On the extraordinary use of the method of Interpolation in the doctrine of Series*

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In the method of interpolation a relation of such a kind between the two variables x and y is in question that, if to the one successively the given values

$$a, b, c, d \quad \text{etc.}$$

are attributed, the other y hence also obtains the given values

$$p, q, r, s \quad \text{etc.},$$

or what reduces to the same, an equation for a curved line of such a kind is in question, which goes trough arbitrarily many given points. Therefore, the greater the number of these points was, the more the curved line is limited; nevertheless, I already observed on another occasion, even if the number of points is augmented to infinity, that always still infinitely many curved lines can be exhibited, which equally will go through all the same points. Since the method of interpolation for each case yields a determined curved line, this solution is always to be considered as highly particular; but this circumstance itself implies a certain singular nature of the found solution, which deserves a more accurate consideration. But this nature of the solution especially depends on the method, by which this interpolation is done, or on the form, which is attributed to the general form, in which the equation in question must be

*original title: „De eximio usu methodi interpolationum in serierum doctrina“, first published in „*Opuscula Analytica* 1, 1783, pp. 157-210“, reprinted in „*Opera Omnia*: Series 1, Volume 15, pp. 435 - 497“, Eneström-Number E555, translated by: Alexander Aycock for the project „Euler-Kreis Mainz“

contained. Since this form can be constituted in infinitely many ways, I will restrict my investigation to this form

$$y = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \epsilon x^9 + \text{etc.},$$

which certainly only contains odd powers of x, such that, which values of y correspond to any arbitrary positive values of x, the same taken negatively also correspond to the same negative values of x; by this innumerable other curved lines are excluded, which would go through the same points.

Problem 1

§1 *To find an equation between the two variables x and y of this form*

$$y = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \epsilon x^9 + \text{etc.},$$

that, if to x the given values

$$a, b, c, d \ \text{ etc.}$$

are attributed, the other variable y likewise obtains the values

$$p, q, r, s \ \text{ etc.}$$

Solution

That the general assumed equation can be accommodated to this case more easily, exhibit it in this form

$$\begin{aligned} y = Ax + Bx(xx - aa) &+ Cx(xx-aa)(xx-bb) \\ &+ Dx(xx-aa)(xx-bb)(xx-cc) \\ &+ Ex(xx-aa)(xx-bb)(xx-cc)(xx-dd) \\ &+ \text{etc.}, \end{aligned}$$

which, even though it might proceed to infinity, if the number of conditions is infinite, of course, nevertheless for the single propounded conditions yields the following finite equations:

$$
\begin{aligned}
&\text{I.} && p = Aa, \\
&\text{II.} && q = Ab + Bb(bb-aa), \\
&\text{III.} && r = Ac + Bc(cc-aa) + Cc(cc-aa)(cc-bb), \\
&\text{IV.} && s = Ad + Bd(dd-aa) + Cd(dd-aa)(dd-bb), \\
& && \qquad + Dd(dd-aa)(dd-bb)(dd-cc), \\
& && \qquad \text{etc.,}
\end{aligned}
$$

which shall be represented this way

$$
\begin{aligned}
&\text{I.} && \frac{p}{a} = A, \\
&\text{II.} && \frac{q}{b} = A + B(bb-aa), \\
&\text{III.} && \frac{r}{c} = A + B(cc-aa) + C(cc-aa)(cc-bb), \\
&\text{IV.} && \frac{s}{d} = A + B(dd-aa) + C(dd-aa)(dd-bb) \\
& && \qquad + D(dd-aa)(dd-bb)(dd-cc) \\
& && \qquad \text{etc.}
\end{aligned}
$$

Now subtract the first from the single following ones and divide the differences by the coefficients of B, that these equations arise:

$$
\begin{aligned}
\frac{aq-bp}{ab(bb-aa)} &= q' = B, \\
\frac{ar-cp}{ac(cc-aa)} &= r' = B + C(cc-bb), \\
\frac{as-dp}{ad(dd-aa)} &= s' = B + C(dd-bb) + D(dd-bb)(dd-cc) \\
&\text{etc.}
\end{aligned}
$$

Now, in similar manner subtracting the first from the following ones and dividing the them by the coefficients of C we will get to these equations:

$$\frac{r'-q'}{cc-bb} = r'' = C,$$
$$\frac{s'-q'}{dd-bb} = d'' = C + D(dd-cc)$$
$$\text{etc.}$$

and further to this one

$$\frac{s''-r''}{dd-cc} = D.$$

Therefore, from the given quantities a, b, c, d etc. and p, q, r, s etc. the coefficients A, B, C, D etc. will be determined most conveniently this way: First, from the given quantities derive these

$$P = \frac{p}{a}, \quad Q = \frac{q}{b}, \quad R = \frac{r}{c}, \quad S = \frac{s}{d} \quad \text{etc.}$$

and hence form these:

$$\begin{aligned}
Q' &= \frac{Q-P}{bb-aa}, \quad & R' &= \frac{R-P}{cc-aa}, \quad & S' &= \frac{S-P}{dd-aa}, \quad & T' &= \frac{T-P}{ee-aa} \quad \text{etc.,}\\
R'' &= \frac{R'-Q'}{cc-bb}, \quad & S'' &= \frac{S'-Q'}{dd-bb}, \quad & T'' &= \frac{T'-Q'}{ee-bb}, \quad \text{etc.,}\\
S''' &= \frac{S''-R''}{dd-cc}, \quad & T''' &= \frac{T''-R''}{ee-cc}, \quad \text{etc.,}\\
T'''' &= \frac{T'''-S'''}{ee-dd}, \quad \text{etc.,}
\end{aligned}$$

Having found these values we will have

$$A = P, \quad B = Q', \quad C = R'', \quad D = S''', \quad E = T'''' \quad \text{etc.}$$

COROLLARY 1

§2 Since it is $P = \frac{p}{a}$, the first coefficient will be

$$A = \frac{p}{a};$$

for the following on the other hand because of

$$Q' = \frac{aq - bp}{ab(bb - aa)}, \quad R' = \frac{ar - cp}{ac(cc - aa)}, \quad S' = \frac{as - dp}{ad(dd - aa)}, \quad T' = \frac{at - ep}{ae(ee - aa)} \quad \text{etc.}$$

the second coefficient will be

$$B = \frac{aq - bp}{ab(bb - aa)}$$

or

$$B = \frac{p}{a(bb - aa)} + \frac{q}{b(bb - aa)}.$$

COROLLARY 2

§3 Further, because it is

$$R'' = \frac{ar - cp}{ac(cc - aa)(cc - bb)} - \frac{aq - bp}{ab(bb - aa)(cc - bb)},$$

it will be

$$C = \frac{p}{a(aa - bb)(aa - cc)} + \frac{q}{b(bb - aa)(bb - cc)} + \frac{r}{c(cc - aa)(cc - bb)}$$

COROLLARY 3

§4 In similar manner by prosecuting the calculation further it will be found

$$\begin{aligned} D = & \frac{p}{a(aa - bb)(aa - cc)(aa - dd)} + \frac{q}{b(bb - aa)(bb - cc)(bb - dd)} \\ & + \frac{r}{c(cc - aa)(cc - bb)(cc - dd)} + \frac{s}{d(dd - aa)(dd - bb)(dd - cc)}, \end{aligned}$$

whence it is possible to conjecture the form of the following quantities E, F, G etc. already quite safely.

SCHOLIUM 1

§5 But in most cases the values of the single coefficients A, B, C, D, E etc. are defined from the preceding ones. For, from the fundamental equations the following formulas are deduced:

$$\begin{aligned}
&A = \frac{p}{a},\\
&B = \frac{q-bA}{b(bb-aa)},\\
&C = \frac{r-cA}{c(c-aa)(cc-bb)} - \frac{B}{cc-bb},\\
&D = \frac{s-dA}{d(dd-aa)(dd-bb)(dd-cc)} - \frac{B}{(dd-bb)(dd-cc)} - \frac{C}{dd-cc},\\
&E = \frac{t-eA}{e(ee-aa)(ee-bb)(ee-cc)(ee-dd)} - \frac{B}{(ee-bb)(ee-cc)(ee-dd)}\\
&\quad - \frac{C}{(ee-cc)(ee-dd)} - \frac{D}{ee-dd}\\
&\quad \text{etc.},
\end{aligned}$$

where in most cases soon a structure of such a kind is observed, whence the following ones cane easily be derived, as it will become plain from the following problems, in which I will accommodate this method to certain particular cases.

Scholium 2

§6 But before I expand cases of this kind, it will be helpful to have observed in general that, if for a certain case a satisfying equation between the two variables x and y was found, which I will denote this way

$$y = X,$$

such that it is

$$X = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \text{etc.},$$

that then hence easily an equation extending much further any equally satisfying can be formed

$$Q = x \cdot \frac{xx-aa}{aa} \cdot \frac{xx-bb}{bb} \cdot \frac{xx-cc}{cc} \cdot \frac{xx-dd}{dd} \cdot \text{etc.},$$

which quantity vanishes for all propounded values of x

$$x = 0, \quad x = \pm a, \quad x = \pm b, \quad x = \pm c \quad \text{etc.},$$

and all functions of Q vanishing together with Q itself will do the same; from this it is manifest, if one sets

$$y = X + Q$$

or

$$y = X + f : Q,$$

that all conditions are equally satisfied. Therefore, since this function $f : Q$ is completely arbitrary, as along as vanishes for $Q = 0$, this equation

$$y = X + f : Q$$

is to be considered to exhibit the most general solution.

PROBLEM 2

§7 *Let a, b, c, d etc. be any circular arcs while the radius is $= 1$, but let the values p, q, r, s etc. be the sines of the same arcs, since in this case this property holds that to negative arcs the same sines taken negatively correspond, hence to define the ratio of the diameter to the circumference approximately.*

SOLUTION

Since here it is

$$p = \sin a, \quad q = \sin b, \quad r = \sin c \quad \text{etc.},$$

the equation between x and y will be of such a nature, that having taken x for the circular arc the quantity yy will approximately be expressed by its sine and it is

$$y = \sin x.$$

Therefore, having defined the coefficients

$$A, \quad B, \quad C, \quad D \quad \text{etc.}$$

by means of the preceding problem one will have this equation

$$\sin x = Ax + Bx(xx - aa) + Cx(xx - aa)(xx - bb) + \text{etc.},$$

which therefore agrees with the truth, as often as it was

$$\text{either} \quad x = 0 \quad \text{or} \quad x = \pm a \quad \text{or} \quad x = \pm b \quad \text{or} \quad x = \pm c \quad \text{etc.}$$

Now let us set the arc x infinitely small, and since then its sine, $\sin x$, becomes equal to the arc x, this equation will arise

$$1 = A - Baa + Caabb - Daabbcc + Eaabbccdd - \text{etc.}$$

Let us substitute the values found above for the letters A, B, F, D etc. here and we will get to this equation

$$\begin{aligned}
1 = {} & \frac{p}{a}\left(1 - \frac{aa}{aa-bb} + \frac{aabb}{(aa-bb)(aa-cc)} - \frac{aabbcc}{(aa-bb)(aa-cc)(aa-dd)} + \text{etc.}\right) \\
& - \frac{q}{b}\left(\frac{aa}{bb-aa} - \frac{aabb}{(bb-aa)(bb-cc)} + \frac{aabbcc}{(bb-aa)(bb-cc)(bb-dd)} - \text{etc.}\right) \\
& + \frac{r}{c}\left(\frac{aabb}{(cc-aa)(cc-bb)} - \frac{aabbcc}{(cc-aa)(cc-bb)(cc-dd)} + \text{etc.}\right) \\
& - \frac{s}{d}\left(\frac{aabbcc}{(dd-aa)(dd-bb)(dd-dd)} - \text{etc.}\right) \\
& + \text{etc.},
\end{aligned}$$

which is reduced to this one, in which all series are similar to each other

$$\begin{aligned}
1 = {} & \frac{p}{a}\left(1 - \frac{aa}{aa-bb} + \frac{aabb}{(aa-bb)(aa-cc)} - \frac{aabbcc}{(aa-bb)(aa-cc)(aa-dd)} + \text{etc.}\right) \\
& - \frac{aaq}{b(bb-aa)}\left(1 + \frac{bb}{cc-bb} + \frac{bbcc}{(cc-bb)(dd-bb)} + \frac{bbccdd}{(cc-bb)(dd-bb)(ee-bb)} + \text{etc.}\right) \\
& + \frac{aabbr}{c(cc-bb)(cc-bb)}\left(1 + \frac{cc}{dd-cc} + \frac{ccdd}{(dd-cc)(ee-cc)} + \text{etc.}\right) \\
& - \frac{aabbccs}{d(dd-aa)(dd-bb)(dd-cc)}\left(1 + \frac{dd}{ee-dd} + \text{etc.}\right) \\
& + \text{etc.}
\end{aligned}$$

But every single one of these series is immediately summable; for; the terms of the first series combined give

$$\frac{bb}{bb-aa};$$

but if to it the third is added, it arises

$$\frac{bbcc}{(bb-aa)(cc-aa)}$$

and hence further the fourth term added yields

$$\frac{bbccdd}{(bb-aa)(cc-aa)(dd-aa)}$$

and so forth, such that the first series of our equation becomes

$$\frac{p}{a}\cdot\frac{bb}{bb-aa}\cdot\frac{cc}{cc-aa}\cdot\frac{dd}{dd-aa}\cdot\frac{ee}{ee-aa}\cdot\text{etc.}$$

But in similar manner it is found for the second

$$-\frac{q}{b}\cdot\frac{aa}{bb-aa}\cdot\frac{cc}{cc-bb}\cdot\frac{dd}{dd-bb}\cdot\frac{ee}{ee-bb}\cdot\text{etc.}$$

and so our equation is finally reduced to this form

$$\begin{aligned}1 = {} & \frac{p}{q}\cdot\frac{bb}{bb-aa}\cdot\frac{cc}{cc-aa}\cdot\frac{dd}{dd-aa}\cdot\frac{ee}{ee-aa}\cdot\text{etc.}\\ & +\frac{q}{b}\cdot\frac{aa}{bb-aa}\cdot\frac{cc}{cc-bb}\cdot\frac{dd}{dd-bb}\cdot\frac{ee}{ee-bb}\cdot\text{etc.}\\ & +\frac{r}{c}\cdot\frac{aa}{aa-cc}\cdot\frac{bb}{bb-cc}\cdot\frac{dd}{dd-cc}\cdot\frac{ee}{ee-cc}\cdot\text{etc.}\\ & +\frac{s}{d}\cdot\frac{aa}{aa-dd}\cdot\frac{bb}{bb-dd}\cdot\frac{cc}{cc-dd}\cdot\frac{ee}{ee-dd}\cdot\text{etc.}\\ & +\frac{t}{e}\cdot\frac{aa}{aa-ee}\cdot\frac{bb}{bb-ee}\cdot\frac{cc}{cc-ee}\cdot\frac{dd}{dd-ee}\cdot\text{etc.}\\ & +\text{etc.},\end{aligned}$$

whence, if the given arcs a, b, c, d etc. have a known ratio to half of the circumference π, the value of this quantity π will be defined

Corollary 1

§8 If the number of these arcs a, b, c, d etc. was finite, then the circumference of the circle will be defined the more accurately, the greater that number is and at the same time the smaller arcs occur among them. But having augmented the amount of propounded arcs to infinity the true ratio of the circumference to the diameter will be derived from this.

§9 In similar manner the sine of the indefinite arc x can be defined in general. For, having substituted the found values instead of the coefficients A, B, C, D etc. the equation will be reduced to this form

$$\begin{aligned}\frac{\sin x}{x} &= \frac{p}{a}\cdot\frac{bb-xx}{bb-aa}\cdot\frac{cc-xx}{cc-aa}\cdot\frac{dd-xx}{dd-aa}\cdot\text{etc.}\\ &+\frac{q}{b}\cdot\frac{aa-xx}{aa-bb}\cdot\frac{cc-xx}{cc-bb}\cdot\frac{dd-xx}{dd-bb}\cdot\text{etc.}\\ &+\frac{r}{c}\cdot\frac{aa-xx}{aa-cc}\cdot\frac{bb-xx}{bb-cc}\cdot\frac{dd-xx}{dd-cc}\cdot\text{etc.}\\ &+\frac{s}{d}\cdot\frac{aa-xx}{aa-dd}\cdot\frac{bb-xx}{bb-dd}\cdot\frac{cc-xx}{cc-dd}\cdot\text{etc.}\\ &+\text{etc.},\end{aligned}$$

which equation having taken a vanishing arc x goes over into that one.

Corollary 3

§10 But this reduction extends a lot further, not having taken into account the arcs. For, if an equation of such a kind between the two variables x and y is in question, that having taken

$$x = 0,\ \ a,\ \ b,\ \ c,\ \ d,\ \ e\ \ \text{etc.}$$

it is

$$x = 0,\ \ p,\ \ q,\ \ r,\ \ s,\ \ t\ \ \text{etc.},$$

this equation can be represented in general this way

$$\begin{aligned}\frac{y}{x} &= \frac{p}{a}\cdot\frac{bb-xx}{bb-aa}\cdot\frac{cc-xx}{cc-aa}\cdot\frac{dd-xx}{dd-aa}\cdot\frac{ee-xx}{ee-aa}\cdot\text{etc.}\\ &+\frac{q}{b}\cdot\frac{aa-xx}{aa-bb}\cdot\frac{cc-xx}{cc-bb}\cdot\frac{dd-xx}{dd-bb}\cdot\frac{ee-xx}{ee-bb}\cdot\text{etc.}\\ &+\frac{r}{c}\cdot\frac{aa-xx}{aa-cc}\cdot\frac{bb-xx}{bb-cc}\cdot\frac{dd-xx}{dd-cc}\cdot\frac{ee-xx}{ee-cc}\cdot\text{etc.}\\ &+\frac{s}{d}\cdot\frac{aa-xx}{aa-dd}\cdot\frac{bb-xx}{bb-dd}\cdot\frac{cc-xx}{cc-dd}\cdot\frac{ee-xx}{ee-dd}\cdot\text{etc.}\\ &+\text{etc.},\end{aligned}$$

from which form it is manifest at the same time, how the single conditions are satisfied.

SCHOLIUM

§11 I do not spend more time on the cases, in which the number of of prescribed conditions a, b, c, d etc. is assumed as finite, since hence only approximations for the measure of the circle are obtained. Nevertheless, it will not be off topic to have observed, if only four arcs are taken, which shall be

$$a = \varphi, \quad b = 2\varphi, \quad c = 3\varphi, \quad d = 4\varphi,$$

that from the solution of the problem it will be

$$\begin{aligned}\varphi = &\frac{\sin\varphi}{1} \cdot \frac{2\cdot 2}{1\cdot 3} \cdot \frac{3\cdot 3}{2\cdot 4} \cdot \frac{4\cdot 4}{3\cdot 5} \\ - &\frac{\sin 2\varphi}{2} \cdot \frac{1\cdot 1}{1\cdot 3} \cdot \frac{3\cdot 3}{1\cdot 5} \cdot \frac{4\cdot 4}{2\cdot 6} \\ + &\frac{\sin 3\varphi}{3} \cdot \frac{1\cdot 1}{2\cdot 4} \cdot \frac{2\cdot 2}{1\cdot 5} \cdot \frac{4\cdot 4}{1\cdot 7} \\ - &\frac{\sin 4\varphi}{4} \cdot \frac{1\cdot 1}{3\cdot 5} \cdot \frac{2\cdot 2}{2\cdot 6} \cdot \frac{3\cdot 3}{1\cdot 7}\end{aligned}$$

$$= \frac{8}{5}\sin\varphi - \frac{2}{5}\sin 2\varphi + \frac{8}{105}\sin 3\varphi - \frac{1}{140}\sin 4\varphi,$$

which expression comes the closer to the truth the smaller the angle φ is taken; nevertheless, even though it is augmented up to the quadrant, that it is

$$\varphi = \frac{\pi}{2},$$

the error does not become enormous; for, it arises

$$\frac{\pi}{2} = \frac{8}{5} - \frac{8}{105} = \frac{32}{21}$$

and so

$$\pi = 3\frac{1}{21}.$$

But if we take

$$\varphi = 30^\circ = \frac{\pi}{6},$$

it is

$$\frac{\pi}{6} = \frac{8}{5} \cdot \frac{1}{2} - \frac{2}{5} \cdot \frac{\sqrt{3}}{2} + \frac{8}{105} - \frac{1}{140} \cdot \frac{\sqrt{3}}{2}$$

or

$$\pi = \frac{184}{35} - \frac{171\sqrt{3}}{140},$$

which value differs from the true one by the hundred-thousandth parts of the unit. But having put aside this consideration I want to go through some cases, where the number of propounded arcs a, b, c, d etc. proceeding in a certain law is infinite.

EXAMPLE I

§12 *Let the arcs a, b, c, d etc. proceed according to the series of natural numbers and let*

$$a = \varphi, \quad b = 2\varphi, \quad c = 3\varphi, \quad a = 4\varphi, \quad \text{etc.} \quad \textit{to infinity};$$

from their sines p, q, r etc. the truth longitude of the arc φ is to be determined.

Therefore, the solution of the problem for this case yields this equation

$$\begin{aligned}
\varphi = & \frac{\sin\varphi}{1} \cdot \frac{2\cdot 2}{1\cdot 3} \cdot \frac{3\cdot 3}{2\cdot 4} \cdot \frac{4\cdot 4}{3\cdot 5} \cdot \frac{5\cdot 5}{4\cdot 6} \cdot \text{etc.} \\
& - \frac{\sin 2\varphi}{2} \cdot \frac{1\cdot 1}{1\cdot 3} \cdot \frac{3\cdot 3}{1\cdot 5} \cdot \frac{4\cdot 4}{2\cdot 6} \cdot \frac{5\cdot 5}{3\cdot 7} \cdot \text{etc.} \\
& + \frac{\sin 3\varphi}{3} \cdot \frac{1\cdot 1}{2\cdot 4} \cdot \frac{2\cdot 2}{1\cdot 5} \cdot \frac{4\cdot 4}{1\cdot 7} \cdot \frac{5\cdot 5}{2\cdot 8} \cdot \text{etc.} \\
& - \frac{\sin 4\varphi}{4} \cdot \frac{1\cdot 1}{3\cdot 5} \cdot \frac{2\cdot 2}{2\cdot 6} \cdot \frac{3\cdot 3}{1\cdot 7} \cdot \frac{5\cdot 5}{1\cdot 9} \cdot \text{etc.} \\
& + \frac{\sin 5\varphi}{5} \cdot \frac{1\cdot 1}{4\cdot 6} \cdot \frac{2\cdot 2}{3\cdot 7} \cdot \frac{3\cdot 3}{2\cdot 8} \cdot \frac{4\cdot 4}{1\cdot 9} \cdot \text{etc.} \\
& + \text{etc.};
\end{aligned}$$

but all these products are found to have the same value $= 2$, such that it is

$$\frac{1}{2}\varphi = \sin\varphi - \frac{1}{2}\sin 2\varphi + \frac{1}{3}\sin 3\varphi - \frac{1}{4}\sin 4\varphi + \frac{1}{5}\sin 5\varphi - \text{etc.},$$

the truth of which series in the case, in which the angle φ is infinitely small, is manifest per se. Therefore, let us expand the following cases:

1.Let

$$\varphi = 90^\circ = \frac{\pi}{2}$$

and the Leibniz series arises

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.}$$

2. Let

$$\varphi = 45^\circ = \frac{\pi}{4}$$

and this series will arise

$$\frac{\pi}{8} = \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{3\sqrt{2}} * - \frac{1}{5\sqrt{2}} + \frac{1}{6} - \frac{1}{7\sqrt{2}} * + \frac{1}{9\sqrt{2}} - \frac{1}{10} + \frac{1}{11\sqrt{2}} - \text{etc.},$$

which is resolved into these two

$$\frac{\pi}{8} = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.}\right)$$

$$-\frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}\right),$$

such that it is

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.} = \frac{\pi}{2\sqrt{2}}.$$

3. Let

$$\varphi = 60^\circ = \frac{\pi}{3}$$

and it will be

$$\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} * + \frac{1}{4} \cdot \frac{\sqrt{3}}{2} - \frac{1}{5} \cdot \frac{\sqrt{3}}{2} + \text{etc.}$$

or

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \text{etc.}$$

4. Let

$$\varphi = 30^\circ = \frac{\pi}{6}$$

and it will be

$$\frac{\pi}{12} = \frac{1}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} - \frac{1}{4} \cdot \frac{\sqrt{3}}{2} + \frac{1}{5} \cdot \frac{1}{2} * - \frac{1}{7} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{\sqrt{3}}{2} - \text{etc.}$$

or

$$\frac{\pi}{12} = \frac{1}{2}\left(1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{19} - \frac{1}{23} + \text{etc.}\right)$$
$$-\frac{\sqrt{3}}{4}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \text{etc.}\right)$$
$$+\frac{1}{3}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}\right),$$

the last of which sums becomes $= \frac{\pi}{12}$; hence it is concluded

$$1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \text{etc.} = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \text{etc.}\right).$$

But both series become equal to the arc $\frac{\pi}{3}$, which is certainly already manifest in the first from the Leibniz series.

COROLLARY 1

§13 From the equation found here

$$\frac{1}{2}\varphi = \sin\varphi - \frac{1}{2}\sin 2\varphi + \frac{1}{3}\sin 3\varphi - \frac{1}{4}\sin 4\varphi + \text{etc.}$$

many other not less remarkable ones can be derived. As having done a differentiation it arises

$$\frac{1}{2} = \cos\varphi - \cos 2\varphi + \cos 3\varphi - \cos\varphi + \text{etc.},$$

the reason for which is manifest from that that by multiplying both sides by $2\cos\frac{1}{2}\varphi$ the identical equation $\cos\frac{1}{2}\varphi = \cos\frac{1}{2}\varphi$ arises.

COROLLARY 2

§14 But if we integrate that equation multiplied by $-d\varphi$, it arises

$$C - \frac{1}{4}\varphi\varphi = \cos\varphi - \frac{1}{4}\cos 2\varphi + \frac{1}{9}\cos 3\varphi - \frac{1}{16}\cos 4\varphi + \text{etc.},$$

where from the case $\varphi = 0$ the constant entering by integration is determined, namely

$$C = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \text{etc.} = \frac{\pi\pi}{12},$$

such that it is

$$\frac{\pi\pi}{12}-\frac{\varphi\varphi}{4}=\cos\varphi-\frac{1}{4}\cos 2\varphi+\frac{1}{9}\cos 3\varphi-\frac{1}{16}\cos 4\varphi+\text{etc.},$$

which series therefore having taken $\varphi=\frac{\pi}{\sqrt{3}}$ becomes $=0$. But it approximately is

$$\frac{\pi}{\sqrt{3}}=103^{\circ}55'23'' \quad \text{and} \quad \cos\frac{\pi}{\sqrt{3}}=-0,2406185.$$

COROLLARY 3

§15 If integrate this equation multiplied by $d\varphi$ again, it will arise

$$\frac{1}{12}\pi\pi\varphi-\frac{1}{12}\varphi^{3}=\sin\varphi-\frac{1}{8}\sin 2\varphi+\frac{1}{27}\sin 3\varphi-\frac{1}{64}\sin 4\varphi+\text{etc.},$$

whence having taken the arc

$$\varphi=90^{\circ}=\frac{\pi}{2}$$

it is obtained

$$\frac{1}{32}\pi^{3}=1-\frac{1}{27}+\frac{1}{125}-\frac{1}{343}+\text{etc.},$$

as is is already known from elsewhere.

SCHOLIUM

§16 About the found series

$$\frac{1}{2}\varphi=\sin\varphi-\frac{1}{2}\sin 2\varphi+\frac{1}{3}\sin 3\varphi-\frac{1}{4}\sin 4\varphi+\text{etc.}$$

there could be some doubt that having taken the arc $\varphi=180^{\circ}=\pi$ the single terms of the series vanish and hence the sum can not become equal to $\frac{1}{2}\pi$. But to resolve this doubt first set $\varphi=\pi-\omega$ and this equation will result

$$\frac{\pi-\omega}{2}=\sin\omega+\frac{1}{2}\sin 2\omega+\frac{1}{3}\sin 3\omega+\frac{1}{4}\sin 4\omega+\text{etc.}$$

but now assume the arc ω to be infinitely small, whence this is obtained

$$\frac{\pi-\omega}{2}=\omega+\omega+\omega+\omega+\omega+\text{etc.},$$

which does not any longer contain anything absurd. The same is to be said if we want to take $\varphi=2\pi$ or $\varphi=3\pi$ etc.

EXAMPLE II

§17 *If the arcs a, b, c, d constitute an arbitrary arithmetic progression that it is*

$$a = n\varphi, \quad b = (n+1)\varphi, \quad c = (n+2)\varphi, \quad d = (n+3)\varphi \quad \text{etc.}$$

from their sines to define the longitude of the arc φ.

The general solution exhibited before for this case gives

$$\begin{aligned}
\varphi = & \frac{\sin n\varphi}{n} \cdot \frac{(n+1)^2}{1(1+2n)} \cdot \frac{(n+2)^2}{2(2+2n)} \cdot \frac{(n+3)^2}{3(3+2n)} \cdot \frac{(n+4)^2}{4(4+2n)} \cdot \frac{(n+5)^2}{5(5+2n)} \cdot \text{etc.} \\
- & \frac{\sin(n+1)\varphi}{n+1} \cdot \frac{n^2}{1(1+2n)} \cdot \frac{(n+2)^2}{1(3+2n)} \cdot \frac{(n+3)^2}{2(4+2n)} \cdot \frac{(n+4)^2}{3(5+2n)} \cdot \frac{(n+5)^2}{4(6+2n)} \cdot \text{etc.} \\
+ & \frac{\sin(n+2)\varphi}{n+2} \cdot \frac{n^2}{2(2+2n)} \cdot \frac{(n+1)^2}{1(3+2n)} \cdot \frac{(n+3)^2}{1(5+2n)} \cdot \frac{(n+4)^2}{2(6+2n)} \cdot \frac{(n+5)^2}{3(7+2n)} \cdot \text{etc.} \\
- & \frac{\sin(n+3)\varphi}{n+3} \cdot \frac{n^2}{3(3+2n)} \cdot \frac{(n+1)^2}{2(4+2n)} \cdot \frac{(n+2)^2}{1(5+2n)} \cdot \frac{(n+4)^2}{1(7+2n)} \cdot \frac{(n+5)^2}{2(8+2n)} \cdot \text{etc.} \\
+ & \frac{\sin(n+4)\varphi}{n+4} \cdot \frac{n^2}{4(4+2n)} \cdot \frac{(n+1)^2}{3(5+2n)} \cdot \frac{(n+2)^2}{2(6+2n)} \cdot \frac{(n+3)^2}{1(7+2n)} \cdot \frac{(n+5)^2}{1(9+2n)} \cdot \text{etc.} \\
& + \text{etc.}
\end{aligned}$$

But to investigate the values of these infinite products for the sake of brevity let us put

$$\varphi = \mathfrak{A}\frac{\sin n\varphi}{n} - \mathfrak{B}\frac{\sin(n+1)\varphi}{n+1} + \mathfrak{C}\frac{\sin(n+2)\varphi}{n+2} - \mathfrak{D}\frac{\sin(n+3)\varphi}{n+3} + \text{etc.}$$

and compare these coefficients to each other the following way

$$\frac{\mathfrak{A}}{\mathfrak{B}} = \frac{nn}{(n+1)^2} \cdot \frac{2(2+2n)}{1(3+2n)} \cdot \frac{3(3+2n)}{2(4+2n)} \cdot \frac{4(4+2n)}{3(5+2n)} \cdot \text{etc.}$$

which value is reduced to

$$\frac{nn}{(n+1)^2} \cdot \frac{(i-1)(2+2n)}{1(i+2n)},$$

while i denotes an infinite number and so it will be

$$\frac{\mathfrak{A}}{\mathfrak{B}} = \frac{2nn}{n+1}.$$

In similar manner it is concluded

$$\frac{\mathfrak{C}}{\mathfrak{B}} = \frac{1(1+2n)}{2(2+2n)} \cdot \frac{(n+1)^2}{(n+2)^2} \cdot \frac{(i-3)((4+2n)}{1(i+2n)} = \frac{(n+1)(2n+1)}{2(n+2)},$$

but then further

$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(n+2)(2n+2)}{3(n+3)}, \quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(n+3)(2n+3)}{4(n+4)}$$

and so forth; hence it follows that it will be

$$\begin{aligned}
\mathfrak{B} &= \frac{2nn}{1(n+1)}\mathfrak{A}, \\
\mathfrak{C} &= \frac{2nn(2n+1)}{1\cdot 2(n+1)}\mathfrak{A}, \\
\mathfrak{D} &= \frac{2nn(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)}\mathfrak{A}, \\
\mathfrak{E} &= \frac{2nn(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4(n+4)}\mathfrak{A}
\end{aligned}$$

etc.

and so the whole task goes back to the invention of the first letter

$$\mathfrak{A} = \frac{(n+1)^2}{1(2n+1)} \cdot \frac{(n+2)^2}{2(2n+2)} \cdot \frac{(n+3)^2}{3(2n+3)} \cdot \frac{(n+4)^2}{4(2n+4)} \cdot \text{etc.}$$

But I already proved a long time ago that the value of this general product

$$\frac{a(b+c)}{b(a+c)} \cdot \frac{(a+d)(b+c+d)}{(b+d)(a+c+d)} \cdot \frac{(a+2d)(b+c+2d)}{(b+2d)(a+c+2d)} \cdot \text{etc.}$$

is expressed in such a way that it is

$$= \frac{\int x^{b-1}dx(1-x^d)^{\frac{c-d}{d}}}{\int x^{a-1}dx(1-x^d)^{\frac{c-d}{d}}},$$

having extended the integration from the boundary $x=0$ to $x=1$, of course. Since from this for our case one has to take

$$a = n+1, \quad b+c = n+1, \quad b = 1, \quad c = n \quad \text{and} \quad d = 1,$$

we will have

$$\mathfrak{A} = \frac{\int \mathrm{d}x(1-x)^{n-1}}{\int x^n \mathrm{d}x(1-x)^{n-1}} = \frac{1}{n\int x^n \mathrm{d}x(1-x)^{n-1}}$$

and hence the following expression for the arc φ

$$\varphi \int x^n \mathrm{d}x(1-x)^{n-1} = \frac{1}{nn}\sin n\varphi - \frac{2n}{1(n+1)^2}\sin(n+1)\varphi$$

$$+\frac{2n(2n+1)}{1\cdot 2(n+2)^2}\sin(n+2)\varphi - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2}\sin(n+3)\varphi$$

$$+\frac{2n(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4(n+4)^2}\sin(n+4)\varphi + \text{etc.}$$

This series deserves even more attention, since it involve the integral formula $\int x^n dx(1-x)^{n-1}$.

COROLLARY 1

§18 It will be helpful to have noted at first about this integral formula

$$\int x^n \mathrm{d}x(1-x)^{n-1},$$

if in the case $n=\lambda$ it was $\triangle$, that it then in the case

$$n = \lambda + 1$$

will be

$$= \frac{\lambda}{2(2\lambda+1)}\triangle\,.$$

So, since in the case $n=1$ it is

$$\int x\mathrm{d}x = \frac{1}{2},$$

it will be

$$\int x^2\mathrm{d}x(1-x) = \frac{1}{2}\cdot\frac{1}{2\cdot 3}, \quad \int x^3\mathrm{d}x(1-x)^2 = \frac{1}{2}\cdot\frac{1}{2\cdot 3}\cdot\frac{2}{2\cdot 5} \quad \text{etc.}$$

COROLLARY 2

§19 Therefore, if in general it is put

$$\int x^n dx(1-x)^{n-1} = f:n,$$

since its value can be considered as a function of n, it will be

$$f:1=\frac{1}{2},\quad f:2=\frac{1}{2}\cdot\frac{1}{6},\quad f:3=\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{2}{10},\quad f:4=\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{2}{10}\cdot\frac{3}{14}$$

and in general

$$f:(n+1)=\frac{n}{2(2n+1)}f:n.$$

Hence, as often as n is an integer number, the value of this formula $f:n$ is easily assigned.

COROLLARY 3

§20 Now let $n=\frac{1}{2}$ and it will be

$$f:\frac{1}{2}=\int\frac{dx\sqrt{x}}{\sqrt{1-x}}=2\int\frac{yydy}{\sqrt{1-yy}},$$

having put $x=yy$; but

$$\int\frac{yydy}{\sqrt{1-yy}}=\frac{1}{2}\int\frac{dy}{\sqrt{1-yy}}=\frac{\pi}{4},$$

whence it is

$$f:\frac{1}{2}=\frac{\pi}{2}$$

and hence further

$$f:\frac{3}{2}=\frac{1}{8}\cdot\frac{\pi}{2},\quad f:\frac{5}{2}=\frac{1}{8}\cdot\frac{3}{16}\cdot\frac{\pi}{2},\quad f:\frac{7}{2}=\frac{1}{8}\cdot\frac{3}{16}\cdot\frac{5}{24}\cdot\frac{\pi}{2}\quad\text{etc.}$$

But if in general it is $n=\frac{\mu}{\nu}$, it is found

$$f:\frac{\mu}{\nu}=\int x^{\frac{\mu}{\nu}}dx(1-x)^{\frac{\mu}{\nu}-1}=\mu\int y^{\mu+\nu-1}dy(1-y^\nu)^{\frac{\mu}{\nu}-1},$$

having put $x=y^\nu$ and hence having done the reduction

$$f:\frac{\mu}{\nu}=\frac{\nu}{2}\int y^{\nu-1}dy(1-y^\nu)^{\frac{\mu}{\nu}-1},$$

which form involves transcendental quantities of each class.

COROLLARY 4

§21 The value of the integral formula

$$\int x^n \mathrm{d}x(1-x)^{n-1}$$

in the case $x = 1$ is vice versa sufficiently elegantly determined from the found series; for, having done a differentiation by considering only the arc φ as a variable it arises

$$\int x^n \mathrm{d}x(1-x)^{n-1} = \frac{1}{n}\cos n\varphi - \frac{2n}{1(n+1)}\cos(n+1)\varphi + \frac{2n(2n+1)}{1\cdot 2(n+2)}\cos(n+2)\varphi$$

$$-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+2)}\cos(n+3)\varphi + \text{etc.},$$

which series is therefore equal to this one arsing from the usual expansion itself

$$\int x^n \mathrm{d}x(1-x)^{n-1} = \frac{1}{n+1} - \frac{n-1}{1(n+2)} + \frac{(n-1)(n-2)}{1\cdot 2(n+3)} - \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3(n+4)} + \text{etc.}$$

.

SCHOLIUM 1

§22 Since we expanded the case $n = 1$ in the preceding example, let us here mainly consider the case

$$n = \frac{1}{2}$$

in which we saw that it is

$$\int x^n \mathrm{d}x(1-x)^{n-1} = \frac{\pi}{2},$$

and it will therefore be

$$\frac{\pi\varphi}{2} = \frac{4}{1}\sin\frac{1}{2}\varphi - \frac{4}{9}\sin\frac{3}{2}\varphi + \frac{4}{25}\sin\frac{5}{2}\varphi - \frac{4}{49}\sin\frac{7}{2}\varphi + \text{etc.}$$

Let us put $\varphi = 2\omega$ and this more convenient series will arise

$$\frac{\pi\omega}{4} = \frac{1}{1}\sin\omega - \frac{1}{9}\sin 3\omega + \frac{1}{25}\sin 5\omega - \frac{1}{49}\sin 7\omega + \text{etc.},$$

which first, if a vanishing arc ω is assumed, gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.}$$

But let

$$\omega = \frac{\pi}{2}$$

and this also known series arises

$$\frac{\pi\pi}{8} = \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.}$$

But having taken the arc

$$\omega = 45^{\circ} = \frac{\pi}{4}$$

it arises

$$\frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{9} - \frac{1}{25} + \frac{1}{49} + \frac{1}{81} - \frac{1}{121} - \frac{1}{169} + \text{etc.}$$

Let

$$\omega = 30^{\circ} = \frac{\pi}{6};$$

it will be

$$\begin{aligned}\frac{\pi\pi}{24} = \frac{1}{2}\left(1 + \frac{1}{7^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{25^2} + \text{etc.}\right) \\ -\ 1\left(\frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \text{etc.}\right) \\ +\ \frac{1}{2}\left(\frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{17^2} + \frac{1}{23^2} + \text{etc.}\right),\end{aligned}$$

where the middle one is $= \frac{\pi\pi}{72}$ and the reason for the remaining ones is perspicuous. Further, the differentiation of our series yields this remarkable form

$$\frac{\pi}{4} = \frac{1}{1}\cos\omega - \frac{1}{3}\cos 3\omega + \frac{1}{5}\cos 5\omega - \frac{1}{7}\cos 7\omega + \text{etc.},$$

since completely all arcs assumed for ω yield the same sum. But then an iterated differentiation yields

$$0 = \sin\omega - \sin 3\omega + \sin 5\omega - \sin 7\omega + \text{etc.}$$

But by means of integration we find

$$C - \frac{\pi\omega^2}{8} = \frac{1}{1}\cos\omega - \frac{1}{3^3}\cos 3\omega + \frac{1}{5^3}\cos 5\omega - \frac{1}{7^3}\cos 7\omega + \text{etc.},$$

where, since having taken $\omega = 0$ it is

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.} = \frac{\pi^3}{32},$$

it will be

$$C = \frac{\pi^3}{32},$$

such that it is

$$\frac{\pi}{8}\left(\frac{\pi\pi}{4} - \omega\omega\right) = \frac{1}{1}\cos\omega - \frac{1}{3^3}\cos 3\omega + \frac{1}{5^3}\cos 5\omega - \frac{1}{7^3}\cos 7\omega + \text{etc.}$$

SCHOLIUM 2

§23 Now let us in general put

$$\varphi = \pi,$$

and since it is

$$\sin(n+1)\pi = -\sin n\pi, \quad \sin(n+2)\pi = +\sin n\pi \quad \text{etc.}$$

our equation divided by $\sin n\pi$ will obtain this form

$$\frac{\pi}{\sin n\pi}\int x^n dx(1-x)^{n-1} = \frac{1}{n^2} + \frac{2n}{1(n+1)^2} + \frac{2n(2n+1)}{1\cdot 2(n+2)^2}$$
$$+ \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2} + \text{etc.};$$

but having taken

$$\varphi = 2\pi$$

in similar manner it will be

$$\frac{2\pi}{\sin 2n\pi}\int x^n dx(1-x)^{n-1} = \frac{1}{n^2} - \frac{2n}{1(n+1)^2} + \frac{2n(2n+1)}{1\cdot 2(n+2)^2}$$

$$-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2}+\text{etc.},$$

that one of these series therefore divided by this one yields the quotient $=\cos n\pi$, which seems to be wrong, since the quotient is smaller than unity. But we already resolved a similar difficulty above, which arose from the position $\varphi=2\pi$; for, if we would have put $\varphi=3\pi$, the first series itself would emerge having the sum

$$=\frac{3\pi}{\sin 3\pi}\int x^n\mathrm{d}x(1-x)^{n-1},$$

which is only equal to that one, if n is a vanishing ratio. Hence only the first series is to be considered to hold; to investigate its sum from its nature itself, let us set

$$s=\frac{1}{n^2}t^n+\frac{2n}{(n+1)^2}t^{n+1}+\frac{2n(2n+1)}{1\cdot 2(n+2)^2}t^{n+2}+\text{etc.}$$

and it will hence be

$$\frac{\mathrm{d}.t\mathrm{d}s}{\mathrm{d}t^2}=1t^{n-1}+\frac{2n}{1}t^n+\frac{2n(2n+1)}{1\cdot 2}t^{n+1}+\text{etc},$$

the sum of which series manifestly is

$$=t^{n-1}(1-t)^{-2n},$$

such that it is

$$\frac{t\mathrm{d}s}{\mathrm{d}t}=\int t^{n-1}\mathrm{d}t(1-t)^{-2n}$$

and

$$s=\int\frac{\mathrm{d}t}{t}\int\frac{t^{n-1}\mathrm{d}t}{(1-t)^{2n}},$$

and so having put $x=1$ after the integration one will have

$$\frac{\pi}{\sin n\pi}\int x^n\mathrm{d}x(1-x)^{n-1}=\int\frac{\mathrm{d}t}{t}\int\frac{t^{n-1}\mathrm{d}t}{(1-t)^{2n}}.$$

The comparison of these two integral formulas is even more memorable, since among many others, which have been discovered, no one of this class is found,

SCHOLIUM 3

§24 Let us put in general

$$\varphi = \frac{\pi}{2}$$

and it will be

$$\sin n\varphi = \sin\frac{n\pi}{2}, \quad \sin(n+1)\varphi = \cos\frac{n\pi}{2},$$
$$\sin(n+2)\varphi = -\sin\frac{n\pi}{2}, \quad \sin(n+3)\varphi = -\cos\frac{n\pi}{2} \quad \text{etc.},$$

whence this equation results

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1} = \sin\frac{n\pi}{2}\left(\frac{1}{nn} - \frac{2n(2n+1)}{1\cdot 2(n+2)^2} + \frac{2n(2n+1)(2n+2)(2n+4)}{1\cdot 2\cdot 3\cdot 4(n+4)^2} - \text{etc.}\right)$$
$$-\cos\frac{n\pi}{2}\left(\frac{2n}{1(n+1)^2} - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2} + \text{etc.}\right).$$

But from the superior reduction it is manifest that it will be

$$1 - \frac{2n(2n+1)}{1\cdot 2}t^2 + \frac{2n(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4}t^4 - \text{etc.}$$
$$= \frac{(1+t\sqrt{-1})^{-2n} + (1-t\sqrt{-1})^{-2n}}{2},$$
$$\frac{2n}{1}t - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}t^3 + \text{etc.}$$
$$= \frac{(1+t\sqrt{-1})^{-2n} - (1-t\sqrt{-1})^{-2n}}{2\sqrt{-1}}$$

and hence it is concluded

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1}$$

$$= \frac{1}{2}\sin\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1+t\sqrt{-1})^{2n}} + \frac{1}{2}\sin\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t\sqrt{-1})^{2n}},$$
$$-\frac{1}{2\sqrt{-1}}\cos\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1+t\sqrt{-1})^{2n}} + \frac{1}{2\sqrt{-1}}\cos\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t\sqrt{-1})^{2n}},$$

where after the integration is must be put $t = 1$. But to free this expression from imaginary quantities, let us put

$$t = \tan\omega = \frac{\sin\omega}{\cos\omega};$$

it will be

$$\mathrm{d}t = \frac{\mathrm{d}\omega}{\cos^2\omega}, \quad \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}\omega}{\sin\omega\cos\omega}, \quad t^{n-1}\mathrm{d}t = \frac{\mathrm{d}\omega\sin^{n-1}\omega}{\cos^{n+1}\omega},$$

but then

$$\begin{aligned}(1 + t\sqrt{-1})^{-2n} &= \cos^{2n}\omega(\cos\omega + \sqrt{-1}\cdot\sin\omega)^{-2n}\\ &= \cos^{2n}\omega(\cos 2n\omega - \sqrt{-1}\cdot\sin 2n\omega),\\ (1 - t\sqrt{-1})^{-2n} &= \cos^{2n}\omega(\cos\omega - \sqrt{-1}\cdot\sin\omega)^{-2n}\\ &= \cos^{2n}\omega(\cos 2n\omega + \sqrt{-1}\cdot\sin 2n\omega).\end{aligned}$$

Having substituted which values the imaginary quantities will cancel each other and this equation will arise

$$\begin{aligned}\frac{\pi}{2}\int x^n\mathrm{d}x(1-x)^{n-1} = \sin\frac{n\pi}{2}\int\frac{\mathrm{d}\omega}{\sin\omega\cos\omega}\int\mathrm{d}\omega\sin^{n-1}\omega\cos^{n-1}\omega\cos 2n\omega\\ + \cos\frac{n\pi}{2}\int\frac{\mathrm{d}\omega}{\sin\omega\cos\omega}\int\mathrm{d}\omega\sin^{n-1}\omega\cos^{n-1}\omega\sin 2n\omega,\end{aligned}$$

which is contracted to this simpler one

$$\frac{\pi}{2}\int x^n\mathrm{d}x(1-x)^{n-1} = \int\frac{\mathrm{d}\omega}{\sin\omega\cos\omega}\int\mathrm{d}\omega\sin^{n-1}\omega\cos^{n-1}\omega\sin\left(\frac{n\pi}{2} + 2n\omega\right)$$

or because of $\sin\omega\cos\omega = \frac{1}{2}\sin 2\omega$ into this one

$$\frac{\pi}{2}\int x^n\mathrm{d}x(1-x)^{n-1} = \frac{1}{2^n}\int\frac{2\mathrm{d}\omega}{\sin 2\omega}\int 2\mathrm{d}\omega\sin^{n-1}2\omega\sin\left(\frac{n\pi}{2} + 2n\omega\right).$$

Now let the angle be $2\omega = \theta$, that it more conveniently is

$$\frac{\pi}{2}\int x^n\mathrm{d}x(1-x)^{n-1} = \frac{1}{2^n}\int\frac{2\mathrm{d}\theta}{\sin\theta}\int\mathrm{d}\theta\sin^{n-1}\theta\sin n\left(\frac{\pi}{2} + \theta\right),$$

where after the integration one must set $\theta = 90^\circ = \frac{\pi}{2}$, that then it is $\omega = 45^\circ$ and $t = \tan\omega = 1$.

EXAMPLE III

§25 *If the arcs a, b, c, d etc. constitute an interrupted arithmetic progression, that it is*

$$a = m\varphi, \quad b = n\varphi, \quad c = (1+m]\varphi, \quad d = (1+n)\varphi,$$
$$e = (2+m)\varphi, \quad f = (2+n)\varphi, \quad etc.,$$

to define the longitude of the arc φ from their sines.

The general solution given above (§ 7) yields this equation

$$\begin{aligned}
\varphi = {} & \frac{\sin m\varphi}{m} \cdot \frac{nn}{(n-m)(n+m)} \cdot \frac{(1+m)^2}{1(1+2m)} \cdot \frac{(1+n)^2}{(1+n-m)(1+n+m)} \\
& \cdot \frac{(2+m)^2}{2(2+2m)} \cdot \frac{(2+n)^2}{(2+n-m)(2+n+m)} \cdot \text{etc.} \\
& - \frac{\sin n\varphi}{n} \cdot \frac{mm}{(n-m)(n+m)} \cdot \frac{(1+m)^2}{(1+m-n)(1+m+n)} \cdot \frac{(1+n)^2}{1(1+2n)} \\
& \cdot \frac{(2+m)^2}{(2+m-n)(2+m+n)} \cdot \frac{(2+n)^2}{2(2+2n)} \cdot \text{etc.} \\
& + \frac{\sin(1+m)\varphi}{1+m} \cdot \frac{mm}{1(1+2m)} \cdot \frac{nn}{(1+m-n)(1+m+n)} \cdot \frac{(1+n)^2}{(n-m)(2+m+n)} \\
& \cdot \frac{(2+m)^2}{1(3+2m)} \cdot \frac{(2+n)^2}{(1+n-m)(3+n+m)} \cdot \text{etc.} \\
& - \frac{\sin(1+n)\varphi}{1+n} \cdot \frac{mm}{(1+n-m)(1+n+m)} \cdot \frac{nn}{1(1+2n)} \cdot \frac{(1+m)^2}{(n-m)(2+n+m)} \\
& \cdot \frac{(2+m)^2}{(1+m-n)(3+m+n)} \cdot \frac{(2+n)^2}{1(3+2n)} \cdot \text{etc.} \\
& + \frac{\sin(2+m)\varphi}{2+m} \cdot \frac{mm}{2(2+2m)} \cdot \frac{nn}{(2+m-n)(2+m+n)} \cdot \frac{(1+m)^2}{1(3+2m)} \\
& \cdot \frac{(1+n)^2}{(1+m-n)(3+m+n)} \cdot \frac{(2+n)^2}{(n-m)(4+m+n)} \cdot \text{etc.} \\
& - \text{etc.}
\end{aligned}$$

But hence it is not possible to conclude anything worth one's attention in general; hence I will expand the especially remarkable case, in which it is

$$n = 1 - m;$$

for this for the sake of brevity I set

$$\varphi = \frac{\mathfrak{A}\sin m\varphi}{m} - \frac{\mathfrak{B}\sin(1-m)\varphi}{1-m} + \frac{\mathfrak{C}\sin(1+m)\varphi}{1+m} - \frac{\mathfrak{D}\sin(2-m)\varphi}{2-m} + \text{etc.},$$

such that it is

$$\begin{aligned}
\mathfrak{A} &= \frac{(1-m)^2}{1(1-2m)} \cdot \frac{(1+m)^2}{1(1+2m)} \cdot \frac{(2-m)^2}{2(2-m)} \cdot \frac{(2+m)^2}{2(2+m)} \cdot \frac{(3-m)^2}{3(2+m)} \cdot \text{etc.},\\
\frac{\mathfrak{B}}{\mathfrak{A}} &= \frac{mm}{(1-m)^2} \cdot \frac{1(1+2m)}{2\cdot 2m} \cdot \frac{2(2-2m)}{1(3-2m)} \cdot \frac{2(2+2m)}{3(1+2m)} \cdot \frac{3(3-2m)}{2(3-2m)} \cdot \text{etc.},\\
\frac{\mathfrak{C}}{\mathfrak{B}} &= \frac{1(1-2m)}{1(1+2m)} \cdot \frac{(1-m)^2}{(1+m)^2} \cdot \frac{1(3-2m)}{3(1-2m)} \cdot \frac{3(1+2m)}{1(3+2m)} \cdot \frac{2(4-2m)}{4(2-2m)} \cdot \text{etc.},\\
\frac{\mathfrak{D}}{\mathfrak{C}} &= \frac{1(1+2m)}{2(2-2m)} \cdot \frac{2\cdot 2m}{1(3-2m)} \cdot \frac{(1+m)^2}{(2-m)^2} \cdot \frac{1(3+2m)}{4\cdot 2m} \cdot \frac{4(2-2m)}{1(5-2m)} \cdot \text{etc.},\\
\frac{\mathfrak{E}}{\mathfrak{D}} &= \frac{2(2-2m)}{2(2+2m)} \cdot \frac{1(3-2m)}{3(1+2m)} \cdot \frac{3(1-2m)}{1(3+2m)} \cdot \frac{(2-m)^2}{(2+m)^2} \cdot \frac{1(5-2m)}{5(1-2m)} \cdot \text{etc.}
\end{aligned}$$

etc.

But from the superior reduction one finds

$$\mathfrak{A} = \frac{\int x^{m-1}dx(1-x)^{-2m}}{m\int x^m dx(1-x)^{m-1} \cdot \int x^{m-1}dx(1-x)^{-m}};$$

but then for the remaining ones from the form of the products itself one concludes

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{m}{1-m}, \quad \frac{\mathfrak{C}}{\mathfrak{B}} = \frac{1-m}{1+m}, \quad \frac{\mathfrak{D}}{\mathfrak{C}} = \frac{1+m}{2-m}, \quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{2-m}{2+m} \quad \text{etc.},$$

such that it is

$$\mathfrak{B} = \frac{m}{1-m}\mathfrak{A}, \quad \mathfrak{C} = \frac{m}{1+m}\mathfrak{A}, \quad \mathfrak{D} = \frac{m}{2-m}\mathfrak{A}, \quad \mathfrak{E} = \frac{m}{2+m}\mathfrak{A}, \quad \text{etc.}$$

Therefore, for the sake of brevity let us put

$$\int x^m dx(1-x)^{m-1} \cdot \frac{\int x^{m-1}dx(1-x)^{-m}}{\int x^{m-1}dx(1-x)^{-2m}} = M$$

and it will be as follows

$$M\varphi = \frac{\sin m\varphi}{m^2} - \frac{\sin(1-m)\varphi}{(1-m)^2} + \frac{\sin(1+m)\varphi}{(1+m)^2} - \frac{\sin(2-m)\varphi}{(2-m)^2} + \frac{\sin(2+m)\varphi}{(2+m)^2} - \text{etc.},$$

whence by differentiating we conclude that it will be

$$M = \frac{\cos m\varphi}{m} - \frac{\cos(1-m)\varphi}{1-m} + \frac{\cos(1+m)\varphi}{1+m} - \frac{\cos(2-m)\varphi}{2-m} + \frac{\cos(2+m)\varphi}{2+m} - \text{etc.},$$

which series because of the extraordinary simplicity is especially remarkable, since by putting $\varphi = 0$ we hence deduce

$$M = \frac{1}{m} - \frac{1}{1-m} + \frac{1}{1+m} - \frac{1}{2-m} + \frac{1}{2+m} - \frac{1}{3-m} + \frac{1}{3+m} - \text{etc.},$$

the sum of which series I already once showed to be

$$M = \frac{\pi\cos m\pi}{\sin m\pi},$$

whence we calculate this elegant comparison

$$\int x^m dx(1-x)^{m-1} = \frac{\pi\cos m\pi}{\sin m\pi}\cdot\frac{\int x^{m-1}dx(1-x)^{-2m}}{\int x^{m-1}dx(1-x)^{-m}},$$

which is further reduced to this one

$$\int x^m dx(1-x)^{m-1} = \frac{(1-m)\pi\cos m\pi}{\sin m\pi}\cdot\frac{\int x^{m}dx(1-x)^{-2m}}{\int x^{m}dx(1-x)^{-m}}$$

or to this even more convenient one

$$\int x^{m-1} dx(1-x)^{m-1} = \frac{2\pi\cos m\pi}{\sin m\pi}\cdot\frac{\int x^{m-1}dx(1-x)^{-2m}}{\int x^{m-1}dx(1-x)^{-m}}.$$

COROLLARY 1

§26 Therefore, lo and behold some extraordinary theorems, which the expansions of this example gives us, the first of which is:

If φ denotes an arbitrary angle, it will be

$$\frac{\pi\cos m\pi}{\sin m\pi} = \frac{\cos m\varphi}{m} - \frac{\cos(1-m)\varphi}{1-m} + \frac{\cos(1+m)\varphi}{1+m} - \frac{\cos(2-m)\varphi}{2-m} + \text{etc.},$$

which equality can also be exhibited this way, that it is

$$\frac{\pi\cos m\pi}{\sin m\pi} = \cos m\varphi\left(\frac{1}{m} - \frac{2m\cos\varphi}{1-mm} - \frac{2m\cos 2\varphi}{4-mm} - \frac{2m\cos 3\varphi}{9-mm} - \text{etc.}\right)$$
$$-2\sin\varphi\left(\frac{\sin\varphi}{1-mm} + \frac{2\sin 2\varphi}{4-mm} + \frac{3\sin 3\varphi}{9-mm} + \frac{4\sin 4\varphi}{16-mm} + \text{etc.}\right),$$

whence, if it is

$$m\varphi = 90^\circ = \frac{\pi}{2} \quad \text{and hence} \quad \varphi = \frac{\pi}{2m},$$

it will be

$$-\frac{\pi\cos m\pi}{\sin m\pi} = \frac{\sin\frac{\pi}{2m}}{1-mm} + \frac{2\sin\frac{2\pi}{2m}}{4-mm} + \frac{3\sin\frac{3\pi}{2m}}{9-mm} + \frac{4\sin\frac{4\pi}{2m}}{16-mm} + \text{etc.}$$

COROLLARY 2

§27 The second theorem is enunciated this way:

If φ denotes an arbitrary angle, it will be

$$\frac{\pi\varphi\cos m\pi}{\sin m\pi} = \frac{\sin m\varphi}{mm} - \frac{\sin(1-m)\varphi}{(1-m)^2} + \frac{\sin(1+m)\varphi}{(1+m)^2} - \frac{\sin(2-m)\varphi}{(2-m)^2} + \text{etc.}$$

Hence having taken $\varphi = \pi$ it will be

$$\frac{\pi\pi\cos m\pi}{\sin m\pi} = \frac{\sin m\pi}{mm} - \frac{\sin m\pi}{(1-m)^2} - \frac{\sin m\pi}{(1+m)^2} + \frac{\sin m\pi}{(2-m)^2} + \frac{\sin m\pi}{(2+m)^2} - \text{etc.}$$

or

$$\frac{\pi\pi}{\sin m\pi\tan m\pi} = \frac{1}{m^2} - \frac{1}{(1-m)^2} - \frac{1}{(1+m)^2} + \frac{1}{(2-m)^2} + \frac{1}{(2+m)^2} - \text{etc.}$$

But having put

$$m\varphi = \pi$$

one will have

$$\frac{\pi\pi\cos m\pi}{m\sin m\pi} = \frac{\sin\frac{\pi}{m}}{(1-m)^2} - \frac{\sin\frac{\pi}{m}}{(1+m)^2} - \frac{\sin\frac{2\pi}{m}}{(2-m)^2} - \frac{\sin\frac{2\pi}{m}}{(2+m)^2} + \text{etc.}$$

or this way

$$\frac{\pi\pi\cos m\pi}{4mm\sin m\pi} = \frac{1\sin\frac{\pi}{m}}{(1-mm)^2} + \frac{2\sin\frac{2\pi}{m}}{(4-mm)^2} + \frac{3\sin\frac{3\pi}{m}}{(9-mm)^2} + \text{etc.}$$

COROLLARY 3

§28 The third theorem concerns the comparison of integral formulas and is enunciated this way:

If the integration of the following formulas is extended from the boundary $x = 0$ to the boundary $x = 1$, it will always be

$$\int x^{m-1}\mathrm{d}x(1-x)^{m-1} \cdot \int x^{m-1}\mathrm{d}x(1-x)^{-m} = \frac{2\pi\cos m\pi}{\sin m\pi}\int x^{m-1}\mathrm{d}x(1-x)^{-2m},$$

or if one puts $m = \frac{\lambda}{n}$ and $x = y^n$, it will be

$$\int \frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{n-\lambda}}} \cdot \int \frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{\lambda}}} = \frac{2\pi\cos\frac{\lambda\pi}{n}}{n\sin\frac{\lambda\pi}{n}} \cdot \int \frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}}$$

SCHOLIUM

§29 The demonstration of this last theorem seems to be very difficult; nevertheless, by means of the things I once published on integral formulas of this kind, its truth can be shown the following way. For, let us indicate, as I did there, this integral formula

$$\int \frac{y^{p-1}}{\sqrt[n]{(1-y^n)^{n-q}}}$$

by this character $\left(\frac{p}{q}\right)$ and it is to be demonstrated that it is

$$\left(\frac{\lambda}{\lambda}\right)\left(\frac{\lambda}{n-\lambda}\right) = \frac{2\pi\cos\frac{\lambda\pi}{n}}{n\sin\frac{\lambda\pi}{n}}\left(\frac{\lambda}{n-2\lambda}\right).$$

Now, first I demonstrated, if it was

$$q + r = n,$$

that then it will be

$$\left(\frac{q}{r}\right) = \frac{\pi}{n\sin\frac{q\pi}{n}},$$

whence it immediately follows

$$\left(\frac{\lambda}{n-\lambda}\right) = \int \frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^\lambda}} = \frac{\pi}{n\sin\frac{\lambda\pi}{n}},$$

such that it remains to prove that it is

$$\left(\frac{\lambda}{\lambda}\right) = 2\cos\frac{\lambda\pi}{n}\left(\frac{\lambda}{n-2\lambda}\right).$$

But on the same occasions I showed, if it was

$$p+q+r=n,$$

that it will be

$$\frac{1}{\sin\frac{r\pi}{n}}\left(\frac{p}{q}\right) = \frac{1}{\sin\frac{q\pi}{n}}\left(\frac{p}{r}\right) = \frac{1}{\sin\frac{p\pi}{n}}\left(\frac{q}{r}\right).$$

Therefore, let us take

$$p=\lambda, \quad q=\lambda$$

and it will be

$$r=n-2\lambda,$$

from which because of

$$\sin\frac{(n-2\lambda)\pi}{n} = \sin\frac{2\lambda\pi}{n}$$

we conclude

$$\frac{1}{\sin\frac{2\lambda\pi}{n}}\left(\frac{\lambda}{\lambda}\right) = \frac{1}{\sin\frac{\lambda\pi}{n}}\left(\frac{\lambda}{n-2\lambda}\right),$$

such that because of

$$\sin\frac{2\lambda\pi}{n} = 2\sin\frac{\lambda\pi}{n}\cos\frac{\lambda\pi}{n}$$

it indeed is

$$\left(\frac{\lambda}{\lambda}\right) = 2\cos\frac{\lambda\pi}{n}\left(\frac{\lambda}{n-2\lambda}\right).$$

But a lot more strange theorem was found above (§ 28), which for the same integration boundaries is

$$\frac{\pi}{\sin n\pi}\int x^n\mathrm{d}x(1-x)^{n-1} = \int\frac{\mathrm{d}x}{x}\int\frac{x^{n-1}\mathrm{d}x}{(1-x)^{2n}}$$

or

$$\frac{\pi}{2\sin n\pi}\int x^{n-1}\mathrm{d}x(1-x)^{n-1}=\int\frac{\mathrm{d}x}{x}\int\frac{x^{n-1}\mathrm{d}x}{(1-x)^{2n}};$$

to reduce this equation to that form, instead of n let us write $\frac{\lambda}{n}$ and let $x=y^n$, whence it is

$$\frac{\pi}{2n\sin\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{n-\lambda}}}=\int\frac{\mathrm{d}y}{y}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}}.$$

But we just saw that it is

$$\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{n-\lambda}}}=2\cos\frac{\lambda\pi}{n}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}}$$

and so via the theorem we conclude that it is

$$\frac{\pi}{n\tan\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}}=\int\frac{\mathrm{d}y}{y}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}},$$

and hence further this not less remarkable theorem

$$\frac{\pi}{n\tan\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}}=-\int\frac{y^{\lambda-1}\mathrm{d}y\cdot\log y}{\sqrt[n]{(1-y^n)^{2\lambda}}},$$

whence having taken $\lambda=1$ we find the following proportion

$$\frac{\pi}{n}:\tan\frac{\pi}{n}=\int\frac{\mathrm{d}y\log\frac{1}{y}}{\sqrt[n]{(1-y^n)^2}}:\int\frac{\mathrm{d}y}{\sqrt[n]{(1-y^n)^2}}.$$

PROBLEM 3

§30 *To find an equation of such a kind for the curved line between two variables, the abscissa x and the ordinate y, that to the abscissas taken in an arithmetic progression given ordinates correspond, namely:*

If it is

$$x=n\theta,\quad (n+1)\theta,\quad (n+2)\theta,\quad (n+3)\theta,\quad (n+4)\theta,\quad \text{etc.}$$

that it is

$$y=p,\qquad q,\qquad r,\qquad s,\qquad t\qquad \text{etc.}$$

SOLUTION

Let us put in general

$$x = \theta\omega$$

and from the general solution given in § 10 we obtain this equation

$$\begin{aligned}
\frac{y}{\omega} = \frac{p}{n} &\cdot \frac{(n+1-\omega)(n+1+\omega)}{1(2n+1)} \cdot \frac{(n+2-\omega)(n+2+\omega)}{2(2n+2)} \cdot \frac{(n+3-\omega)(n+3+\omega)}{3(2n+3)} \cdot \text{etc.} \\
-\frac{p}{n+1} &\cdot \frac{(n-\omega)(n+\omega)}{1(2n+1)} \cdot \frac{(n+2-\omega)(n+2+\omega)}{1(2n+3)} \cdot \frac{(n+3-\omega)(n+3+\omega)}{2(2n+4)} \cdot \text{etc.} \\
+\frac{r}{n+2} &\cdot \frac{(n-\omega)(n+\omega)}{2(2n+2)} \cdot \frac{(n+1-\omega)(n+1+\omega)}{1(2n+3)} \cdot \frac{(n+3-\omega)(n+3+\omega)}{1(2n+5)} \cdot \text{etc.}
\end{aligned}$$

$$-\text{etc.},$$

which equation for the sake of brevity we want to represent this way

$$\frac{y}{\omega} = \mathfrak{A} \cdot \frac{p}{n} - \mathfrak{B} \cdot \frac{q}{n+1} + \mathfrak{C} \cdot \frac{r}{n+2} - \mathfrak{D} \cdot \frac{s}{n+3} + \text{etc.};$$

and for finding the value of $\mathfrak{A}$ from the general form mentioned in § 17 we will have for this case

$$a = n+1-\omega, \quad b = 1, \quad c = n-\omega \quad \text{and} \quad d = 1,$$

whence by means of integral formulas to be extended from the boundary $z = 0$ to $z = 1$ we conclude

$$\mathfrak{A} = \frac{\int dz(1-z)^{n-\omega-1}}{\int z^{n-\omega}dz(1-z)^{n-\omega-1}} = \frac{1}{(n-\omega)\int z^{n-\omega}dz(1-z)^{n-\omega-1}}$$

or

$$\mathfrak{A} = \frac{2}{(n-\omega)\int z^{n-\omega-1}dz(1-z)^{n-\omega-1}},$$

having conceding this integration the remaining ones are easily handled. From it will be as above in § 17

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{(n-\omega)(n+\omega)}{(n+1-\omega)(n+1+\omega)} \cdot (2+2n) = \frac{2(n+1)(n-\omega)(n+\omega)}{n+1-\omega)(n+1+\omega)},$$
$$\frac{\mathfrak{C}}{\mathfrak{B}} = \frac{(n+1-\omega)(n+1+\omega)}{(n+2-\omega)(n+2+\omega)} \cdot \frac{(1+2n)(2+n)}{2(n+1)},$$
$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(n+2-\omega)(n+3+\omega)}{(n+3-\omega)(n+3+\omega)} \cdot \frac{(2+2n)(3+n)}{3(n+2)},$$
$$\frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(n+3-\omega)(n+3+\omega)}{(n+4-\omega)(n+4+\omega)} \cdot \frac{(3+2n)(4+n)}{4(n+3)}$$

etc.

Therefore, let us set the integral formula

$$\int z^{n-\omega-1}dz(1-z)^{n-\omega-1} = \triangle,$$

that it is

$$\mathfrak{A} = \frac{2}{(n-\omega)\triangle},$$

and the remaining coefficients will be defined by means of $\mathfrak{A}$ this way:

$$\mathfrak{B} = \frac{2(n+1)}{1} \cdot \frac{nn-\omega\omega}{(n+1)^2-\omega\omega}\mathfrak{A},$$
$$\mathfrak{C} = \frac{2(n+2)(2n+1)}{1\cdot 2} \cdot \frac{nn-\omega\omega}{(n+2)^2-\omega\omega}\mathfrak{A},$$
$$\mathfrak{D} = \frac{2(n+3)(2n+1)(2n+2)}{1\cdot 2\cdot 3} \cdot \frac{nn-\omega\omega}{(n+3)^2-\omega\omega}\mathfrak{A},$$
$$\mathfrak{E} = \frac{2(n+4)(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{nn-\omega\omega}{(n+4)^2-\omega\omega}\mathfrak{A}$$

etc.

Therefore, the equation is question between y and $x = \theta\omega$ will be of this nature:

$$\frac{n\triangle y}{2(n+\omega)\omega} = \frac{p}{nn-\omega\omega} - \frac{2n}{1} \cdot \frac{q}{(n+1)^2-\omega\omega}$$
$$+\frac{2n(2n+1)}{1\cdot 2} \cdot \frac{r}{(n+2)^2-\omega\omega} - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3} \cdot \frac{s}{(n+3)^2-\omega\omega} + \text{etc.},$$

whence for each value of $x = \vartheta\omega$ a corresponding value of y is defined and this by means of the ordinates p, q, r etc., which are assumed to correspond to the abscissas $n\theta$, $(n+1)\theta$, $(n+2)\theta$ etc. Here it must certainly be noted, if ω is taken equal to a certain term of the progression n, $n+1$, $n+2$ etc., that then the denominator of the given corresponding ordinate vanishes, such that with respect to the term, certainly infinite, the remaining ones vanish. But then at the same time also the value $\triangle$ arises as infinite and precisely of such a kind, that it then either is $y = p$ and $y = q$ or $y = r$ etc., as the nature of the subject demands it.

COROLLARY 1

§31 If the propounded abscissas denote circular arcs, the ordinates on the other hand their sines, that it is

$$p = \sin n\theta, \quad q = \sin(n+1)\theta, \quad q = \sin(n+1)\theta, \quad r = \sin(n+2)\theta, \quad \text{etc.},$$

it will be

$$y = \sin\omega\theta,$$

whence this general equation results

$$\frac{n \triangle \sin\omega\theta}{2(n+\omega)\omega} = \frac{\sin n\theta}{nn-\omega\omega} - \frac{2n}{1}\cdot\frac{\sin(n+1)\theta}{(n+1)^2-\omega^2} + \frac{2n(2n+1)}{1\cdot 2}\cdot\frac{\sin(n+2)\theta}{(n+2)^2-\omega^2}$$
$$-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{\sin(n+3)\theta}{(n+3)^2-\omega^2} + \text{etc.},$$

where it is especially remarkable that the three letters, n, θ and ω can be assumed arbitrarily.

COROLLARY 2

§32 Therefore, if we take

$$\theta = \pi,$$

that all sines of the series are reduced to the same $\sin n\theta$, it will be

$$\frac{n \triangle \sin\omega\theta}{2(n+\omega)\omega\sin n\pi} = \frac{1}{nn-\omega\omega} + \frac{2n}{1}\cdot\frac{1}{(n+1)^2-\omega^2} + \frac{2n(2n+1)}{1\cdot 2}\cdot\frac{1}{(n+2)^2-\omega^2}$$

$$+\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{1}{(n+3)^2-\omega^2}+\text{etc.}$$

Hence, if it is

$$n=\frac{1}{2}\quad\text{und}\quad \triangle=\int z^{-\omega-\frac{1}{2}}dz(1-z)^{-\omega-\frac{1}{2}}$$

or

$$\triangle=2\int\frac{z^{\frac{1}{2}-\omega}dz}{(1-z)^{\frac{1}{2}+\omega}},$$

one will have

$$\frac{\triangle\sin\omega\pi}{8(1+2\omega)\omega}=\frac{1}{1-4\omega^2}+\frac{1}{9-4\omega^2}+\frac{1}{25-4\omega^2}+\frac{1}{49-4\omega^2}+\text{etc.},$$

the sum of which series I showed to be

$$=\frac{\pi}{8\omega}\tan\omega\pi,$$

such that it is

$$\frac{\triangle\sin\omega\pi}{8(1+2\omega)\omega}=\frac{\pi}{8\omega}\tan\omega\pi$$

and hence

$$\triangle=\frac{(1+2\omega)\pi}{\cos\omega\pi}.$$

SCHOLIUM 1

§33 But it is not possible to trust these conclusions too much for the reason mentioned above already. For, having put the ordinates

$$p=\sin n\theta,\quad q=\sin(n+1)\theta,\quad r=\sin(n+2)\theta\quad\text{etc.},$$

while the arcs $n\theta$, $(n+1)\theta$, $(n+2)\theta$ etc. are considered as abscissas, the found equation yields a curved line of such a kind, which goes through all these points. And it does hence not follow that this curve is the line of sines, since infinitely many other curved lines passing through that same infinitely many points are given. Hence having kept the letter y for indicating the corresponding ordinate of the abscissa $x=\theta\omega$ our solution give this equation for the curve in question

$$\frac{n\triangle y}{2(n+\omega)}=\frac{\sin n\theta}{n^2-\omega^2}-\frac{2n}{1}\cdot\frac{\sin(n+1)\theta}{(n+1)^2-\omega^2}+\frac{2n(2n+1)}{1\cdot 2}\cdot\frac{\sin(n+2)\theta}{(n+2)^2-\omega^2}$$

$$-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{\sin(n+3)\theta}{(n+3)^2-\omega^2}+\text{etc.},$$

such that to the abscissa

$$x=(n\pm i)\theta$$

this ordinate corresponds

$$y=\sin(n\pm i)\theta,$$

if only i is an arbitrary integer. On the other hand it could also happen, that for other abscissas, where i is not an integer number and hence generally, if $x=\omega\theta$, the ordinate is not $y=\sin\omega$. That this is seen more clearly, let us investigate the general equation for completely all lines passing through given points, and let the value found up to now be

$$y=\Theta$$

and find a function vanishing for all given abscissas, of which kind this is

$$\omega(nn-\omega\omega)\frac{((n+1)^2-\omega^2)}{1(2n+1)}\frac{((n+2)^2-\omega^2)}{2(2n+2)}\frac{((n+3)^2-\omega^2)}{3(2n+3)}\ \text{etc.},$$

which by means of the thing mentioned above is

$$=\omega(nn-\omega\omega)\mathfrak{A}=\frac{2\omega(n+\omega)}{\triangle}.$$

Call this quantity $=\Omega$ and let $f:\Omega$ be a function of Ω of such kind, which vanishes, if $\Omega=0$, and the general equation for all satisfying curved lines will be

$$y=\Theta+f:\Omega=\Omega+f:\frac{2\omega(n+\omega)}{\triangle}.$$

And now without any doubt it is certain that in this equation the equation $y=\sin\omega\theta$ is contained having put $x=\omega\theta$, since this equation satisfies all prescribed conditions. From this it could completely happen that the equation $y=\Theta$ was different from this one $y=\sin\omega\theta$; this can especially depend on the values attributed to the letters θ and n, such the in the one case the found equation $y=\Theta$ agrees with this one $y=\sin\omega\theta$, but in others differs from the same.

Scholium 2

§34 We want to apply these to the case, in which it is

$$\theta = \pi \quad \text{and} \quad n = \frac{1}{2}$$

and

$$\triangle = 2\int \frac{z^{\frac{1}{2}-\omega}\mathrm{d}z}{(1-z)^{\frac{1}{2}+\omega}};$$

and since the sum of the found series is

$$= \frac{\pi}{8\omega}\tan\omega\pi$$

one will have this general equation

$$\frac{\triangle y}{8(1+2\omega)\omega} = \frac{\pi}{8\omega}\tan\omega\pi + \frac{\triangle}{8(1+2\omega)}f : \frac{\omega(1+2\omega)}{2\triangle}$$

or

$$y = \frac{\pi(1+2\omega)}{\triangle}\tan\omega\pi + f : \frac{\omega(1+2\omega)}{2\triangle},$$

where the added function in general is of such a nature that it vanishes in the cases

$$\omega = 0, \quad \omega = \pm\frac{1}{2}, \quad \omega = \pm\frac{3}{2}, \quad \omega = \pm\frac{5}{2} \quad \text{etc.}$$

of which kind these formulas are

$$\sin 2\omega\pi, \quad \omega\cos\omega\pi, \quad \text{likewise} \quad \sin 2i\omega\pi \quad \text{and} \quad \omega\cos(2i-1)\omega\pi,$$

while i denotes an arbitrary integer number; hence it is possible to combine any arbitrary number of these formulas. Therefore, a certain function of this kind will be given, which shall be φ, that it is

$$y = \sin\omega\pi$$

and hence

$$\sin\omega\pi = \frac{\pi(1+2\omega)}{\triangle}\tan\omega\pi + \varphi$$

or

$$\triangle = \frac{\pi(1+2\omega)\tan\omega\pi}{\sin\omega\pi - \varphi} = 2\int \frac{z^{\frac{1}{2}-\omega}\mathrm{d}z}{(1-z)^{\frac{1}{2}+\omega}}.$$

Therefore, since in the case $\omega = 0$ the function φ certainly vanishes, it will be $\pi = \triangle$, of course, which is an indication that the function φ contains ω^λ, whose exponent λ is greater than unity, since otherwise having taken $\omega = 0$ the quantity φ would not vanish with respect to $\sin \omega\pi$. And for this reason the conclusions of the preceding problem are to be considered as true.

PROBLEM 4

§35 *To find an equation of such a kind for a curve line between the abscissa x and the ordinate y, that to the abscissas proceeding in an interrupted arithmetic progression given ordinates correspond, namely*

$$x = n\theta, \qquad (1-n)\theta, \qquad (1+n)\theta, \qquad (2-n)\theta, \qquad (2+n)\theta, \qquad (3-n)\theta \qquad \text{etc.,}$$

and

$$y = p, \qquad q, \qquad r, \qquad s, \qquad t, \qquad u \qquad \text{etc.}$$

SOLUTION

Let us in general put the abscissa

$$x = \theta\omega$$

and for the equation between x and y let us set this equation

$$\frac{y}{\omega} = \mathfrak{A} \cdot \frac{p}{n} - \mathfrak{B} \cdot \frac{q}{1-n} + \mathfrak{C} \cdot \frac{r}{1+n} - \mathfrak{D} \cdot \frac{s}{2-n} + \mathfrak{E} \cdot \frac{t}{2+n} - \mathfrak{F} \cdot \frac{u}{3-n} + \text{etc.}$$

and from paragraph 25 extended to this general case one will have

$$\mathfrak{A} = \tfrac{(1-n-\omega)(1-n+\omega)}{1(1-2n)} \cdot \tfrac{(1+n-\omega)(1+n+\omega)}{1(1+2n)} \cdot \tfrac{(2-n-\omega)(2-n+\omega)}{2(2-2n)} \cdot \tfrac{(2+n-\omega)(2+n+\omega)}{2(2+2n)} \cdot \text{etc.}$$

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{(n-\omega)(n+\omega)}{(1-n-\omega)(1-n+\omega)} \cdot \frac{1-n}{n}, \quad \frac{\mathfrak{C}}{\mathfrak{B}} = \frac{(1-n-\omega)(1-n+\omega)}{(1+n-\omega)(1+n+\omega)} \cdot \frac{1+n}{1-n},$$
$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(1+n-\omega)(1+n+\omega)}{(2-n-\omega)(2-n+\omega)} \cdot \frac{2-n}{1+n}, \quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(2-n-\omega)(2-n+\omega)}{(2+n-\omega)(2+n+\omega)} \cdot \frac{2+n}{2-n}$$

etc.

Let us expand the value of $\mathfrak{A}$ into two products

$$\mathfrak{P}=\frac{(1-n-\omega)(1-n+\omega)}{1(1-2n)}\cdot\frac{(2-n-\omega)(2-n+\omega)}{2(2-2n)}\cdot\frac{(3-n-\omega)(3-n+\omega)}{3(3-2n)}\cdot\text{etc.},$$
$$\mathfrak{Q}=\frac{(1+n-\omega)(1+n+\omega)}{1(1+2n)}\cdot\frac{(2+n-\omega)(2+n+\omega)}{2(2+2n)}\cdot\frac{(3+n-\omega)(3+n+\omega)}{3(3+2n)}\cdot\text{etc.},$$

that it is

$$\mathfrak{A}=\mathfrak{P}\mathfrak{Q},$$

and let us define the value of both by means of integral formulas according to the prescriptions in § 17. And at first for the infinite product $\mathfrak{P}$ let us set

$$a=1-n-\omega,\quad b=1,\quad c=-n+\omega\quad\text{and}\quad d=1$$

and it will be

$$\mathfrak{P}=\frac{\int dx(1-x)^{-1-n+\omega}}{\int x^{-n-\omega}dx(1-x)^{-1-n+\omega}}=\frac{1}{\omega-n}\cdot\frac{1}{\int x^{-n-\omega}dx(1-x)^{-1-n+\omega}},$$

if it certainly is

$$\omega-n>0.$$

For the other infinite product only by taking n negatively it will be

$$\mathfrak{Q}==\frac{1}{\omega+n}\cdot\frac{1}{\int x^{n-\omega}dx(1-x)^{n+\omega-1}}.$$

But that the condition $\omega-n>0$ is not necessary, let us use another distribution and let

$$\mathfrak{P}=\frac{(1+n+\omega)(1-n-\omega)}{1\cdot1}\cdot\frac{(2+n+\omega)(2-n-\omega)}{2\cdot2}\cdot\frac{(3+n+\omega)(3-n-\omega)}{3\cdot3}\cdot\text{etc.},$$
$$\mathfrak{Q}=\frac{(1+n-\omega)(1-n+\omega)}{(1-2n)(1+2n)}\cdot\frac{(2+n-\omega)(2-n+\omega)}{(2-2n)(2+2n)}\cdot\frac{(3+n-\omega)(3-n+\omega)}{(3-2n)(3+2n)}\cdot\text{etc.},$$

and let is set for $\mathfrak{P}$

$$a = 1 - n - \omega, \quad b = 1, \quad c = n + \omega, \quad d = 1,$$

for $\mathfrak{Q}$ on the other hand

$$a = 1 + n - \omega, \quad b = 1 - 2n, \quad c = n + \omega, \quad \text{und} \quad d = 1$$

and it will be

$$\mathfrak{P} = \frac{\int dx(1-x)^{-1+n+\omega}}{\int x^{-n-\omega}dx(1-x)^{-1+n+\omega}} = \frac{1}{n+\omega} \cdot \frac{1}{\int x^{-n-\omega}dx(1-x)^{-1+n+\omega}},$$

$$\mathfrak{Q} = \frac{\int x^{-2n}dx(1-x)^{-1+n+\omega}}{\int x^{n-\omega}dx(1-x)^{-1+n+\omega}}.$$

But it will be

$$\int x^m dx(1-x)^{k-1} = \frac{m+k+1}{k} \int x^m dx(1-x)^k,$$

therefore

$$\int x^{-n-\omega}dx(1-x)^{-1+n+\omega} = \frac{1}{n+\omega} \int x^{-n-\omega}dx(1-x)^{n+\omega}$$

$$= \frac{1}{n+\omega} \int y^{n+\omega}dy(1-y)^{-n-\omega},$$

$$\int x^{-2n}dx(1-x)^{-1+n+\omega} = \frac{1-n+\omega}{n+\omega} \int x^{-2n}dx(1-x)^{n+\omega}$$

$$\frac{1-n+\omega}{n+\omega} \int y^{n+\omega}dy(1-y)^{-2n},$$

$$\int x^{n-\omega}dx(1-x)^{-1+n+\omega} = \frac{1+2n}{n+\omega} \int x^{n-\omega}dx(1-x)^{n+\omega}$$

$$= \frac{1+2n}{n+\omega} \int y^{n+\omega}dy(1-y)^{n-\omega},$$

whence it is concluded

$$\mathfrak{A} = \mathfrak{P}\mathfrak{Q} = \frac{(1-n-\omega)\int y^{n+\omega}dy(1-y)^{-2n}}{\int y^{n+\omega}dy(1-y)^{-n+\omega} \cdot \int y^{n+\omega}dy(1-y)^{n-\omega}}$$

or

$$\mathfrak{A} = \frac{\int y^{n+\omega-1}dy(1-y)^{-2n}}{\int y^{n+\omega}dy(1-y)^{-n-\omega} \cdot \int y^{n+\omega-1}dy(1-y)^{n-\omega}}$$

or

$$\mathfrak{A} = \frac{\int y^{n+\omega-1}dy(1-y)^{-2n}}{(n+\omega)\int y^{n+\omega-1}dy(1-y)^{-n-\omega}\cdot\int y^{n+\omega-1}dy(1-y)^{n-\omega}}.$$

Therefore, since it is

$$\mathfrak{B} = \frac{1-n}{n}\cdot\frac{nn-\omega\omega}{(1-n)^2-\omega\omega}\mathfrak{A},\quad \mathfrak{C} = \frac{1+n}{n}\cdot\frac{nn-\omega\omega}{(1+n)^2-\omega\omega}\mathfrak{A},$$
$$\mathfrak{D} = \frac{2-n}{n}\cdot\frac{nn-\omega\omega}{(2-n)^2-\omega\omega}\mathfrak{A},\quad \mathfrak{E} = \frac{2+n}{n}\cdot\frac{nn-\omega\omega}{(2+n)^2-\omega\omega}\mathfrak{A}$$
$$\text{etc.},$$

by means of a sufficiently convenient series it will be

$$\frac{y}{\mathfrak{A}\omega} = \frac{p}{n} - \frac{(nn-\omega\omega)q}{n((1-n)^2-\omega^2)} + \frac{(nn-\omega\omega)r}{n((1+n)^2-\omega^2)} - \frac{(nn-\omega\omega)s}{n((2-n)^2-\omega^2)} + \text{etc.}$$

or

$$\frac{ny}{\mathfrak{A}\omega(nn-\omega\omega} = \frac{p}{n^2-\omega^2} - \frac{q}{(1-n)^2-\omega^2} + \frac{r}{(1+n)^2-\omega^2} - \text{etc.}$$

But be resubstituting the integral formula for $\mathfrak{A}$, where for the sake of distinction I will denote the new variable by the letter z, this same series is equal to this expression

$$\frac{ny}{(n-\omega)\omega}\cdot\frac{\int z^{n+\omega-1}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}}{\int z^{n+\omega-1}dz(1-z)^{-2n}},$$

the integration of which formulas is to be understood to be extended from the boundary $z=0$ to $z=1$.

Corollary 1

§36 Therefore, if for the sake of brevity we put this general integral formula

$$\frac{\int z^{n+\omega-1}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}}{\int z^{n+\omega-1}dz(1-z)^{-2n}} = \triangle$$

and resolve the single terms of the series into two terms, we will have

$$\begin{aligned}\frac{2n\triangle y}{n-\omega} = &+\frac{p}{n-\omega} - \frac{q}{1-n-\omega} + \frac{r}{1+n-\omega} - \frac{s}{2-n-\omega} + \frac{t}{2+n-\omega} - \text{etc.}\\ &-\frac{p}{n+\omega} + \frac{q}{1-n+\omega} - \frac{r}{1+n+\omega} + \frac{s}{2-n+\omega} - \frac{t}{2+n+\omega} + \text{etc.}\end{aligned}$$

COROLLARY 2

§37 Therefore, this equation defines a curves lines, in which to the abscissas

$$x = 0, \quad n\theta, \qquad (1-n)\theta, \qquad (1+n)\theta, \qquad (2-n)\theta, \qquad (2+n)\theta \qquad \text{etc.}$$

these ordinates correspond

$$y = 0, \quad p, \qquad q, \qquad r, \qquad s, \qquad t \qquad \text{etc.},$$

but to the same abscissas taken negatively the same ordinates taken negatively correspond. But in general here the abscissa was put $x = \theta\omega$.

COROLLARY 3

§38 Since here the letter θ goes out of the letter, it would be possible to write the unity for it, that the letter ω denotes the abscissa itself. But if we want to make the application to arc and their sine, it is convenient to retain the letter θ in the calculation.

SCHOLIUM

§39 The use of this problem is especially seen, if as above the abscissas are considered as circular arcs and the given abscissas are taken in such a way that the ordinates p, q, r, s, t etc. become equal to each other, whether positive or negative. Therefore, that it becomes clear in these cases, whether the found series can be summed from elsewhere, recall, what I once published on similar series, whence the sums of the following two series are calculated

$$\frac{1}{\alpha} - \frac{1}{\beta-\alpha} + \frac{1}{\beta+\alpha} - \frac{1}{2\beta-\alpha} + \frac{1}{2\beta+\alpha} - \text{etc.} = \frac{\pi}{\beta\tan\frac{\alpha\pi}{\beta}},$$
$$\frac{1}{\alpha} + \frac{1}{\beta-\alpha} - \frac{1}{\beta+\alpha} - \frac{1}{2\beta-\alpha} + \frac{1}{2\beta+\alpha} + \text{etc.} = \frac{\pi}{\beta\sin\frac{\alpha\pi}{\beta}}.$$

Therefore, hence for our problem we deduce the following for summations

$$\text{I.}\quad \frac{1}{n-\omega}-\frac{1}{1-n+\omega}+\frac{1}{1+n-\omega}-\frac{1}{2-n+\omega}+\frac{1}{2+n-\omega}-\text{etc.}=\frac{\pi}{\tan(n-\omega)\pi},$$

$$\text{II.}\quad \frac{1}{n-\omega}+\frac{1}{1-n+\omega}-\frac{1}{1+n-\omega}-\frac{1}{2-n+\omega}+\frac{1}{2+n-\omega}+\text{etc.}=\frac{\pi}{\sin(n-\omega)\pi},$$

$$\text{III.}\quad \frac{1}{n+\omega}-\frac{1}{1-n-\omega}+\frac{1}{1+n+\omega}-\frac{1}{2-n-\omega}+\frac{1}{2+n+\omega}-\text{etc.}=\frac{\pi}{\tan(n+\omega)\pi},$$

$$\text{IV.}\quad \frac{1}{n+\omega}+\frac{1}{1-n-\omega}-\frac{1}{1+n+\omega}-\frac{1}{2-n-\omega}+\frac{1}{2+n+\omega}+\text{etc.}=\frac{\pi}{\sin(n+\omega)\pi}.$$

Having observed these let us expand the cases, which by means of these summations can be reduced to finite expressions.

Example I

§40 *Let the ordinate which correspond to the abscissas*

$$x=0,\quad n\theta,\quad (1-n)\theta,\quad (1+n)\theta,\quad (2-n)\theta,\quad (2+n)\theta\quad \text{etc.}$$

be

$$p=f,\quad q=f,\quad r=-f,\quad s=-f,\quad t=+f,\quad u=+f\quad \text{etc.}$$

and by means of a finite equation investigate the relation between the ordinate y and the abscissa $x=\theta\omega$.

Solution

The first corollary for this case yields this equation

$$\begin{aligned}\frac{2n\triangle y}{f(n-\omega)}=&+\frac{1}{n-\omega}-\frac{1}{1-n-\omega}-\frac{1}{1+n-\omega}+\frac{1}{2-n-\omega}+\frac{1}{2+n-\omega}-\text{etc.},\\ &-\frac{1}{n+\omega}+\frac{1}{1-n+\omega}+\frac{1}{1+n+\omega}-\frac{1}{2-n+\omega}-\frac{1}{2+n+\omega}+\text{etc.},\end{aligned}$$

which two series are reduced by means of the four mentioned above, whose summation is known, to II minus IV, and hence the equation in question in finite form will behave this way

$$\frac{2n \triangle y}{f(n-\omega)} = \frac{\pi}{\sin(n-\omega)\pi} - \frac{\pi}{\sin(n+\omega)\pi},$$

which expression is reduced to this one

$$\frac{2\pi \cos n\pi \sin \omega\pi}{\sin(n-\omega)\pi \cdot \sin(n+\omega)\pi} = \frac{4\pi \cos n\pi \cdot \sin \omega\pi}{\cos 2\omega\pi - \cos 2n\pi},$$

such that for our curve one has this equation

$$\frac{n \triangle y}{f(n-\omega)} = \frac{\pi \cos n\pi \sin \omega\pi}{\sin(n-\omega)\pi \cdot \sin(n+\omega)\pi}.$$

We gave the value of $\triangle$ expressed by means of integral formulas before; but since from the superior things it is

$$\triangle = \frac{1}{\mathfrak{A}(n+\omega)},$$

by means of an infinite product we will have

$$\triangle = \frac{1}{n+\omega} \cdot \frac{1(1-2n)}{(1-n)^2-\omega^2} \cdot \frac{1(1+2n)}{(1+n)^2-\omega^2} \cdot \frac{2(2-2n)}{(2-n)^2-\omega^2} \cdot \frac{2(2+2n)}{(2+n)^2-\omega^2} \cdot \text{etc.},$$

where it is more clear than from the integral formulas that the value $\triangle$ becomes infinite, as often as it was

$$\omega = \pm(i \pm n),$$

while i denotes an arbitrary integer number, but the same value $\triangle$ vanishes in the cases, in which it is

$$n = \pm\frac{1}{2}.$$

But then it will also be helpful to have noted, if, while ω goes over into $1+\omega$, the value of $\triangle$ goes over into $\triangle'$

$$\triangle' = -\frac{(1-n-\omega)\triangle}{n-\omega}.$$

And if in similar manner $\triangle''$ corresponds to the value $2+\omega$ assumed instead of ω, it will be

$$\triangle'' = \frac{-(2-n+\omega)\triangle'}{-(1-n+\omega)} = \frac{-(2-n+\omega)\triangle}{n-\omega}.$$

COROLLARY 1

§41 If the quantity $\triangle$ depends on ω, consider a function of it and denote it this way

$$\triangle = f : \omega;$$

therefore, then it will be

$$f : (1+\omega) = \frac{n-1-\omega}{n-\omega} f : \omega$$

and

$$f : (2+\omega) = \frac{n-2-\omega}{n-\omega} f : \omega$$

etc.

Hence, if ω denotes an arbitrary integer number, one will have this theorem

$$\frac{f : (i+\omega)}{n-i-\omega} = \frac{f : \omega}{n-\omega}.$$

COROLLARY 2

§42 Further, since having taken a negative ω it is

$$f : (\omega) = \frac{n+\omega}{n-\omega} f : \omega,$$

it will be

$$\frac{f : -\omega}{n+\omega} = \frac{f : \omega}{n-\omega}$$

hence also in general

$$\frac{f : (i-\omega)}{n-i+\omega} = \frac{f : \omega}{n-\omega}.$$

SCHOLIUM

§43 This case corresponds to that one, which we expanded above in § 25, where the given ordinates also were the sines of the abscissas; and for the present case one must put

$$\theta = \pi,$$

that it is

$$f = \sin n\pi$$

and all given points lie on a line of sines. But hence it does not follow that the curve itself, which the found equation exhibits, is a line of sines, since innumerable other curves can go through the same given points. Hence it is still by no means certain that the value of y corresponding to the abscissa $y = \sin \pi\omega$ and defined by this equation

$$\frac{n \triangle y}{(n-\omega)\sin n\pi} = \frac{\pi \cos n\pi \cdot \sin \omega\pi}{\sin(n-\omega)\pi \cdot \sin(n+\omega)\pi}$$

become equal to the sine of the arc $\pi\omega$, that it is $y = \sin \pi\omega$, even though this is true in the cases $\omega = \pm(i \pm n)$ and $\omega = 0$. But above we certainly saw that even in the case, in which ω is a very small quantity, the equation agrees with the truth by taking $y = \sin \pi\omega$, such that it is

$$\triangle = \frac{\pi \cos n\pi}{\sin n\pi},$$

while it is

$$\triangle = \frac{\int z^{n-1}\mathrm{d}z(1-z)^{-n} \cdot \int z^{n-1}\mathrm{d}z(1-z)^{n}}{\int z^{n-1}\mathrm{d}z(1-z)^{-2n}},$$

as I also showed there. That this subject can be explored more easily in general, for expressing the value $\triangle$ more conveniently I observe that it is

$$\frac{\int z^{n+\omega-1}\mathrm{d}z(1-z)^{-n-\omega}}{\int z^{n+\omega-1}\mathrm{d}z(1-z)^{-2n}} = \frac{\int z^{\omega-n}\mathrm{d}z(1-z)^{-n-\omega}}{\int \mathrm{d}z(1-z)^{-2n}} = (1-2n)\int z^{\omega-n}\mathrm{d}z(1-z)^{-n-\omega},$$

while it is

$$n < \frac{1}{2},$$

whence it will be

$$\triangle = (1-2n)\int z^{\omega-n}\mathrm{d}z(1-z)^{-n-\omega} \cdot \int z^{n+\omega-1}\mathrm{d}z(1-z)^{n-\omega}.$$

But if it was in general

$$y = \sin \omega\pi,$$

it would also be

$$\triangle = \frac{(n-\omega)\pi \sin n\pi \cos n\pi}{n \sin(n-\omega)\pi \cdot \sin(n+\omega)\pi}.$$

Therefore, the question reduces to this, whether this equation

$$(1-2n)\int z^{\omega-n}\mathrm{d}z(1-z)^{-n-\omega} \cdot \int z^{n+\omega-1}\mathrm{d}z(1-z)^{n-\omega}$$

$$= \frac{(n-\omega)\pi \sin n\pi \cos n\pi}{n \sin(n-\omega)\pi \cdot \sin(n+\omega)\pi}$$

is also true in other cases than the ones mentioned above or not. For this aim, let us consider the case, in which it is

$$n = \frac{1}{4} \quad \text{and} \quad \omega = \frac{1}{2},$$

where certainly the second part becomes

$$= \frac{-\frac{1}{4} \cdot \pi\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}}}{-\frac{1}{4} \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}}} = \pi;$$

the left hand side on the other hand will be

$$= \frac{1}{2}\int \frac{z^{\frac{1}{4}}\mathrm{d}z}{(1-z)^{\frac{3}{4}}} \cdot \int \frac{z^{-\frac{1}{4}}\mathrm{d}z}{(1-z)^{\frac{1}{4}}},$$

which having put

$$z = v^4$$

goes over into this form

$$8\int \frac{v^4\mathrm{d}v}{\sqrt[4]{(1-v^4)^3}} \cdot \int \frac{v^2\mathrm{d}v}{\sqrt[4]{(1-v^4)}} = 4\int \frac{\mathrm{d}v}{\sqrt[4]{(1-v^4)^3}} \cdot \int \frac{vv\mathrm{d}v}{\sqrt[4]{(1-v^4)}},$$

whose value by means of the things, which I demonstrated in formulas of this kind, indeed becomes $= \pi$, which therefore is a testimony for the truth of our equation, which can be demonstrated perfectly in the following way.

THEOREM

§44 *However the two numbers n and ω are assumed, this equation agrees with the truth*

$$(1-2n)\int\frac{z^{\omega-n}\mathrm{d}z}{(1-z)^{n+\omega}}\cdot\int\frac{z^{n+\omega-1}\mathrm{d}z}{(1-z)^{\omega-n}}=\frac{(n-\omega)\pi\sin n\pi\cdot\cos n\pi}{n\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi},$$

if certainly the integration of those formulas is extended from the boundary $z=0$ *to the boundary* $z=1$.

PROOF

To reduce these formulas to a form, which I treated, let us out

$$n+\omega=\frac{\mu}{\lambda}\quad\text{and}\quad\omega-n=\frac{\nu}{\lambda},$$

that it is

$$2n=\frac{\mu-\nu}{\lambda}$$

and this equation must be proved

$$\frac{\lambda-\mu+\nu}{\lambda}\int\frac{z^{\frac{\nu}{\lambda}}\mathrm{d}z}{\sqrt[\lambda]{(1-z)^{\mu}}}\cdot\int\frac{z^{\frac{\mu-\lambda}{\lambda}}\mathrm{d}z}{\sqrt[\lambda]{(1-z)^{\nu}}}=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\nu\pi}{\lambda}\cdot\sin\frac{\mu\pi}{\lambda}}.$$

Now put $z=v^{\lambda}$ and one will have

$$\lambda(\lambda-\mu+\nu)\int\frac{v^{\lambda+\nu-1}\mathrm{d}v}{\sqrt[\lambda]{(1-v^{\lambda})^{\mu}}}\cdot\int\frac{v^{\mu-1}\mathrm{d}v}{\sqrt[\lambda]{(1-v^{\lambda})^{\nu}}}=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\nu\pi}{\lambda}\cdot\sin\frac{\mu\pi}{\lambda}};$$

and in the way to express these integral formulas there the left hand side will be represented this way

$$\lambda(\lambda-\mu-\nu)\left(\frac{\lambda+\nu}{\lambda-\mu}\right)\left(\frac{\mu}{\lambda-\nu}\right),$$

which by means of the first reduction

$$\left(\frac{p}{q}\right)=\frac{p-\lambda}{p+q-\lambda}\left(\frac{p-\lambda}{q}\right),$$

goes over into

$$\lambda\nu\left(\frac{\nu}{\lambda-\mu}\right)\left(\frac{\mu}{\lambda-\nu}\right)=\lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right).$$

But this reduction on the other hand

$$\left(\frac{\lambda-q}{p}\right)\left(\frac{\lambda+p-q}{q}\right)=\frac{\pi}{\lambda p\sin\frac{q\pi}{\lambda}}$$

having taken

$$p=\mu-\nu \quad \text{and} \quad q=\mu$$

gives

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)=\frac{\pi}{\lambda(\mu-\nu)\sin\frac{\mu\pi}{\lambda}}.$$

But it also is

$$\left(\frac{\lambda-\nu}{\nu}\right)=\frac{\pi}{\lambda\sin\frac{\nu\pi}{\lambda}},$$

whose product is

$$\left(\frac{\lambda-\nu}{\mu}\right)\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\mu}{\mu-\nu}\right)=\frac{\pi\pi}{\lambda\lambda(\mu-\nu)\sin\frac{\mu\pi}{\lambda}\cdot\sin\frac{\nu\pi}{\lambda}}.$$

Further, since it is in general

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right),$$

by taking

$$p=\lambda-\mu, \quad q=\mu-\nu, \quad \text{and} \quad r=\nu$$

it will be

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\nu}\right)=\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\mu+\nu}{\mu-\nu}\right)$$

and because of

$$\left(\frac{\lambda-p}{p}\right)=\frac{\pi}{\lambda\sin\frac{p\pi}{\lambda}}$$

having taken

$$p=\mu-\nu$$

it will be

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\nu}\right)=\left(\frac{\lambda-\mu}{\nu}\right)\cdot\frac{\pi}{\lambda\sin\frac{\mu-\nu}{\lambda}\pi}$$

and hence

$$\left(\frac{\lambda-\nu}{\mu}\right)\left(\frac{\lambda-\mu}{\nu}\right)\cdot\frac{\pi}{\lambda\sin\frac{\mu-\nu}{\lambda}\pi}=\frac{\pi\pi}{\lambda\lambda(\mu-\nu)\sin\frac{\mu\pi}{\lambda}\cdot\sin\frac{\nu\pi}{\lambda}};$$

from which the left hand side reduces to this form

$$\lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\mu\pi}{\lambda}\cdot\sin\frac{\nu\pi}{\lambda}},$$

which is the equation to be demonstrated above.

COROLLARY 1

§45 Therefore, in the doctrine of integral formulas of this kind

$$\int\frac{v^{p-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\lambda-q}}},$$

which is denote by this character

$$\left(\frac{p}{q}\right),$$

to which $\left(\frac{p}{q}\right)$ is equivalent, this reduction is of high importance, in which I demonstrated that it is

$$\lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\mu\pi}{\lambda}\cdot\sin\frac{\nu\pi}{\lambda}},$$

such that the product of such integral formulas $\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)$ can be exhibited by means of angles alone.

COROLLARY 2

§46 If in the value found first for $\triangle$ one equally puts

$$n+\omega=\frac{\mu}{\nu}\quad\text{and}\quad\omega-n=\frac{\nu}{\lambda},$$

but then

$$z = v^\lambda,$$

it will be

$$\triangle = \lambda \int \frac{v^{\mu-1}\mathrm{d}v}{\sqrt[\lambda]{(1-v^\lambda)^\mu}} \cdot \int \frac{v^{\mu-1}\mathrm{d}v}{\sqrt[\lambda]{(1-v^\lambda)^\nu}} : \int \frac{v^{\mu-1}\mathrm{d}v}{\sqrt[\lambda]{(1-v^\lambda)^{\mu-\nu}}}$$

and hence in this way of notation

$$\triangle = \frac{\lambda\left(\frac{\mu}{\lambda-\mu}\right)\left(\frac{\mu}{\lambda-\nu}\right)}{\left(\frac{\mu}{\lambda-\mu+\nu}\right)}$$

or

$$\triangle = \frac{\lambda\left(\frac{\lambda-\mu}{\mu}\right)\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}.$$

Therefore, the same value also is

$$\triangle = \frac{\nu\pi}{\mu-\nu} \cdot \frac{\sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\mu\pi}{\lambda} \cdot \sin\frac{\nu\pi}{\lambda}}.$$

COROLLARY 3

§47 Therefore, since for this last formula it immediately is

$$\left(\frac{\lambda-\mu}{\mu}\right) = \frac{\pi}{\lambda \sin\frac{\mu\pi}{\lambda}},$$

it will be

$$\frac{\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)} = \frac{\nu}{\mu-\nu} \cdot \frac{\sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\nu\pi}{\lambda}},$$

whose truth is shown from the following general theorem

$$\frac{\left(\frac{p}{q}\right)}{\left(\frac{r}{p}\right)} = \frac{\left(\frac{p+r}{q}\right)}{\left(\frac{p+q}{r}\right)};$$

for, it will be

$$\frac{\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)} = \frac{\left(\frac{\lambda+\nu}{\lambda-\nu}\right)}{\left(\frac{\lambda+\mu-\nu}{\lambda-\mu+\nu}\right)} = \frac{\nu}{\mu-\nu} \cdot \frac{\left(\frac{\nu}{\lambda-\nu}\right)}{\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right)}$$

because of

$$\left(\frac{\lambda+\nu}{\lambda-\nu}\right)=\frac{\nu}{\lambda}\left(\frac{\nu}{\lambda-\nu}\right)\quad\text{and}\quad\left(\frac{\lambda+\mu-\nu}{\lambda-\mu+\nu}\right)=\frac{\mu-\nu}{\lambda}\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right);$$

it then is

$$\left(\frac{\nu}{\lambda-\nu}\right)=\frac{\pi}{\lambda\sin\frac{\nu\pi}{\lambda}}\quad\text{and}\quad\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right)=\frac{\pi}{\lambda\sin\frac{\mu-\nu}{\lambda}\pi}.$$

EXAMPLE II

§48 *Let the ordinates, which correspond to the abscissas*

$$n\theta,\quad (1-n)\theta,\quad (1+n)\theta,\quad (2-n)\theta,\quad (2+n)\theta\quad \text{etc.}$$

be

$$p=f,\quad q=-f,\quad r=+f,\quad s=-f,\quad t=+f,\quad u=-f\quad \text{etc.},$$

and by means of a finite equation investigate the relation between the abscissa $x=\theta\omega$ *and the ordinate* $=y$ *in general.*

The general equation of paragraph 36 accommodated to this case yields

$$\begin{aligned}\frac{2n\triangle y}{f(n-\omega)}=&\frac{1}{n-\omega}+\frac{1}{1-n-\omega}+\frac{1}{1+n-\omega}+\frac{1}{2-n-\omega}+\frac{1}{2+n-\omega}+\text{etc.}\\&-\frac{1}{n+\omega}-\frac{1}{1-n+\omega}-\frac{1}{1+n+\omega}-\frac{1}{2-n+\omega}-\frac{1}{2+n+\omega}-\text{etc.},\end{aligned}$$

where we now certainly know that it is

$$\triangle=\frac{(n-\omega)\pi\sin 2n\pi}{2n\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}.$$

But that series from § 39 becomes

$$\text{I minus III}=\frac{\pi}{\tan(n-\omega)\pi}-\frac{\pi}{\tan(n+\omega)\pi}=\frac{\pi\sin 2\omega\pi}{\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}$$

having substituted which sum

$$\frac{y}{f}\cdot\frac{\pi\sin 2n\pi}{\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}=\frac{\pi\sin 2\omega\pi}{\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}$$

or

$$y=\frac{f\sin 2\omega\pi}{\sin 2n\pi}=\frac{f\sin\frac{2x\pi}{\theta}}{\sin 2n\pi}.$$

Therefore, this curve again is a line of sines, and if one takes $\theta=2\pi$, that it is $f=\sin 2n\pi$, the ordinate will be $y=\sin x$.

Corollary 1

§49 If one takes

$$\theta = \pi \quad \text{and} \quad f = \tan n\theta = \tan n\pi,$$

the given points will be on a line of tangents; and nevertheless the found curve itself will not be a line of tangents; but its nature will be expressed by this equation

$$y = \frac{\tan n\pi \cdot \sin 2x}{\sin 2n\pi} = \frac{\sin 2x}{2\cos^2 n\pi} = \frac{\sin 2x}{1 + \cos 2n\pi}$$

and here it will be $y = \tan x$, as often as it was $x = \pm(i \pm n)\pi$.

Corollary 2

§50 If in the solution of the first example, in which it was

$$p = f, \quad q = f, \quad r = -f, \quad s = -f, \quad t = f, \quad u = f \quad \text{etc.},$$

instead of $\triangle$ we would have immediately put the found value, this equation would have arisen

$$y = \frac{f \sin \omega\pi}{\sin n\pi}.$$

Hence it would have been perspicuous that having taken $\theta = \pi$ and $f = \sin n\pi$ the curve itself will be a line of sines.

Scholium

§51 It especially deserves to be mentioned that in problem 4, where the given abscissas constitute an interrupted arithmetic progression, the value of the quantity $\triangle$ can be exhibited absolutely by means of angles, although nevertheless in problem 3, where the given abscissas constituted a true arithmetic progression, the integral formula $\triangle$ in general cannot be expressed by angles by any means. For, since there it was

$$\triangle = \int z^{n-\omega-1} dz (1-z)^{n-\omega-1},$$

this formulas having put $n - \omega = \frac{\nu}{\lambda}$ and $z = v^{\lambda}$ goes over into

$$\triangle = \lambda \int \frac{v^{\nu-1} dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\lambda-\nu}}} \quad \text{or} \quad \triangle = \lambda \left(\frac{\nu}{\nu}\right),$$

which formula can imply highly transcendental formulas. As if in that problem the given ordinates are sets

$$p = f, \quad q = -f, \quad r = f, \quad s = -f, \quad t = f, \quad u = -f \quad \text{etc.}$$

and it was $n = \frac{1}{2}$, the equation for the curve passing through these points will be

$$\frac{\triangle y}{2(1+2\omega)\omega f} = \frac{4}{1-4\omega\omega} + \frac{4}{9-4\omega\omega} + \frac{4}{1-4\omega\omega} + \text{etc.}$$

or

$$\frac{\triangle y}{2f\omega(1+2\omega)} = \frac{\pi}{2\omega}\tan\omega\pi,$$

such that it is

$$y = \frac{\pi f(1+2\omega)\tan\omega\pi}{\triangle},$$

whence, even though one takes

$$\theta = \pi \quad \text{and} \quad f = \sin n\theta = \sin\frac{1}{2}\pi = 1,$$

it manifestly does not follow that it will be $y = \sin\theta\omega = \sin\omega\pi$. Since in the first example it is already certain that it is

$$y = \frac{f\sin\omega\pi}{\sin n\pi},$$

let us expand the same case from the first problem in such a way that we investigate the values of the single coefficients A, B, C, D etc.

PROBLEM 5

§52 *The general equation constituted above in Problem 1 in such a way that to these abscissas*

$$x \quad = n\theta, \qquad (1-n)\theta, \qquad (1+n)\theta, \qquad (2-n)\theta, \qquad (2+n)\theta \qquad \text{etc.}$$

these ordinates correspond

$$y = +f, \qquad +f, \qquad -f, \qquad -f, \qquad +f, \qquad \text{etc.,}$$

SOLUTION

As before set $x = \theta\omega$ and consider the equation in question in this form

$$y = A\omega + B\omega(\omega\omega - nn) + C\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)$$
$$+D\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)(\omega\omega - (1+n)^2)$$
$$+E\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)(\omega\omega - (1+n)^2)(\omega\omega - (2-n)^2)$$
$$+\text{etc},$$

whence these equations are deduced

$$\begin{aligned}
\frac{f}{n} &= A,\\
\frac{f}{1-n} &= A \ + B \cdot 1(1-2n),\\
\frac{-f}{1+n} &= A \ + B \cdot 1(1-2n) \ + C \cdot (1+2n) \cdot 2 \cdot 2n,\\
\frac{-f}{2-n} &= A \ + B \cdot 1(1-2n) \ + C \cdot 2(2-2n) \cdot 1(3-2n)\\
&\qquad + D \cdot 2(2-2n) \cdot 1(3-2n) \cdot 3(1-2n)
\end{aligned}$$

$$\text{etc.}$$

and hence the following values of the coefficients

$$A = \frac{f}{n}, \quad B = \frac{-f}{n(1-n)}, \quad C = \frac{f}{2n(1-n)(1+n)}, \quad D = \frac{-f}{6n(1-n)(1+n)(2-n)},$$

$$D = \frac{f}{24n(1-n)(1+n)(2-n)(2+n)} \quad \text{etc.};$$

since this progression is sufficiently simple, our series for the value of y, which we already to be

$$= \frac{f \sin\omega\pi}{\sin n\pi},$$

deserves even greater attention

$$\frac{\sin\omega\pi}{\sin n\pi} = \frac{\omega}{n} - \frac{\omega}{n} \cdot \frac{\omega\omega - nn}{1(1-n)} + \frac{\omega}{n} \cdot \frac{\omega\omega - nn}{1(1-n)} \cdot \frac{\omega\omega - (1-n)^2}{2(1+n)}$$

$$-\frac{\omega}{n}\cdot\frac{\omega\omega-nn}{1(1-n)}\cdot\frac{\omega\omega-(1-n)^2}{2(1+n)}\cdot\frac{\omega\omega-(1+n)^2}{3(2-n)}+\text{etc.},$$

or if Π always denotes the preceding term, the whole while be

$$\frac{\sin\omega\pi}{\sin n\pi}=\frac{\omega}{n}-\Pi\cdot\frac{\omega\omega-nn}{1(1-n)}+\Pi\cdot\frac{\omega\omega-(1-n)^2}{2(1+n)}-\Pi\cdot\frac{\omega\omega-(1+n)^2}{3(2-n)}$$

$$+\Pi\cdot\frac{\omega\omega-(2-n)^2}{4(2+n)}-\Pi\cdot\frac{\omega\omega-(2+n)^2}{5(3-n)}+\text{etc.}$$

If all terms affected with the same sign are desired, it will be

$$\begin{aligned}\frac{\sin\omega\pi}{\sin n\pi}=\frac{\omega}{n}&+\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}+\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}\cdot\frac{(1-n)^2-\omega\omega}{2(1+n)}\\&+\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}\cdot\frac{(1-n)^2-\omega\omega}{2(1+n)}\cdot\frac{(1+n)^2-\omega\omega}{3(2-n)}\\&+\frac{\omega}{n}\cdot\frac{nn-\omega\omega}{1(1-n)}\cdot\frac{(1-n)^2-\omega\omega}{2(1+n)}\cdot\frac{(1+n)^2-\omega\omega}{3(2-n)}\cdot\frac{(2-n)^2-\omega\omega}{4(2+n)}\\&\text{etc.}\end{aligned}$$

Therefore, this series seem even more remarkable, since it recedes from the usual form of a series and in it even the two arbitrary numbers n and ω occur.

COROLLARY 1

§53 If the number ω vanishes, that it is $\sin\omega\pi=\omega\pi$, having divided by ω one will have the equation

$$\frac{\pi}{\sin n\pi}=\frac{1}{n}+\frac{n}{1(1-n)}+\frac{n(1-n)}{1\cdot 2(1+n)}+\frac{n(1-n)(1+n)}{1\cdot 2\cdot 3(2-n)}$$

$$+\frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4(2+n)}+\text{etc.},$$

whence having taken $n=\frac{1}{2}$ because of $\sin\frac{\pi}{2}=1$ it will be

$$\pi=2+1+\frac{1\cdot 1\cdot 2}{2\cdot 4\cdot 3}+\frac{1\cdot 1\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 3}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 8\cdot 5}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 2}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 5}+\text{etc.}$$

or

$$\pi=2+\frac{1}{2\cdot 2^1\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 2^3\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^5\cdot 7}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 82^7\cdot 9}+\text{etc.}$$

$$+1+\frac{1}{2\cdot 2^2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 2^4\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^6\cdot 7}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 82^8\cdot 9}+\text{etc.};$$

since the second of these series is the half of the first, the sum of the second will be $=\frac{\pi}{3}$, the reason for what is certainly also clear from that that it is

$$\int\frac{dx}{\sqrt{1-xx}}=\arcsin x=x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\text{etc.},$$

whence that series becomes $=\frac{\arcsin x}{x}$ for $x=\frac{1}{2}$ and hence $=2\frac{\pi}{6}=\frac{\pi}{3}$.

COROLLARY 2

§54 If the other number n vanishes that it is $\sin n\pi=n\pi$ and the equation is multiplied by n, it will arise

$$\frac{\sin\omega\pi}{\pi}=\omega-\frac{\omega^3}{1}+\frac{\omega^3(\omega^2-1)}{1\cdot 2\cdot 1^2}-\frac{\omega^3(\omega^2-1)(\omega^2-1)}{1\cdot 2\cdot 3\cdot 1^2\cdot 2}+\frac{\omega^3(\omega^2-1)(\omega^2-1)(\omega^2-4)}{1\cdot 2\cdot 3\cdot 4\cdot 1^2\cdot 2^2}$$
$$-\frac{\omega^3(\omega^2-1)(\omega^2-1)(\omega^2-4)(\omega^2-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 1^2\cdot 2^2\cdot 3}+\text{etc.},$$

which series divided by ω is resolved into the following two

$$\frac{\sin\omega\pi}{\omega\pi}=1+\frac{\omega^2(\omega^2-1)}{1\cdot 2\cdot 1^2}+\frac{\omega^2(\omega^2-1)^2(\omega^2-4)}{1\cdot 2\cdot 3\cdot 4\cdot 1^2\cdot 2^2}+\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2(\omega^2-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 1^2\cdot 2^2\cdot 3^2}+\text{etc.}$$
$$-\frac{\omega^2}{1}-\frac{\omega^2(\omega^2-1)^2}{1\cdot 2\cdot 3\cdot 1^2\cdot 2}-\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 1^2\cdot 2^2\cdot 3}-\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2(\omega^2-9)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 1^2\cdot 2^2\cdot 3^2\cdot 4}-\text{etc.}$$

Let us set $\omega=\frac{1}{2}$ here; it will be

$$\frac{2}{\pi}=1-\frac{1\cdot 1\cdot 3}{1\cdot 1\cdot 1\cdot 2^5}-\frac{1\cdot 1\cdot 3\cdot 1\cdot 3\cdot 5}{1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 2^{10}}-\frac{1\cdot 1\cdot 3\cdot 1\cdot 3\cdot 5\cdot 3\cdot 5\cdot 7}{1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 2^{15}}-\text{etc.}$$
$$-\frac{1\cdot 1}{2^2}-\frac{1\cdot 1\cdot 1\cdot 3\cdot 1\cdot 3}{1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 2^6}-\frac{1\cdot 1\cdot 1\cdot 3\cdot 1\cdot 3\cdot 3\cdot 5\cdot 3\cdot 5}{1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 5\cdot 2^{10}}-\text{etc.},$$

which last series can be represented this way.

$$-\frac{1}{2^2}-\frac{1\cdot 1\cdot 1\cdot 3}{1\cdot 1\cdot 2\cdot 2^7}-\frac{1\cdot 1\cdot 3\cdot 3\cdot 3\cdot 5}{1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3\cdot 2^{12}}-\frac{1\cdot 1\cdot 3\cdot 3\cdot 3\cdot 5\cdot 5\cdot 5\cdot 7}{1\cdot 1\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 4\cdot 2^{17}}-\text{etc.}$$

COROLLARY 3

§55 If it was $n = \frac{1}{2}$ that it is $\sin n\pi = 1$, the factors, from which the single terms of the series must be formed, will be

$$\frac{2\omega}{1} \cdot \frac{1-4\omega\omega}{1\cdot 2} \cdot \frac{1-4\omega\omega}{3\cdot 4} \cdot \frac{9-4\omega\omega}{3\cdot 6} \cdot \frac{9-4\omega\omega}{5\cdot 8} \cdot \frac{25-4\omega\omega}{5\cdot 10} \cdot \frac{25-4\omega\omega}{7\cdot 12} \cdot \text{etc.}$$

and the sum of the series will be $\sin\omega\pi$, namely

$$\sin\omega\pi = 2\omega + \frac{2\omega(1-4\omega\omega)}{1\cdot 2} + \frac{2\omega(1-4\omega\omega)^2}{1\cdot 2\cdot 3\cdot 4} + \frac{2\omega(1-4\omega\omega)^2(9-4\omega\omega)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.},$$

whence having taken $\omega = 1$ it must be

$$0 = 2 - 3 + \frac{3}{2^2} + \frac{5}{2^3\cdot 3} + \frac{5}{2^6\cdot 3} + \frac{7}{2^7\cdot 3} + \frac{7}{2^9\cdot 5} + \frac{9}{2^{10}\cdot 7} + \frac{5\cdot 9}{2^{14}\cdot 7} + \text{etc.},$$

whose truth will become plain to any one performing the calculation.

SCHOLIUM

§55 For this cases also the solution found above deserves it to be considered with more attention, which from § 36 because of

$$\triangle = \frac{(n-\omega)\pi\sin 2n\pi}{2n\sin(n-\omega)\pi\cdot\sin(n-\omega)\pi} \quad \text{und} \quad y = \frac{f\sin\omega\pi}{\sin n\pi},$$

since it is

$$p = f, \quad q = f, \quad r = -f, \quad s = -f, \quad t = f, \quad u = f \quad \text{etc.},$$

is contained in this equation

$$\frac{\pi\cos n\pi\cdot\sin\omega\pi}{\omega\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}$$

$$= \frac{1}{nn-\omega\omega} - \frac{1}{(1-n)-\omega^2} - \frac{1}{(1+n)-\omega^2} + \frac{1}{(2-n)-\omega^2} + \frac{1}{(2+n)-\omega^2} + \text{etc.},$$

which series deviates a lot from the one we just found. But I observe the following things on this series:

I. If ω vanishes, it will be

$$\frac{\pi\pi\cos n\pi}{(\sin n\pi)^2} = \frac{1}{nn} - \frac{1}{(1-n)^2} - \frac{1}{(1+n)^2} + \frac{1}{(2-n)^2} + \frac{1}{(2+n)^2} - \frac{1}{(3-n)^2} - \text{etc.};$$

but if additionally n vanishes, because of $\sin n\pi = n\pi$ the following inconvenience arises

$$\frac{1}{nn} = \frac{1}{nn} - \frac{2}{1} + \frac{2}{4} - \frac{2}{9} + \frac{2}{16} - \text{etc.}$$

But to get rid of this, let us not consider the number n only as vanishing, and since it is

$$\cos n\pi = 1 - \frac{1}{2}nn\pi\pi$$

and also

$$\sin n\pi = n\pi - \frac{1}{6}n^3\pi^3 = n\pi\left(1 - \frac{1}{6}nn\pi\pi\right),$$

it will be

$$\frac{\cos n\pi}{(\sin n\pi)^2} = \frac{1-\frac{1}{2}nn\pi\pi}{nn\pi\pi(1-\frac{1}{3}nn\pi\pi)} = \frac{1-\frac{1}{6}nn\pi\pi}{nn\pi\pi};$$

hence this true equation is obtained

$$\frac{1}{nn} - \frac{1}{6}\pi\pi = \frac{1}{nn} - \frac{2}{1} + \frac{2}{4} - \frac{2}{9} + \frac{2}{16} - \frac{2}{25} + \text{etc.}$$

For, it is

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} = \frac{1}{12}\pi\pi.$$

II. Now let us put $n = 0$ and we will have

$$-\frac{\pi}{\omega\sin\omega\pi} = -\frac{1}{\omega^2} - \frac{1}{1-\omega^2} - \frac{1}{1-\omega^2} + \frac{1}{4-\omega^2} + \frac{1}{4-\omega^2} - \frac{1}{9-\omega^2} - \frac{1}{9-\omega^2} + \text{etc.}$$

or

$$\frac{\pi}{\omega\sin\omega\pi} = \frac{1}{\omega^2} + \frac{2}{1-\omega^2} - \frac{2}{4-\omega^2} + \frac{2}{9-\omega^2} - \frac{2}{16-\omega^2} + \frac{2}{25-\omega^2} - \text{etc.}$$

whence we obtain this memorable summation

$$\frac{1}{1-\omega^2} - \frac{1}{4-\omega^2} + \frac{1}{9-\omega^2} - \frac{1}{16-\omega^2} + \text{etc.} = \frac{\pi}{2\omega\sin\omega\pi} - \frac{1}{2\omega\omega},$$

whose truth I demonstrated elsewhere. But hence having taken ω infinitely small because of

$$\sin\omega\pi = \omega\pi\left(1 - \frac{1}{6}\omega^2\pi^2\right)$$

the sum of the series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \text{etc.}$$

as before is calculated to be

$$\frac{1}{2\omega\omega(1 - \frac{1}{6}\omega^2\pi^2)} - \frac{1}{2\omega\omega} = \frac{1}{12}\pi\pi.$$

III. If one takes $n = \frac{1}{2}$, because of $\cos n\pi = 0$ also the series itself vanishes, while all terms indeed cancel each other. But that this happens, if n differs infinitely less from $\frac{1}{2}$, differentiate with respect to the variable n, whence it is

$$-\frac{n\pi\sin n\pi\sin\omega\pi(1+\cos(n-\omega)\pi\cdot\cos(n+\omega)\pi)}{\omega(\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi)^2} = -\frac{2n}{(nn-\omega\omega)^2} - \frac{2(1-n)}{((1-n)^2-\omega^2)^2}$$

$$\frac{2(1+n)}{((1+n)^2-\omega^2)^2} + \frac{2(2-n)}{((2-n)^2-\omega^2)^2} - \frac{2(2+n)}{((2+n)^2-\omega^2)^2} - \text{etc.}$$

Therefore, now take $n = \frac{1}{2}$ and it will be

$$-\frac{\pi\pi\sin\omega\pi}{\omega(\cos\omega\pi)^2} = -\frac{16}{(1-4\omega^2)^2} - \frac{16}{(1-4\omega^2)^2} + \frac{3\cdot 16}{(9-4\omega^2)^2} + \frac{3\cdot 16}{(9-4\omega^2)^2} - \text{etc.}$$

or

$$\frac{\pi\pi\sin\omega\pi}{32\omega(\cos\omega\pi)^2} = \frac{1}{(1-4\omega^2)^2} - \frac{3}{(9-4\omega^2)^2} + \frac{5}{(25-4\omega^2)^2} - \frac{7}{(49-4\omega^2)^2} + \text{etc.},$$

where having taken $\omega = 0$ it follows that it will be

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.},$$

which is certainly known from elsewhere.

But the series found in the preceding problem seems to be a lot more difficult. Yes, even the cases expanded in corollary 1, even though it is highly particular, deserves a more diligent expansion, which I will try to give in the following problem.

Problem 6

§57 *If n is an arbitrary number, to find the sum of this series*

$$s = \frac{1}{n} + \frac{n}{1(1-n)} + \frac{n(1-n)}{1\cdot 2(1+n)} + \frac{n(1-n)(1+n)}{1\cdot 2\cdot 3(2-n)} + \frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4(2+n)} + \text{etc.},$$

which we certainly found before [§ 53] to be

$$s = \frac{\pi}{\sin n\pi}.$$

Solution

Since in this series the law of progression is interrupted, it will be convenient to split it into two parts. Therefore, let us set

$$P = \frac{1}{n} + \frac{n(1-n)}{1\cdot 2(1+n)}$$

$$+ \frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4(2+n)} + \frac{n(1-n)(1+n)(2-n)(2+n)(3-n)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6(3+n)} + \text{etc.},$$

$$Q = \frac{n}{1(1-n)} + \frac{n(1-n)(1+n)}{1\cdot 2\cdot 3(2-n)} + \frac{n(1-n)(1+n)(2-n)(2+n)}{1\cdot 2\cdot 3\cdot 4\cdot 5(3-n)} + \text{etc.},$$

such that it is

$$s = P + Q.$$

Now, investigating the sum of these series I recall the following series derived from the doctrine of angle

$$\frac{\cos\mu\varphi}{\cos\varphi} = 1 + \frac{(1-\mu)(1+\mu)}{1\cdot 2}\sin^2\varphi + \frac{(1-\mu)(1+\mu)(3-\mu)(3+\mu)}{1\cdot 2\cdot 3\cdot 4}\sin^4\varphi + \text{etc.},$$

$$\frac{\sin\nu}{\cos\varphi} = \nu\sin\varphi + \frac{\nu(2-\nu)(2+\nu)}{1\cdot 2\cdot 3}\sin^3\varphi + \frac{\nu(2-\nu)(2+\nu)(4-\nu)(4+\nu)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\sin^5\varphi + \text{etc.}$$

and first I will accommodate that to the first form P. Therefore, since these fractions

$$\frac{(1-\mu)(1+\mu)}{n(1-n)},\quad \frac{(3-\mu)(3+\mu)}{(1+n)(2-n)},\quad \frac{(5-\mu)(5+\mu)}{(2+n)(3-n)}\quad \text{etc.}$$

must be equal, I conclude that one has to take $\mu = 1 - 2n$, whence it will be

$$\frac{\cos(1-2n)\varphi}{\cos\varphi} = 1 + \frac{n(1-n)}{1\cdot 2}\cdot 2^2\sin^2\varphi + \frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4}\cdot 2^4\sin^4\varphi + \text{etc.}$$

Let us multiply by $d\varphi\sin^{2n-1}\varphi\cos\varphi$ and integrate, it will become

$$\int \mathrm{d}\varphi\sin^{2n-1}\varphi\,\cos(1-2n)\varphi = \frac{1}{2n}\cdot\sin^{2n}\varphi + \frac{n(1-n)}{1\cdot 2(n+1)}\cdot 2\sin^{2n+2}\varphi$$

$$+\frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4(n+2)}\cdot 2^3\sin^{2n+4}\varphi + \text{etc.}$$

Now after the integration set $\sin\varphi = \frac{1}{2}$ or $\varphi = 30^\circ$ and it will be

$$P = 2^{2n+1}\int \mathrm{d}\varphi\sin^{2n-1}\varphi\,\cos(1-2n)\varphi;$$

the series Q on the other hand will easily be deduced from the other known one by taking $\nu = 2n$, whence it is

$$\frac{\sin 2n\varphi}{\cos\varphi} = n\cdot 2\sin\varphi + \frac{n(1-n)(1+n)}{1\cdot 2\cdot 3}\cdot 2^3\sin^3\varphi$$

$$+\frac{n(1-n)(1+n)(2-n)(2+n)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot 2^5\sin^5\varphi + \text{etc.}$$

Multiply by $\mathrm{d}\varphi\sin^{-2n}\varphi\ \cos\varphi$ and integrate; it will be

$$\int \mathrm{d}\varphi\sin^{-2n}\varphi\ \sin 2n\varphi = \frac{n}{1(1-n)}\cdot\sin^{2-2n}\varphi + \frac{n(1-n)(1+n)}{1\cdot 2\cdot 3(2-n)}\cdot 2^2\sin^{4-2n}\varphi + \text{etc.}$$

Equally having done the integration set $\sin\varphi = \frac{1}{2}$ or $\varphi = 30^\circ$ and this series will arise

$$Q = 2^{2-2n}\int \mathrm{d}\varphi\sin^{-2n}\varphi\ \sin 2n\varphi.$$

Therefore, the sum of the propounded series will expressed in such a way that it is

$$s = 2^{2n+1}\int \mathrm{d}\varphi\sin^{2n-1}\varphi\,\cos(1-2n)\varphi + 2^{2-2n}\int \mathrm{d}\varphi\sin^{-2n}\varphi\ \sin 2n\varphi,$$

and since this sum is already known from elsewhere, one will have

$$\frac{\pi}{\sin n\pi} = 4\int \mathrm{d}\varphi\,\cos(1-2n)\varphi\ (2\sin\varphi)^{2n-1} + 4\int \mathrm{d}\varphi\ \sin 2n\varphi\ (2\sin\varphi)^{-2n}.$$

Corollary 1

§58 If one put $2n = \frac{1-\lambda}{2}$, it will be $1 - 2n = \frac{1+\lambda}{2}$m having put what our equation becomes more convenient, and it will be

$$\frac{\pi}{\sin\frac{1-\lambda}{4}\pi} = 4\int\frac{\mathrm{d}\varphi\cos\frac{1+\lambda}{2}\varphi}{(2\sin\varphi)^{\frac{1+\lambda}{2}}} + 4\int\frac{\mathrm{d}\varphi\sin\frac{1-\lambda}{2}\varphi}{(2\sin\varphi)^{\frac{1-\lambda}{2}}} = \frac{\pi\sqrt{2}}{\cos\frac{\lambda\pi}{4} - \sin\frac{\lambda\pi}{4}},$$

having put $\varphi = 30^{\circ}$ after the integration.

Corollary 2

§59 In a similar manner having taken λ negatively it will be

$$\frac{\pi}{\sin\frac{1+\lambda}{4}\pi} = 4\int\frac{\mathrm{d}\varphi\cos\frac{1-\lambda}{2}\varphi}{(2\sin\varphi)^{\frac{1-\lambda}{2}}} + 4\int\frac{\mathrm{d}\varphi\sin\frac{1+\lambda}{2}\varphi}{(2\sin\varphi)^{\frac{1+\lambda}{2}}} = \frac{\pi\sqrt{2}}{\cos\frac{\lambda\pi}{4} + \sin\frac{\lambda\pi}{4}},$$

where it will be helpful to have noted that in all cases, which can be expanded, the same value of these integral formulas, which we exhibited here, is actually found.