Various Artifices to investigate the Nature of Series *

Leonhard Euler

Many times series of such a kind occur, whose origin might be sufficiently perspicuous, but whose law of progression and nature is highly mysterious and whose properties can only be investigated applying extraordinary analytic artifices. It certainly is hardly possible to propound artifices of this kind in general in such a way that their use is clearly seen; but their applicability is rather shown most conveniently in examples, whence at the same time the reason and necessity to invent them is understood a lot more clearly. Therefore, I will contemplate a completely singular series or progression of numbers here, which results, if the powers of the trinomial $1 + x + xx$ are expanded and from each expansion only the middle term is taken, i.e. the one with the largest coefficient; for, this way an even more remarkable series of numbers results, since the law of progression is not obvious. But having explored it, most beautiful properties will be revealed, which is the main task of analytical artifices. But this memorable series especially exhibits an example, how careful we have to be when using the method of induction, which is usually applied in investigations of this kind, since here an induction of such a kind occurs, which, even though it seems to be correct, nevertheless leads to an error.

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EXPANSION OF THE POWERS OF THE TRINOMIAL

$$1+x+x^2$$
$$1+2x+3x^2+2x^3+x^4$$
$$1+3x+6x^2+7x^3+6x^4+3x^5+x^6$$
$$1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$$
$$1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$$
$$1+6x+21x^2+50x^3+90x^4+126x^5+141x^6+126x^7+90x^8+50x^9+21x^{10}+6x^{11}+x^{12}$$
$$\text{etc.}$$

From these formulas I will only take the middle terms, which yield this progression

$$x+3x^2+7x^3+19x^4+51x^5+141x^6+\text{etc.}$$

whose nature I want to investigate here, where, omitting all powers of x, the whole task is reduced to this numerical progression

$$1, \quad 3, \quad 7, \quad 19, \quad 51, \quad 141, \quad 393 \quad \text{etc.}$$

CONSIDERATION I

1. Considering this series, it comes to mind that each term is well-approximated by the triple of its predecessor, since this series, if continued to infinity, by nature must obviously be confounded with the tripled geometric progression. Therefore, I write the tripled preceding terms under the terms of the series moved one term further to the right, but I denote the indices this way:

Indices	0	1	2	3	4	5	6	7	8	9
A	1	1	3	7	19	51	141	393	1107	3139
B		3	3	9	21	57	153	423	1179	3321
C		2	0	2	2	6	12	30	72	182
D		1	0	1	1	3	6	15	36	91
E		1	0	1	1	2	3	5	8	13

where the series A is the propounded series itself, which subtracted from the series B, its triple, leaves behind the series C; but having split this series into two parts the series D results, whose single terms are the triangular numbers, under which I wrote the respective indices, whence the series E resulted.

2. In this series E the structure seems be of such a kind that each term is equal to the sum of the two preceding ones, and this conclusion based on inspection, since it is confirmed by ten terms of the series, seems so certain to be correct that one can neither doubt that all terms of the series D are the triangular numbers, nor that their indices constitute that simple recurring series, in which each term is the aggregate of the two preceding ones. We certainly often trust induction, even if based on a less solid foundation, in investigations of this kind.

3. If this induction would be true, it would be a discovery of highest importance, since hence the general term of the propounded series A could be assigned; of course, the term corresponding to the index n would be

$$\frac{1}{10}3^n + \frac{1}{10}(-1)^n + \frac{1}{5}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{3-\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

and our progression would arise from the following three recurring series:

										Relation Scale
A	1	1	5	13	41	121	365	1093	3281	$2 \quad +3$
B	2	3	7	18	47	123	322	843	2207	$3 \quad -1$
C	2	1	3	4	7	11	18	29	47	$1 \quad +1$
D	5	5	15	35	95	255	705	1965	5535	and divinding by 5
E	1	1	3	7	19	51	141	393	1107	etc.

For, from the recurring series A, B, C by addition of the single terms the series D results, whose terms divided by 5 produce our progression our series, at least up to ten terms.

4. It is important to teach, how I found the expression of this general term, since the above induction, no matter how well-grounded it might seem, nevertheless is false. For, if our progression is continued and the operations applied in par. 1 are done, we find:

Indices	5	6	7	8	9	10	11
A	51	141	393	1107	3139	8953	25653
B	57	153	423	1179	3321	9417	26859
C	6	12	30	72	182	464	1206
D	3	6	15	36	91	232	603
E	2	3	5	8	13	–	–

in the series D the terms 232 and 603 are not triangular numbers and hence the law of the series E is not valid anymore. Therefore, this example of an erroneous induction is even more remarkable, since I have never found a case of such a kind before, in which a so specious induction is actually wrong.

Consideration II

5. Therefore, leaving all induction aside, I attempt to investigate the nature of our series from its origin. And first it is certainly clear, if in this series

$$x, \quad 3x^2, \quad 7x^3, \quad 19x^4, \quad 51x^5, \quad 141x^6, \quad 391x^7 \quad \text{etc.}$$

the term corresponding to the index n is put

$$= Nx^n,$$

that Nx^n will be the term of this power of x, which results from the expansion of $(1+x+xx)^n$. Therefore, I, having combined the first two terms, treat the trinomial $1+x+xx$ as a binomial and it will be

$$(1+x+xx)^n = (1+x)^n + \frac{n}{1}xx(1+x)^{n-1} + \frac{n(n-1)}{1\cdot 2}x^4(1+x)^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^6(1+x)^{n-3} + \text{etc.}$$

from whose single terms one has to take the power x^n, and hence the sum of all together will give our term Nx^n in question.

6. But from the first term $(1+x)^n$ or $(x+1)^n$ after the expansion it results

$$x^n;$$

but for the second member one has to take the second term from the expansion of the formula $(x+1)^{n-1}$, i.e.

$$\frac{n-1}{1}x^{n-2}.$$

Further, for the third member $(x+1)^{n-2}$ the third term

$$\frac{(n-2)(n-3)}{1\cdot 2}x^{n-4}$$

multiplied by the factor $\frac{n(n-1)}{1\cdot 2}x^4$ yields

$$\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}x^n$$

and likewise for all remaining members; hence we obtain

$$N = 1 + \frac{n(n-1)}{1\cdot 1} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3} + \text{etc.},$$

the number of which parts to be added is finite for each integer number n; and so the value of the term N can be easily assigned. But the same expression is found more easily, if the power of the trinomial is expanded this way

$$(x(1+x)+1)^n = x^n(1+x)^n + \frac{n}{1}x^{n-1}(1+x)^{n-1}$$
$$+\frac{n(n-1)}{1\cdot 2}x^{n-2}(1+x)^{n-2} + \text{etc.},$$

where the coefficient of the power x^n from the first term is seen to be 1, from the second $\frac{n}{1}\cdot\frac{n-1}{1}$, from the third $\frac{n(n-1)}{1\cdot 2}\cdot\frac{(n-2)(n-3)}{1\cdot 2}$ etc. as above.

Consideration III

7. Having found an expression, by which the coefficient of the power x^n is defined in general in our progression, first I observe that it can not be simplified by any means, i.e. be reduced to a finite expression. For, even though the invention of the number N can be reduced to a differential equation of second order, it nevertheless is of such a nature, that it can not be solved by any means. Therefore, since all efforts to find a more convenient form of the expression for N would be pointless, I will focus on finding the law, by which an arbitrary term in our progression can be defined from several preceding ones.

8. For this purpose I represent our progression this way

$$x,\quad 3x^2,\quad 7x^3,\quad 19x^4,\quad 51x^5,\quad \cdots\quad px^{n-2},\quad qx^{n-1},\quad rx^n,$$

and will investigate, how the number r can be determined in terms of the preceding q and p. But one has the values p, q, r from the series found above for N, which values, in order to be able to manipulate them analytically, I express as follows:

$$p = 1 + \frac{(n-2)(n-3)}{1\cdot 1}z^2 + \frac{(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2}z^4 + \text{ etc.}$$

$$q = 1 + \frac{(n-1)(n-2)}{1\cdot 1}z^2 + \frac{(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2\cdot 2}z^4 + \text{ etc.}$$

$$r = 1 + \frac{n(n-1)}{1\cdot 1}z^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}z^4 + \text{etc.},$$

whence, by subtracting each one from the following one, we first conclude

$$\frac{q-p}{2} = \frac{n-2}{1}z^2 + \frac{(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2}z^4 + \frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1\cdot 1\cdot 2\cdot 2\cdot 3}z^6 + \text{etc.}$$

$$\frac{r-q}{2} = \frac{n-1}{1}z^2 + \frac{(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2}z^4 + \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2\cdot 3}z^6 + \text{etc.}$$

9. But having differentiated the values q and r we obtain

$$\frac{dq}{2dz} = \frac{(n-1)(n-2)}{1}z + \frac{(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2}z^3 + \text{ etc.}$$

$$\frac{dr}{2dz} = \frac{n(n-1)}{1}z + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2}z^3 + \text{etc.},$$

which series is easily compared to the preceding ones, since obviously

$$\frac{(b-1)(q-p)}{2} = \frac{zdq}{2dz} \quad \text{and} \quad \frac{n(r-q)}{2} = \frac{zdr}{2dz},$$

whence we conclude that it will be

$$dq = (n-1)(q-p) \cdot \frac{dz}{z}$$

and

$$dr = n(r-q) \cdot \frac{dz}{z}.$$

10. Further, if differentiated, the last formulas of the preceding paragraph yield:

$$\frac{dq-dp}{4dz} = (n-2)z + \frac{(n-2)(n-3)(n-4)}{1 \cdot 1}z^3 + \frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1 \cdot 1 \cdot 2 \cdot 2}z^5 + \text{etc.}$$

$$\frac{dr-dq}{4dz} = (n-1)z + \frac{(n-1)(n-2)(n-3)}{1 \cdot 1}z^3 + \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1 \cdot 1 \cdot 2 \cdot 2}z^5 + \text{etc.},$$

which differ from the first formulas only in that regard that here the coefficients have one more factor; but by means of differentiation the same factors can easily added there as follows

$$\frac{d.pz^{2-n}}{dz} = -(n-2)z^{1-n} - \frac{(n-2)(n-3)(n-4)}{1 \cdot 1}z^{3-n}$$
$$-\frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1 \cdot 1 \cdot 2 \cdot 2}z^{5-n} - \text{etc.}$$

$$\frac{d.qz^{1-n}}{dz} = -(n-1)z^{-n} - \frac{(n-1)(n-2)(n-3)}{1 \cdot 1}z^{2-n}$$
$$-\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1 \cdot 1 \cdot 2 \cdot 2}z^{4-n} - \text{etc.},$$

whence it is obvious that

$$\frac{dq-dp}{4dz} + \frac{z^n d.pz^{2-n}}{dz} = 0$$

and

$$\frac{dr - dq}{4dz} + \frac{z^{n+1}d.q^{1-n}}{dz} = 0$$

and after the expansion

$$\begin{aligned} dq - dp + 4zzdp - 4(n-2)pzdz &= 0, \\ dr - dq + 4zzdq - 4(n-1)qzdz &= 0. \end{aligned}$$

11. Therefore, since above we found the differentials dq and dr expressed in terms of dz, if we substitute these values in the last equation, we will obtain

$$\frac{n(r-q)}{z} - \frac{(n-1)(q-p)}{z} + 4(n-1)(q-p) - 4(n-1)qz = 0,$$

so that, having removed the differentials, we found a finite relation among p, q and r, which reads as follows

$$n(r-q) = (n-1)(q-p)(1-4zz) + 4(n-1)qzz$$

or

$$n(r-q) = (n-1)(q + p(4zz-1)).$$

12. Therefore, we found a relation among the three consecutive values p, q, r, by means of which, given two of them, the third is easily defined and this a lot more general than it is necessary for our case, since this relation holds for any number z. Therefore, since in our case $z = 1$, it will be

$$n(r-q) = (n-1)(q+3p)$$

or

$$r = q + \frac{n-1}{n}(q+3p),$$

by means of which formula our progression can easily be continued arbitrarily far as follows:

A	1	3	7	19	51	141	393	1107	3139
B		3	9	21	57	153	423	1179	
C		6	16	40	108	294	816	2286	
D		3	4	5	6	7	8	9	
E		2	4	8	18	42	102	254	
F		4	12	32	90	252	714	2032	

Of course, the series A, as far as it had already been continued, is written over the same terms tripled and moved to the right one place, leading to series B; then the sum $A + B$ will give the series C, which is written over the the arithmetic progression D; the division $C : D$ yields the series E, whence $C - E$ gives the series F, whose terms added to the corresponding term of the series A gives its next term.

13. Therefore, let us continue our progression this way:

A	1107	3139	8953	25653	73789	212941	616227
B	1179	3321	9417	26859	76959	221367	
C	2286	6460	18370	52512	150748	434308	
D	9	10	11	12	13	14	
E	254	646	1670	4376	11596	31022	
F	2032	5814	16700	48136	139152	403286	

whence having added the powers of x, since the term corresponding to x^0 certainly is 1, as it also follows from the law of progression we found, our progression will look as follows:

$$1, \quad 1x, \quad 3x^2, \quad 7x^3, \quad 19x^4, \quad 51x^5, \quad 141x^6, \quad 393x^7, \quad 1107x^8, 3139x^9, \quad 8953x^{10},$$
$$25653x^{11}, \quad 73789x^{12}, \quad 212941x^{13}, \quad 616227x^{14} \quad \text{etc.}$$
$$\cdots px^{n-2}, \quad qx^{n-1}, \quad rx^n$$

and the law of progression is of such a nature that

$$r = q + \frac{n-1}{n}(q+3p) = 2q + 3p - \frac{1}{n}(q+3p).$$

14. But here especially the artifice, by which we got to this relation among three consecutive terms through differentials, should be noted, since in the final formula indeed no variable is contained anymore. Now it is easy to see that the same relation can be found without differentiation, if in the three series of par. 8. one uses this multiplication that $(A + azz)p + Bq + Cr = 0$. For, it will quickly become clear that one can attribute values of such a kind to the letters A, a, B and C, that all powers of z vanish, which then leads to the above relation. But considering this at the beginning certainly is a less obvious thing to do.

Consideration IV

15. Having found this law of progression, this raises the equally interesting question, in which the sum of the same progression continued to infinity is investigated. Therefore, let us put

$$s = 1 + x + 3x^2 + 7x^3 + \cdots + px^{n-2} + qx^{n-1} + rx^n + \text{etc.},$$

and since we found

$$n(r - 2q - 3p) + q + 3p = 0,$$

differentiating this equality let us introduce it as follows:

$$\begin{array}{rcl}
\dfrac{ds}{dx} & = & 1 + 6x + 21x^2 + \cdots + (n-2)px^{n-3} + (n-1)qx^{n-2} + nrx^{n-1} \\
-\dfrac{2d.xs}{dx} & = & -2 - 4x - 18x^2 - \cdots \qquad - 2(n-1)px^{n-2} - 2nqx^{n-1} \\
-\dfrac{3d.x^2s}{dx} & = & \quad - 6x - 9x^2 - \cdots \qquad - 3npx^{n-1} \\
\hline
s & = & 1 + x + 3x^2 + \cdots \qquad + qx^{n-1} \\
3xs & = & \quad 3x + 3x^2 + \cdots \qquad + 3px^{n-1}
\end{array}$$

whence we obtain

$$\frac{ds - 2d.xs - 3d.x^2s}{dx} + s + 3xs = 0$$

or

$$(1 - 2x - 3xx)ds - sdx - 3xsdx = 0.$$

From this equation it follows

$$\frac{ds}{s} = \frac{dx + 3xdx}{1 - 2x - 3xx}$$

and hence by integration

$$s = \frac{1}{\sqrt{1 - 2x - 3xx}} = \frac{1}{\sqrt{(1 + x)(1 - 3x)}}.$$

16. Therefore, lo and behold the new origin of our series, which now results from the expansion of this form

$$(1 - 2x - 3xx)^{-\frac{1}{2}},$$

whence after the calculation this series is detected to result

$$1 + x + 3x^2 + 7x^3 + 19x^4 + 51x^5 + 141x^6 + \text{etc.}$$

But at the same time hence it is clear, how large the sum of this series, if continued to infinity, will be for each value of x; here certainly it has to be noted, if either $x = -1$ and $x = \frac{1}{3}$, that the sum will be infinite; but if $x > \frac{1}{3}$, the sum is imaginary. But the sum will be finite, if x is contained within the limits $\frac{1}{3}$ and -1; but outside these limits always an imaginary sum results. So, having taken $x = \frac{1}{4}$, it will be

$$1 + \frac{1}{4} + \frac{3}{4^2} + \frac{7}{4^3} + \frac{19}{4^4} + \frac{51}{4^5} + \text{etc.} = \frac{4}{\sqrt{5}}.$$

CONSIDERATION V

17. This investigation can be extended to a series of middle terms taken from the expansion of the more general trinomial $a + bx + cx^2$. For, having in general put Nx^n for the middle term of the power $(a+bx+cxx)^n$, the value of the coefficient N can be determined this way: Since

$$(x(b+cx)+a)^n = x^n(b+cx)^n + \frac{n}{1}ax^{n-1}(b+cx)^{n-1}$$
$$+\frac{n(n-1)}{1\cdot 2}a^2x^{n-2}(b+cx)^{n-2} + \text{etc.},$$

collect the coefficients of the power x^n from the single terms, and one will find

$$N = b^n + \frac{n(n-1)}{1\cdot 1}ab^{n-2}c + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}a^2b^{n-4}c^2 + \text{etc.}$$

or, for the sake of brevity having put,

$$\frac{ac}{bb} = g,$$

it will be

$$N = b^n\left(1 + \frac{n(n-1)}{1\cdot 1}g + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}g^2 + \text{etc.}\right).$$

Hence, since having taken $n=0$ we have $N=1$, if we represent this progression this way

$$1 + Ax + Bx^2 + Cx^3 + Dx^4 + \cdots + Nx^n + \text{etc.},$$

these coefficients will be:

$$\begin{aligned}
A &= b,\\
B &= b^2(1+2g),\\
C &= b^3(1+6g),\\
D &= b^4(1+12g+6gg),\\
E &= b^5(1+20g+30gg),\\
F &= b^6(1+30g+90gg+20g^3).
\end{aligned}$$

18. To investigate, how each term is determined by the two preceding ones, let us represent the series this way

$$1, \quad bx, \quad (1+2g)b^2x^2, \quad (1+6g)b^3x^3, \quad \cdots \quad pb^{n-2}x^{n-2}, \quad qb^{n-1}x^{n-1}, \quad rb^nx^n$$

and writing g for zz we will have

$$p = 1 + \frac{(n-2)(n-3)}{1\cdot 1}z^2 + \frac{(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2}z^4 + \text{ etc.}$$
$$q = 1 + \frac{(n-1)(n-2)}{1\cdot 1}z^2 + \frac{(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2\cdot 2}z^4 + \text{ etc.}$$
$$r = 1 + \frac{n(n-1)}{1\cdot 1}z^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}z^4 + \text{etc.},$$

which series are the same we treated above; and therefore, it will be

$$n(r-q) = (n-1)(q + p(4zz-1)).$$

Substituting g for zz in our series the term r is determined by the two preceding ones in such a way that

$$r = q + \frac{n-1}{n}(q + (4g-1)p)$$

or

$$r = 2q + (4g-1)p - \frac{1}{n}(q + (4g-1)p).$$

19. Let us put $4g - 1 = h$ that

$$h = \frac{4ac - bb}{bb},$$

and since the law of progression yields

$$r = 2q + hp - \frac{1}{n}(q + hp)$$

and omitting the powers b^nx^n the two initial terms are 1 and 1, our progression will be

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ 1, & 1, & \dfrac{3+h}{2}, & \dfrac{5+3h}{2}, & \dfrac{35+30h+3hh}{8}, & \dfrac{63+70h+15hh}{8}, \end{array}$$

whence having taken $h = 3$ the series treated before results. But if one takes $h = -1$ or $g = 0$, all terms will go over into 1, which follows from the relation

$$n(r-q) = (n-1)(q-p);$$

for, if two contiguous terms p and q are equal, all are necessary also equal to them.

CONSIDERATION VI

20. Let us generalize the investigation of the sum of this progression a lot more and let

$$s = A + Bx + Cx^2 + \cdots + px^{n-2} + qx^{n-1} + rx^n + \text{etc.},$$

the law of progression of which series is understood to be of such a nature that

$$n(ap + bq + cr) = fp + gq,$$

and repeating the calculation from above in par. 15 we will have:

$$\begin{array}{rcl} \dfrac{ad.xxs}{dx} &=& \phantom{bA +{}} 2aAx + 3aBxx + \cdots + napx^{n-1} \\ \dfrac{bd.xs}{dx} &=& bA + 2bBx + 3bCxx + \cdots + nbqx^{n-1} \\ \dfrac{cds}{dx} &=& cB + 2cCx + 3cDxx + \cdots + ncrx^{n-1} \\ \hline gs &=& Ag + gBx + gCxx + \cdots + gqx^{n-1} \\ fxs &=& \phantom{Ag +{}} fAx + fBxx + \cdots + fpx^{n-1} \end{array}$$

Therefore, from the nature of the series it necessarily is

$$\frac{ad.xxs + bd.xs + cds}{dx} - (fx + g)s = (b - g)A + cB$$

or

$$\frac{ds(axx + bx + c)}{dx} + s((2a - f)x + (b - g)) = (b - g)A + cB.$$

21. Therefore, since we have

$$ds + \frac{sdx((2a - f)x - (b - g))}{axx + bx + c} = \frac{(b - g)Adx + cBdx}{axx + bx + c},$$

the integration of this equation has to be done in such a way that for $x = 0$ we have $s = A$, from which this summation has no difficulty. Therefore, let us accommodate this to the series found before, which was

$$s = 1 + x + \frac{3 + h}{2}x^2 + \frac{5 + 3h}{2}x^3 + \cdots + px^{n-2} + qx^{n-1} + rx^n + \text{etc.},$$

for which

$$n(hp + 2q - r) = hp + q$$

and

$$A = 1, \quad B = 1,$$

and after the application it will be

$$a = h, \quad b = 2, \quad c = -1, \quad f = h, \quad g = 1,$$

whence the value of the sum s has to be defined from this equation

$$ds + \frac{sdx(hx + 1)}{hxx + 2x - 1} = \frac{Adx - Bdx}{hxx + 2x - 1} = 0$$

and hence one concludes

$$s\sqrt{hxx + 2x - 1} = \sqrt{-1}$$

or

$$s = \frac{1}{\sqrt{1 - 2x - hxx}}.$$

22. Let us substitute the value assumed in par. 19 again

$$h = 4g - 1 = \frac{4ac - bb}{bb}$$

and let us write bx instead of x so that this series is to be summed

$$s = 1 + bx + (bb + 2ac)x^2 + (b^3 + 6abc)x^3 + (b^4 + 12abbc + 6aacc)x^4 + \text{etc.},$$

and its sum will be

$$s = \frac{1}{\sqrt{1 - 2bx + (bb - 4ac)xx}}$$

or

$$s = \frac{1}{\sqrt{(1 - bx)^2 - 4acxx}}.$$

But the origin of this series is that its single terms are the middle terms taken from the powers $(a + bx + cxx)^n$. But then the law of progression is of such a nature that, having put three consecutive terms

$$px^{n-2}, \quad qx^{n-1}, \quad rx^n,$$

the coefficient r is determined by the other two in such a way that

$$r = bq + \frac{n-1}{n}(bq + (4ac - bb)p)$$

or

$$r = \frac{2n-1}{n}bq + \frac{n-1}{n}(4ac - bb)p.$$

23. But if one puts $bb = 4ac$ so that

$$a + bx + cxx = (\sqrt{a} + x\sqrt{c})^2,$$

each term of our progression is determined by the preceding one alone so that

$$r = \frac{2n-1}{n} \cdot 2q\sqrt{ac}.$$

In this case put

$$a = 1, \quad c = 1 \quad \text{and} \quad b = 2,$$

that our series consists of the middle terms of the powers

$$(1 + 2x + xx)^n \quad \text{or} \quad (1 + x)^{2n},$$

and it will be

$$r = \frac{2(2n-1)}{n} q$$

and the series itself

$$s = 1 + 2x + \frac{2 \cdot 6}{1 \cdot 2} x^2 + \frac{2 \cdot 6 \cdot 10}{1 \cdot 2 \cdot 3} x^3 + \frac{2 \cdot 6 \cdot 10 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \text{etc.},$$

whose sum is

$$s = \frac{1}{\sqrt{1-4x}},$$

as it is manifest per se.

CONSIDERATION VII

24. From the sum of the preceding series

$$s = 1 + bx + (bb + 2ac)x^2 + (b^3 + 6abc)x^3 + \text{etc.}$$

we found before, i.e.

$$s = ((1-bx)^2 - 4acxx)^{-\frac{1}{2}},$$

one can vice versa find the general term or the coefficient of the power x^n. For, since, having done the expansion in the usual way,

$$s = \frac{1}{1-bx} + \frac{1}{2} \cdot \frac{acxx}{(1-bx)^3} + \frac{2 \cdot 6}{1 \cdot 2} \cdot \frac{a^2c^2x^4}{(1-bx)^5} + \frac{2 \cdot 6 \cdot 10}{1 \cdot 2 \cdot 3} \cdot \frac{a^3c^3x^6}{(1-bx)^7} + \text{etc.},$$

collect the single powers x^n from the single terms; from the first term it results

$$b^n x^n,$$

from the second

$$\frac{2}{1}\cdot\frac{n(n-1)}{1\cdot 2}acb^{n-2}x^n,$$

from the third

$$\frac{2\cdot 6}{1\cdot 2}\cdot\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^2c^2b^{n-4}x^n,$$

which collected into one sum will give

$$b^nx^n\left(1+\frac{n(n-1)}{1\cdot 1}\frac{ac}{bb}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}\frac{aacc}{b^4}+\text{etc.}\right),$$

precisely as we found from the origin of this series.