.$$

COROLLARY 1

§35 So if $i = 1$, we will have

$$\int dx \sqrt[4]{\left(\log \frac{1}{x}\right)^{-3}} = \sqrt[4]{4^3 \int \frac{dx}{\sqrt[4]{(1-x^4)^3}} \cdot \int \frac{xdx}{\sqrt[4]{(1-x^4)^3}} \cdot \int \frac{x^2dx}{\sqrt[4]{(1-x^4)^3}}};$$

if this expression is denoted by the letter P, it will be in general

$$\int dx \sqrt[4]{\left(\log \frac{1}{x}\right)^{4n-3}} = \frac{1}{4} \cdot \frac{5}{4} \cdot \frac{9}{4} \cdots \frac{4n-3}{4} P.$$

COROLLARY 2

§36 For the other principal case let us take $i = 3$ and it will be

$$\int dx \sqrt[4]{\left(\log \frac{1}{x}\right)^{-1}} = \sqrt[4]{2 \cdot 4^3 \int \frac{x^2dx}{\sqrt[4]{1-x^4}} \cdot \int \frac{x^5dx}{\sqrt[4]{1-x^4}} \cdot \int \frac{x^8dx}{\sqrt[4]{1-x^4}}}$$

or after a reduction to simpler forms

$$\int dx \sqrt[4]{\left(\log \frac{1}{x}\right)^{-1}} = \sqrt[4]{8 \int \frac{xxdx}{\sqrt[4]{1-x^4}} \cdot \int \frac{xdx}{\sqrt[4]{1-x^4}} \cdot \int \frac{dx}{\sqrt[4]{1-x^4}}};$$

if the expression is denoted by the letter Q, it will be in general

$$\int dx \sqrt[4]{\left(\log \frac{1}{x}\right)^{4n-1}} = \frac{3}{4} \cdot \frac{7}{4} \cdot \frac{11}{4} \cdots \frac{4n-1}{4} Q.$$

Remark

§37 If we indicate the integral formula $\int \frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}}$ by the sign $\left(\frac{p}{q}\right)$, the solution in general will behave as follows

$$\int dx \sqrt[4]{\log\left(\frac{1}{x}\right)^{i-4}} = \sqrt[4]{1\cdot 2\cdot 3\cdots(i-1)4^3\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)}$$

and for the two cases expanded before

$$P = \sqrt[4]{4^3\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)} \quad \text{and} \quad Q = \sqrt[4]{8\left(\frac{3}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)}.$$

Now let us put for the formulas depending on the circle

$$\left(\frac{3}{1}\right) = \frac{\pi}{4\sin\frac{\pi}{4}} = \alpha \quad \text{and} \quad \left(\frac{2}{2}\right) = \frac{\pi}{4\sin\frac{2\pi}{4}} = \beta,$$

but for transcendental ones of higher order

$$\left(\frac{2}{1}\right) = \int \frac{xdx}{\sqrt[4]{(1-x^4)^3}} = \int \frac{dx}{\sqrt[2]{1-x^4}} = A,$$

on which all remaining depend, and we will find

$$P = \sqrt[4]{4^3\frac{\alpha\alpha}{\beta}AA} \quad \text{and} \quad Q = \sqrt[4]{4\alpha\alpha\beta\frac{1}{AA}},$$

whence it is clear that

$$PQ = 4\alpha = \frac{\pi}{\sin\frac{\pi}{4}}.$$

But because it is $\alpha = \frac{\pi}{2\sqrt{2}}$ and $\beta = \frac{\pi}{4}$, it will be

$$P = \sqrt[4]{32\pi AA} \quad \text{and} \quad Q = \sqrt[4]{\frac{\pi^3}{8AA}} \quad \text{and} \quad \frac{P}{Q} = \frac{4A}{\sqrt{\pi}}.$$

Problem 5

§38 *While i denotes a positive integer, to define the value of the integral formula* $\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{i-5}}$*, having extended the integration from* $x=0$ *to* $x=1$.

SOLUTION

From the preceding solutions it is already perspicuous enough, that for this case one will finally reach this formula

$$\frac{\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{i-5}}}{\sqrt[5]{1\cdot 2\cdot 3\cdots(i-1)}}=\sqrt[5]{5^4\int\frac{x^{i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}}\cdot\int\frac{x^{3i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}}\cdot\int\frac{x^{4i-1}dx}{\sqrt[5]{(1-x^5)^{5-i}}}},$$

which integral formulas are to be referred to the fifth class of my dissertation mentionend above. Hence if in they same way as there the sign $\left(\frac{p}{q}\right)$ denotes this formula $\int\frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}}$, the value looked for can be more conveniently expressed so, that

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{i-5}}=\sqrt[5]{1\cdot 2\cdot 3\cdots(i-1)5^4\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\left(\frac{4i}{i}\right)},$$

where it indeed suffices, to have assigned values smaller than five to i; but whenever the numerators exceed five, it is to be noted that

$$\left(\frac{5+m}{i}\right)=\frac{m}{m+i}\left(\frac{m}{i}\right),$$

but then further

$$\left(\frac{10+m}{i}\right)=\frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\left(\frac{m}{i}\right),$$

$$\left(\frac{15+m}{i}\right)=\frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\cdot\frac{m+10}{m+i+10}\left(\frac{m}{i}\right).$$

Furthermore for this class indeed two formulas involve the quadrature of the circle, which are

$$\left(\frac{4}{1}\right)=\frac{\pi}{5\sin\frac{\pi}{5}}=\alpha\quad\text{and}\quad\left(\frac{3}{2}\right)=\frac{\pi}{5\sin\frac{2\pi}{5}}=\beta,$$

but two contain higher quadratures, which we want to put

$$\left(\frac{3}{1}\right)=\int\frac{xxdx}{\sqrt[5]{(1-x^5)^4}}=\int\frac{dx}{\sqrt[5]{(1-x^5)^2}}=A\quad\text{and}\quad\left(\frac{2}{2}\right)=\int\frac{xdx}{\sqrt[5]{(1-x^5)^3}}=B,$$

and from these I assigned the values of all remaining formulas of this class, of course

$$
\begin{array}{lllll}
\left(\frac{5}{1}\right) = 1, & \left(\frac{5}{2}\right) = \frac{1}{2}, & \left(\frac{5}{3}\right) = \frac{1}{3} & \left(\frac{5}{4}\right) = \frac{1}{4}, & \left(\frac{5}{5}\right) = \frac{1}{5}; \\
\left(\frac{4}{1}\right) = \alpha, & \left(\frac{4}{2}\right) = \frac{\beta}{A}, & \left(\frac{4}{3}\right) = \frac{\beta}{2B} & \left(\frac{4}{4}\right) = \frac{\alpha}{3A}; & \\
\left(\frac{3}{1}\right) = A, & \left(\frac{3}{2}\right) = \beta, & \left(\frac{3}{3}\right) = \frac{\beta\beta}{\alpha B}; & & \\
\left(\frac{2}{1}\right) = \frac{\alpha B}{\beta}, & \left(\frac{2}{2}\right) = B; & & & \\
\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}. & & & &
\end{array}
$$

COROLLARY 1

Having taken the exponent $i = 1$, it will be

$$\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{-4}} = \sqrt[5]{5^4 \left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)} = \sqrt[5]{5^4 \frac{\alpha^3}{\beta^2} A^2 B},$$

whence we conclude in general, that while n denotes any positive integer

$$\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-4}} = \frac{1}{5}\cdot\frac{6}{5}\cdot\frac{11}{5}\cdots\frac{5n-4}{5}\sqrt[5]{5^4\frac{\alpha^3}{\beta}A^2B}.$$

COROLLARY 2

§40 Now let $i = 2$, and because it arises

$$\int dx \sqrt[5]{\left(\log\frac{1}{x}\right)^{-3}} = \sqrt[5]{5^4 \left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)\left(\frac{8}{2}\right)},$$

because of

$$\left(\frac{6}{2}\right) = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{1}\right) \quad \text{and} \quad \left(\frac{8}{2}\right) = \frac{3}{3}\left(\frac{3}{2}\right)$$

it will be this expression

$$\sqrt[5]{5^3\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)}=\sqrt[5]{5^3\alpha\beta\frac{BB}{A}}$$

and in general

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-3}}=\frac{2}{5}\cdot\frac{7}{5}\cdot\frac{12}{5}\cdots\frac{5n-3}{5}\sqrt[5]{5^3\alpha\beta\frac{BB}{A}}.$$

COROLLARY 3

§41 Let $i=3$ and the form found

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{-2}}=\sqrt[5]{2\cdot 5^4\left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{9}{3}\right)\left(\frac{12}{3}\right)}$$

because of

$$\left(\frac{6}{3}\right)=\frac{1}{4}\left(\frac{3}{1}\right),\quad\left(\frac{9}{3}\right)=\frac{4}{7}\left(\frac{4}{3}\right),\quad\left(\frac{12}{3}\right)=\frac{2}{5}\cdot\frac{7}{10}\left(\frac{3}{2}\right)$$

changes into

$$\sqrt[5]{2\cdot 5^2\left(\frac{3}{3}\right)\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)\left(\frac{3}{2}\right)}=\sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}},$$

whence it is concluded in general

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-2}}=\frac{3}{5}\cdot\frac{8}{5}\cdot\frac{13}{5}\cdots\frac{5n-2}{5}\sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}}.$$

COROLLARY 4

§42 Finally for $i=4$ our form

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{-1}}=\sqrt[5]{6\cdot 5^4\left(\frac{4}{4}\right)\left(\frac{8}{4}\right)\left(\frac{12}{4}\right)\left(\frac{16}{4}\right)}$$

because of

$$\binom{8}{4} = \frac{3}{7}\binom{4}{3}, \quad \binom{12}{4} = \frac{2}{6}\cdot\frac{7}{11}\binom{4}{2}, \quad \binom{16}{4} = \frac{1}{5}\cdot\frac{6}{10}\cdot\frac{11}{15}\binom{4}{1}$$

will be transformed into this one

$$\sqrt[5]{6\cdot 5\left(\frac{4}{4}\right)\left(\frac{4}{3}\right)\left(\frac{4}{2}\right)\left(\frac{4}{1}\right)} = \sqrt[5]{5\frac{\alpha\alpha\beta\beta}{AAB}},$$

so that it is in general

$$\int dx\sqrt[5]{\left(\log\frac{1}{x}\right)^{5n-1}} = \frac{4}{5}\cdot\frac{9}{5}\cdot\frac{14}{5}\cdots\frac{5n-1}{5}\sqrt[5]{5\alpha\alpha\beta\beta\frac{1}{AAB}}.$$

REMARK

§43 If we represent the value of the integral formula $\int dx\left(\log\frac{1}{x}\right)^{\lambda}$ by the sign $[\lambda]$, the cases expanded up to now yield

$$\left[-\frac{4}{5}\right] = \sqrt[5]{5^4\frac{\alpha^3}{\beta^2}\cdot A^2B}, \qquad \left[+\frac{1}{5}\right] = \frac{1}{5}\sqrt[5]{5^4\frac{\alpha^3}{\beta^2}\cdot A^2B},$$

$$\left[-\frac{3}{5}\right] = \sqrt[5]{5^3\alpha\beta\cdot\frac{BB}{A}}, \qquad \left[+\frac{2}{5}\right] = \frac{2}{5}\sqrt[5]{5^3\alpha\beta\frac{BB}{A}},$$

$$\left[-\frac{2}{5}\right] = \sqrt[5]{5^2\frac{\beta^4}{\beta}\cdot\frac{A}{BB}}, \qquad \left[+\frac{3}{5}\right] = \frac{3}{5}\sqrt[5]{5^2\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}},$$

$$\left[-\frac{1}{5}\right] = \sqrt[5]{5\alpha^2\beta^2\cdot\frac{1}{AAB}}, \qquad \left[+\frac{4}{5}\right] = \frac{4}{5}\sqrt[5]{5\alpha^2\beta^2\cdot\frac{1}{AAB}},$$

whence by combining two, whose indices added give 0, we conclude

$$\left[+\frac{1}{5}\right] \cdot \left[-\frac{1}{5}\right] = \alpha \quad = \frac{\pi}{5\sin\frac{\pi}{5}},$$

$$\left[+\frac{2}{5}\right] \cdot \left[-\frac{2}{5}\right] = 2\beta = \frac{2\pi}{5\sin\frac{2\pi}{5}},$$

$$\left[+\frac{3}{5}\right] \cdot \left[-\frac{3}{5}\right] = 3\beta = \frac{3\pi}{5\sin\frac{3\pi}{5}},$$

$$\left[+\frac{4}{5}\right] \cdot \left[-\frac{4}{5}\right] = 4\alpha = \frac{\pi}{5\sin\frac{4\pi}{5}}.$$

But from the preceding problem we deduce in the same way

$$\begin{aligned} \left[-\frac{3}{4}\right] &= P = \sqrt[4]{4^3\frac{\alpha\alpha}{\beta}\cdot AA}, \quad & \left[+\frac{1}{4}\right] &= \frac{1}{4}\sqrt[4]{4^3\frac{\alpha\alpha}{\beta}\cdot AA}, \\ \left[-\frac{1}{4}\right] &= Q = \sqrt[4]{4\alpha\alpha\beta\cdot\frac{1}{AA}}, \quad & \left[+\frac{3}{4}\right] &= \frac{3}{4}\sqrt[4]{4\alpha\alpha\beta\cdot\frac{1}{AA}} \end{aligned}$$

and hence

$$\begin{aligned} \left[+\frac{1}{4}\right] \cdot \left[-\frac{1}{4}\right] &= \alpha \quad = \frac{\pi}{4\sin\frac{\pi}{4}}, \\ \left[+\frac{3}{4}\right] \cdot \left[-\frac{3}{4}\right] &= 3\alpha = \frac{3\pi}{4\sin\frac{3\pi}{4}}, \end{aligned}$$

whence in general we obtain this theorem, that it is

$$[\lambda] \cdot [-\lambda] = \frac{\lambda\pi}{\sin\lambda\pi},$$

the reason for which can be given from the interpolation method once exposed as follows. Because it is

$$[\lambda] = \frac{1^{1-\lambda}\cdot 2^{\lambda}}{1+\lambda} \cdot \frac{2^{1-\lambda}\cdot 3^{\lambda}}{2+\lambda} \cdot \frac{3^{1-\lambda}\cdot 4^{\lambda}}{3+\lambda} \cdot \text{etc.},$$

it will be

$$[-\lambda] = \frac{1^{1+\lambda} \cdot 2^{-\lambda}}{1-\lambda} \cdot \frac{2^{1+\lambda} \cdot 3^{-\lambda}}{2-\lambda} \cdot \frac{3^{1+\lambda} \cdot 4^{-\lambda}}{3-\lambda} \cdot \text{etc.}$$

and hence

$$[\lambda] \cdot [-\lambda] = \frac{1 \cdot 1}{1-\lambda\lambda} \cdot \frac{2 \cdot 2}{4-\lambda\lambda} \cdot \frac{3 \cdot 3}{9-\lambda\lambda} \cdot \text{etc.} = \frac{\lambda\pi}{\sin\lambda\pi},$$

as I demonstrated elsewhere.

Problem 6 - General Problem

§44 *If the letters i and n denote positive integers, to define the value of the integral formula*

$$\int dx \left(\log\frac{1}{x}\right)^{\frac{i-n}{n}} \quad \textit{or} \quad \int dx \sqrt[n]{\left(\log\frac{1}{x}\right)^{i-n}}$$

having extended the integration from $x = 0$ to $x = 1$.

Solution

The used method will exhibit the value looked for expressed by means of quadratures of algebraic curves in the following way

$$\frac{\int dx \sqrt[n]{\left(\log\frac{1}{x}\right)^{i-n}}}{\sqrt[n]{1 \cdot 2 \cdot 3 \cdots (i-1)}} = \sqrt[n]{n^{n-1} \int \frac{x^{i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}} \cdot \int \frac{x^{2i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}} \cdots \int \frac{x^{(n-1)i-1}dx}{\sqrt[n]{(1-x^n)^{n-i}}}}$$

So if we for the sake of brevity denote the integral formula $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ by this character $\left(\frac{p}{q}\right)$, but on the other hand the formula $\int dx \sqrt[n]{\left(\log\frac{1}{x}\right)^m}$ by this $\left[\frac{m}{n}\right]$, so that $\left[\frac{m}{n}\right]$ denotes the value of this indefinite product $1 \cdot 2 \cdot 3 \cdots z$, while $z = \frac{m}{n}$, the value looked for will expressed more succinctly as follows

$$\left[\frac{i-n}{n}\right] = \sqrt[n]{1 \cdot 2 \cdot 3 \cdots (i-1) n^{n-1} \left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right) \cdots \left(\frac{ni-i}{i}\right)},$$

whence it is also concluded

$$\left[\frac{i}{n}\right] = \frac{i}{n}\sqrt[n]{1\cdot 2\cdot 3\cdots (i-1)n^{n-1}\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\left(\frac{ni-i}{i}\right)}.$$

Here it will always be sufficient to have taken the number i smaller than n, because it is known for greater ones, that

$$\left[\frac{i+n}{n}\right] = \frac{i+n}{n}\left[\frac{i}{n}\right], \quad \text{in the same way} \quad \left[\frac{i+2n}{n}\right] = \frac{i+n}{n}\cdot\frac{i+2n}{n}\left[\frac{i}{n}\right] \quad \text{etc.,}$$

and so the whole investigation is reduced to those cases only, in which the numerator i of the fraction $\frac{i}{n}$ is smaller than the denominator n. In addition it will be helpful to have noted the following on the integral formulas

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \left(\frac{p}{q}\right):$$

I. The letters p and q are permutable, that

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right).$$

II. If one of the two numbers p or q is equal to the exponent n, the value of the integral formula will be algebraic, of course

$$\left(\frac{n}{p}\right) = \left(\frac{p}{n}\right) = \frac{1}{p} \quad \text{or} \quad \left(\frac{n}{q}\right) = \left(\frac{q}{n}\right) = \frac{1}{q}.$$

III. If the sum of the numbers $p+q$ is equal to the exponent n, the value of the integral formula $\left(\frac{p}{q}\right)$ can be exhibited by means of the circle, because

$$\left(\frac{p}{n-p}\right) = \left(\frac{n-p}{p}\right) = \frac{\pi}{n\sin\frac{p\pi}{n}} \quad \text{and} \quad \left(\frac{q}{n-q}\right) = \left(\frac{n-q}{q}\right) = \frac{\pi}{n\sin\frac{q\pi}{n}}.$$

IV. If one of the numbers p or q is greater then the expoent n, the integral formula $\left(\frac{p}{q}\right)$ can be reduced to another one, whose terms are smaller than n, what happens by means of this reduction

$$\left(\frac{p+n}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right).$$

V. Among many of the integral formulas of this kind there consists such a relation, that

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),$$

by means of which one finds all reductions, that I exposed in my observations on these formulas.

COROLLARY 1

§45 If in this way by means of reduction IV we accommodate the found formula to the single cases, we will be able to exhibit them in the most simple way in the following manner. And for the case $n=2$, in which no further reduction is neccessary, we will have

$$\left[\frac{1}{2}\right]=\frac{1}{2}\sqrt[2]{2\left(\frac{1}{1}\right)}=\frac{1}{2}\sqrt[2]{\frac{\pi}{\sin\frac{\pi}{2}}}=\frac{1}{2}\sqrt{\pi}.$$

COROLLARY 2

§46 For the cases $n=3$ we will have these reductions

$$\left[\frac{1}{3}\right]=\frac{1}{3}\sqrt[3]{3^2\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)}$$
$$\left[\frac{2}{3}\right]=\frac{2}{3}\sqrt[3]{3\cdot 1\left(\frac{2}{2}\right)\left(\frac{1}{2}\right)}.$$

COROLLARY 3

§47 For the case $n=4$ one obtains these three reductions

$$\left[\frac{1}{4}\right]=\frac{1}{4}\sqrt[4]{4^3\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)},$$
$$\left[\frac{2}{4}\right]=\frac{2}{4}\sqrt[4]{4^2\cdot 2\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)}=\frac{1}{2}\sqrt[2]{4\left(\frac{2}{2}\right)}$$

because of $\left(\frac{4}{2}\right) = \frac{1}{2}$,

$$\left[\frac{3}{4}\right] = \frac{3}{4}\sqrt{4 \cdot 1 \cdot 2 \left(\frac{3}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)};$$

because in the middle it is $\left(\frac{2}{2}\right) = \left(\frac{4-2}{2}\right) = \frac{\pi}{4}$, it will be as before, of course

$$\left[\frac{2}{4}\right] = \left[\frac{1}{2}\right] = \frac{1}{2}\sqrt{\pi}.$$

COROLLARY 4

§48 Now let n be $= 5$ and these four reductions arise

$$\left[\frac{1}{5}\right] = \frac{1}{5}\sqrt[5]{5^4\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)},$$
$$\left[\frac{2}{5}\right] = \frac{2}{5}\sqrt[5]{5^3 \cdot 2\left(\frac{2}{2}\right)\left(\frac{4}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)},$$
$$\left[\frac{3}{5}\right] = \frac{3}{5}\sqrt[5]{5^2 \cdot 1 \cdot 2\left(\frac{3}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)},$$
$$\left[\frac{4}{5}\right] = \frac{4}{5}\sqrt[5]{5 \cdot 1 \cdot 2 \cdot 3\left(\frac{4}{4}\right)\left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right)}.$$

COROLLARY 5

§49 Let $n = 6$ and we will have these reductions

$$\left[\frac{1}{6}\right] = \frac{1}{6}\sqrt[6]{6^5\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)},$$
$$\left[\frac{2}{4}\right] = \frac{2}{6}\sqrt[6]{6^4 \cdot 2\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2\left(\frac{6}{2}\right)} = \frac{1}{3}\sqrt[3]{6^2\left(\frac{3}{2}\right)\left(\frac{4}{2}\right)},$$
$$\left[\frac{3}{6}\right] = \frac{3}{6}\sqrt[6]{6^3 \cdot 3 \cdot 3\left(\frac{3}{3}\right)^3\left(\frac{6}{3}\right)^2} = \frac{1}{2}\sqrt[2]{6\left(\frac{3}{3}\right)},$$

$$\left[\frac{4}{6}\right] = \frac{4}{6}\sqrt[8]{6^2 \cdot 2 \cdot 4 \cdot 2 \left(\frac{4}{4}\right)^2 \left(\frac{2}{4}\right)^2 \left(\frac{6}{4}\right)} = \frac{2}{3}\sqrt[3]{6 \cdot 2 \left(\frac{4}{4}\right) \left(\frac{2}{4}\right)},$$

$$\left[\frac{5}{6}\right] = \frac{5}{6}\sqrt[6]{6 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \left(\frac{5}{5}\right) \left(\frac{4}{5}\right) \left(\frac{3}{5}\right) \left(\frac{2}{5}\right) \left(\frac{1}{5}\right)}.$$

COROLLARY 6

§50 For $n = 7$ the following six equations arise

$$\left[\frac{1}{7}\right] = \frac{1}{7}\sqrt[7]{7^6 \left(\frac{1}{1}\right) \left(\frac{2}{1}\right) \left(\frac{3}{1}\right) \left(\frac{4}{1}\right) \left(\frac{5}{1}\right) \left(\frac{6}{1}\right)},$$

$$\left[\frac{2}{7}\right] = \frac{2}{7}\sqrt[7]{7^5 \cdot 1 \left(\frac{2}{2}\right) \left(\frac{4}{2}\right) \left(\frac{6}{2}\right) \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) \left(\frac{5}{2}\right)},$$

$$\left[\frac{3}{7}\right] = \frac{3}{7}\sqrt[7]{7^4 \cdot 1 \cdot 2 \left(\frac{3}{3}\right) \left(\frac{6}{3}\right) \left(\frac{2}{3}\right) \left(\frac{5}{3}\right) \left(\frac{1}{3}\right) \left(\frac{4}{3}\right)},$$

$$\left[\frac{4}{7}\right] = \frac{4}{7}\sqrt[7]{7^3 \cdot 1 \cdot 2 \cdot 3 \left(\frac{4}{4}\right) \left(\frac{1}{4}\right) \left(\frac{5}{4}\right) \left(\frac{2}{4}\right) \left(\frac{6}{4}\right) \left(\frac{3}{4}\right)},$$

$$\left[\frac{5}{7}\right] = \frac{5}{7}\sqrt[7]{7^2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \left(\frac{5}{5}\right) \left(\frac{3}{5}\right) \left(\frac{1}{5}\right) \left(\frac{6}{5}\right) \left(\frac{4}{5}\right) \left(\frac{2}{5}\right)},$$

$$\left[\frac{6}{7}\right] = \frac{6}{7}\sqrt[7]{7 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \left(\frac{6}{6}\right) \left(\frac{5}{6}\right) \left(\frac{4}{6}\right) \left(\frac{3}{6}\right) \left(\frac{2}{6}\right) \left(\frac{1}{6}\right)}.$$

COROLLARY 7

§51 Now let n be $= 8$ and one will get to these seven reductions

$$\left[\frac{1}{8}\right] = \frac{1}{8}\sqrt[8]{8^7 \left(\frac{1}{1}\right) \left(\frac{2}{1}\right) \left(\frac{3}{1}\right) \left(\frac{4}{1}\right) \left(\frac{5}{1}\right) \left(\frac{6}{1}\right) \left(\frac{7}{1}\right)},$$

$$\left[\frac{2}{8}\right] = \frac{2}{8}\sqrt[8]{8^6 \cdot 2 \left(\frac{2}{2}\right)^2 \left(\frac{4}{2}\right)^2 \left(\frac{6}{2}\right)^2 \left(\frac{8}{2}\right)} = \frac{1}{4}\sqrt[4]{8^3 \left(\frac{2}{2}\right) \left(\frac{4}{2}\right) \left(\frac{6}{2}\right)},$$

$$\left[\frac{3}{8}\right] = \frac{3}{8}\sqrt[8]{8^5\cdot 1\cdot 2\left(\frac{3}{3}\right)\left(\frac{6}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)},$$

$$\left[\frac{4}{8}\right] = \frac{4}{8}\sqrt[8]{8^4\cdot 4\cdot 4\cdot 4\left(\frac{4}{4}\right)^4\left(\frac{8}{4}\right)^3} = \frac{1}{2}\sqrt[2]{8\left(\frac{4}{4}\right)},$$

$$\left[\frac{5}{8}\right] = \frac{5}{8}\sqrt[8]{8^3\cdot 1\cdot 2\cdot 3\cdot 4\left(\frac{5}{5}\right)\left(\frac{2}{5}\right)\left(\frac{7}{5}\right)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)\left(\frac{6}{5}\right)\left(\frac{3}{5}\right)},$$

$$\left[\frac{6}{8}\right] = \frac{6}{8}\sqrt[8]{8^2\cdot 4\cdot 2\cdot 6\cdot 4\cdot 2\left(\frac{6}{6}\right)^2\left(\frac{4}{6}\right)^2\left(\frac{2}{6}\right)^2\left(\frac{8}{6}\right)} = \frac{3}{4}\sqrt[4]{8\cdot 2\cdot 4\left(\frac{6}{6}\right)\left(\frac{4}{6}\right)\left(\frac{2}{6}\right)},$$

$$\left[\frac{7}{8}\right] = \frac{7}{8}\sqrt[8]{8\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\left(\frac{7}{7}\right)\left(\frac{6}{7}\right)\left(\frac{5}{7}\right)\left(\frac{4}{7}\right)\left(\frac{3}{7}\right)\left(\frac{2}{7}\right)\left(\frac{1}{7}\right)}.$$

Remark

§52 It would be superfluous to expand these cases any further, because from the ones listed the structure of these formulas is already seen well enough. If in the propounded formula $\left[\frac{m}{n}\right]$ the numbers m and n are prime to each other, the law is manifest, because

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)};$$

but if these numbers m and n have a common divisor, it will indeed be useful, to reduce this $\frac{m}{n}$ to the smallest form and extract the searched value from the preceding cases; nevertheless the operation can also be executed as follows. Because the searched expression certainly has this form

$$\left[\frac{m}{n}\right] = \frac{m}{n}\sqrt[n]{n^{n-m}PQ},$$

where Q is the product of the $n-1$ integral formulas, P on the other hand the product of some absolute numbers, for finding that product Q just continue this series of formulas $\left(\frac{m}{m}\right)\left(\frac{2m}{m}\right)\left(\frac{3m}{m}\right)$etc., until the numerator exceeds the exponent n, and instead of this numerator write its excess over n; if this is put $=\alpha$, that our formula is $\left(\frac{\alpha}{m}\right)$, this numerator α will give a factor of a product P; then from there on further put the series of formulas $\left(\frac{\alpha}{m}\right)\left(\frac{\alpha+m}{m}\right)\left(\frac{\alpha+2m}{m}\right)$etc., until one again reaches a numerator greater than the exponent n and the formula $\left(\frac{n+\beta}{m}\right)$, instead of which one has to write $\left(\frac{\beta}{m}\right)$, and from there the factor β is inferred into to product and like this one has to continue, until $n-1$ formulas for Q will have arosen.

To understand these operations in an easier way, let us expand the case of the formula

$$\left[\frac{9}{12}\right] = \frac{9}{12}\sqrt[12]{12^3 PQ}$$

in this manner, where the investigation of the letters P and Q is executed as follows:

$$\begin{array}{llllllllllll} \text{for} \quad Q\ldots & \left(\frac{9}{9}\right) & \left(\frac{6}{9}\right) & \left(\frac{3}{9}\right) & \left(\frac{12}{9}\right) & \left(\frac{9}{9}\right) & \left(\frac{6}{9}\right) & \left(\frac{3}{9}\right) & \left(\frac{12}{9}\right) & \left(\frac{9}{9}\right) & \left(\frac{6}{9}\right) & \left(\frac{3}{9}\right), \\ \text{for} \quad P\ldots & & 6 \cdot & 3 & & 9 \cdot & 6 \cdot & 3 & & 9 \cdot & 6 \cdot & 3 \end{array}$$

and so one finds

$$Q = \left(\frac{9}{9}\right)^3 \left(\frac{6}{9}\right)^3 \left(\frac{3}{9}\right)^3 \left(\frac{12}{9}\right)^2 \quad \text{and} \quad P = 6^3 \cdot 3^3 \cdot 9^2.$$

Because it is $\left(\frac{12}{9}\right) = \frac{1}{9}$, it is $PQ = 6^3 \cdot 3^3 \left(\frac{9}{9}\right)^3 \left(\frac{6}{9}\right)^3 \left(\frac{3}{9}\right)^3$ and hence

$$\left[\frac{9}{12}\right] = \frac{3}{4}\sqrt[4]{12 \cdot 6 \cdot 3 \left(\frac{9}{9}\right)\left(\frac{6}{9}\right)\left(\frac{3}{9}\right)}.$$

Theorem

§53 *No matter which positive numbers are indicated by the letters m and n, in the used notation explained before it will always be*

$$\left[\frac{m}{n}\right] = \frac{m}{m}\sqrt[n]{n^{n-m} \cdot 1 \cdot 2 \cdot 3 \cdots (m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)}.$$

Demonstration

For the cases, in which m and n are numbers, that are prime to each other, the validity of this theorem was evicted in the preceding theorems; but that it also holds, if those numbers m and n enjoy a common divisor, is not evident from there; but from this itself, that for the cases, in which m and n are relatively prime, the validity is confirmed, one can without doubt conclude, that this theorem is true in general. I certainly by no means deny, that this kind to conclude something is completely singular and has to seem suspect to most people. Hence to leave no room for doubt, because for the cases, in which the numbers m and n are composite, we obtained two expressions, it will be useful to have shown the aggreement for the cases expanded before. But the cases $m = n$ already provides a huge confirmation, in which case our formula manifestly produces the unity.

COROLLARY 1

§54 The first cases, demanding a demonstration of the agreement, is that one, in which $m = 2$ and $n = 4$, for which we found above (§47)

$$\left[\frac{2}{4}\right] = \frac{2}{4}\sqrt[4]{4^2\left(\frac{2}{2}\right)^2};$$

but now via the theorem it is

$$\left[\frac{2}{4}\right] = \frac{2}{4}\sqrt[4]{4^2\cdot 1\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\left(\frac{3}{2}\right)},$$

where by comparison it is

$$\left(\frac{2}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right),$$

whose validity was confirmed in my observations mentioned above.

COROLLARY 2

§55 If $m = 2$ and $n = 6$, it is from the things above (§49)

$$\left[\frac{2}{6}\right] = \frac{2}{6}\sqrt[6]{6^4\left(\frac{2}{2}\right)^2\left(\frac{4}{2}\right)^2};$$

now on the other hand by means of the theorem

$$\left[\frac{2}{6}\right] = \frac{2}{6}\sqrt[6]{6^4 \cdot 1 \left(\frac{1}{2}\right) \left(\frac{2}{2}\right) \left(\frac{3}{2}\right) \left(\frac{4}{2}\right) \left(\frac{5}{2}\right)}$$

and therefore it has to be

$$\left(\frac{2}{2}\right) \left(\frac{4}{2}\right) = \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) \left(\frac{5}{2}\right),$$

whose validity is clear from the same source.

COROLLARY 3

§56 If $m = 3$ and $n = 6$, one gets to this equation

$$\left(\frac{3}{3}\right)^2 = 1 \cdot 2 \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) \left(\frac{4}{3}\right) \left(\frac{5}{3}\right);$$

but if $m = 4$ and $n = 6$, it is in similar manner

$$2^2 \left(\frac{4}{4}\right) \left(\frac{2}{4}\right) = 1 \cdot 2 \cdot 3 \left(\frac{1}{4}\right) \left(\frac{3}{4}\right) \left(\frac{5}{4}\right)$$

or

$$\left(\frac{4}{4}\right) \left(\frac{2}{4}\right) = \frac{3}{2} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right) \left(\frac{5}{4}\right),$$

which is also detected to be true.

COROLLARY 4

§57 The case $m = 2$ and $n = 8$ yields this equality

$$\left(\frac{2}{2}\right) \left(\frac{4}{2}\right) \left(\frac{6}{2}\right) = \left(\frac{1}{2}\right) \left(\frac{3}{2}\right) \left(\frac{5}{2}\right) \left(\frac{7}{2}\right),$$

but the case $m = 4$ and $n = 8$ this one

$$\left(\frac{4}{4}\right)^2 = 1 \cdot 2 \cdot 3 \left(\frac{1}{4}\right) \left(\frac{2}{4}\right) \left(\frac{3}{4}\right) \left(\frac{5}{4}\right) \left(\frac{6}{4}\right) \left(\frac{7}{4}\right)$$

and finally the case $m = 6$ and $n = 8$ this equation

$$2\cdot 4\left(\frac{6}{6}\right)\left(\frac{4}{6}\right)\left(\frac{2}{6}\right)=1\cdot 3\cdot 5\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right),$$

which also agree with the truth.

Remark

§58 But if in general the numbers m and n have the common factor 2 and the propounded formula is $\left[\frac{2m}{2n}\right]=\left[\frac{m}{n}\right]$, because it is

$$\left[\frac{m}{n}\right]=\frac{m}{n}\sqrt[n]{n^{n-m}\cdot 1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right)},$$

after having reduced the same to the exponent $2n$ it will be

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m}\cdot 2^2\cdot 4^2\cdot 6^2\cdots(2m-2)^2\left(\frac{2}{2m}\right)^2\left(\frac{4}{2m}\right)^2\left(\frac{6}{2m}\right)^2\cdots\left(\frac{2n-2}{2m}\right)^2}.$$

By the theorem the same expression on the other hand becomes

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m}\cdot 1\cdot 2\cdot 3\cdots(2m-1)\left(\frac{1}{2m}\right)\left(\frac{2}{2m}\right)\left(\frac{3}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right)},$$

whence for the exponent $2n$ it will be

$$\begin{aligned}&2\cdot 4\cdot 6\cdots(2m-2)\left(\frac{2}{2m}\right)\left(\frac{4}{2m}\right)\left(\frac{6}{2m}\right)\cdots\left(\frac{2n-2}{2m}\right)\\ &=1\cdot 3\cdot 5\cdots(2m-1)\left(\frac{1}{2m}\right)\left(\frac{3}{2m}\right)\left(\frac{5}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right).\end{aligned}$$

If in the same way the common divisor is 3, one will find for the exponent $3n$

$$\begin{aligned}&3^2\cdot 6^2\cdot 9^2\cdots(3m-3)^2\left(\frac{3}{3m}\right)^2\left(\frac{6}{3m}\right)^2\left(\frac{9}{3m}\right)^2\cdots\left(\frac{3n-3}{3m}\right)^2\\ &=1\cdot 2\cdot 4\cdot 5\cdots(3m-2)(3m-1)\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\cdots\left(\frac{3n-1}{3m}\right),\end{aligned}$$

which equation can be more conveniently exhibited as follows

$$\frac{1\cdot 2\cdot 4\cdot 5\cdot 7\cdot 8\cdot 10\cdots(3m-2)(3m-1)}{3^2\cdot 6^2\cdot 9^2\cdots(3m-3)^2}=\frac{\left(\frac{3}{3m}\right)^2\left(\frac{6}{3m}\right)^2\cdots\left(\frac{3n-3}{3m}\right)^2}{\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\left(\frac{7}{3m}\right)\cdots\left(\frac{3n-2}{3m}\right)\left(\frac{3n-1}{3m}\right)}.$$

But if in general the common divisor is d and the exponent dn, one will have

$$\left(d\cdot 2d\cdot 3d\cdots(dm-d)\left(\frac{d}{dm}\right)\left(\frac{2d}{dm}\right)\left(\frac{3d}{dm}\right)\cdots\left(\frac{dn-d}{dm}\right)\right)^d$$
$$=1\cdot 2\cdot 3\cdot 4\cdots(dm-1)\left(\frac{1}{dm}\right)\left(\frac{2}{dm}\right)\left(\frac{3}{dm}\right)\cdots\left(\frac{dn-1}{dm}\right),$$

which equation can easily be accommodated to any cases, whence the following theorem deserves it to be noted.

THEOREM

§59 *If α was a common divisor of the numbers m and n and the formula $\left(\frac{p}{q}\right)$ denotes the value of the integral $\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ extended from $x=0$ to $x=1$, it will be*

$$\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots(m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n-\alpha}{m}\right)\right)^\alpha$$
$$=1\cdot 2\cdot 3\cdots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right).$$

DEMONSTRATION

From the preceding remark the validity of this theorem is already seen; while there the commom divisor was $=d$ and the two propounded numbers were dm and dn, here I just wrote m and n instead of them, but instead of their divisor d I wrote the letter α, which kind of divisor the enunciated equality contains in such a way, that one assumes the numbers m and n and therefore also $n-\alpha$ and $n-\alpha$ to occur in the continued arithmetic progression α, 2α, 3α etc. In addition I am forced to confess that this demonstration is of course mainly based on induction and cannot be seen to be rigorous by any means; but because we are nevertheless convicted of its truth, this theorem seems to be worthy of even greater attention; there is nevertheless no doubt, that a further expansion of integral formulas of this kind will finally give a perfect demonstration; but that it was possible for us, to see its truth before, hence shines a extraordinary specimen of analytical investigation.

COROLLARY 1

§60 So if we instead of the used signs substitute the integral formulas itself, our theorem will behave as follows, that

$$\alpha \cdot 2\alpha \cdot 3\alpha \cdots (m-\alpha) \int \frac{x^{\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdot \int \frac{x^{2\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdots \int \frac{x^{n-\alpha-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}}$$

$$= \sqrt[\alpha]{1 \cdot 2 \cdot 3 \cdots (m-1) \int \frac{dx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdot \int \frac{xdx}{\sqrt[n]{(1-x^n)^{n-m}}} \cdots \int \frac{x^{n-2}dx}{\sqrt[n]{(1-x^n)^{n-m}}}}.$$

COROLLARY 2

§61 Or if we for abbreviation set $\sqrt[n]{(1-x^n)^{n-m}} = X$, it will be

$$\alpha \cdot 2\alpha \cdot 3\alpha \cdots (m-\alpha) \int \frac{x^{\alpha-1}dx}{X} \cdot \int \frac{x^{2\alpha-1}dx}{X} \cdots \int \frac{x^{n-\alpha-1}dx}{X}$$

$$= \sqrt[\alpha]{1 \cdot 2 \cdot 3 \cdots (m-1) \int \frac{dx}{X} \cdot \int \frac{xdx}{X} \cdot \int \frac{x^2dx}{X} \cdots \int \frac{x^{n-2}dx}{X}}.$$

General Theorem

§62 *If the divisors of the two numbers m and n are α, β, γ etc. and the formula $\left(\frac{p}{q}\right)$ denotes the value of the integral $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ extended from $x=0$ to $x=1$, the following expressions formed from integral formulas of this kind will be equal to each other*

$$\begin{aligned}
&\left(\alpha \cdot 2\alpha \cdot 3\alpha \cdots (m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n-\alpha}{m}\right)\right)^{\alpha} \\
=&\left(\beta \cdot 2\beta \cdot 3\beta \cdots (m-\beta)\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\left(\frac{n-\beta}{m}\right)\right)^{\beta} \\
=&\left(\gamma \cdot 2\gamma \cdot 3\gamma \cdots (m-\gamma)\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\left(\frac{n-\gamma}{m}\right)\right)^{\gamma}
\end{aligned}$$

etc.

Demonstration

From the preceding theorem the validity of this theorem manifestly follows, because every single of these expressions is equal to this one

$$1 \cdot 2 \cdot 3 \cdots (m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n-1}{m}\right),$$

which corresponds to the unity as smallest common divisor of the numbers m and n. Therefore so many expressions of this kind can be equal to each other, as there were common divisors of the two numbers m and n.

Corollary 1

§63 Because this formula $\left(\frac{n}{m}\right)$ is $=\frac{1}{m}$ and hence $m\left(\frac{n}{m}\right)=1$, our equal expressions can be more succinctly represented as follows

$$\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha}$$
$$=\left(\beta\cdot 2\beta\cdot 3\beta\cdots m\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\beta}$$
$$=\left(\gamma\cdot 2\gamma\cdot 3\gamma\cdots m\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\gamma}.$$

For even if the number of factors was increased here, nevertheless the way of the composition easily meets the eye.

Corollary 2

§64 So if $m = 6$ and $n = 12$, because of the common divisors of these numbers, 6, 3, 2, 1 one will have the following for forms all equal to each other

$$=\left(6\left(\frac{6}{6}\right)\left(\frac{12}{6}\right)\right)^{6}=\left(3\cdot 6\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\right)^{3}$$
$$=\left(2\cdot 4\cdot 6\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)\right)^{2}$$
$$=1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{3}{6}\right)\cdots\left(\frac{12}{6}\right).$$

Corollary 3

§65 If the last is combined with the penultimate, this equation will arise

$$\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}=\frac{\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{9}{6}\right)\left(\frac{11}{6}\right)},$$

but the last compared to the penpenultimate yields

$$\frac{1\cdot 2\cdot 4\cdot 5}{3\cdot 3\cdot 6\cdot 6}=\frac{\left(\frac{3}{6}\right)\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{11}{6}\right)}.$$

Remark

§66 Hence infinitely many relations between the integral formulas of the form

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \left(\frac{p}{q}\right)$$

follow, which are even more remarkable, because we were led to them by a completely singular method. And if someone still doubts their truth, he should consult my observations on these integral formulas and will then hence for any case easily be convinced of their truth. But even if that treatment serve for this confirmation, the relations found here are nevertheless of even greater importance, because in them a certain structure is noticed and they are in easy fashion continued throughout all classes, no matter how one wants to assume the exponent n, whereas in the first treatment the calculation for the higher classes becomes continuously more labourious and intricate.

Supplement containing the demonstration of the theorem propounded in § 53

It is convenient to derive this demonstration from the things mentioned above; just take the equation given in §25, which for $f=1$ having changed the letters is

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{\nu-1}\cdot\int dx\left(\log\frac{1}{x}\right)^{\mu-1}}{\int dx\left(\log\frac{1}{x}\right)^{\nu+\mu-1}} = \varkappa\int\frac{x^{\varkappa\mu-1}dx}{(1-x^{\varkappa})^{1-\nu}},$$

and by known reductions represent it in this form

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{\nu}\cdot\int dx\left(\log\frac{1}{x}\right)^{\mu}}{\int dx\left(\log\frac{1}{x}\right)^{\nu+\mu}} = \frac{\varkappa\mu\nu}{\mu+\nu}\int\frac{x^{\varkappa\mu-1}dx}{(1-x^{\varkappa})^{1-\nu}}.$$

Now set $\nu=\frac{m}{n}$ and $\mu=\frac{\lambda}{n}$, but then $\varkappa=n$, that we have

$$\frac{\int dx\left(\log\frac{1}{x}\right)^{\frac{m}{n}}\cdot\int dx\left(\log\frac{1}{x}\right)^{\frac{\lambda}{n}}}{\int dx\left(\log\frac{1}{x}\right)^{\frac{\lambda+m}{n}}} = \frac{\lambda m}{\lambda+m}\int\frac{x^{\lambda-1}dx}{\sqrt[\varkappa]{(1-x^n)^{n-m}}},$$

which, for the sake of brevity having used the way from above, is more conveniently expressed as follows

$$\frac{\left[\frac{m}{n}\right]\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{n}\right]}=\frac{\lambda m}{\lambda+m}\left(\frac{\lambda}{m}\right).$$

Now instead of λ successively write the numbers 1, 2, 3, $4\ldots n$ and multiply all these equations, whose number is $=n$, and the resulting equation will be

$$\begin{aligned}&\left[\frac{m}{n}\right]^n\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+1}{n}\right]\left[\frac{m+2}{n}\right]\left[\frac{m+3}{n}\right]\cdots\left[\frac{m+n}{n}\right]}\\
&=m^n\frac{1}{m+1}\cdot\frac{2}{m+2}\cdot\frac{3}{m+3}\cdots\frac{n}{m+n}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)\\
&=m^n\frac{1\cdot 2\cdot 3\cdots m}{(n+1)(n+2)(n+3)\cdots(m+n)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right).\end{aligned}$$

But in similar manner just transform the first part, that

$$\left[\frac{m}{n}\right]^n\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+1}{n}\right]\left[\frac{n+2}{n}\right]\left[\frac{n+3}{n}\right]\cdots\left[\frac{n+m}{n}\right]},$$

whose agreement with the preceding is revealed by cross multiplication. But because it is from the nature of these formulas

$$\left[\frac{n+1}{n}\right]=\frac{n+1}{n}\left[\frac{1}{n}\right],\quad\left[\frac{n+2}{n}\right]=\frac{n+2}{2}\left[\frac{2}{n}\right],\quad\left[\frac{n+3}{n}\right]=\frac{n+3}{n}\left[\frac{3}{n}\right]\quad\text{etc.,}$$

because of these m formulas this first part will become

$$\left[\frac{m}{n}\right]^n\frac{n^m}{(n+1)(n+2)(n+3)\cdots(n+m)};$$

because this one is equal to the other part exhibited before

$$m^n\frac{1\cdot 2\cdot 3\cdots m}{(n+1)(n+2)(n+3)\cdots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

we obtain this equation

$$\left[\frac{m}{n}\right]^n=\frac{m^n}{n^n}1\cdot 2\cdot 3\cdots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

so that

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1\cdot 2\cdot 3\cdots m}{n^m}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)},$$

which because $\left(\frac{n}{m}\right) = \frac{1}{m}$ completely agrees with the one propounded in §53, whence its truth was now indeed evicted from most certain principles.

Demonstration of the theorem propounded in §59

Also this theorem needs a more solid demonstration, which I give from the equation established before

$$\frac{\left[\frac{m}{n}\right]\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{n}\right]} = \frac{\lambda m}{\lambda+m}\left(\frac{\lambda}{m}\right)$$

as follows. While α is a common divisor of the number m and n successively write the number α, 2α, 3α etc. up to n instead of λ, whose amount is $= \frac{n}{\alpha}$, and now multiply all equalities resulting in this manner, that this equation arises

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\cdots\left[\frac{m+n}{n}\right]}$$
$$= m^{\frac{n}{\alpha}}\frac{\alpha}{m+\alpha}\cdot\frac{2\alpha}{m+2\alpha}\cdot\frac{3\alpha}{m+3\alpha}\cdots\frac{n}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Now transform the first part in this one equal to it

$$\left[\frac{m}{m}\right]^{\frac{n}{\alpha}} \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+\alpha}{n}\right]\left[\frac{n+2\alpha}{n}\right]\left[\frac{n+3\alpha}{n}\right]\cdots\left[\frac{n+m}{n}\right]},$$

which because of $\left[\frac{n+\alpha}{n}\right] = \frac{n+\alpha}{n}\left[\frac{\alpha}{n}\right]$ and so for the remaining is reduced to this one

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \frac{n}{n+\alpha}\cdot\frac{n}{n+2\alpha}\cdot\frac{n}{n+3\alpha}\cdots\frac{n}{n+m},$$

The second part of the equation on the other hand is in similar manner transformed into this one

$$m^{\frac{n}{\alpha}}\frac{\alpha}{n+\alpha}\cdot\frac{2\alpha}{n+2\alpha}\cdot\frac{3\alpha}{n+3\alpha}\cdots\frac{m}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right),$$

whence this equation arise

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} n^{\frac{m}{\alpha}} = n^{\frac{n}{\alpha}}\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)$$

and hence

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1}{n^m}\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha}}$$

which equaion compared to the preceding yields this equation

$$\left(\alpha\cdot 2\alpha\cdot 3\alpha\cdots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\left(\frac{n}{m}\right)\right)^{\alpha} = 1\cdot 2\cdot 3\cdots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right),$$

what is to be understood about all common divisors of the two numbers m and n.