Observations on harmonic Progressions *

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§1 Under the name of harmonic progressions all series of fractions are understood, whose numerators are equal to each other, but whose denominators on the other hand constitute an arithmetic progression. Therefore, a general form of this kind is

$$\frac{c}{a}, \quad \frac{c}{a+b}, \quad \frac{c}{c+2b}, \quad \frac{c}{a+3b} \quad \text{etc.}$$

For, each three contiguous terms, as

$$\frac{c}{a+b}, \quad \frac{c}{a+2b}, \quad \frac{c}{a+3b},$$

have this property that the differences of the outer ones from the middle term are proportional to the outer terms themselves. Of course, it is

$$\frac{c}{a+b}-\frac{c}{a+2b}:\frac{c}{a+2b}-\frac{c}{a+3b}=\frac{c}{a+c}:\frac{c}{a+3b}.$$

But because this is the property of the harmonic proportion, series of fractions of this kind were called *harmonic progressions*. The could also have been called reciprocals of first order, since in the general term $\frac{c}{a+(n-1)b}$ the index n has one, more precisely one negative, dimension.

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§2 Although in these series the terms continuously decrease, the sum of a series of this kind continued to infinity is nevertheless always infinite. To demonstrate this no method to sum these series is necessary, but the validity will easily become clear from the following principle. A series, which continued to infinity has a finite sum, even if it is extended twice as far, will obtain no augmentation, but that, what is added after infinity by cogitation, will in reality be infinitely small. For, if this would not be the case, the sum of series, even if continued to infinity, would not be determined and therefore not finite. From this it follows, if that, what arises from the continuation beyond the infinitesimal term, is of finite magnitude, the sum of the series necessarily must be infinite. Therefore, from this principle we will be able to decide, whether the sum of a given series is infinite or finite.

§3 Therefore, let the series

$$\frac{c}{a}, \quad \frac{c}{a+b}, \quad \frac{c}{a+2b} \quad \text{etc.}$$

be continued to infinity and the infinitesimal term $\frac{c}{a+(i-1)b}$, while i is an infinite number, which is the index of this term. Now, continue this series further from the term $\frac{c}{a+ib}$ until the term $\frac{c}{a+(ni-1)b}$, whose exponent is ni. Therefore, the number of additionally added terms is $(n-1)i$. But their sum will be smaller than

$$\frac{(n-1)ic}{a+ib},$$

but larger than

$$\frac{(n-1)ic}{a+(ni-1)b}.$$

But because i is infinitely large, a vanishes in each of both denominators. Hence the sum will be greater than

$$\frac{(n-1)c}{nb},$$

but smaller than

$$\frac{(n-1)c}{b}.$$

From this it is perspicuous that this sum is finite and as a logical consequence the sum of the propounded series $\frac{c}{a}$, $\frac{c}{a+b}$ continued to infinity is infinitely large.

§4 But closer limits of this sum of the terms from i to ni are found from the following properties of the harmonic proportion. Of course, every harmonic proportion is of such nature that the middle term is smaller than the third part of the sum of all three. Therefore, the middle term between $\frac{c}{a+ib}$ and $\frac{c}{a+(ni-1)b}$, which is $\frac{c}{a+\frac{ni+i-1}{2}b}$, multiplied by the number of terms $(n-i)i$ or

$$\frac{(n-1)ic}{a+\frac{ni+i-1}{2}b}$$

will be smaller than the sum of the terms. Or the sum of the terms hence will be greater than

$$\frac{2(n-1)c}{(n+1)b}$$

because of the infinite i. Furthermore, the arithmetic mean of the most outer terms is greater than the third part of the sum of the terms. From this it follows that also in the harmonic series the sum of terms will be smaller than $(n-1)i$ times the arithmetic mean of the most outer terms, which is

$$\frac{(2a+(ni+i-1)b)c}{2(a+ib)(a+(ni-1)b)} \quad \text{or} \quad \frac{(n+1)c}{2nib}.$$

Hence the sum will be smaller than

$$\frac{n^2-1)c}{2nb},$$

such that these two limits are

$$\frac{2(n-1)c}{(n+1)b} \quad \text{and} \frac{(n^2-1)c}{2nb}$$

and hence the sum approximately

$$= \frac{(n-1)c}{b\sqrt{n}},$$

which is the proportional middle between the limits.

§5 From these things is it possible to conclude in which cases this more universal series

$$\frac{c}{a}, \quad \frac{c}{a+b}, \quad \frac{c}{a+2^{\alpha}b} \quad \text{etc. to infinity until} \quad \frac{c}{a+i^{\alpha}b}$$

has a finite or infinite sum. For, let $(n-1)i$ terms follow after the last term and the sum of these will be smaller than

$$\frac{(n-1)c}{i^{\alpha-1}b},$$

but greater than

$$\frac{(n-1)c}{n^{\alpha}i^{\alpha-1}b}.$$

Hence, if α was a number greater than unity, the sum of these following terms will be $=0$ and therefore the sum of the progression will be finite. But if it is $\alpha < 1$, the sum of the following terms will be infinite, which is why the sum of the progression itself will be infinite of an infinitely larger degree. Therefore, among these progressions only the harmonic, in which it is $\alpha = 1$, has this property that the sum of it continued to infinity is infinitely large but the sum of the following terms after the infinitesimal term on the other hand is finite.

§6 But how large the sum of terms from the term of the index i to the term of the index ni is, I investigate in the following way. Put the sum of the series

$$\frac{c}{a}, \quad \frac{c}{a+b}, \quad \dots \quad, \frac{c}{a+(i-1)b}$$

until the term of the index $i = s$, which is a quantity to be determined from a, b, c and i. Let i grow by the unity and s will have the following term $\frac{c}{a+ib}$ as augmentation. Hence it will be

$$di : ds = 1 : \frac{c}{a+ib} \quad \text{or} \quad ds = \frac{cdi}{a+ib}.$$

Hence one finds

$$s = C + \frac{c}{b}\log(a+ib),$$

while C denotes a certain constant quantity. But it is also clear from this form that the sum of the same series continued from the beginning to the term of the index ni will be

$$= C + \frac{c}{b}\log(a+nib).$$

Therefore, the difference of these sums

$$\frac{c}{b}\log\frac{a+nib}{a+ib} = \frac{c}{b}\log n \quad \text{(while } a \text{ vanishes)}$$

will give the sum of the terms from $\frac{c}{a+ib}$ to $\frac{c}{a+nib}$. But because we assigned the limits of this sum above, $\frac{c}{b}\log n$ will be greater than $\frac{2(n-1)c}{(n+1)b}$ and smaller than $\frac{(n^2-1)c}{2nb}$, or

$$\log > \frac{2(n-1)}{n+1} \quad \text{and} \quad \log n < \frac{n^2-1}{2n}.$$

§7 Below we will show that that quantity C is finite and we will try to determine it. Therefore, C will vanish in the sum and the sum of the progression

$$\frac{c}{a}, \quad \frac{c}{a+b}, \quad \cdots \quad \frac{c}{a+(i-1)b}.$$

while the number of times is infinite $= i$, will become

$$= \frac{c}{b}\log(a+ib) = \frac{c}{b}\log i.$$

Therefore, the sum will be as the logarithm of the number of terms and hence infinitely smaller than the root of arbitrary large power of the number of terms; nevertheless it is infinitely large.

§8 From this consideration innumerable series arise to denote the logarithms of certain numbers. At first, let us take this harmonic progression

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.},$$

for which it is $a = 1$, $b = 1$, $c = 1$. Therefore, the difference between this series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i}$$

continued to the term of the index i and the same

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{ni}$$

continued to the term of the index ni will be $= \log n$. hence that series subtracted from this one leaves $\log n$. But since the number of terms of this series is n times greater than the number of the latter, from n term of the series

$$1 + \frac{1}{2} + \cdots + \frac{1}{ni}$$

one has to subtract one of the other series

$$1 + \frac{1}{2} + \cdots + \frac{1}{i},$$

that the subtraction to infinity can be done in the same way. Hence it will be

$$\begin{array}{l}
\log n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \cdots + \frac{1}{3n} + \text{etc.} \\
\qquad\qquad\qquad\quad -1 \qquad\qquad\qquad\quad -\frac{1}{2} \qquad\qquad\qquad\quad -\frac{1}{3}
\end{array}$$

Therefore, if the single terms of the inferior series are actually subtracted from the terms written over them of the superior series and for n the integer number 2, 3, 4 $\cdots$ etc. are written, we will successively obtain the following series of logarithms:

$$\begin{aligned}
\log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \text{etc.}, \\
\log 3 &= 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} - \frac{1}{10} + \frac{1}{11} - \frac{2}{12} + \text{etc.}, \\
\log 4 &= 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} - \frac{3}{12} + \text{etc.}, \\
\log 5 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{4}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{4}{10} + \frac{1}{11} + \frac{2}{12} + \text{etc.}, \\
\log 6 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{5}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} - \frac{5}{12} + \text{etc.}
\end{aligned}$$

$$\text{etc.},$$

Hence for the logarithm of each number a convergent series is easily found.

§9 From these series others of the same form, which have a rational sum, can be derived. For, since the double of the series $= \log 2$ is $\log 4$, if the series

$$1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \text{etc.}$$

is subtracted from this one

$$2 - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \text{etc.},$$

the remainder, namely, this series

$$1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \text{etc.},$$

will be $= 0$, or

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{3}{10} + \text{etc.}$$

Similarly, if the series exhibiting $\log 6$ is subtracted from the sum of the series exhibiting $\log 2$ and $\log 3$, the residue, namely

$$1 - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{1}{5} + \frac{2}{6} + \frac{1}{7} - \frac{1}{8} - \frac{2}{9} - \frac{1}{10} + \text{etc.},$$

will be $= 0$, or

$$1 = \frac{1}{2} + \frac{2}{3} + \frac{1}{4} - \frac{1}{5} - \frac{2}{6} - \frac{1}{7} + \frac{1}{8} + \frac{2}{9} + \frac{1}{10} - \text{etc.}$$

In similar manner, one will be able to find innumerable other series of this kind.

§10 Those series expressing the logarithms certainly converge, but very slowly, whence, that by means of them the logarithms can conveniently be found, a certain auxiliary tool is required. To find this it must be noted that these series do not proceed uniformly, but have certain revolutions, which are absolved in so many terms as n has unities; therefore, I will call that many terms taken simultaneously one member of the series. So in the series for $\log 2$ two terms will constitute one member, in the series for $\log 3$ three, in the series for $\log 4$ and so forth. Therefore, the members will constitute an equal series and to find logarithms it is necessary to add several members. For, let us put that m members were added to find the logarithm of two and instead of all the following ones one will be able to add $\frac{1}{4m}$, which will come the closer to the truth, the greater the number m was. To find $\log 3$ to m already added members instead of all the following ones add $\frac{1}{9m}$. In similar manner for $\log 4$ one must add $\frac{1}{16m}$ and so forth. These remarks follow from the method of summing applied in § 6; since in this m must be a very large quantity, I neglected the numbers added to m in the differential, that the integration does not depend on logarithms.

§11 But to determine the sum, even though it is infinite, of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{i}$ accurately, I express the single terms in the following way. It is

$$1 = \log 2 + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \text{etc.}$$

and

$$\begin{aligned}
\frac{1}{2} &= \log\frac{3}{2} + \frac{1}{2\cdot 4} - \frac{1}{3\cdot 8} + \frac{1}{4\cdot 16} - \frac{1}{5\cdot 32} + \text{etc.},\\
\frac{1}{3} &= \log\frac{4}{3} + \frac{1}{2\cdot 9} - \frac{1}{3\cdot 27} + \frac{1}{4\cdot 81} - \frac{1}{5\cdot 243} + \text{etc.},\\
\frac{1}{4} &= \log\frac{5}{4} + \frac{1}{2\cdot 16} - \frac{1}{3\cdot 64} + \frac{1}{4\cdot 256} - \frac{1}{5\cdot 1024} + \text{etc.}\\
&\vdots\\
\frac{1}{i} &= \log\frac{i+1}{i} + \frac{1}{i^2} - \frac{1}{3\cdot i^3} + \frac{1}{4\cdot i^4} - \frac{1}{5\cdot i^5} + \text{etc.}
\end{aligned}$$

Having added these series it will arise

$$\begin{aligned}
1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i} = \log(i+1) &+ \frac{1}{2}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}\right)\\
&- \frac{1}{3}\left(1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\text{etc.}\right)\\
&+ \frac{1}{4}\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\text{etc.}\right)\text{etc.}
\end{aligned}$$

Since these series are convergent, if they are summed approximately, it will arise

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i}=\log(i+1)+0.577218.$$

If the sum is called s, it will be, as we did it above,

$$ds=\frac{di}{i+1} \quad \text{and hence} \quad s==\log(i+1)+C.$$

Therefore, we detected the value of this constant C, which is $C=0.577218$.

§12 If the series $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i}$ is continued further to infinity and subdivided into members, of which each as the series itself contains i terms, the member contained within $\frac{1}{i}$ and $\frac{1}{2i}$ will be $=\log 2$, the following $=\log\frac{3}{2}$, the third $=\log\frac{4}{3}$ etc. And because the series of the sum itself is the logarithm of infinity, one can analogously put $\log\frac{1}{0}$. And this way we will obtain the following rather curious scheme:

Series	$1+\frac{1}{2}+\cdots+\frac{1}{i}$	$+\cdots+\frac{1}{2i}$	$+\cdots+\frac{1}{3i}$	$+\cdots+\frac{1}{4i}$	$+\cdots+\frac{1}{5i}$	etc.
Sums	$\log\frac{1}{0}$	$\log\frac{2}{1}$	$\log\frac{3}{2}$	$\log\frac{4}{3}$	$\log\frac{5}{4}$	

§13 It might certainly seem to be difficult to find these same properties of harmonic and logarithmic expressions analytically and in the same way I used elsewhere to sum series. But to anyone considering the subject with more attention it will become clear that this cannot only be done but can even be done in much more generality. For, I consider not the simple harmonic progression but the one connected a geometric progression, of which kind this one is

$$\frac{cx}{a}+\frac{cx^2}{a+b}+\frac{cx^3}{a+2b}+\frac{cx^4}{a+3b}+\text{etc.}$$

I put its sum s and having multiplied both by $bx^{\frac{a-b}{b}}$ it will be

$$bx^{\frac{a-b}{b}}s=\frac{bcx^{\frac{a-b}{b}}}{a}+\frac{bcx^{\frac{a+b}{b}}}{a+b}+\frac{bcx^{\frac{a+2b}{b}}}{a+2b}+\text{etc.}$$

And having taken differential one will have

$$bD.x^{\frac{a-b}{b}}s = dx\left(cx^{\frac{a-b}{b}} + cx^{\frac{a}{b}} + cx^{\frac{a+b}{b}} + \text{etc.}\right) = \frac{cx^{\frac{a-b}{b}}dx}{1-x}.$$

Having taken integrals again it will be

$$bx^{\frac{a-b}{b}}s = c\int \frac{x^{\frac{a-b}{b}}dx}{1-x}$$

and

$$s = \frac{c}{bx^{\frac{a-b}{b}}}\int \frac{x^{\frac{a-b}{b}}dx}{1-x}.$$

From this series I now subtract this one

$$\frac{fx^m}{g} + \frac{fx^{2m}}{g+h} + \frac{fx^{3m}}{g+2h} + \text{etc.},$$

whose sum shall be t. Multiply it by

$$\frac{h}{m}x^{\frac{m(g-h)}{h}};$$

it will be

$$\frac{h}{m}x^{\frac{m(g-h)}{h}}t = \frac{fhx^{\frac{mg}{h}}}{mg} + \frac{fhx^{\frac{m(g+h)}{h}}}{m(g+h)} + \frac{fhx^{\frac{m(g+2h)}{h}}}{m(g+2h)} + \text{etc.}$$

And having taken differentials it will be

$$\frac{h}{m}D.x^{\frac{m(g-h)}{h}}t = dx\left(fx^{\frac{mg-h}{h}} + fx^{\frac{m(g+h)-h}{h}} + fx^{\frac{m(g+2h)-h}{h}} + \text{etc.}\right) = \frac{fx^{\frac{mg-h}{h}}dx}{1-x^m}.$$

Hence one will have

$$t = \frac{fm}{hx^{\frac{m(g-h)}{h}}}\int \frac{x^{\frac{mg-h}{h}}dx}{1-x^m}.$$

And hence

$$s - t = \frac{c}{bx^{\frac{a-b}{b}}}\int \frac{x^{\frac{a-b}{a}}dx}{1-x} - \frac{fm}{hx^{\frac{m(g-h)}{h}}}\int \frac{x^{\frac{mg-h}{h}}dx}{1-x^m}.$$

But this subtraction has to be done in such a way that from the term of the index m of the series s the first term of the series t is subtracted and from the term of the index $2m$ of that series the second of this series and so forth.

§14 To find our logarithmic series, let it be $a = b$ and $g = h$. Having done this it will be

$$s = \frac{c}{b} \int \frac{dx}{1-x} = \frac{c}{b} \log \frac{1}{1-x}$$

and

$$t = \frac{f}{h} \int \frac{mx^{m-1}dx}{1-x^m} = \frac{f}{h} \log \frac{1}{1-x^m}.$$

Therefore

$$s - t = \log \frac{(1-x^m)^{\frac{f}{h}}}{(1-x)^{\frac{c}{b}}}.$$

But that this expression becomes finite for $x = 1$, it must be $\frac{f}{h} = \frac{c}{b}$; therefore, let all this letters become $= 1$ and it will be

$$s - t = \log \frac{1-x^m}{1-x} = \log(1 + x + x^2 + \cdots + x^{m-1}).$$

This expression gives the difference between these series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \text{etc.} \quad \text{and} \quad \frac{x^m}{1} + \frac{x^{2m}}{2} + \frac{x^{3m}}{3} + \text{etc.}$$

Hence, if it is $m = 2$, it will be

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.};$$

if it is $m = 3$, it will be

$$\log(1+x+x^2) = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \text{etc.}$$

and in similar manner

$$\log(1+x+x^2+x^3) = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \text{etc.}$$

If in these it is $x=1$, the same series for the logarithms of natural numbers as those we gave before [§ 8] will arise.

§15 If it is $h=2g$, it will be

$$t=\frac{fx^{\frac{m}{2}}}{h}\int\frac{mx^{\frac{m-2}{2}}dx}{1-x^m}.$$

Put $x^m=y$; it will be

$$t=\frac{f\sqrt{y}}{h}\int\frac{dy}{(1-y)\sqrt{y}}=\frac{f\sqrt{y}}{h}\log\frac{1+\sqrt{y}}{1-\sqrt{y}}=\frac{fx^{\frac{m}{2}}}{h}\log\frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}.$$

If furthermore it is $a=b$, it will be

$$s=\frac{c}{b}\log\frac{1}{1-x}.$$

But s is the sum of this series

$$\frac{cx}{a}+\frac{cx^2}{2a}+\frac{cx^3}{3c}+\text{etc.}$$

and

$$tx^{\frac{-m}{2}}=\frac{f}{h}\log\frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}$$

gives this series

$$\frac{fx^{\frac{m}{2}}}{g}+\frac{fx^{\frac{3m}{2}}}{3g}+\frac{fx^{\frac{5m}{2}}}{5g}+\text{etc.}$$

Let $a=1$ and $g=1$; it will be

$$s-tx^{\frac{-m}{2}}=c\log\frac{1}{1-x}-\frac{f}{2}\log\frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}=\log\frac{\left(1-x^{\frac{m}{2}}\right)^{\frac{f}{2}}}{(1-x)^c\left(1+x^{\frac{m}{2}}\right)^{\frac{f}{2}}}.$$

That this expression becomes finite, if it is $x=1$, it is necessary that it is $\frac{f}{2}=c$ or $f=2c$. Therefore, let $c=1$ and $m=2n$; the differences of the series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \text{etc.}$$

and

$$\frac{2x^n}{1} + \frac{2x^{3n}}{3} + \frac{2x^{5n}}{5} + \text{etc.}$$

will be

$$= \log \frac{1 - x^n}{(1-x)(1+x^n)}.$$

Put $n = 2$; the difference will be $= \log \frac{1+x}{1+x^2}$ and for $x = 1$ it will be $= 0$, whence this series

$$1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \frac{1}{7} + \text{etc.}$$

it will be $= 0$, as we already found above [§ 9].

§16 One can now find infinitely many other series of this kind having a rational sum from this form itself $\log \frac{1+x}{1+xx}$ by assuming other similar forms, which vanish for $x = 1$. For, from this form $\log \frac{1+x}{1+x^2}$, if it is expressed by means of a series, the found series itself immediately results. For, it is

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}$$

and

$$\log(1+x^2) = \frac{x^2}{1} - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \text{etc.}$$

Therefore, this series subtracted from the superior one leaves this one behind

$$\frac{x}{1} - \frac{3x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{3x^6}{6} + \text{etc.},$$

whose sum will be $\log \frac{1+x}{1+x^2}$. In similar manner, $\log \frac{1+x}{1+x^3}$ will give this series

$$\frac{x}{1} - \frac{x^2}{2} - \frac{2x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \frac{2x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} - \frac{2x^9}{9} - \text{etc.}$$

Therefore, having put $x = 1$ it will be

$$0 = 1 - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{1}{5} + \frac{2}{6} + \frac{1}{7} - \frac{1}{8} - \frac{2}{9} - \text{etc.},$$

which same we found already in § 9.

§17 In this way one will be able to find the sums of all irregular series of this kind which nevertheless proceed regularly in members; for, they are always to be considered as the difference of two series. Let, e.g., this series be propounded

$$1 - \frac{2}{2} + \frac{1}{3} + \frac{1}{4} - \frac{2}{5} + \frac{1}{6} + \frac{1}{7} - \frac{2}{8} + \text{etc.}$$

This is the difference of these series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \text{etc.}$$

and

$$\frac{3x^2}{2} + \frac{3x^5}{5} + \frac{3x^8}{8} + \text{etc.}$$

for $x = 1$. But the of that series is $\log \frac{1}{1-x}$, the sum of the first on the other hand is $\int \frac{3xdx}{1-x^3}$ or

$$\log \frac{1}{1-x} + \frac{1}{2}\log(x^2+x+1) + \frac{\sqrt{-3}}{2}\log\frac{2x+1-\sqrt{-3}}{2x+1+\sqrt{-3}} - \frac{\sqrt{-3}}{2}\log\frac{1-\sqrt{-3}}{1+\sqrt{-3}}.$$

Therefore, having subtracted this one from that one and having put $x = 1$

$$-\frac{1}{2}\log 3 + \frac{\sqrt{-3}}{2}\log\frac{3+\sqrt{-3}}{3-\sqrt{-3}} - \frac{\sqrt{-3}}{2}\log\frac{1+\sqrt{-3}}{1-\sqrt{-3}}$$

will arise for the sum of the propounded progression. But $\frac{\sqrt{-3}}{2}\log\frac{3+\sqrt{-3}}{3-\sqrt{-3}}$ is indeed the circumference of the circle divided by $\sqrt{3}$ having put the diameter $= 1$ and $\frac{\sqrt{-3}}{2}\log\frac{1-\sqrt{-3}}{1+\sqrt{-3}}$ is its half.

§18 But even if the members themselves go in non uniformly, the sum is assigned more difficultly. Let us take this series

$$1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{2}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{3}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}-\frac{1}{14}+\text{etc.}$$

This is the difference between these series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{i\cdot\frac{i+3}{2}}$$

and

$$\frac{2}{2}+\frac{3}{5}+\frac{4}{9}+\frac{5}{14}+\cdots+\frac{i+1}{i\cdot\frac{i+3}{2}}$$

continued to infinity in such a way that the most outer terms have the same denominator $i\cdot\frac{i+3}{2}$. But the sum of the first of these series is $C+\log i+\log(i+3)-\log 2$, where C denotes the constant found in § 11, namely 0.577218. The other series which is to be subtracted is resolved into these two

$$\frac{2}{3}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{i}\right)$$

and

$$\frac{4}{3}\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{i+3}\right)$$

The sum of that one $\frac{2}{3}C+\frac{2}{3}\log i$, the sum of the first on the other hand is $\frac{4}{3}C-\frac{22}{9}+\frac{4}{3}\log(i+3)$. These two subtracted from that sum $C+\log i+\log(i+3)-\log 2$ leaves ${}_c+\frac{22}{9}-\log 2$ or approximately 1.174078 for the sum of the propounded series.