On the extraordinary use of the method of Interpolation in the doctrine of Series*

Leonhard Euler

In the method of interpolation a relation of such a kind among the two variables x and y is in question that, if to the one x successively the given values

$$a, b, c, d \text{ etc.}$$

are attributed, the other y hence also obtains the given values

$$p, q, r, s \text{ etc.},$$

or what reduces to the same, an equation for a curved line is in question, which passes trough arbitrarily many given points. Therefore, the greater the number of these points was, the more restricted the curved line is; nevertheless, I already observed on another occasion[1], even if the number of points is augmented to infinity, that always still infinitely many curved lines passing through all the same points in like manner can be found. Since the method of interpolation for each case yields a determined curved line, this solution is always to be considered as highly particular; but this circumstance itself implies a certain singular nature of the found solution, which deserves a more accurate consideration. But this nature of the solution especially depends on

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[1]Euler refers to his paper "De serierum determinatione seu nova methodus inveniendi terminos generales serierum". This is E189 in the Eneström-Index.

the method, how this interpolation is done, or on the form, which is attributed to the general equation, in which the equation in question must be contained. Since this form can be constituted in infinitely many ways, I will restrict my investigations to this form

$$y = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \epsilon x^9 + \text{etc.},$$

which certainly only contains odd powers of x so that y is a centrally symmetric function; hence innumerable other curved lines are excluded, which would pass through the same points.

PROBLEM 1

§1 *To find an equation between the two variables x and y of this form*

$$y = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \epsilon x^9 + \text{etc.},$$

that, if to x the given values

$$a, b, c, d \quad \text{etc.}$$

are attributed, the other variable y likewise obtains the values

$$p, q, r, s \quad \text{etc.}$$

SOLUTION

In order to apply the general equation to this case more easily, exhibit it in this form

$$\begin{aligned} y = Ax + Bx(xx-aa) &+ Cx(xx-aa)(xx-bb) \\ &+ Dx(xx-aa)(xx-bb)(xx-cc) \\ &+ Ex(xx-aa)(xx-bb)(xx-cc)(xx-dd) \\ &+ \text{etc.}, \end{aligned}$$

which, even though it might proceed to infinity, if the number of conditions is infinite, of course, nevertheless yields the following finite equations for the propounded conditions:

$$\begin{aligned}
&\text{I.} \quad p = Aa,\\
&\text{II.} \quad q = Ab + Bb(bb-aa),\\
&\text{III.} \; r = Ac + Bc(cc-aa) + Cc(cc-aa)(cc-bb),\\
&\text{IV.} \; s = Ad + Bd(dd-aa) + Cd(dd-aa)(dd-bb),\\
&\qquad\qquad\qquad\qquad + Dd(dd-aa)(dd-bb)(dd-cc),\\
&\qquad\qquad\qquad\qquad \text{etc.},
\end{aligned}$$

which we want to represent this way

$$\begin{aligned}
&\text{I.} \quad \frac{p}{a} = A,\\
&\text{II.} \quad \frac{q}{b} = A + B(bb-aa),\\
&\text{III.} \; \frac{r}{c} = A + B(cc-aa) + C(cc-aa)(cc-bb),\\
&\text{IV.} \; \frac{s}{d} = A + B(dd-aa) + C(dd-aa)(dd-bb)\\
&\qquad\qquad + D(dd-aa)(dd-bb)(dd-cc)\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

Now subtract the first from the following ones and divide the differences by the coefficients of B so that these equations result:

$$\begin{aligned}
&\frac{aq-bp}{ab(bb-aa)} = q' = B,\\
&\frac{ar-cp}{ac(cc-aa)} = r' = B + C(cc-bb),\\
&\frac{as-dp}{ad(dd-aa)} = s' = B + C(dd-bb) + D(dd-bb)(dd-cc)\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

Now, in like manner subtracting the first from the following ones and dividing the them by the coefficients of C we will get to these equations:

$$\frac{r'-q'}{cc-bb}=r''=C,$$
$$\frac{s'-q'}{dd-bb}=d''=C+D(dd-cc)$$
$$\text{etc.}$$

and further to this one

$$\frac{s''-r''}{dd-cc}=D.$$

Therefore, the coefficients A, B, C, D etc. will be determined most conveniently this way from the given quantities a, b, c, d etc. and p, q, r, s etc.: First, from the given quantities derive these

$$P=\frac{p}{a},\quad Q=\frac{q}{b},\quad R=\frac{r}{c},\quad S=\frac{s}{d}\quad \text{etc.}$$

and hence form these:

$$\begin{aligned}
Q' &= \frac{Q-P}{bb-aa}, & R' &= \frac{R-P}{cc-aa}, & S' &= \frac{S-P}{dd-aa}, & T' &= \frac{T-P}{ee-aa} \quad \text{etc.},\\
R'' &= \frac{R'-Q'}{cc-bb}, & S'' &= \frac{S'-Q'}{dd-bb}, & T'' &= \frac{T'-Q'}{ee-bb}, \quad \text{etc.,}\\
S''' &= \frac{S''-R''}{dd-cc}, & T''' &= \frac{T''-R''}{ee-cc}, \quad \text{etc.,}\\
T'''' &= \frac{T'''-S'''}{ee-dd}, \quad \text{etc.,}
\end{aligned}$$

Having found these values we will have

$$A=P,\quad B=Q',\quad C=R'',\quad D=S''',\quad E=T''''\quad \text{etc.}$$

COROLLARY 1

§2 Since it is $P=\frac{p}{a}$, the first coefficient will be

$$A=\frac{p}{a};$$

for the following on the other hand because of

$$Q' = \frac{aq - bp}{ab(bb - aa)}, \quad R' = \frac{ar - cp}{ac(cc - aa)}, \quad S' = \frac{as - dp}{ad(dd - aa)}, \quad T' = \frac{at - ep}{ae(ee - aa)} \quad \text{etc.}$$

the second coefficient will be

$$B = \frac{aq - bp}{ab(bb - aa)}$$

or

$$B = \frac{p}{a(bb - aa)} + \frac{q}{b(bb - aa)}.$$

COROLLARY 2

§3 Further, because it is

$$R'' = \frac{ar - cp}{ac(cc - aa)(cc - bb)} - \frac{aq - bp}{ab(bb - aa)(cc - bb)},$$

it will be

$$C = \frac{p}{a(aa - bb)(aa - cc)} + \frac{q}{b(bb - aa)(bb - cc)} + \frac{r}{c(cc - aa)(cc - bb)}$$

COROLLARY 3

§4 In like manner by continuing the calculation it will be found

$$\begin{aligned} D = {} & \frac{p}{a(aa - bb)(aa - cc)(aa - dd)} + \frac{q}{b(bb - aa)(bb - cc)(bb - dd)} \\ & + \frac{r}{c(cc - aa)(cc - bb)(cc - dd)} + \frac{s}{d(dd - aa)(dd - bb)(dd - cc)}, \end{aligned}$$

whence it is already possible to conjecture the form of the following quantities E, F, G etc.

SCHOLIUM 1

§5 But in most cases the values of the single coefficients A, B, C, D, E etc. are defined by the preceding ones. For, the following formulas are deduced from the fundamental equations:

$$\begin{aligned}
A &= \frac{p}{a},\\
B &= \frac{q-bA}{b(bb-aa)},\\
C &= \frac{r-cA}{c(c-aa)(cc-bb)} - \frac{B}{cc-bb},\\
D &= \frac{s-dA}{d(dd-aa)(dd-bb)(dd-cc)} - \frac{B}{(dd-bb)(dd-cc)} - \frac{C}{dd-cc},\\
E &= \frac{t-eA}{e(ee-aa)(ee-bb)(ee-cc)(ee-dd)} - \frac{B}{(ee-bb)(ee-cc)(ee-dd)}\\
&\quad - \frac{C}{(ee-cc)(ee-dd)} - \frac{D}{ee-dd}\\
&\text{etc.,}
\end{aligned}$$

where in most cases immediately a structure of such a kind is observed, whence the following ones can easily be derived, as it will be seen in the following problems, in which I will apply this method to certain particular cases.

SCHOLIUM 2

§6 But before I expand cases of this kind, it will be helpful to have observed in general that, if for a certain case a satisfying equation between the two variables x and y was found, which I will denote this way

$$y = X,$$

so that it is

$$X = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \text{etc.},$$

that then hence easily an equation extending much further and equally fulfilling the prescribed conditions can be formed. For, set

$$Q = x \cdot \frac{xx-aa}{aa} \cdot \frac{xx-bb}{bb} \cdot \frac{xx-cc}{cc} \cdot \frac{xx-dd}{dd} \cdot \text{etc.},$$

which quantity vanishes for all propounded values of x

$$x = 0, \quad x = \pm a, \quad x = \pm b, \quad x = \pm c \quad \text{etc.,}$$

and all functions of Q vanishing alongside with Q itself will do the same; hence it is manifest, if one sets

$$y = X + Q$$

or

$$y = X + f : Q,$$

that all conditions are equally fulfilled. Therefore, since this function $f : Q$ is completely arbitrary, as long as vanishes for $Q = 0$, this equation

$$y = X + f : Q$$

is to be considered to exhibit the most general solution.

PROBLEM 2

§7 *Let a, b, c, d etc. be any circular arcs while the radius of the circle is $= 1$, but let the values p, q, r, s etc. be the sines of the same arcs, since in this case this property holds that to negative arcs the same sines taken negatively correspond, hence to define the ratio of the diameter to the circumference approximately.*

SOLUTION

Since here it is

$$p = \sin a, \quad q = \sin b, \quad r = \sin c \quad \text{etc.},$$

the equation between x and y will be of such a nature that having taken x for the circular arc the quantity y will approximately be expressed by its sine and it is

$$y = \sin x.$$

Therefore, having defined the coefficients

$$A, \quad B, \quad C, \quad D \quad \text{etc.}$$

by means of the preceding problem one will have this equation

$$\sin x = Ax + Bx(xx - aa) + Cx(xx - aa)(xx - bb) + \text{etc.},$$

which is therefore true, as often as it was

$$\text{either} \quad x=0 \quad \text{or} \quad x=\pm a \quad \text{or} \quad x=\pm b \quad \text{or} \quad x=\pm c \quad \text{etc.}$$

Now let us assume the arc x to be infinitely small, and since then its sine, $\sin x$, becomes equal to the arc x, this equation will result

$$1 = A - Baa + Caabb - Daabbcc + Eaabbccdd - \text{etc.}$$

Let us substitute the values found above for the letters A, B, F, D etc. here and we will get to this equation

$$\begin{aligned}
1 = {} & \frac{p}{a}\left(1 - \frac{aa}{aa-bb} + \frac{aabb}{(aa-bb)(aa-cc)} - \frac{aabbcc}{(aa-bb)(aa-cc)(aa-dd)} + \text{etc.}\right) \\
& - \frac{q}{b}\left(\frac{aa}{bb-aa} - \frac{aabb}{(bb-aa)(bb-cc)} + \frac{aabbcc}{(bb-aa)(bb-cc)(bb-dd)} - \text{etc.}\right) \\
& + \frac{r}{c}\left(\frac{aabb}{(cc-aa)(cc-bb)} - \frac{aabbcc}{(cc-aa)(cc-bb)(cc-dd)} + \text{etc.}\right) \\
& - \frac{s}{d}\left(\frac{aabbcc}{(dd-aa)(dd-bb)(dd-dd)} - \text{etc.}\right) \\
& + \text{etc.},
\end{aligned}$$

which can be reduced to this one, in which all series are similar to each other

$$\begin{aligned}
1 = {} & \frac{p}{a}\left(1 - \frac{aa}{aa-bb} + \frac{aabb}{(aa-bb)(aa-cc)} - \frac{aabbcc}{(aa-bb)(aa-cc)(aa-dd)} + \text{etc.}\right) \\
& - \frac{aaq}{b(bb-aa)}\left(1 + \frac{bb}{cc-bb} + \frac{bbcc}{(cc-bb)(dd-bb)} + \frac{bbccdd}{(cc-bb)(dd-bb)(ee-bb)} + \text{etc.}\right) \\
& + \frac{aabbr}{c(cc-bb)(cc-bb)}\left(1 + \frac{cc}{dd-cc} + \frac{ccdd}{(dd-cc)(ee-cc)} + \text{etc.}\right) \\
& - \frac{aabbccs}{d(dd-aa)(dd-bb)(dd-cc)}\left(1 + \frac{dd}{ee-dd} + \text{etc.}\right) \\
& + \text{etc.}
\end{aligned}$$

But every single one of these series is immediately summable; for, combining terms of the first series we will find

$$\frac{bb}{bb-aa};$$

but if the third is added to it, it will be

$$\frac{bbcc}{(bb-aa)(cc-aa)}$$

and hence further adding the fourth term yields

$$\frac{bbccdd}{(bb-aa)(cc-aa)(dd-aa)}$$

and so forth, so that the first series of our equation becomes

$$\frac{p}{a}\cdot\frac{bb}{bb-aa}\cdot\frac{cc}{cc-aa}\cdot\frac{dd}{dd-aa}\cdot\frac{ee}{ee-aa}\cdot\text{etc.}$$

But in like manner this product is found for the second

$$-\frac{q}{b}\cdot\frac{aa}{bb-aa}\cdot\frac{cc}{cc-bb}\cdot\frac{dd}{dd-bb}\cdot\frac{ee}{ee-bb}\cdot\text{etc.}$$

and so our equation is finally reduced to this form

$$\begin{aligned}
1 = {} & \frac{p}{q}\cdot\frac{bb}{bb-aa}\cdot\frac{cc}{cc-aa}\cdot\frac{dd}{dd-aa}\cdot\frac{ee}{ee-aa}\cdot\text{etc.}\\
& +\frac{q}{b}\cdot\frac{aa}{bb-aa}\cdot\frac{cc}{cc-bb}\cdot\frac{dd}{dd-bb}\cdot\frac{ee}{ee-bb}\cdot\text{etc.}\\
& +\frac{r}{c}\cdot\frac{aa}{aa-cc}\cdot\frac{bb}{bb-cc}\cdot\frac{dd}{dd-cc}\cdot\frac{ee}{ee-cc}\cdot\text{etc.}\\
& +\frac{s}{d}\cdot\frac{aa}{aa-dd}\cdot\frac{bb}{bb-dd}\cdot\frac{cc}{cc-dd}\cdot\frac{ee}{ee-dd}\cdot\text{etc.}\\
& +\frac{t}{e}\cdot\frac{aa}{aa-ee}\cdot\frac{bb}{bb-ee}\cdot\frac{cc}{cc-ee}\cdot\frac{dd}{dd-ee}\cdot\text{etc.}\\
& +\text{etc.},
\end{aligned}$$

whence, if the given arcs a, b, c, d etc. have a known ratio to half of the circumference π, the value of this quantity π will be defined

COROLLARY 1

§8 If the number of these arcs a, b, c, d etc. was finite, then the circumference of the circle will be defined the more accurately, the larger that number is and at the same time the smaller arcs occur among them. But having augmented the amount of propounded arcs to infinity the true ratio of the circumference to the diameter can be derived this way.

§9 In like manner the sine of the indefinite arc x can be defined in general. For, having substituted the found values for the coefficients A, B, C, D etc. the equation will be reduced to this form

$$\begin{aligned}\frac{\sin x}{x} = & \frac{p}{a}\cdot\frac{bb-xx}{bb-aa}\cdot\frac{cc-xx}{cc-aa}\cdot\frac{dd-xx}{dd-aa}\cdot\text{etc.}\\ &+\frac{q}{b}\cdot\frac{aa-xx}{aa-bb}\cdot\frac{cc-xx}{cc-bb}\cdot\frac{dd-xx}{dd-bb}\cdot\text{etc.}\\ &+\frac{r}{c}\cdot\frac{aa-xx}{aa-cc}\cdot\frac{bb-xx}{bb-cc}\cdot\frac{dd-xx}{dd-cc}\cdot\text{etc.}\\ &+\frac{s}{d}\cdot\frac{aa-xx}{aa-dd}\cdot\frac{bb-xx}{bb-dd}\cdot\frac{cc-xx}{cc-dd}\cdot\text{etc.}\\ &+\text{etc.},\end{aligned}$$

which equation having taken a vanishing arc for x goes over into the one we found before.

Corollary 3

§10 But this reduction extends a lot further, since it is not restricted to arcs. For, if an equation of such a kind between the two variables x and y is in question, that having taken

$$x = 0,\quad a,\quad b,\quad c,\quad d,\quad e\ \text{ etc.}$$

it is

$$x = 0,\quad p,\quad q,\quad r,\quad s,\quad t\ \text{ etc.},$$

this equation can be represented in general this way

$$\begin{aligned}\frac{y}{x} = & \frac{p}{a}\cdot\frac{bb-xx}{bb-aa}\cdot\frac{cc-xx}{cc-aa}\cdot\frac{dd-xx}{dd-aa}\cdot\frac{ee-xx}{ee-aa}\cdot\text{etc.}\\ &+\frac{q}{b}\cdot\frac{aa-xx}{aa-bb}\cdot\frac{cc-xx}{cc-bb}\cdot\frac{dd-xx}{dd-bb}\cdot\frac{ee-xx}{ee-bb}\cdot\text{etc.}\\ &+\frac{r}{c}\cdot\frac{aa-xx}{aa-cc}\cdot\frac{bb-xx}{bb-cc}\cdot\frac{dd-xx}{dd-cc}\cdot\frac{ee-xx}{ee-cc}\cdot\text{etc.}\\ &+\frac{s}{d}\cdot\frac{aa-xx}{aa-dd}\cdot\frac{bb-xx}{bb-dd}\cdot\frac{cc-xx}{cc-dd}\cdot\frac{ee-xx}{ee-dd}\cdot\text{etc.}\\ &+\text{etc.};\end{aligned}$$

from this form it is manifest at the same time, how the single conditions are fulfilled.

SCHOLIUM

§11 I do not spend more time on the cases, in which the number of of prescribed conditions a, b, c, d etc. is finite, since hence only approximations for the measure of the circle are obtained. Nevertheless, it will not be out of place to have observed, if only four arcs are taken, which we want to call

$$a = \varphi, \quad b = 2\varphi, \quad c = 3\varphi, \quad d = 4\varphi,$$

that from the solution of the problem it will be

$$\begin{aligned}\varphi = &\frac{\sin\varphi}{1} \cdot \frac{2\cdot 2}{1\cdot 3} \cdot \frac{3\cdot 3}{2\cdot 4} \cdot \frac{4\cdot 4}{3\cdot 5}\\ &- \frac{\sin 2\varphi}{2} \cdot \frac{1\cdot 1}{1\cdot 3} \cdot \frac{3\cdot 3}{1\cdot 5} \cdot \frac{4\cdot 4}{2\cdot 6}\\ &+ \frac{\sin 3\varphi}{3} \cdot \frac{1\cdot 1}{2\cdot 4} \cdot \frac{2\cdot 2}{1\cdot 5} \cdot \frac{4\cdot 4}{1\cdot 7}\\ &- \frac{\sin 4\varphi}{4} \cdot \frac{1\cdot 1}{3\cdot 5} \cdot \frac{2\cdot 2}{2\cdot 6} \cdot \frac{3\cdot 3}{1\cdot 7}\end{aligned}$$

$$= \frac{8}{5}\sin\varphi - \frac{2}{5}\sin 2\varphi + \frac{8}{105}\sin 3\varphi - \frac{1}{140}\sin 4\varphi,$$

which expression comes the closer to the truth the smaller the angle φ is taken; nevertheless, even if one takes

$$\varphi = \frac{\pi}{2},$$

the error does not become enormous; for, it results

$$\frac{\pi}{2} = \frac{8}{5} - \frac{8}{105} = \frac{32}{21}$$

and so

$$\pi = 3\frac{1}{21}.$$

But if we take

$$\varphi = 30^\circ = \frac{\pi}{6},$$

it is

$$\frac{\pi}{6} = \frac{8}{5}\cdot\frac{1}{2} - \frac{2}{5}\cdot\frac{\sqrt{3}}{2} + \frac{8}{105} - \frac{1}{140}\cdot\frac{\sqrt{3}}{2}$$

or

$$\pi = \frac{184}{35} - \frac{171\sqrt{3}}{140},$$

which value differs from the true one by the hundred-thousandth part of 1. But having put aside this consideration I want to go through some cases, where the number of propounded arcs a, b, c, d etc. proceeding in a certain way is infinite.

EXAMPLE I

§12 *Let the arcs a, b, c, d etc. proceed according to the series of natural numbers and let*

$$a=\varphi, \quad b=2\varphi, \quad c=3\varphi, \quad a=4\varphi, \quad \textit{etc.} \quad \textit{to infinity};$$

to determine the true length of the arc φ from their sines p, q, r etc.

Therefore, the solution of the problem for this case yields this equation

$$\begin{aligned}\varphi = &\frac{\sin\varphi}{1}\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\text{etc.}\\ &-\frac{\sin 2\varphi}{2}\cdot\frac{1\cdot 1}{1\cdot 3}\cdot\frac{3\cdot 3}{1\cdot 5}\cdot\frac{4\cdot 4}{2\cdot 6}\cdot\frac{5\cdot 5}{3\cdot 7}\cdot\text{etc.}\\ &+\frac{\sin 3\varphi}{3}\cdot\frac{1\cdot 1}{2\cdot 4}\cdot\frac{2\cdot 2}{1\cdot 5}\cdot\frac{4\cdot 4}{1\cdot 7}\cdot\frac{5\cdot 5}{2\cdot 8}\cdot\text{etc.}\\ &-\frac{\sin 4\varphi}{4}\cdot\frac{1\cdot 1}{3\cdot 5}\cdot\frac{2\cdot 2}{2\cdot 6}\cdot\frac{3\cdot 3}{1\cdot 7}\cdot\frac{5\cdot 5}{1\cdot 9}\cdot\text{etc.}\\ &+\frac{\sin 5\varphi}{5}\cdot\frac{1\cdot 1}{4\cdot 6}\cdot\frac{2\cdot 2}{3\cdot 7}\cdot\frac{3\cdot 3}{2\cdot 8}\cdot\frac{4\cdot 4}{1\cdot 9}\cdot\text{etc.}\\ &+\text{etc.};\end{aligned}$$

but all these products are found to have the same value $=2$, so that it is

$$\frac{1}{2}\varphi=\sin\varphi-\frac{1}{2}\sin 2\varphi+\frac{1}{3}\sin 3\varphi-\frac{1}{4}\sin 4\varphi+\frac{1}{5}\sin 5\varphi-\text{etc.};$$

the correctness of these series in the case, in which the angle φ is infinitely small, is manifest per se. Therefore, let us expand the following cases:

1. Let

$$\varphi=90^{\circ}=\frac{\pi}{2}$$

and the Leibniz series results

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}$$

2. Let

$$\varphi = 45^\circ = \frac{\pi}{4}$$

and this series will result

$$\frac{\pi}{8} = \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{3\sqrt{2}} * - \frac{1}{5\sqrt{2}} + \frac{1}{6} - \frac{1}{7\sqrt{2}} * + \frac{1}{9\sqrt{2}} - \frac{1}{10} + \frac{1}{11\sqrt{2}} - \text{etc.},$$

which is resolved into these two

$$\frac{\pi}{8} = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.}\right)$$

$$-\frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}\right),$$

such that it is

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.} = \frac{\pi}{2\sqrt{2}}.$$

3. Let

$$\varphi = 60^\circ = \frac{\pi}{3}$$

and it will be

$$\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} * + \frac{1}{4} \cdot \frac{\sqrt{3}}{2} - \frac{1}{5} \cdot \frac{\sqrt{3}}{2} + \text{etc.}$$

or

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \text{etc.}$$

4. Let

$$\varphi = 30^\circ = \frac{\pi}{6}$$

and it will be

$$\frac{\pi}{12} = \frac{1}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} - \frac{1}{4} \cdot \frac{\sqrt{3}}{2} + \frac{1}{5} \cdot \frac{1}{2} * - \frac{1}{7} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{\sqrt{3}}{2} - \text{etc.}$$

or

$$\frac{\pi}{12} = \frac{1}{2}\left(1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{19} - \frac{1}{23} + \text{etc.}\right)$$
$$-\frac{\sqrt{3}}{4}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \text{etc.}\right)$$
$$+\frac{1}{3}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}\right);$$

the last of these sums becomes $= \frac{\pi}{12}$; hence it is concluded

$$1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \text{etc.} = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \text{etc.}\right).$$

But both series become equal to the arc $\frac{\pi}{3}$, which is certainly already manifest in the first using the Leibniz series.

COROLLARY 1

§13 From the equation found here

$$\frac{1}{2}\varphi = \sin\varphi - \frac{1}{2}\sin 2\varphi + \frac{1}{3}\sin 3\varphi - \frac{1}{4}\sin 4\varphi + \text{etc.}$$

many other not less remarkable ones can be derived. For, after a differentiation this equation results

$$\frac{1}{2} = \cos\varphi - \cos 2\varphi + \cos 3\varphi - \cos\varphi + \text{etc.};$$

the reason this is manifest from the observation that by multiplying both sides by $2\cos\frac{1}{2}\varphi$ the identical equation $\cos\frac{1}{2}\varphi = \cos\frac{1}{2}\varphi$ results.

COROLLARY 2

§14 But if we multiply this equation by $-d\varphi$ and integrate it afterwards, it results

$$C - \frac{1}{4}\varphi\varphi = \cos\varphi - \frac{1}{4}\cos 2\varphi + \frac{1}{9}\cos 3\varphi - \frac{1}{16}\cos 4\varphi + \text{etc.},$$

where the constant entering by integration is determined from the case $\varphi = 0$, it is found to be

$$\mathrm{C} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \text{etc.} = \frac{\pi\pi}{12},$$

so that it is

$$\frac{\pi\pi}{12}-\frac{\varphi\varphi}{4}=\cos\varphi-\frac{1}{4}\cos 2\varphi+\frac{1}{9}\cos 3\varphi-\frac{1}{16}\cos 4\varphi+\text{etc.},$$

which series therefore, having taken $\varphi=\frac{\pi}{\sqrt{3}}$, becomes $=0$. But it approximately is

$$\frac{\pi}{\sqrt{3}}=103^{\circ}55'23'' \quad \text{and} \quad \cos\frac{\pi}{\sqrt{3}}=-0,2406185.$$

Corollary 3

§15 If we multiply by $d\varphi$ and integrate again, it will result

$$\frac{1}{12}\pi\pi\varphi-\frac{1}{12}\varphi^3=\sin\varphi-\frac{1}{8}\sin 2\varphi+\frac{1}{27}\sin 3\varphi-\frac{1}{64}\sin 4\varphi+\text{etc.},$$

whence having taken the arc

$$\varphi=90^{\circ}=\frac{\pi}{2}$$

this sum is obtained

$$\frac{1}{32}\pi^3=1-\frac{1}{27}+\frac{1}{125}-\frac{1}{343}+\text{etc.},$$

as it is already known from other sources[2].

Scholium

§16 It could cause some doubts that the sum of the series

$$\frac{1}{2}\varphi=\sin\varphi-\frac{1}{2}\sin 2\varphi+\frac{1}{3}\sin 3\varphi-\frac{1}{4}\sin 4\varphi+\text{etc.}$$

for $\varphi=180^{\circ}=\pi$ can not become equal to $\frac{1}{2}\pi$, since all the terms vanish. But to clear these doubts first set $\varphi=\pi-\omega$ and this equation will result

$$\frac{\pi-\omega}{2}=\sin\omega+\frac{1}{2}\sin 2\omega+\frac{1}{3}\sin 3\omega+\frac{1}{4}\sin 4\omega+\text{etc.}$$

but now assume the arc ω to be infinitely small, whence this equation is obtained

$$\frac{\pi-\omega}{2}=\omega+\omega+\omega+\omega+\omega+\text{etc.},$$

which does not any longer contain anything paradoxical. The same is true if we want to take $\varphi=2\pi$ or $\varphi=3\pi$ etc.

[2]Euler finds this series, e.g., in his paper "De summis serierum reciprocarum". This is E41 in the Eneström-Index.

EXAMPLE II

§17 *If the arcs a, b, c, d constitute an arbitrary arithmetic progression that it is*

$$a = n\varphi, \quad b = (n+1)\varphi, \quad c = (n+2)\varphi, \quad d = (n+3)\varphi \quad \text{etc.}$$

to define the longitude of the arc φ from their sines.

The general solution exhibited before for this case gives

$$\begin{aligned}
\varphi = & \frac{\sin n\varphi}{n} \cdot \frac{(n+1)^2}{1(1+2n)} \cdot \frac{(n+2)^2}{2(2+2n)} \cdot \frac{(n+3)^2}{3(3+2n)} \cdot \frac{(n+4)^2}{4(4+2n)} \cdot \frac{(n+5)^2}{5(5+2n)} \cdot \text{etc.} \\
& - \frac{\sin(n+1)\varphi}{n+1} \cdot \frac{n^2}{1(1+2n)} \cdot \frac{(n+2)^2}{1(3+2n)} \cdot \frac{(n+3)^2}{2(4+2n)} \cdot \frac{(n+4)^2}{3(5+2n)} \cdot \frac{(n+5)^2}{4(6+2n)} \cdot \text{etc.} \\
& + \frac{\sin(n+2)\varphi}{n+2} \cdot \frac{n^2}{2(2+2n)} \cdot \frac{(n+1)^2}{1(3+2n)} \cdot \frac{(n+3)^2}{1(5+2n)} \cdot \frac{(n+4)^2}{2(6+2n)} \cdot \frac{(n+5)^2}{3(7+2n)} \cdot \text{etc.} \\
& - \frac{\sin(n+3)\varphi}{n+3} \cdot \frac{n^2}{3(3+2n)} \cdot \frac{(n+1)^2}{2(4+2n)} \cdot \frac{(n+2)^2}{1(5+2n)} \cdot \frac{(n+4)^2}{1(7+2n)} \cdot \frac{(n+5)^2}{2(8+2n)} \cdot \text{etc.} \\
& + \frac{\sin(n+4)\varphi}{n+4} \cdot \frac{n^2}{4(4+2n)} \cdot \frac{(n+1)^2}{3(5+2n)} \cdot \frac{(n+2)^2}{2(6+2n)} \cdot \frac{(n+3)^2}{1(7+2n)} \cdot \frac{(n+5)^2}{1(9+2n)} \cdot \text{etc.} \\
& + \text{etc.}
\end{aligned}$$

But to investigate the values of these infinite products, for the sake of brevity, let us put

$$\varphi = \mathfrak{A}\frac{\sin n\varphi}{n} - \mathfrak{B}\frac{\sin(n+1)\varphi}{n+1} + \mathfrak{C}\frac{\sin(n+2)\varphi}{n+2} - \mathfrak{D}\frac{\sin(n+3)\varphi}{n+3} + \text{etc.}$$

and compare these coefficients to each other the following way

$$\frac{\mathfrak{A}}{\mathfrak{B}} = \frac{nn}{(n+1)^2} \cdot \frac{2(2+2n)}{1(3+2n)} \cdot \frac{3(3+2n)}{2(4+2n)} \cdot \frac{4(4+2n)}{3(5+2n)} \cdot \text{etc.}$$

which value is reduced to

$$\frac{nn}{(n+1)^2} \cdot \frac{(i-1)(2+2n)}{1(i+2n)},$$

while i denotes an infinite number and so it will be

$$\frac{\mathfrak{A}}{\mathfrak{B}} = \frac{2nn}{n+1}.$$

In like manner it is concluded

$$\frac{\mathfrak{C}}{\mathfrak{B}} = \frac{1(1+2n)}{2(2+2n)} \cdot \frac{(n+1)^2}{(n+2)^2} \cdot \frac{(i-3)((4+2n)}{1(i+2n)} = \frac{(n+1)(2n+1)}{2(n+2)},$$

but then further

$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(n+2)(2n+2)}{3(n+3)}, \quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(n+3)(2n+3)}{4(n+4)}$$

and so forth; hence it follows that it will be

$$\begin{aligned}
\mathfrak{B} &= \frac{2nn}{1(n+1)}\mathfrak{A}, \\
\mathfrak{C} &= \frac{2nn(2n+1)}{1\cdot 2(n+1)}\mathfrak{A}, \\
\mathfrak{D} &= \frac{2nn(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)}\mathfrak{A}, \\
\mathfrak{E} &= \frac{2nn(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4(n+4)}\mathfrak{A}
\end{aligned}$$

etc.

and so the whole task reduces to the invention of the first letter

$$\mathfrak{A} = \frac{(n+1)^2}{1(2n+1)} \cdot \frac{(n+2)^2}{2(2n+2)} \cdot \frac{(n+3)^2}{3(2n+3)} \cdot \frac{(n+4)^2}{4(2n+4)} \cdot \text{etc.}$$

But I already proved a long time ago[3] that the value of this general product

$$\frac{a(b+c)}{b(a+c)} \cdot \frac{(a+d)(b+c+d)}{(b+d)(a+c+d)} \cdot \frac{(a+2d)(b+c+2d)}{(b+2d)(a+c+2d)} \cdot \text{etc.}$$

is expressed in such a way that it is

$$= \frac{\int x^{b-1}\mathrm{d}x(1-x^d)^{\frac{c-d}{d}}}{\int x^{a-1}\mathrm{d}x(1-x^d)^{\frac{c-d}{d}}},$$

having extended the integration from the lower limit $x=0$ to the upper limit $x=1$, of course. Since using this theorem for our case one has to take

$$a = n+1, \quad b+c = n+1, \quad b = 1, \quad c = n \quad \text{and} \quad d = 1,$$

[3]Euler did so in his paper "De productis ex infinitis factoribus ortis". This is E122 in the Eneström-Index.

we will have

$$\mathfrak{A} = \frac{\int \mathrm{d}x(1-x)^{n-1}}{\int x^n \mathrm{d}x(1-x)^{n-1}} = \frac{1}{n\int x^n \mathrm{d}x(1-x)^{n-1}}$$

and hence the following expression for the arc φ

$$\varphi \int x^n \mathrm{d}x(1-x)^{n-1} = \frac{1}{nn}\sin n\varphi - \frac{2n}{1(n+1)^2}\sin(n+1)\varphi$$

$$+\frac{2n(2n+1)}{1\cdot 2(n+2)^2}\sin(n+2)\varphi - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2}\sin(n+3)\varphi$$

$$+\frac{2n(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4(n+4)^2}\sin(n+4)\varphi + \text{etc.}$$

This series deserves even more attention, since it involves the integral formula $\int x^n \mathrm{d}x(1-x)^{n-1}$.

COROLLARY 1

§18 It will be helpful to have noted at first about this integral formula

$$\int x^n \mathrm{d}x(1-x)^{n-1},$$

if it was $\triangle$ in the case $n=\lambda$, that it then in the case

$$n = \lambda + 1$$

will be

$$= \frac{\lambda}{2(2\lambda+1)}\triangle.$$

So, since in the case $n=1$ it is

$$\int x\mathrm{d}x = \frac{1}{2},$$

it will be

$$\int x^2 \mathrm{d}x(1-x) = \frac{1}{2}\cdot\frac{1}{2\cdot 3}, \quad \int x^3 \mathrm{d}x(1-x)^2 = \frac{1}{2}\cdot\frac{1}{2\cdot 3}\cdot\frac{2}{2\cdot 5} \quad \text{etc.}$$

Corollary 2

§19 Therefore, if in general it is put

$$\int x^n \mathrm{d}x(1-x)^{n-1} = f : n,$$

since its value can be considered as a function of n, it will be

$$f:1 = \frac{1}{2}, \quad f:2 = \frac{1}{2}\cdot\frac{1}{6}, \quad f:3 = \frac{1}{2}\cdot\frac{1}{6}\cdot\frac{2}{10}, \quad f:4 = \frac{1}{2}\cdot\frac{1}{6}\cdot\frac{2}{10}\cdot\frac{3}{14}$$

and in general

$$f:(n+1) = \frac{n}{2(2n+1)} f:n.$$

Hence, as often as n is an integer number, the value of this formula $f:n$ is easily assigned.

Corollary 3

§20 Now let $n = \frac{1}{2}$ and it will be

$$f:\frac{1}{2} = \int \frac{\mathrm{d}x\sqrt{x}}{\sqrt{1-x}} = 2\int \frac{yy\mathrm{d}y}{\sqrt{1-yy}},$$

having put $x = yy$; but

$$\int \frac{yy\mathrm{d}y}{\sqrt{1-yy}} = \frac{1}{2}\int \frac{\mathrm{d}y}{\sqrt{1-yy}} = \frac{\pi}{4},$$

whence it is

$$f:\frac{1}{2} = \frac{\pi}{2}$$

and hence further

$$f:\frac{3}{2} = \frac{1}{8}\cdot\frac{\pi}{2}, \quad f:\frac{5}{2} = \frac{1}{8}\cdot\frac{3}{16}\cdot\frac{\pi}{2}, \quad f:\frac{7}{2} = \frac{1}{8}\cdot\frac{3}{16}\cdot\frac{5}{24}\cdot\frac{\pi}{2} \quad \text{etc.}$$

But if in general it is $n = \frac{\mu}{\nu}$, it is found

$$f:\frac{\mu}{\nu} = \int x^{\frac{\mu}{\nu}}\mathrm{d}x(1-x)^{\frac{\mu}{\nu}-1} = \mu\int y^{\mu+\nu-1}\mathrm{d}y(1-y^\nu)^{\frac{\mu}{\nu}-1},$$

then having put $x = y^\nu$ and after some simplification one finds

$$f:\frac{\mu}{\nu} = \frac{\nu}{2}\int y^{\nu-1}\mathrm{d}y(1-y^\nu)^{\frac{\mu}{\nu}-1},$$

which form involves transcendental quantities of several species.

COROLLARY 4

§21 The value of the integral formula

$$\int x^n \mathrm{d}x (1-x)^{n-1}$$

in the case $x = 1$ is vice versa elegantly determined using the found series; for, after a differentiation with respect to the arc φ as a variable it results

$$\int x^n \mathrm{d}x (1-x)^{n-1} = \frac{1}{n}\cos n\varphi - \frac{2n}{1(n+1)}\cos(n+1)\varphi + \frac{2n(2n+1)}{1\cdot 2(n+2)}\cos(n+2)\varphi$$

$$-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+2)}\cos(n+3)\varphi + \text{etc.},$$

which series is therefore equal to this one resulting from the usual expansion itself

$$\int x^n \mathrm{d}x (1-x)^{n-1} = \frac{1}{n+1} - \frac{n-1}{1(n+2)} + \frac{(n-1)(n-2)}{1\cdot 2(n+3)} - \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3(n+4)} + \text{etc.}$$

.

SCHOLIUM 1

§22 Since we expanded the case $n = 1$ in the preceding example, let us here mainly consider the case

$$n = \frac{1}{2}$$

in which we saw that it is

$$\int x^n \mathrm{d}x (1-x)^{n-1} = \frac{\pi}{2},$$

and it will therefore be

$$\frac{\pi\varphi}{2} = \frac{4}{1}\sin\frac{1}{2}\varphi - \frac{4}{9}\sin\frac{3}{2}\varphi + \frac{4}{25}\sin\frac{5}{2}\varphi - \frac{4}{49}\sin\frac{7}{2}\varphi + \text{etc.}$$

Let us put $\varphi = 2\omega$ and this more convenient series will result

$$\frac{\pi\omega}{4} = \frac{1}{1}\sin\omega - \frac{1}{9}\sin 3\omega + \frac{1}{25}\sin 5\omega - \frac{1}{49}\sin 7\omega + \text{etc.},$$

which first, if a vanishing arc ω is assumed, gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.}$$

But let

$$\omega = \frac{\pi}{2}$$

and this also known series results

$$\frac{\pi\pi}{8} = \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.}$$

But having taken the arc

$$\omega = 45^\circ = \frac{\pi}{4}$$

it results

$$\frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{9} - \frac{1}{25} + \frac{1}{49} + \frac{1}{81} - \frac{1}{121} - \frac{1}{169} + \text{etc.}$$

Let

$$\omega = 30^\circ = \frac{\pi}{6};$$

it will be

$$\begin{aligned}\frac{\pi\pi}{24} = \frac{1}{2}\left(1 + \frac{1}{7^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{25^2} + \text{etc.}\right) \\ - 1\left(\frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \text{etc.}\right) \\ + \frac{1}{2}\left(\frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{17^2} + \frac{1}{23^2} + \text{etc.}\right),\end{aligned}$$

where the middle one is $= \frac{\pi\pi}{72}$ and the reason for the remaining ones is perspicuous. Further, a differentiation of our series yields this remarkable form

$$\frac{\pi}{4} = \frac{1}{1}\cos\omega - \frac{1}{3}\cos 3\omega + \frac{1}{5}\cos 5\omega - \frac{1}{7}\cos 7\omega + \text{etc.};$$

this is an extraordinary result, since completely all arcs assumed for ω yield the same sum. But then an iterated differentiation yields

$$0 = \sin\omega - \sin 3\omega + \sin 5\omega - \sin 7\omega + \text{etc.}$$

But by means of integration we find

$$C - \frac{\pi\omega^2}{8} = \frac{1}{1}\cos\omega - \frac{1}{3^3}\cos 3\omega + \frac{1}{5^3}\cos 5\omega - \frac{1}{7^3}\cos 7\omega + \text{etc.},$$

where, since having taken $\omega = 0$ it is

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.} = \frac{\pi^3}{32},$$

it will be

$$C = \frac{\pi^3}{32},$$

so that it is

$$\frac{\pi}{8}\left(\frac{\pi\pi}{4} - \omega\omega\right) = \frac{1}{1}\cos\omega - \frac{1}{3^3}\cos 3\omega + \frac{1}{5^3}\cos 5\omega - \frac{1}{7^3}\cos 7\omega + \text{etc.}$$

SCHOLIUM 2

§23 Now let us in general put

$$\varphi = \pi,$$

and since it is

$$\sin(n+1)\pi = -\sin n\pi, \quad \sin(n+2)\pi = +\sin n\pi \quad \text{etc.}$$

our equation divided by $\sin n\pi$ will obtain this form

$$\frac{\pi}{\sin n\pi}\int x^n dx(1-x)^{n-1} = \frac{1}{n^2} + \frac{2n}{1(n+1)^2} + \frac{2n(2n+1)}{1\cdot 2(n+2)^2}$$
$$+ \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2} + \text{etc.};$$

but having taken

$$\varphi = 2\pi$$

in like manner it will be

$$\frac{2\pi}{\sin 2n\pi}\int x^n dx(1-x)^{n-1} = \frac{1}{n^2} - \frac{2n}{1(n+1)^2} + \frac{2n(2n+1)}{1\cdot 2(n+2)^2}$$

$$-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2}+\text{etc.};$$

hence the first of these series divided by this one yields the quotient $= \cos n\pi$, which seems to be wrong, since the quotient is smaller than 1. But we already resolved a similar difficulty above, which resulted from putting $\varphi = 2\pi$; for, if we would have put $\varphi = 3\pi$, the first series would have this sum

$$=\frac{3\pi}{\sin 3\pi}\int x^n \mathrm{d}x(1-x)^{n-1},$$

which is only equal to the other one, if n is infinitely small. Hence only the first series is to be considered to hold; in order to investigate its sum, let us set

$$s=\frac{1}{n^2}t^n+\frac{2n}{(n+1)^2}t^{n+1}+\frac{2n(2n+1)}{1\cdot 2(n+2)^2}t^{n+2}+\text{etc.}$$

and it will hence be

$$\frac{\mathrm{d}.t\mathrm{d}s}{\mathrm{d}t^2}=1t^{n-1}+\frac{2n}{1}t^n+\frac{2n(2n+1)}{1\cdot 2}t^{n+1}+\text{etc.};$$

the sum of this series manifestly is

$$=t^{n-1}(1-t)^{-2n},$$

so that it is

$$\frac{t\mathrm{d}s}{\mathrm{d}t}=\int t^{n-1}\mathrm{d}t(1-t)^{-2n}$$

and

$$s=\int\frac{\mathrm{d}t}{t}\int\frac{t^{n-1}\mathrm{d}t}{(1-t)^{2n}},$$

and so having put $x = 1$ after the integration one will have

$$\frac{\pi}{\sin n\pi}\int x^n\mathrm{d}x(1-x)^{n-1}=\int\frac{\mathrm{d}t}{t}\int\frac{t^{n-1}\mathrm{d}t}{(1-t)^{2n}}.$$

The comparison of these two integral formulas is even more memorable, since among many others, which have been discovered, no one of this kind is found.

SCHOLIUM 3

§24 Let us put in general

$$\varphi = \frac{\pi}{2}$$

and it will be

$$\sin n\varphi = \sin\frac{n\pi}{2}, \quad \sin(n+1)\varphi = \cos\frac{n\pi}{2},$$
$$\sin(n+2)\varphi = -\sin\frac{n\pi}{2}, \quad \sin(n+3)\varphi = -\cos\frac{n\pi}{2} \text{ etc.},$$

whence this equation results

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1} = \sin\frac{n\pi}{2}\left(\frac{1}{nn} - \frac{2n(2n+1)}{1\cdot 2(n+2)^2} + \frac{2n(2n+1)(2n+2)(2n+4)}{1\cdot 2\cdot 3\cdot 4(n+4)^2} - \text{etc.}\right)$$
$$-\cos\frac{n\pi}{2}\left(\frac{2n}{1(n+1)^2} - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)^2} + \text{etc.}\right).$$

But from the reduction given above it is manifest that it will be

$$1 - \frac{2n(2n+1)}{1\cdot 2}t^2 + \frac{2n(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4}t^4 - \text{etc.}$$
$$= \frac{(1+t\sqrt{-1})^{-2n} + (1-t\sqrt{-1})^{-2n}}{2},$$
$$\frac{2n}{1}t - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}t^3 + \text{etc.}$$
$$= \frac{(1+t\sqrt{-1})^{-2n} - (1-t\sqrt{-1})^{-2n}}{2\sqrt{-1}}$$

and hence it is concluded

$$\frac{\pi}{2}\int x^n dx(1-x)^{n-1}$$

$$= \frac{1}{2}\sin\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1+t\sqrt{-1})^{2n}} + \frac{1}{2}\sin\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t\sqrt{-1})^{2n}}$$
$$-\frac{1}{2\sqrt{-1}}\cos\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1+t\sqrt{-1})^{2n}} + \frac{1}{2\sqrt{-1}}\cos\frac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t\sqrt{-1})^{2n}},$$

where after the integration one has to put $t = 1$. But in order to get rid of the imaginary quantities in this expression, let us put

$$t = \tan\omega = \frac{\sin\omega}{\cos\omega};$$

it will be

$$\mathrm{d}t = \frac{\mathrm{d}\omega}{\cos^2\omega}, \quad \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}\omega}{\sin\omega\cos\omega}, \quad t^{n-1}\mathrm{d}t = \frac{\mathrm{d}\omega\sin^{n-1}\omega}{\cos^{n+1}\omega},$$

but then

$$\begin{aligned}(1+t\sqrt{-1})^{-2n} &= \cos^{2n}\omega(\cos\omega + \sqrt{-1}\cdot\sin\omega)^{-2n}\\ &= \cos^{2n}\omega(\cos 2n\omega - \sqrt{-1}\cdot\sin 2n\omega),\\ (1-t\sqrt{-1})^{-2n} &= \cos^{2n}\omega(\cos\omega - \sqrt{-1}\cdot\sin\omega)^{-2n}\\ &= \cos^{2n}\omega(\cos 2n\omega + \sqrt{-1}\cdot\sin 2n\omega).\end{aligned}$$

Having substituted these values the imaginary quantities will cancel each other and this equation will result

$$\begin{aligned}\frac{\pi}{2}\int x^n\mathrm{d}x(1-x)^{n-1} = \sin\frac{n\pi}{2}\int\frac{\mathrm{d}\omega}{\sin\omega\cos\omega}\int\mathrm{d}\omega\sin^{n-1}\omega\cos^{n-1}\omega\cos 2n\omega\\ + \cos\frac{n\pi}{2}\int\frac{\mathrm{d}\omega}{\sin\omega\cos\omega}\int\mathrm{d}\omega\sin^{n-1}\omega\cos^{n-1}\omega\sin 2n\omega,\end{aligned}$$

which can be contracted into this simpler one

$$\frac{\pi}{2}\int x^n\mathrm{d}x(1-x)^{n-1} = \int\frac{\mathrm{d}\omega}{\sin\omega\cos\omega}\int\mathrm{d}\omega\sin^{n-1}\omega\cos^{n-1}\omega\sin\left(\frac{n\pi}{2}+2n\omega\right)$$

or because of $\sin\omega\cos\omega = \frac{1}{2}\sin 2\omega$ into this one

$$\frac{\pi}{2}\int x^n\mathrm{d}x(1-x)^{n-1} = \frac{1}{2^n}\int\frac{2\mathrm{d}\omega}{\sin 2\omega}\int 2\mathrm{d}\omega\sin^{n-1}2\omega\sin\left(\frac{n\pi}{2}+2n\omega\right).$$

Now let the angle be $2\omega = \theta$, so that it more conveniently is

$$\frac{\pi}{2}\int x^n\mathrm{d}x(1-x)^{n-1} = \frac{1}{2^n}\int\frac{2\mathrm{d}\theta}{\sin\theta}\int\mathrm{d}\theta\sin^{n-1}\theta\sin n\left(\frac{\pi}{2}+\theta\right),$$

where after the integration one must set $\theta = 90^\circ = \frac{\pi}{2}$, so that then it is $\omega = 45^\circ$ and $t = \tan\omega = 1$.

Example III

§25 *If the arcs a, b, c, d etc. constitute an interrupted arithmetic progression, so that it is*

$$a = m\varphi, \quad b = n\varphi, \quad c = (1+m]\varphi, \quad d = (1+n)\varphi,$$
$$e = (2+m)\varphi, \quad f = (2+n)\varphi, \quad etc.,$$

to define the longitude of the arc φ from their sines.

The general solution given above (§ 7) yields this equation

$$\begin{aligned}
\varphi = {} & \frac{\sin m\varphi}{m} \cdot \frac{nn}{(n-m)(n+m)} \cdot \frac{(1+m)^2}{1(1+2m)} \cdot \frac{(1+n)^2}{(1+n-m)(1+n+m)} \\
& \cdot \frac{(2+m)^2}{2(2+2m)} \cdot \frac{(2+n)^2}{(2+n-m)(2+n+m)} \cdot \text{etc.} \\
- {} & \frac{\sin n\varphi}{n} \cdot \frac{mm}{(n-m)(n+m)} \cdot \frac{(1+m)^2}{(1+m-n)(1+m+n)} \cdot \frac{(1+n)^2}{1(1+2n)} \\
& \cdot \frac{(2+m)^2}{(2+m-n)(2+m+n)} \cdot \frac{(2+n)^2}{2(2+2n)} \cdot \text{etc.} \\
+ {} & \frac{\sin(1+m)\varphi}{1+m} \cdot \frac{mm}{1(1+2m)} \cdot \frac{nn}{(1+m-n)(1+m+n)} \cdot \frac{(1+n)^2}{(n-m)(2+m+n)} \\
& \cdot \frac{(2+m)^2}{1(3+2m)} \cdot \frac{(2+n)^2}{(1+n-m)(3+n+m)} \cdot \text{etc.} \\
- {} & \frac{\sin(1+n)\varphi}{1+n} \cdot \frac{mm}{(1+n-m)(1+n+m)} \cdot \frac{nn}{1(1+2n)} \cdot \frac{(1+m)^2}{(n-m)(2+n+m)} \\
& \cdot \frac{(2+m)^2}{(1+m-n)(3+m+n)} \cdot \frac{(2+n)^2}{1(3+2n)} \cdot \text{etc.} \\
+ {} & \frac{\sin(2+m)\varphi}{2+m} \cdot \frac{mm}{2(2+2m)} \cdot \frac{nn}{(2+m-n)(2+m+n)} \cdot \frac{(1+m)^2}{1(3+2m)} \\
& \cdot \frac{(1+n)^2}{(1+m-n)(3+m+n)} \cdot \frac{(2+n)^2}{(n-m)(4+m+n)} \cdot \text{etc.} \\
& -\text{etc.}
\end{aligned}$$

But hence it is not possible to conclude anything worth one's attention in general; hence I will expand the especially remarkable case, in which it is

$$n = 1 - m;$$

for this, for the sake of brevity, I set

$$\varphi = \frac{\mathfrak{A}\sin m\varphi}{m} - \frac{\mathfrak{B}\sin(1-m)\varphi}{1-m} + \frac{\mathfrak{C}\sin(1+m)\varphi}{1+m} - \frac{\mathfrak{D}\sin(2-m)\varphi}{2-m} + \text{etc.},$$

so that it is

$$\begin{aligned}
\mathfrak{A} &= \frac{(1-m)^2}{1(1-2m)} \cdot \frac{(1+m)^2}{1(1+2m)} \cdot \frac{(2-m)^2}{2(2-m)} \cdot \frac{(2+m)^2}{2(2+m)} \cdot \frac{(3-m)^2}{3(2+m)} \cdot \text{etc.},\\
\frac{\mathfrak{B}}{\mathfrak{A}} &= \frac{mm}{(1-m)^2} \cdot \frac{1(1+2m)}{2\cdot 2m} \cdot \frac{2(2-2m)}{1(3-2m)} \cdot \frac{2(2+2m)}{3(1+2m)} \cdot \frac{3(3-2m)}{2(3-2m)} \cdot \text{etc.},\\
\frac{\mathfrak{C}}{\mathfrak{B}} &= \frac{1(1-2m)}{1(1+2m)} \cdot \frac{(1-m)^2}{(1+m)^2} \cdot \frac{1(3-2m)}{3(1-2m)} \cdot \frac{3(1+2m)}{1(3+2m)} \cdot \frac{2(4-2m)}{4(2-2m)} \cdot \text{etc.},\\
\frac{\mathfrak{D}}{\mathfrak{C}} &= \frac{1(1+2m)}{2(2-2m)} \cdot \frac{2\cdot 2m}{1(3-2m)} \cdot \frac{(1+m)^2}{(2-m)^2} \cdot \frac{1(3+2m)}{4\cdot 2m} \cdot \frac{4(2-2m)}{1(5-2m)} \cdot \text{etc.},\\
\frac{\mathfrak{E}}{\mathfrak{D}} &= \frac{2(2-2m)}{2(2+2m)} \cdot \frac{1(3-2m)}{3(1+2m)} \cdot \frac{3(1-2m)}{1(3+2m)} \cdot \frac{(2-m)^2}{(2+m)^2} \cdot \frac{1(5-2m)}{5(1-2m)} \cdot \text{etc.}
\end{aligned}$$

etc.

But from the reduction given above one finds

$$\mathfrak{A} = \frac{\int x^{m-1}\mathrm{d}x(1-x)^{-2m}}{m\int x^m\mathrm{d}x(1-x)^{m-1} \cdot \int x^{m-1}\mathrm{d}x(1-x)^{-m}};$$

but then for the remaining ones using the form of the products itself one concludes

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{m}{1-m}, \quad \frac{\mathfrak{C}}{\mathfrak{B}} = \frac{1-m}{1+m}, \quad \frac{\mathfrak{D}}{\mathfrak{C}} = \frac{1+m}{2-m}, \quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{2-m}{2+m} \quad \text{etc.},$$

so that it is

$$\mathfrak{B} = \frac{m}{1-m}\mathfrak{A}, \quad \mathfrak{C} = \frac{m}{1+m}\mathfrak{A}, \quad \mathfrak{D} = \frac{m}{2-m}\mathfrak{A}, \quad \mathfrak{E} = \frac{m}{2+m}\mathfrak{A}, \quad \text{etc.}$$

Therefore, for the sake of brevity let us put

$$\int x^m\mathrm{d}x(1-x)^{m-1} \cdot \frac{\int x^{m-1}\mathrm{d}x(1-x)^{-m}}{\int x^{m-1}\mathrm{d}x(1-x)^{-2m}} = M$$

and it will be as follows

$$M\varphi = \frac{\sin m\varphi}{m^2} - \frac{\sin(1-m)\varphi}{(1-m)^2} + \frac{\sin(1+m)\varphi}{(1+m)^2} - \frac{\sin(2-m)\varphi}{(2-m)^2} + \frac{\sin(2+m)\varphi}{(2+m)^2} - \text{etc.},$$

whence by differentiating we conclude that it will be

$$M = \frac{\cos m\varphi}{m} - \frac{\cos(1-m)\varphi}{1-m} + \frac{\cos(1+m)\varphi}{1+m} - \frac{\cos(2-m)\varphi}{2-m} + \frac{\cos(2+m)\varphi}{2+m} - \text{etc.},$$

which series because of its extraordinary simplicity is especially remarkable, since by putting $\varphi = 0$ we hence deduce

$$M = \frac{1}{m} - \frac{1}{1-m} + \frac{1}{1+m} - \frac{1}{2-m} + \frac{1}{2+m} - \frac{1}{3-m} + \frac{1}{3+m} - \text{etc.},$$

the sum of which series I already once showed[4] to be

$$M = \frac{\pi \cos m\pi}{\sin m\pi},$$

whence we deduce this elegant comparison

$$\int x^m dx(1-x)^{m-1} = \frac{\pi \cos m\pi}{\sin m\pi} \cdot \frac{\int x^{m-1}dx(1-x)^{-2m}}{\int x^{m-1}dx(1-x)^{-m}},$$

which is further reduced to this one

$$\int x^m dx(1-x)^{m-1} = \frac{(1-m)\pi \cos m\pi}{\sin m\pi} \cdot \frac{\int x^m dx(1-x)^{-2m}}{\int x^m dx(1-x)^{-m}}$$

or to this even more convenient one

$$\int x^{m-1} dx(1-x)^{m-1} = \frac{2\pi \cos m\pi}{\sin m\pi} \cdot \frac{\int x^{m-1}dx(1-x)^{-2m}}{\int x^{m-1}dx(1-x)^{-m}}.$$

COROLLARY 1

§26 Therefore, lo and behold these extraordinary theorems the expansions of this example give us; the first of these theorems is:

[4]Euler gave this series in his paper "De seriebus quibusdam considerationes". This is E130 in the Eneström-Index.

If φ denotes an arbitrary angle, it will be

$$\frac{\pi\cos m\pi}{\sin m\pi}=\frac{\cos m\varphi}{m}-\frac{\cos(1-m)\varphi}{1-m}+\frac{\cos(1+m)\varphi}{1+m}-\frac{\cos(2-m)\varphi}{2-m}+\text{etc.},$$

which equality can also be exhibited in such a way that it is

$$\frac{\pi\cos m\pi}{\sin m\pi}=\cos m\varphi\left(\frac{1}{m}-\frac{2m\cos\varphi}{1-mm}-\frac{2m\cos 2\varphi}{4-mm}-\frac{2m\cos 3\varphi}{9-mm}-\text{etc.}\right)$$

$$-2\sin\varphi\left(\frac{\sin\varphi}{1-mm}+\frac{2\sin 2\varphi}{4-mm}+\frac{3\sin 3\varphi}{9-mm}+\frac{4\sin 4\varphi}{16-mm}+\text{etc.}\right),$$

whence, if it is

$$m\varphi=90^\circ=\frac{\pi}{2}\quad\text{and hence}\quad\varphi=\frac{\pi}{2m},$$

it will be

$$-\frac{\pi\cos m\pi}{\sin m\pi}=\frac{\sin\frac{\pi}{2m}}{1-mm}+\frac{2\sin\frac{2\pi}{2m}}{4-mm}+\frac{3\sin\frac{3\pi}{2m}}{9-mm}+\frac{4\sin\frac{4\pi}{2m}}{16-mm}+\text{etc.}$$

COROLLARY 2

§27 The second theorem can be formulated this way:

If φ denotes an arbitrary angle, it will be

$$\frac{\pi\varphi\cos m\pi}{\sin m\pi}=\frac{\sin m\varphi}{mm}-\frac{\sin(1-m)\varphi}{(1-m)^2}+\frac{\sin(1+m)\varphi}{(1+m)^2}-\frac{\sin(2-m)\varphi}{(2-m)^2}+\text{etc.}$$

Hence having taken $\varphi=\pi$ it will be

$$\frac{\pi\pi\cos m\pi}{\sin m\pi}=\frac{\sin m\pi}{mm}-\frac{\sin m\pi}{(1-m)^2}-\frac{\sin m\pi}{(1+m)^2}+\frac{\sin m\pi}{(2-m)^2}+\frac{\sin m\pi}{(2+m)^2}-\text{etc.}$$

or

$$\frac{\pi\pi}{\sin m\pi\tan m\pi}=\frac{1}{m^2}-\frac{1}{(1-m)^2}-\frac{1}{(1+m)^2}+\frac{1}{(2-m)^2}+\frac{1}{(2+m)^2}-\text{etc.}$$

But having put

$$m\varphi=\pi$$

one will have

$$\frac{\pi\pi\cos m\pi}{m\sin m\pi} = \frac{\sin\frac{\pi}{m}}{(1-m)^2} - \frac{\sin\frac{\pi}{m}}{(1+m)^2} - \frac{\sin\frac{2\pi}{m}}{(2-m)^2} - \frac{\sin\frac{2\pi}{m}}{(2+m)^2} + \text{etc.}$$

or this way

$$\frac{\pi\pi\cos m\pi}{4mm\sin m\pi} = \frac{1\sin\frac{\pi}{m}}{(1-mm)^2} + \frac{2\sin\frac{2\pi}{m}}{(4-mm)^2} + \frac{3\sin\frac{3\pi}{m}}{(9-mm)^2} + \text{etc.}$$

COROLLARY 3

§28 The third theorem concerns the comparison of integral formulas and can be stated this way:

If the integration of the following formulas is extended from the lower limit $x=0$ to the upper limit $x=1$, it will always be

$$\int x^{m-1}\mathrm{d}x(1-x)^{m-1}\cdot\int x^{m-1}\mathrm{d}x(1-x)^{-m} = \frac{2\pi\cos m\pi}{\sin m\pi}\int x^{m-1}\mathrm{d}x(1-x)^{-2m},$$

or if one puts $m=\frac{\lambda}{n}$ and $x=y^n$, it will be

$$\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{n-\lambda}}}\cdot\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{\lambda}}} = \frac{2\pi\cos\frac{\lambda\pi}{n}}{n\sin\frac{\lambda\pi}{n}}\cdot\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}}$$

SCHOLIUM

§29 The proof of this last theorem seems to be very difficult; nevertheless, using the results I once published on integral formulas of this kind[5], it can be shown to be true as follows. For, let us indicate, as I did there, this integral formula

$$\int\frac{y^{p-1}}{\sqrt[n]{(1-y^n)^{n-q}}}$$

[5]Euler refers to his paper "Observationes circa integralia formularum $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ posito post integrationem $x=1$". This is E321 in the Eneström-Index.

by this character $\left(\frac{p}{q}\right)$ and it is to be demonstrated that it is

$$\left(\frac{\lambda}{\lambda}\right)\left(\frac{\lambda}{n-\lambda}\right) = \frac{2\pi\cos\frac{\lambda\pi}{n}}{n\sin\frac{\lambda\pi}{n}}\left(\frac{\lambda}{n-2\lambda}\right).$$

Now, first I demonstrated, if it was

$$q + r = n,$$

that then it will be

$$\left(\frac{q}{r}\right) = \frac{\pi}{n\sin\frac{q\pi}{n}},$$

whence it immediately follows

$$\left(\frac{\lambda}{n-\lambda}\right) = \int \frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^\lambda}} = \frac{\pi}{n\sin\frac{\lambda\pi}{n}},$$

so that it remains to prove that it is

$$\left(\frac{\lambda}{\lambda}\right) = 2\cos\frac{\lambda\pi}{n}\left(\frac{\lambda}{n-2\lambda}\right).$$

But on the same occasion I showed, if it was

$$p + q + r = n,$$

that it will be

$$\frac{1}{\sin\frac{r\pi}{n}}\left(\frac{p}{q}\right) = \frac{1}{\sin\frac{q\pi}{n}}\left(\frac{p}{r}\right) = \frac{1}{\sin\frac{p\pi}{n}}\left(\frac{q}{r}\right).$$

Therefore, let us take

$$p = \lambda, \quad q = \lambda$$

and it will be

$$r = n - 2\lambda;$$

hence, because of

$$\sin\frac{(n-2\lambda)\pi}{n} = \sin\frac{2\lambda\pi}{n},$$

we conclude

$$\frac{1}{\sin\frac{2\lambda\pi}{n}}\left(\frac{\lambda}{\lambda}\right) = \frac{1}{\sin\frac{\lambda\pi}{n}}\left(\frac{\lambda}{n-2\lambda}\right),$$

so that because of

$$\sin\frac{2\lambda\pi}{n} = 2\sin\frac{\lambda\pi}{n}\cos\frac{\lambda\pi}{n}$$

it indeed is

$$\left(\frac{\lambda}{\lambda}\right) = 2\cos\frac{\lambda\pi}{n}\left(\frac{\lambda}{n-2\lambda}\right).$$

But a lot more strange theorem was found above (§ 28), which for the same limits of integration says that it is

$$\frac{\pi}{\sin n\pi}\int x^n \mathrm{d}x(1-x)^{n-1} = \int\frac{\mathrm{d}x}{x}\int\frac{x^{n-1}\mathrm{d}x}{(1-x)^{2n}}$$

or

$$\frac{\pi}{2\sin n\pi}\int x^{n-1}\mathrm{d}x(1-x)^{n-1} = \int\frac{\mathrm{d}x}{x}\int\frac{x^{n-1}\mathrm{d}x}{(1-x)^{2n}};$$

to reduce this equation to that form, let us write $\frac{\lambda}{n}$ instead of n and let $x = y^n$, whence it is

$$\frac{\pi}{2n\sin\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{n-\lambda}}} = \int\frac{\mathrm{d}y}{y}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}}.$$

But we just saw that it is

$$\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{n-\lambda}}} = 2\cos\frac{\lambda\pi}{n}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}}$$

and so via the theorem we conclude that it is

$$\frac{\pi}{n\tan\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}} = \int\frac{\mathrm{d}y}{y}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}},$$

and hence further this not less remarkable theorem

$$\frac{\pi}{n\tan\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}\mathrm{d}y}{\sqrt[n]{(1-y^n)^{2\lambda}}} = -\int\frac{y^{\lambda-1}\mathrm{d}y\cdot\log y}{\sqrt[n]{(1-y^n)^{2\lambda}}},$$

whence for $\lambda = 1$ we find the following proportion

$$\frac{\pi}{n} : \tan\frac{\pi}{n} = \int\frac{\mathrm{d}y\log\frac{1}{y}}{\sqrt[n]{(1-y^n)^2}} : \int\frac{\mathrm{d}y}{\sqrt[n]{(1-y^n)^2}}.$$

Problem 3

§30 *To find an equation of such a kind (for a curved line) between two variables, the abscissa x and the ordinate y, that to the abscissas taken in an arithmetic progression given ordinates correspond, namely:*

If it is

$$x = n\theta, \quad (n+1)\theta, \quad (n+2)\theta, \quad (n+3)\theta, \quad (n+4)\theta, \quad \text{etc.}$$

that it is

$$y = p, \qquad q, \qquad r, \qquad s, \qquad t \qquad \text{etc.}$$

Solution

Let us put in general

$$x = \theta\omega$$

and from the general solution given in § 10 we obtain this equation

$$\begin{aligned}\frac{y}{\omega} = {} & \frac{p}{n} \cdot \frac{(n+1-\omega)(n+1+\omega)}{1(2n+1)} \cdot \frac{(n+2-\omega)(n+2+\omega)}{2(2n+2)} \cdot \frac{(n+3-\omega)(n+3+\omega)}{3(2n+3)} \cdot \text{etc.} \\ & - \frac{p}{n+1} \cdot \frac{(n-\omega)(n+\omega)}{1(2n+1)} \cdot \frac{(n+2-\omega)(n+2+\omega)}{1(2n+3)} \cdot \frac{(n+3-\omega)(n+3+\omega)}{2(2n+4)} \cdot \text{etc.} \\ & + \frac{r}{n+2} \cdot \frac{(n-\omega)(n+\omega)}{2(2n+2)} \cdot \frac{(n+1-\omega)(n+1+\omega)}{1(2n+3)} \cdot \frac{(n+3-\omega)(n+3+\omega)}{1(2n+5)} \cdot \text{etc.} \\ & - \text{etc.},\end{aligned}$$

which equation for the sake of brevity we want to represent this way

$$\frac{y}{\omega} = \mathfrak{A} \cdot \frac{p}{n} - \mathfrak{B} \cdot \frac{q}{n+1} + \mathfrak{C} \cdot \frac{r}{n+2} - \mathfrak{D} \cdot \frac{s}{n+3} + \text{etc.};$$

and in order to find the value of $\mathfrak{A}$ from the general form mentioned in § 17 we will have for this case

$$a = n+1-\omega, \quad b = 1, \quad c = n-\omega \quad \text{and} \quad d = 1,$$

whence by means of integral formulas to be extended from the lower limit $z = 0$ to the upper limit $z = 1$ we conclude

$$\mathfrak{A} = \frac{\int dz(1-z)^{n-\omega-1}}{\int z^{n-\omega}dz(1-z)^{n-\omega-1}} = \frac{1}{(n-\omega)\int z^{n-\omega}dz(1-z)^{n-\omega-1}}$$

or

$$\mathfrak{A} = \frac{2}{(n-\omega)\int z^{n-\omega-1}dz(1-z)^{n-\omega-1}};$$

having conceded this integration the remaining ones are easily handled. For, using it it will be as above in § 17

$$\begin{aligned}
\frac{\mathfrak{B}}{\mathfrak{A}} &= \frac{(n-\omega)(n+\omega)}{(n+1-\omega)(n+1+\omega)} \cdot (2+2n) = \frac{2(n+1)(n-\omega)(n+\omega)}{n+1-\omega)(n+1+\omega)},\\
\frac{\mathfrak{C}}{\mathfrak{B}} &= \frac{(n+1-\omega)(n+1+\omega)}{(n+2-\omega)(n+2+\omega)} \cdot \frac{(1+2n)(2+n)}{2(n+1)},\\
\frac{\mathfrak{D}}{\mathfrak{C}} &= \frac{(n+2-\omega)(n+3+\omega)}{(n+3-\omega)(n+3+\omega)} \cdot \frac{(2+2n)(3+n)}{3(n+2)},\\
\frac{\mathfrak{E}}{\mathfrak{D}} &= \frac{(n+3-\omega)(n+3+\omega)}{(n+4-\omega)(n+4+\omega)} \cdot \frac{(3+2n)(4+n)}{4(n+3)}
\end{aligned}$$

etc.

Therefore, let us set the integral formula

$$\int z^{n-\omega-1}dz(1-z)^{n-\omega-1} = \triangle,$$

that it is

$$\mathfrak{A} = \frac{2}{(n-\omega)\triangle},$$

and the remaining coefficients will be defined by $\mathfrak{A}$ this way:

$$\begin{aligned}
\mathfrak{B} &= \frac{2(n+1)}{1} \cdot \frac{nn-\omega\omega}{(n+1)^2-\omega\omega}\mathfrak{A},\\
\mathfrak{C} &= \frac{2(n+2)(2n+1)}{1\cdot 2} \cdot \frac{nn-\omega\omega}{(n+2)^2-\omega\omega}\mathfrak{A},\\
\mathfrak{D} &= \frac{2(n+3)(2n+1)(2n+2)}{1\cdot 2\cdot 3} \cdot \frac{nn-\omega\omega}{(n+3)^2-\omega\omega}\mathfrak{A},\\
\mathfrak{E} &= \frac{2(n+4)(2n+1)(2n+2)(2n+3)}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{nn-\omega\omega}{(n+4)^2-\omega\omega}\mathfrak{A}
\end{aligned}$$

etc.

Therefore, the equation in question between y and $x = \theta\omega$ will be of this nature:

$$\frac{n \triangle y}{2(n+\omega)\omega} = \frac{p}{nn-\omega\omega} - \frac{2n}{1}\cdot\frac{q}{(n+1)^2-\omega\omega}$$
$$+\frac{2n(2n+1)}{1\cdot 2}\cdot\frac{r}{(n+2)^2-\omega\omega} - \frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{s}{(n+3)^2-\omega\omega} + \text{etc.},$$

whence for each value of $x = \vartheta\omega$ a corresponding value of y is defined in terms of the ordinates p, q, r etc., which are assumed to correspond to the abscissas $n\theta$, $(n+1)\theta$, $(n+2)\theta$ etc. Here it has certainly to be noted, if ω is taken equal to a certain term of the progression n, $n+1$, $n+2$ etc., that then the denominator of the given corresponding ordinate vanishes, so that with respect to the term, which is certainly infinite, the remaining ones vanish. But then at the same time also the value $\triangle$ becomes infinite and precisely of such a kind, that it then either is $y = p$ and $y = q$ or $y = r$ etc., as the nature of the question demands it.

COROLLARY 1

§31 If the propounded abscissas denote circular arcs, the ordinates on the other hand their sines, that it is

$$p = \sin n\theta, \quad q = \sin(n+1)\theta, \quad q = \sin(n+1)\theta, \quad r = \sin(n+2)\theta, \quad \text{etc.},$$

it will be

$$y = \sin\omega\theta,$$

whence this general equation results

$$\frac{n \triangle \sin\omega\theta}{2(n+\omega)\omega} = \frac{\sin n\theta}{nn-\omega\omega} - \frac{2n}{1}\cdot\frac{\sin(n+1)\theta}{(n+1)^2-\omega^2} + \frac{2n(2n+1)}{1\cdot 2}\cdot\frac{\sin(n+2)\theta}{(n+2)^2-\omega^2}$$
$$-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{\sin(n+3)\theta}{(n+3)^2-\omega^2} + \text{etc.},$$

where it is especially remarkable that the three letters, n, θ and ω can be assumed arbitrarily.

COROLLARY 2

§32 Therefore, if we take

$$\theta = \pi,$$

that all sines of the series are reduced to the same $\sin n\theta$, it will be

$$\frac{n \triangle \sin \omega\theta}{2(n+\omega)\omega \sin n\pi} = \frac{1}{nn-\omega\omega} + \frac{2n}{1}\cdot\frac{1}{(n+1)^2-\omega^2} + \frac{2n(2n+1)}{1\cdot 2}\cdot\frac{1}{(n+2)^2-\omega^2}$$

$$+\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3}\cdot\frac{1}{(n+3)^2-\omega^2} + \text{etc.}$$

Hence, if it is

$$n = \frac{1}{2} \quad \text{und} \quad \triangle = \int z^{-\omega-\frac{1}{2}}dz(1-z)^{-\omega-\frac{1}{2}}$$

or

$$\triangle = 2\int \frac{z^{\frac{1}{2}-\omega}dz}{(1-z)^{\frac{1}{2}+\omega}},$$

one will have

$$\frac{\triangle \sin \omega\pi}{8(1+2\omega)\omega} = \frac{1}{1-4\omega^2} + \frac{1}{9-4\omega^2} + \frac{1}{25-4\omega^2} + \frac{1}{49-4\omega^2} + \text{etc.};$$

the sum of which series I showed to be[6]

$$= \frac{\pi}{8\omega}\tan \omega\pi,$$

so that it is

$$\frac{\triangle \sin \omega\pi}{8(1+2\omega)\omega} = \frac{\pi}{8\omega}\tan \omega\pi$$

and hence

$$\triangle = \frac{(1+2\omega)\pi}{\cos \omega\pi}.$$

[6] Euler shows this for the first time in E41.

SCHOLIUM 1

§33 But it is not possible to trust these conclusions too much for the reason mentioned above already. For, having put the ordinates

$$p = \sin n\theta, \quad q = \sin(n+1)\theta, \quad r = \sin(n+2)\theta \quad \text{etc.},$$

while the arcs $n\theta$, $(n+1)\theta$, $(n+2)\theta$ etc. are considered as abscissas, the found equation certainly yields a curved line passing through all these points. But hence it does not follow that this curve is the sine, since infinitely many other curved lines passing through that same infinitely many points exist. Therefore, still using the letter y to indicate the ordinate corresponding to the abscissa $x = \theta\omega$ our solution gives this equation for the curve in question

$$\frac{n \triangle y}{2(n+\omega)} = \frac{\sin n\theta}{n^2-\omega^2} - \frac{2n}{1} \cdot \frac{\sin(n+1)\theta}{(n+1)^2-\omega^2} + \frac{2n(2n+1)}{1\cdot 2} \cdot \frac{\sin(n+2)\theta}{(n+2)^2-\omega^2}$$
$$-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3} \cdot \frac{\sin(n+3)\theta}{(n+3)^2-\omega^2} + \text{etc.},$$

so that to the abscissa

$$x = (n \pm i)\theta$$

this ordinate corresponds

$$y = \sin(n \pm i)\theta,$$

if i is an arbitrary integer. On the other hand it could also happen, that for other abscissas, where i is not an integer number and hence generally, if $x = \omega\theta$, the ordinate is not $y = \sin\omega$. To see this more clearly, let us investigate the general equation for completely all lines passing through given points, and let the value found up to now be

$$y = \Theta$$

and find a function vanishing for all given abscissas, of which kind this is

$$\omega(nn-\omega\omega)\frac{((n+1)^2-\omega^2)}{1(2n+1)}\frac{((n+2)^2-\omega^2)}{2(2n+2)}\frac{((n+3)^2-\omega^2)}{3(2n+3)} \quad \text{etc.},$$

which by means of the things mentioned above is

$$= \omega(nn-\omega\omega)\mathfrak{A} = \frac{2\omega(n+\omega)}{\triangle}.$$

Call this quantity $= \Omega$ and let $f : \Omega$ be a function of Ω of such kind, which vanishes, if $\Omega = 0$, and the general equation for all satisfying curved lines will be

$$y = \Theta + f : \Omega = \Theta + f : \frac{2\omega(n+\omega)}{\triangle}.$$

And now without any doubt it is certain that in this equation the equation $y = \sin \omega\theta$ is contained having put $x = \omega\theta$, since this equation satisfies all prescribed conditions. So it could certainly happen that the equation $y = \Theta$ was different from this one $y = \sin \omega\theta$; this depends especially on the values attributed to the letters θ and n, so that in the one case the found equation $y = \Theta$ agrees with this one $y = \sin \omega\theta$, but in others differs from the same.

SCHOLIUM 2

§34 We want to apply these results to the case, in which it is

$$\theta = \pi \quad \text{and} \quad n = \frac{1}{2}$$

and

$$\triangle = 2\int \frac{z^{\frac{1}{2}-\omega}dz}{(1-z)^{\frac{1}{2}+\omega}};$$

and since the sum of the found series is

$$= \frac{\pi}{8\omega}\tan \omega\pi$$

one will have this general equation

$$\frac{\triangle y}{8(1+2\omega)\omega} = \frac{\pi}{8\omega}\tan \omega\pi + \frac{\triangle}{8(1+2\omega)} f : \frac{\omega(1+2\omega)}{2\triangle}$$

or

$$y = \frac{\pi(1+2\omega)}{\triangle}\tan \omega\pi + f : \frac{\omega(1+2\omega)}{2\triangle},$$

where the added function in general is of such a nature that it vanishes in the cases

$$\omega = 0, \quad \omega = \pm\frac{1}{2}, \quad \omega = \pm\frac{3}{2}, \quad \omega = \pm\frac{5}{2} \quad \text{etc.}$$

of which kind these formulas are

$$\sin 2\omega\pi, \quad \omega\cos\omega\pi, \quad \text{likewise} \quad \sin 2i\omega\pi \quad \text{and} \quad \omega\cos(2i-1)\omega\pi,$$

while i denotes an arbitrary integer number; hence it is possible to combine any arbitrary number of these formulas. Therefore, a certain function of this kind will be given, which we want to put φ, that it is

$$y = \sin\omega\pi$$

and hence

$$\sin\omega\pi = \frac{\pi(1+2\omega)}{\triangle}\tan\omega\pi + \varphi$$

or

$$\triangle = \frac{\pi(1+2\omega)\tan\omega\pi}{\sin\omega\pi - \varphi} = 2\int \frac{z^{\frac{1}{2}-\omega}dz}{(1-z)^{\frac{1}{2}+\omega}}.$$

Therefore, since in the case $\omega = 0$ the function φ certainly vanishes, it will be $\pi = \triangle$, of course, which is an indication that the function φ contains ω^λ, whose exponent λ is greater than 1, since otherwise the quantity φ would not vanish with respect to $\sin\omega\pi$ for $\omega = 0$. And for this reason the conclusions of the preceding problem are to be considered to be true.

PROBLEM 4

§35 *To find an equation of such a kind (for a curve line) between the abscissa x and the ordinate y, that to the abscissas proceeding in an interrupted arithmetic progression given ordinates correspond, namely*

$$x = n\theta, \qquad (1-n)\theta, \qquad (1+n)\theta, \qquad (2-n)\theta, \qquad (2+n)\theta, \qquad (3-n)\theta \qquad \text{etc.,}$$

to

$$y = p, \qquad q, \qquad r, \qquad s, \qquad t, \qquad u \qquad \text{etc.}$$

SOLUTION

Let us in general put the abscissa

$$x = \theta\omega$$

and for the equation between x and y let us assume this equation

$$\frac{y}{\omega} = \mathfrak{A}\cdot\frac{p}{n} - \mathfrak{B}\cdot\frac{q}{1-n} + \mathfrak{C}\cdot\frac{r}{1+n} - \mathfrak{D}\cdot\frac{s}{2-n} + \mathfrak{E}\cdot\frac{t}{2+n} - \mathfrak{F}\cdot\frac{u}{3-n} + \text{etc.}$$

and applying the results of § 25 to this case one will have

$$\mathfrak{A} = \frac{(1-n-\omega)(1-n+\omega)}{1(1-2n)} \cdot \frac{(1+n-\omega)(1+n+\omega)}{1(1+2n)} \cdot \frac{(2-n-\omega)(2-n+\omega)}{2(2-2n)} \cdot \frac{(2+n-\omega)(2+n+\omega)}{2(2+2n)} \cdot \text{etc.}$$

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{(n-\omega)(n+\omega)}{(1-n-\omega)(1-n+\omega)} \cdot \frac{1-n}{n}, \quad \frac{\mathfrak{C}}{\mathfrak{B}} = \frac{(1-n-\omega)(1-n+\omega)}{(1+n-\omega)(1+n+\omega)} \cdot \frac{1+n}{1-n},$$
$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(1+n-\omega)(1+n+\omega)}{(2-n-\omega)(2-n+\omega)} \cdot \frac{2-n}{1+n}, \quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(2-n-\omega)(2-n+\omega)}{(2+n-\omega)(2+n+\omega)} \cdot \frac{2+n}{2-n}$$

etc.

Let us expand the value of $\mathfrak{A}$ into two products

$$\mathfrak{P} = \frac{(1-n-\omega)(1-n+\omega)}{1(1-2n)} \cdot \frac{(2-n-\omega)(2-n+\omega)}{2(2-2n)} \cdot \frac{(3-n-\omega)(3-n+\omega)}{3(3-2n)} \cdot \text{etc.},$$
$$\mathfrak{Q} = \frac{(1+n-\omega)(1+n+\omega)}{1(1+2n)} \cdot \frac{(2+n-\omega)(2+n+\omega)}{2(2+2n)} \cdot \frac{(3+n-\omega)(3+n+\omega)}{3(3+2n)} \cdot \text{etc.},$$

that it is

$$\mathfrak{A} = \mathfrak{P}\mathfrak{Q},$$

and let us define the value of both by means of integral formulas according to the prescriptions given in § 17. And at first for the infinite product $\mathfrak{P}$ let us set

$$a = 1-n-\omega, \quad b = 1, \quad c = -n+\omega \quad \text{and} \quad d = 1$$

and it will be

$$\mathfrak{P} = \frac{\int dx(1-x)^{-1-n+\omega}}{\int x^{-n-\omega}dx(1-x)^{-1-n+\omega}} = \frac{1}{\omega-n} \cdot \frac{1}{\int x^{-n-\omega}dx(1-x)^{-1-n+\omega}},$$

if it is

$$\omega - n > 0.$$

For the other infinite product only by taking n negatively it will be

$$\mathfrak{Q} == \frac{1}{\omega+n} \cdot \frac{1}{\int x^{n-\omega}dx(1-x)^{n+\omega-1}}.$$

But that the condition $\omega - n > 0$ is not necessary, let us use another distribution and let

$$\mathfrak{P} = \frac{(1+n+\omega)(1-n-\omega)}{1\cdot 1}\cdot\frac{(2+n+\omega)(2-n-\omega)}{2\cdot 2}\cdot\frac{(3+n+\omega)(3-n-\omega)}{3\cdot 3}\cdot\text{etc.},$$
$$\mathfrak{Q} = \frac{(1+n-\omega)(1-n+\omega)}{(1-2n)(1+2n)}\cdot\frac{(2+n-\omega)(2-n+\omega)}{(2-2n)(2+2n)}\cdot\frac{(3+n-\omega)(3-n+\omega)}{(3-2n)(3+2n)}\cdot\text{etc.},$$

and let is set for $\mathfrak{P}$

$$a = 1-n-\omega,\quad b=1,\quad c=n+\omega,\quad d=1,$$

for $\mathfrak{Q}$ on the other hand

$$a = 1+n-\omega,\quad b=1-2n,\quad c=n+\omega,\quad \text{and}\quad d=1$$

and it will be

$$\mathfrak{P} = \frac{\int dx(1-x)^{-1+n+\omega}}{\int x^{-n-\omega}dx(1-x)^{-1+n+\omega}} = \frac{1}{n+\omega}\cdot\frac{1}{\int x^{-n-\omega}dx(1-x)^{-1+n+\omega}},$$

$$\mathfrak{Q} = \frac{\int x^{-2n}dx(1-x)^{-1+n+\omega}}{\int x^{n-\omega}dx(1-x)^{-1+n+\omega}}.$$

But it will be

$$\int x^m dx(1-x)^{k-1} = \frac{m+k+1}{k}\int x^m dx(1-x)^k,$$

therefore

$$\int x^{-n-\omega}dx(1-x)^{-1+n+\omega} = \frac{1}{n+\omega}\int x^{-n-\omega}dx(1-x)^{n+\omega}$$
$$= \frac{1}{n+\omega}\int y^{n+\omega}dy(1-y)^{-n-\omega},$$
$$\int x^{-2n}dx(1-x)^{-1+n+\omega} = \frac{1-n+\omega}{n+\omega}\int x^{-2n}dx(1-x)^{n+\omega}$$
$$\frac{1-n+\omega}{n+\omega}\int y^{n+\omega}dy(1-y)^{-2n},$$
$$\int x^{n-\omega}dx(1-x)^{-1+n+\omega} = \frac{1+2n}{n+\omega}\int x^{n-\omega}dx(1-x)^{n+\omega}$$

$$= \frac{1+2n}{n+\omega} \int y^{n+\omega} dy (1-y)^{n-\omega},$$

whence it is concluded

$$\mathfrak{A} = \mathfrak{P}\mathfrak{Q} = \frac{(1-n-\omega) \int y^{n+\omega} dy (1-y)^{-2n}}{\int y^{n+\omega} dy (1-y)^{-n+\omega} \cdot \int y^{n+\omega} dy (1-y)^{n-\omega}}$$

or

$$\mathfrak{A} = \frac{\int y^{n+\omega-1} dy (1-y)^{-2n}}{\int y^{n+\omega} dy (1-y)^{-n-\omega} \cdot \int y^{n+\omega-1} dy (1-y)^{n-\omega}}$$

or

$$\mathfrak{A} = \frac{\int y^{n+\omega-1} dy (1-y)^{-2n}}{(n+\omega) \int y^{n+\omega-1} dy (1-y)^{-n-\omega} \cdot \int y^{n+\omega-1} dy (1-y)^{n-\omega}}.$$

Therefore, since it is

$$\mathfrak{B} = \frac{1-n}{n} \cdot \frac{nn-\omega\omega}{(1-n)^2-\omega\omega}\mathfrak{A}, \quad \mathfrak{C} = \frac{1+n}{n} \cdot \frac{nn-\omega\omega}{(1+n)^2-\omega\omega}\mathfrak{A},$$
$$\mathfrak{D} = \frac{2-n}{n} \cdot \frac{nn-\omega\omega}{(2-n)^2-\omega\omega}\mathfrak{A}, \quad \mathfrak{E} = \frac{2+n}{n} \cdot \frac{nn-\omega\omega}{(2+n)^2-\omega\omega}\mathfrak{A}$$

etc.,

by means of a convenient series it will be

$$\frac{y}{\mathfrak{A}\omega} = \frac{p}{n} - \frac{(nn-\omega\omega)q}{n((1-n)^2-\omega^2)} + \frac{(nn-\omega\omega)r}{n((1+n)^2-\omega^2)} - \frac{(nn-\omega\omega)s}{n((2-n)^2-\omega^2)} + \text{etc.}$$

or

$$\frac{ny}{\mathfrak{A}\omega(nn-\omega\omega)} = \frac{p}{n^2-\omega^2} - \frac{q}{(1-n)^2-\omega^2} + \frac{r}{(1+n)^2-\omega^2} - \text{etc.}$$

But by substituting the integral formula for $\mathfrak{A}$ again, where for the sake of distinction I will denote the new variable by the letter z, this same series is equal to this expression

$$\frac{ny}{(n-\omega)\omega} \cdot \frac{\int z^{n+\omega-1} dz (1-z)^{-n-\omega} \cdot \int z^{n+\omega-1} dz (1-z)^{n-\omega}}{\int z^{n+\omega-1} dz (1-z)^{-2n}},$$

and the integration of these formulas is to be understood to be extended from the lower limit $z=0$ to the upper limit $z=1$.

COROLLARY 1

§36 Therefore, if for the sake of brevity we put this general integral formula

$$\frac{\int z^{n+\omega-1}\mathrm{d}z(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}\mathrm{d}z(1-z)^{n-\omega}}{\int z^{n+\omega-1}\mathrm{d}z(1-z)^{-2n}}=\triangle$$

and resolve the single terms of the series into two terms, we will have

$$\begin{aligned}\frac{2n\triangle y}{n-\omega}=&+\frac{p}{n-\omega}-\frac{q}{1-n-\omega}+\frac{r}{1+n-\omega}-\frac{s}{2-n-\omega}+\frac{t}{2+n-\omega}-\text{etc.}\\&-\frac{p}{n+\omega}+\frac{q}{1-n+\omega}-\frac{r}{1+n+\omega}+\frac{s}{2-n+\omega}-\frac{t}{2+n+\omega}+\text{etc.}\end{aligned}$$

COROLLARY 2

§37 Therefore, this equation defines a curved line, in which to the abscissas

$$x=0,\quad n\theta,\qquad (1-n)\theta,\qquad (1+n)\theta,\qquad (2-n)\theta,\qquad (2+n)\theta\qquad \text{etc.}$$

these ordinates correspond

$$y=0,\quad p,\qquad q,\qquad r,\qquad s,\qquad t\qquad \text{etc.},$$

but to the same abscissas taken negatively the same ordinates taken negatively correspond. But in general here the abscissa was put $x=\theta\omega$.

COROLLARY 3

§38 Since here the letter θ goes out of the calculation, it would be possible to write simply 1 for it, so that the letter ω denotes the abscissa itself. But if we want to make the application to arcs and their sines, it is convenient to keep the letter θ in the calculation.

SCHOLIUM

§39 The use of this problem is especially seen, if as above the abscissas are considered as circular arcs and the given abscissas are taken in such a way that the ordinates p, q, r, s, t etc. become equal to each other, whether positive or negative. Therefore, that it becomes clear in these cases, whether the found series can be summed from elsewhere, recall, what I once published on similar series[7], whence the sums of the following two series are calculated

$$\frac{1}{\alpha}-\frac{1}{\beta-\alpha}+\frac{1}{\beta+\alpha}-\frac{1}{2\beta-\alpha}+\frac{1}{2\beta+\alpha}-\text{etc.}=\frac{\pi}{\beta\tan\frac{\alpha\pi}{\beta}},$$
$$\frac{1}{\alpha}+\frac{1}{\beta-\alpha}-\frac{1}{\beta+\alpha}-\frac{1}{2\beta-\alpha}+\frac{1}{2\beta+\alpha}+\text{etc.}=\frac{\pi}{\beta\sin\frac{\alpha\pi}{\beta}}.$$

Therefore, hence for our problem we deduce the following four summations

$$\text{I.}\quad \frac{1}{n-\omega}-\frac{1}{1-n+\omega}+\frac{1}{1+n-\omega}-\frac{1}{2-n+\omega}+\frac{1}{2+n-\omega}-\text{etc.}=\frac{\pi}{\tan(n-\omega)\pi},$$
$$\text{II.}\quad \frac{1}{n-\omega}+\frac{1}{1-n+\omega}-\frac{1}{1+n-\omega}-\frac{1}{2-n+\omega}+\frac{1}{2+n-\omega}+\text{etc.}=\frac{\pi}{\sin(n-\omega)\pi},$$
$$\text{III.}\quad \frac{1}{n+\omega}-\frac{1}{1-n-\omega}+\frac{1}{1+n+\omega}-\frac{1}{2-n-\omega}+\frac{1}{2+n+\omega}-\text{etc.}=\frac{\pi}{\tan(n+\omega)\pi},$$
$$\text{IV.}\quad \frac{1}{n+\omega}+\frac{1}{1-n-\omega}-\frac{1}{1+n+\omega}-\frac{1}{2-n-\omega}+\frac{1}{2+n+\omega}+\text{etc.}=\frac{\pi}{\sin(n+\omega)\pi}.$$

Having observed these things let us expand the cases, which can be reduced to finite expressions by means of these summations.

EXAMPLE I

§40 *Let the ordinate corresponding to the abscissas*

$$x=0,\quad n\theta,\quad (1-n)\theta,\quad (1+n)\theta,\quad (2-n)\theta,\quad (2+n)\theta\quad \text{etc.}$$

be

$$p=f,\quad q=f,\quad r=-f,\quad s=-f,\quad t=+f,\quad u=+f\quad \text{etc.}$$

and by means of a finite equation investigate the relation between the ordinate y and the abscissa $x=\theta\omega$.

[7] Euler refers to E130 again.

SOLUTION

The first corollary for this case yields this equation

$$\begin{aligned}\frac{2n\triangle y}{f(n-\omega)} = &+\frac{1}{n-\omega}-\frac{1}{1-n-\omega}-\frac{1}{1+n-\omega}+\frac{1}{2-n-\omega}+\frac{1}{2+n-\omega}-\text{etc.},\\ &-\frac{1}{n+\omega}+\frac{1}{1-n+\omega}+\frac{1}{1+n+\omega}-\frac{1}{2-n+\omega}-\frac{1}{2+n+\omega}+\text{etc.};\end{aligned}$$

these two series can be reduced to II. minus IV. from the four series mentioned above, whose summation is known, and hence the equation in question in finite form will be

$$\frac{2n\triangle y}{f(n-\omega)} = \frac{\pi}{\sin(n-\omega)\pi}-\frac{\pi}{\sin(n+\omega)\pi},$$

which expression is reduced to this one

$$\frac{2\pi\cos n\pi\sin\omega\pi}{\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi} = \frac{4\pi\cos n\pi\cdot\sin\omega\pi}{\cos 2\omega\pi-\cos 2n\pi},$$

so that for our curve one finds this equation

$$\frac{n\triangle y}{f(n-\omega)} = \frac{\pi\cos n\pi\sin\omega\pi}{\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}.$$

We gave the value of $\triangle$ expressed in terms of integral formulas before; but since from the results mentioned above it is

$$\triangle = \frac{1}{\mathfrak{A}(n+\omega)},$$

by means of an infinite product we will have

$$\triangle = \frac{1}{n+\omega}\cdot\frac{1(1-2n)}{(1-n)^2-\omega^2}\cdot\frac{1(1+2n)}{(1+n)^2-\omega^2}\cdot\frac{2(2-2n)}{(2-n)^2-\omega^2}\cdot\frac{2(2+2n)}{(2+n)^2-\omega^2}\cdot\text{etc.},$$

where it is more clear than from the integral formulas that the value $\triangle$ becomes infinite, as often as it was

$$\omega = \pm(i\pm n),$$

while i denotes an arbitrary integer number, but the same value $\triangle$ vanishes in the cases, in which it is

$$n = \pm\frac{1}{2}.$$

But then it will also be helpful to have noted, if, while ω goes over into $1+\omega$, the value of $\triangle$ goes over into $\triangle'$

$$\triangle' = -\frac{(1-n-\omega)\triangle}{n-\omega}.$$

And if in like manner $\triangle''$ corresponds to the value $2+\omega$ assumed instead of ω, it will be

$$\triangle'' = \frac{-(2-n+\omega)\triangle'}{-(1-n+\omega)} = \frac{-(2-n+\omega)\triangle}{n-\omega}.$$

COROLLARY 1

§41 If the quantity $\triangle$ depends on ω, consider a function of it and denote it this way

$$\triangle = f : \omega;$$

therefore, it will then be

$$f : (1+\omega) = \frac{n-1-\omega}{n-\omega} f : \omega$$

and

$$f : (2+\omega) = \frac{n-2-\omega}{n-\omega} f : \omega$$

etc.

Hence, if ω denotes an arbitrary integer number, one will have this theorem

$$\frac{f : (i+\omega)}{n-i-\omega} = \frac{f : \omega}{n-\omega}.$$

COROLLARY 2

§42 Further, since having taken a negative ω it is

$$f:(-\omega) = \frac{n+\omega}{n-\omega} f:\omega,$$

it will be

$$\frac{f:-\omega}{n+\omega} = \frac{f:\omega}{n-\omega}$$

hence also in general

$$\frac{f:(i-\omega)}{n-i+\omega} = \frac{f:\omega}{n-\omega}.$$

SCHOLIUM

§43 This case corresponds to that one we expanded above in § 25, where the given ordinates also were the sines of the abscissas; and for the present case one has to put

$$\theta = \pi,$$

that it is

$$f = \sin n\pi$$

and all given points lie on sine-curve. But hence it does not follow that the curve the found equation exhibits is a sine, since innumerable other curves can pass through the same given points. Hence it is still certain that the value of y corresponding to the abscissa $x = \pi\omega$ and defined by this equation

$$\frac{n \triangle y}{(n-\omega)\sin n\pi} = \frac{\pi \cos n\pi \cdot \sin \omega\pi}{\sin(n-\omega)\pi \cdot \sin(n+\omega)\pi}$$

becomes equal to the sine of the arc $\pi\omega$ that it is $y = \sin \pi\omega$, even though this is true in the cases $\omega = \pm(i \pm n)$ and $\omega = 0$. But above we certainly saw that even in the case, in which ω is a very small quantity, the equation is true by taking $y = \sin \pi\omega$, so that it is

$$\triangle = \frac{\pi \cos n\pi}{\sin n\pi},$$

while it is

$$\triangle = \frac{\int z^{n-1} dz (1-z)^{-n} \cdot \int z^{n-1} dz (1-z)^{n}}{\int z^{n-1} dz (1-z)^{-2n}},$$

as I also showed there. In order to be able to explore this subject more easily in general and to express the value $\triangle$ more conveniently I observe that it is

$$\frac{\int z^{n+\omega-1}\mathrm{d}z(1-z)^{-n-\omega}}{\int z^{n+\omega-1}\mathrm{d}z(1-z)^{-2n}}=\frac{\int z^{\omega-n}\mathrm{d}z(1-z)^{-n-\omega}}{\int \mathrm{d}z(1-z)^{-2n}}=(1-2n)\int z^{\omega-n}\mathrm{d}z(1-z)^{-n-\omega},$$

while it is

$$n<\frac{1}{2},$$

whence it will be

$$\triangle=(1-2n)\int z^{\omega-n}\mathrm{d}z(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}\mathrm{d}z(1-z)^{n-\omega}.$$

But if it was in general

$$y=\sin\omega\pi,$$

it would also be

$$\triangle=\frac{(n-\omega)\pi\sin n\pi\cos n\pi}{n\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}.$$

Therefore, the question reduces to this, whether this equation

$$(1-2n)\int z^{\omega-n}\mathrm{d}z(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}\mathrm{d}z(1-z)^{n-\omega}$$

$$=\frac{(n-\omega)\pi\sin n\pi\cos n\pi}{n\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}$$

is also true in other cases than the ones mentioned above or not. Hence let us consider the case, in which it is

$$n=\frac{1}{4}\quad\text{and}\quad\omega=\frac{1}{2},$$

where right-hand side becomes

$$=\frac{-\frac{1}{4}\cdot\pi\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}}}{-\frac{1}{4}\cdot\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}}}=\pi;$$

the left-hand side on the other hand will be

$$=\frac{1}{2}\int\frac{z^{\frac{1}{4}}\mathrm{d}z}{(1-z)^{\frac{3}{4}}}\cdot\int\frac{z^{-\frac{1}{4}}\mathrm{d}z}{(1-z)^{\frac{1}{4}}},$$

which having put

$$z = v^4$$

goes over into this form

$$8\int \frac{v^4 \mathrm{d}v}{\sqrt[4]{(1-v^4)^3}} \cdot \int \frac{v^2 \mathrm{d}v}{\sqrt[4]{(1-v^4)}} = 4\int \frac{\mathrm{d}v}{\sqrt[4]{(1-v^4)^3}} \cdot \int \frac{vv\mathrm{d}v}{\sqrt[4]{(1-v^4)}},$$

whose value using the results I demonstrated on formulas of this kind indeed becomes $= \pi$, which therefore is an indication for our equation being true; but it can be proved to be true in the following way.

THEOREM

§44 *However the two numbers n and ω are assumed, this equation is true*

$$(1-2n)\int \frac{z^{\omega-n}\mathrm{d}z}{(1-z)^{n+\omega}} \cdot \int \frac{z^{n+\omega-1}\mathrm{d}z}{(1-z)^{\omega-n}} = \frac{(n-\omega)\pi \sin n\pi \cdot \cos n\pi}{n\sin(n-\omega)\pi \cdot \sin(n+\omega)\pi},$$

if the integration of those formulas is extended from the lower limit $z=0$ to the upper limit $z=1$.

PROOF

To reduce these formulas to a form I treated, let us put

$$n+\omega = \frac{\mu}{\lambda} \quad \text{and} \quad \omega - n = \frac{\nu}{\lambda},$$

that it is

$$2n = \frac{\mu-\nu}{\lambda}$$

and then this equation has to be proved

$$\frac{\lambda-\mu+\nu}{\lambda}\int \frac{z^{\frac{\nu}{\lambda}}\mathrm{d}z}{\sqrt[\lambda]{(1-z)^{\mu}}} \cdot \int \frac{z^{\frac{\mu-\lambda}{\lambda}}\mathrm{d}z}{\sqrt[\lambda]{(1-z)^{\nu}}} = \frac{\nu}{\mu-\nu} \cdot \frac{\pi \sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\nu\pi}{\lambda} \cdot \sin\frac{\mu\pi}{\lambda}}.$$

Now put $z = v^\lambda$ and one will have

$$\lambda(\lambda-\mu+\nu)\int \frac{v^{\lambda+\nu-1}\mathrm{d}v}{\sqrt[\lambda]{(1-v^\lambda)^{\mu}}} \cdot \int \frac{v^{\mu-1}\mathrm{d}v}{\sqrt[\lambda]{(1-v^\lambda)^{\nu}}} = \frac{\nu}{\mu-\nu} \cdot \frac{\pi \sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\nu\pi}{\lambda} \cdot \sin\frac{\mu\pi}{\lambda}};$$

and in the way we expressed these integral formulas there the left-hand side will be represented this way

$$\lambda(\lambda-\mu-\nu)\left(\frac{\lambda+\nu}{\lambda-\mu}\right)\left(\frac{\mu}{\lambda-\nu}\right),$$

which by means of the first reduction

$$\left(\frac{p}{q}\right)=\frac{p-\lambda}{p+q-\lambda}\left(\frac{p-\lambda}{q}\right),$$

goes over into

$$\lambda\nu\left(\frac{\nu}{\lambda-\mu}\right)\left(\frac{\mu}{\lambda-\nu}\right)=\lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right).$$

But this reduction on the other hand

$$\left(\frac{\lambda-q}{p}\right)\left(\frac{\lambda+p-q}{q}\right)=\frac{\pi}{\lambda p\sin\frac{q\pi}{\lambda}}$$

having taken

$$p=\mu-\nu \quad \text{and} \quad q=\mu$$

gives

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)=\frac{\pi}{\lambda(\mu-\nu)\sin\frac{\mu\pi}{\lambda}}.$$

But it also is

$$\left(\frac{\lambda-\nu}{\nu}\right)=\frac{\pi}{\lambda\sin\frac{\nu\pi}{\lambda}},$$

whose product is

$$\left(\frac{\lambda-\nu}{\mu}\right)\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\mu}{\mu-\nu}\right)=\frac{\pi\pi}{\lambda\lambda(\mu-\nu)\sin\frac{\mu\pi}{\lambda}\cdot\sin\frac{\nu\pi}{\lambda}}.$$

Further, since it is in general

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right),$$

by taking

$$p=\lambda-\mu, \quad q=\mu-\nu, \quad \text{and} \quad r=\nu$$

it will be

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\nu}\right)=\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\mu+\nu}{\mu-\nu}\right)$$

and because of

$$\left(\frac{\lambda-p}{p}\right)=\frac{\pi}{\lambda\sin\frac{p\pi}{\lambda}}$$

having taken

$$p=\mu-\nu$$

it will be

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\nu}\right)=\left(\frac{\lambda-\mu}{\nu}\right)\cdot\frac{\pi}{\lambda\sin\frac{\mu-\nu}{\lambda}\pi}$$

and hence

$$\left(\frac{\lambda-\nu}{\mu}\right)\left(\frac{\lambda-\mu}{\nu}\right)\cdot\frac{\pi}{\lambda\sin\frac{\mu-\nu}{\lambda}\pi}=\frac{\pi\pi}{\lambda\lambda(\mu-\nu)\sin\frac{\mu\pi}{\lambda}\cdot\sin\frac{\nu\pi}{\lambda}};$$

hence the left-hand side reduces to this form

$$\lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\mu\pi}{\lambda}\cdot\sin\frac{\nu\pi}{\lambda}},$$

which is the equation to be demonstrated above.

Corollary 1

§45 Therefore, in the doctrine of integral formulas of this kind

$$\int\frac{v^{p-1}dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\lambda-q}}},$$

which I denote by this character

$$\left(\frac{p}{q}\right)$$

and $\left(\frac{p}{q}\right)$ is equivalent to, the following reduction I proved is very important, namely

$$\lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)=\frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\mu\pi}{\lambda}\cdot\sin\frac{\nu\pi}{\lambda}},$$

so that the product of such integral formulas, $\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)$, can be exhibited using only angles.

COROLLARY 2

§46 If in the value found first for $\triangle$ one in like manner sets

$$n+\omega=\frac{\mu}{\nu} \quad \text{and} \quad \omega-n=\frac{\nu}{\lambda},$$

but then

$$z=v^{\lambda},$$

it will be

$$\triangle=\lambda\int\frac{v^{\mu-1}\mathrm{d}v}{\sqrt[\lambda]{(1-v^{\lambda})^{\mu}}}\cdot\int\frac{v^{\mu-1}\mathrm{d}v}{\sqrt[\lambda]{(1-v^{\lambda})^{\nu}}}:\int\frac{v^{\mu-1}\mathrm{d}v}{\sqrt[\lambda]{(1-v^{\lambda})^{\mu-\nu}}}$$

and hence in the notation we introduced it will be

$$\triangle=\frac{\lambda\left(\frac{\mu}{\lambda-\mu}\right)\left(\frac{\mu}{\lambda-\nu}\right)}{\left(\frac{\mu}{\lambda-\mu+\nu}\right)}$$

or

$$\triangle=\frac{\lambda\left(\frac{\lambda-\mu}{\mu}\right)\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}.$$

Therefore, the same value also is

$$\triangle=\frac{\nu\pi}{\mu-\nu}\cdot\frac{\sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\mu\pi}{\lambda}\cdot\sin\frac{\nu\pi}{\lambda}}.$$

COROLLARY 3

§47 Therefore, since for this last formula it immediately is

$$\left(\frac{\lambda-\mu}{\mu}\right)=\frac{\pi}{\lambda\sin\frac{\mu\pi}{\lambda}},$$

it will be

$$\frac{\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}=\frac{\nu}{\mu-\nu}\cdot\frac{\sin\frac{\mu-\nu}{\lambda}\pi}{\sin\frac{\nu\pi}{\lambda}},$$

which formula is shown to be true applying the following theorem

$$\frac{\left(\frac{p}{q}\right)}{\left(\frac{r}{p}\right)}=\frac{\left(\frac{p+r}{q}\right)}{\left(\frac{p+q}{r}\right)};$$

for, it will be

$$\frac{\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)} = \frac{\left(\frac{\lambda+\nu}{\lambda-\nu}\right)}{\left(\frac{\lambda+\mu-\nu}{\lambda-\mu+\nu}\right)} = \frac{\nu}{\mu-\nu}\cdot\frac{\left(\frac{\nu}{\lambda-\nu}\right)}{\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right)}$$

because of

$$\left(\frac{\lambda+\nu}{\lambda-\nu}\right) = \frac{\nu}{\lambda}\left(\frac{\nu}{\lambda-\nu}\right) \quad \text{and} \quad \left(\frac{\lambda+\mu-\nu}{\lambda-\mu+\nu}\right) = \frac{\mu-\nu}{\lambda}\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right);$$

it then is

$$\left(\frac{\nu}{\lambda-\nu}\right) = \frac{\pi}{\lambda\sin\frac{\nu\pi}{\lambda}} \quad \text{and} \quad \left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right) = \frac{\pi}{\lambda\sin\frac{\mu-\nu}{\lambda}\pi}.$$

Example II

§48 *Let the ordinates corresponding to the abscissas*

$$n\theta, \quad (1-n)\theta, \quad (1+n)\theta, \quad (2-n)\theta, \quad (2+n)\theta \quad \text{etc.}$$

be

$$p = f, \quad q = -f, \quad r = +f, \quad s = -f, \quad t = +f, \quad u = -f \quad \text{etc.},$$

and by means of a finite equation investigate the relation among the abscissa $x = \theta\omega$ and the ordinate $= y$ in general.

The general equation of paragraph 36 applied to this case yields

$$\begin{aligned}\frac{2n \triangle y}{f(n-\omega)} = {} & \frac{1}{n-\omega} + \frac{1}{1-n-\omega} + \frac{1}{1+n-\omega} + \frac{1}{2-n-\omega} + \frac{1}{2+n-\omega} + \text{etc.} \\ & - \frac{1}{n+\omega} - \frac{1}{1-n+\omega} - \frac{1}{1+n+\omega} - \frac{1}{2-n+\omega} - \frac{1}{2+n+\omega} - \text{etc.},\end{aligned}$$

where we now certainly know that it is

$$\triangle = \frac{(n-\omega)\pi\sin 2n\pi}{2n\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}.$$

But that series from § 39 becomes

$$\text{I. minus III.} = \frac{\pi}{\tan(n-\omega)\pi} - \frac{\pi}{\tan(n+\omega)\pi} = \frac{\pi\sin 2\omega\pi}{\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi},$$

after this sum was substituted

$$\frac{y}{f}\cdot\frac{\pi\sin 2n\pi}{\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}=\frac{\pi\sin 2\omega\pi}{\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi}$$

or

$$y=\frac{f\sin 2\omega\pi}{\sin 2n\pi}=\frac{f\sin\frac{2x\pi}{\theta}}{\sin 2n\pi}.$$

Therefore, this curve again is a sine, and if one takes $\theta=2\pi$, that it is $f=\sin 2n\pi$, the ordinate will be $y=\sin x$.

COROLLARY 1

§49 If one takes

$$\theta=\pi \quad \text{and} \quad f=\tan n\theta=\tan n\pi,$$

the given points will lie on a tangent-curve; and nevertheless the found curve itself will not be $\tan x$; but its nature will be expressed by this equation

$$y=\frac{\tan n\pi\cdot\sin 2x}{\sin 2n\pi}=\frac{\sin 2x}{2\cos^2 n\pi}=\frac{\sin 2x}{1+\cos 2n\pi}$$

and here it will be $y=\tan x$, as often as it was $x=\pm(i\pm n)\pi$.

COROLLARY 2

§50 If in the solution of the first example, in which it was

$$p=f,\quad q=f,\quad r=-f,\quad s=-f,\quad t=f,\quad u=f\quad \text{etc.},$$

we would have substituted the found value for $\triangle$ immediately, this equation would have resulted

$$y=\frac{f\sin\omega\pi}{\sin n\pi}.$$

Hence it would have been perspicuous that having taken $\theta=\pi$ and $f=\sin n\pi$ the curve itself will be a sine-curve.

SCHOLIUM

§51 It especially deserves to be mentioned that in problem 4, where the given abscissas constitute an interrupted arithmetic progression, the value of the quantity $\triangle$ can be exhibited absolutely in terms of angles, although nevertheless in problem 3, where the given abscissas constituted a true arithmetic progression, the integral formula $\triangle$ in general cannot be expressed in terms of angles by any means. For, since there it was

$$\triangle = \int z^{n-\omega-1} dz (1-z)^{n-\omega-1},$$

this formula having put $n - \omega = \frac{\nu}{\lambda}$ and $z = v^{\lambda}$ goes over into

$$\triangle = \lambda \int \frac{v^{\nu-1} dv}{\sqrt[\lambda]{(1-v^{\lambda})^{\lambda-\nu}}} \quad \text{or} \quad \triangle = \lambda \left(\frac{\nu}{\nu}\right),$$

which formula can involve highly transcendental formulas. If in that problem the given ordinates are set

$$p = f, \quad q = -f, \quad r = f, \quad s = -f, \quad t = f, \quad u = -f \quad \text{etc.}$$

and it is $n = \frac{1}{2}$, the equation for the curve passing through these points will be

$$\frac{\triangle y}{2(1+2\omega)\omega f} = \frac{4}{1-4\omega\omega} + \frac{4}{9-4\omega\omega} + \frac{4}{1-4\omega\omega} + \text{etc.}$$

or

$$\frac{\triangle y}{2f\omega(1+2\omega)} = \frac{\pi}{2\omega} \tan \omega\pi,$$

such that it is

$$y = \frac{\pi f(1+2\omega) \tan \omega\pi}{\triangle},$$

whence, even though one takes

$$\theta = \pi \quad \text{and} \quad f = \sin n\theta = \sin \frac{1}{2}\pi = 1,$$

it manifestly does not follow that it will be $y = \sin \theta\omega = \sin \omega\pi$. Since in the first example it is already certain that it is

$$y = \frac{f \sin \omega\pi}{\sin n\pi},$$

let us expand the same case of the first problem in such a way that we investigate the values of the single coefficients A, B, C, D etc.

Problem 5

§52 *To determine the general equation constituted above in Problem 1 in such a way that to these abscissas*

$$x \quad = n\theta, \qquad (1-n)\theta, \qquad (1+n)\theta, \qquad (2-n)\theta, \qquad (2+n)\theta \qquad \text{etc.}$$

these ordinates correspond

$$y = +f, \qquad +f, \qquad -f, \qquad -f, \qquad +f, \qquad \text{etc.},$$

Solution

As before set $x = \theta\omega$ and consider the equation in question expressed in this form

$$y = A\omega + B\omega(\omega\omega - nn) + C\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)$$
$$+D\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)(\omega\omega - (1+n)^2)$$
$$+E\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)(\omega\omega - (1+n)^2)(\omega\omega - (2-n)^2)$$
$$+\text{etc},$$

whence these equations are deduced

$$\begin{aligned}
\frac{f}{n} &= A, \\
\frac{f}{1-n} &= A \ + B\cdot 1(1-2n), \\
\frac{-f}{1+n} &= A \ + B\cdot 1(1-2n) \ + C\cdot(1+2n)\cdot 2\cdot 2n, \\
\frac{-f}{2-n} &= A \ + B\cdot 1(1-2n) \ + C\cdot 2(2-2n)\cdot 1(3-2n) \\
&\qquad + D\cdot 2(2-2n)\cdot 1(3-2n)\cdot 3(1-2n)
\end{aligned}$$

$$\text{etc.}$$

and hence the following values of the coefficients

$$A = \frac{f}{n}, \quad B = \frac{-f}{n(1-n)}, \quad C = \frac{f}{2n(1-n)(1+n)}, \quad D = \frac{-f}{6n(1-n)(1+n)(2-n)},$$

$$D = \frac{f}{24n(1-n)(1+n)(2-n)(2+n)} \quad \text{etc.;}$$

since this progression is rather simple, our series for the value of y, which we already found to be

$$= \frac{f \sin \omega\pi}{\sin n\pi},$$

deserves even greater attention

$$\frac{\sin \omega\pi}{\sin n\pi} = \frac{\omega}{n} - \frac{\omega}{n} \cdot \frac{\omega\omega - nn}{1(1-n)} + \frac{\omega}{n} \cdot \frac{\omega\omega - nn}{1(1-n)} \cdot \frac{\omega\omega - (1-n)^2}{2(1+n)}$$

$$- \frac{\omega}{n} \cdot \frac{\omega\omega - nn}{1(1-n)} \cdot \frac{\omega\omega - (1-n)^2}{2(1+n)} \cdot \frac{\omega\omega - (1+n)^2}{3(2-n)} + \text{etc.},$$

or if Π always denotes the preceding term, the whole expression will be

$$\frac{\sin \omega\pi}{\sin n\pi} = \frac{\omega}{n} - \Pi \cdot \frac{\omega\omega - nn}{1(1-n)} + \Pi \cdot \frac{\omega\omega - (1-n)^2}{2(1+n)} - \Pi \cdot \frac{\omega\omega - (1+n)^2}{3(2-n)}$$

$$+ \Pi \cdot \frac{\omega\omega - (2-n)^2}{4(2+n)} - \Pi \cdot \frac{\omega\omega - (2+n)^2}{5(3-n)} + \text{etc.}$$

Simplifying this expression in such a way that all terms have the same sign, it will be

$$\begin{aligned} \frac{\sin \omega\pi}{\sin n\pi} = \frac{\omega}{n} + \frac{\omega}{n} \cdot \frac{nn - \omega\omega}{1(1-n)} + \frac{\omega}{n} \cdot \frac{nn - \omega\omega}{1(1-n)} \cdot \frac{(1-n)^2 - \omega\omega}{2(1+n)} \\ + \frac{\omega}{n} \cdot \frac{nn - \omega\omega}{1(1-n)} \cdot \frac{(1-n)^2 - \omega\omega}{2(1+n)} \cdot \frac{(1+n)^2 - \omega\omega}{3(2-n)} \\ + \frac{\omega}{n} \cdot \frac{nn - \omega\omega}{1(1-n)} \cdot \frac{(1-n)^2 - \omega\omega}{2(1+n)} \cdot \frac{(1+n)^2 - \omega\omega}{3(2-n)} \cdot \frac{(2-n)^2 - \omega\omega}{4(2+n)} \end{aligned}$$

$$\text{etc.}$$

Therefore, this series seems to be even more remarkable, since it recedes from the usual form of a series and even the two arbitrary numbers n and ω occur in it.

COROLLARY 1

§53 If the number ω vanishes, that it is $\sin\omega\pi = \omega\pi$, having divided by ω one will have the equation

$$\frac{\pi}{\sin n\pi} = \frac{1}{n} + \frac{n}{1(1-n)} + \frac{n(1-n)}{1\cdot 2(1+n)} + \frac{n(1-n)(1+n)}{1\cdot 2\cdot 3(2-n)}$$
$$+\frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4(2+n)} + \text{etc.},$$

whence for $n = \frac{1}{2}$ because of $\sin\frac{\pi}{2} = 1$ it will be

$$\pi = 2 + 1 + \frac{1\cdot 1\cdot 2}{2\cdot 4\cdot 3} + \frac{1\cdot 1\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 8\cdot 5} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 2}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 5} + \text{etc.}$$

or

$$\pi = 2 + \frac{1}{2\cdot 2^1\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 2^3\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^5\cdot 7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 82^7\cdot 9} + \text{etc.}$$
$$+1 + \frac{1}{2\cdot 2^2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 2^4\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^6\cdot 7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 82^8\cdot 9} + \text{etc.};$$

since the second of these series is the half of the first, the sum of the second will be $= \frac{\pi}{3}$; the reason for this follows from this equation

$$\int \frac{dx}{\sqrt{1-xx}} = \arcsin x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \text{etc.},$$

whence that series becomes $= \frac{\arcsin x}{x}$ for $x = \frac{1}{2}$ and hence $= 2\frac{\pi}{6} = \frac{\pi}{3}$.

COROLLARY 2

§54 If the other number n vanishes and it is $\sin n\pi = n\pi$ and the equation is multiplied by n, this equation will result

$$\frac{\sin\omega\pi}{\pi} = \omega - \frac{\omega^3}{1} + \frac{\omega^3(\omega^2-1)}{1\cdot 2\cdot 1^2} - \frac{\omega^3(\omega^2-1)(\omega^2-1)}{1\cdot 2\cdot 3\cdot 1^2\cdot 2} + \frac{\omega^3(\omega^2-1)(\omega^2-1)(\omega^2-4)}{1\cdot 2\cdot 3\cdot 4\cdot 1^2\cdot 2^2}$$
$$-\frac{\omega^3(\omega^2-1)(\omega^2-1)(\omega^2-4)(\omega^2-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 1^2\cdot 2^2\cdot 3} + \text{etc.},$$

which series divided by ω is resolved into the following two

$$\frac{\sin\omega\pi}{\omega\pi} = 1 + \frac{\omega^2(\omega^2-1)}{1\cdot 2\cdot 1^2} + \frac{\omega^2(\omega^2-1)^2(\omega^2-4)}{1\cdot 2\cdot 3\cdot 4\cdot 1^2\cdot 2^2} + \frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2(\omega^2-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 1^2\cdot 2^2\cdot 3^2} + \text{etc.}$$

$$-\frac{\omega^2}{1}-\frac{\omega^2(\omega^2-1)^2}{1\cdot2\cdot3\cdot1^2\cdot2}-\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2}{1\cdot2\cdot3\cdot4\cdot5\cdot1^2\cdot2^2\cdot3}-\frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2(\omega^2-9)^2}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot1^2\cdot2^2\cdot3^2\cdot4}-\text{etc.}$$

Let us set $\omega=\frac{1}{2}$ here; it will be

$$\frac{2}{\pi}=1-\frac{1\cdot1\cdot3}{1\cdot1\cdot1\cdot2^5}-\frac{1\cdot1\cdot3\cdot1\cdot3\cdot5}{1\cdot1\cdot1\cdot2\cdot2\cdot2\cdot2^{10}}-\frac{1\cdot1\cdot3\cdot1\cdot3\cdot5\cdot3\cdot5\cdot7}{1\cdot1\cdot1\cdot2\cdot2\cdot2\cdot3\cdot3\cdot3\cdot2^{15}}-\text{etc.}$$

$$-\frac{1\cdot1}{2^2}-\frac{1\cdot1\cdot1\cdot3\cdot1\cdot3}{1\cdot1\cdot1\cdot2\cdot2\cdot3\cdot2^6}-\frac{1\cdot1\cdot1\cdot3\cdot1\cdot3\cdot3\cdot5\cdot3\cdot5}{1\cdot1\cdot1\cdot2\cdot2\cdot2\cdot3\cdot3\cdot4\cdot5\cdot2^{10}}-\text{etc.},$$

which last series can be represented this way.

$$-\frac{1}{2^2}-\frac{1\cdot1\cdot1\cdot3}{1\cdot1\cdot2\cdot2^7}-\frac{1\cdot1\cdot3\cdot3\cdot3\cdot5}{1\cdot1\cdot2\cdot2\cdot2\cdot3\cdot2^{12}}-\frac{1\cdot1\cdot3\cdot3\cdot3\cdot5\cdot5\cdot5\cdot7}{1\cdot1\cdot2\cdot2\cdot2\cdot3\cdot3\cdot3\cdot4\cdot2^{17}}-\text{etc.}$$

Corollary 3

§55 If it was $n=\frac{1}{2}$ that it is $\sin n\pi=1$, the factors, from which the single terms of the series must be formed, will be

$$\frac{2\omega}{1}\cdot\frac{1-4\omega\omega}{1\cdot2}\cdot\frac{1-4\omega\omega}{3\cdot4}\cdot\frac{9-4\omega\omega}{3\cdot6}\cdot\frac{9-4\omega\omega}{5\cdot8}\cdot\frac{25-4\omega\omega}{5\cdot10}\cdot\frac{25-4\omega\omega}{7\cdot12}\cdot\text{etc.}$$

and the sum of the series will be $\sin\omega\pi$, namely

$$\sin\omega\pi=2\omega+\frac{2\omega(1-4\omega\omega)}{1\cdot2}+\frac{2\omega(1-4\omega\omega)^2}{1\cdot2\cdot3\cdot4}+\frac{2\omega(1-4\omega\omega)^2(9-4\omega\omega)}{1\cdot2\cdot3\cdot4\cdot5\cdot6}+\text{etc.};$$

hence for $\omega=1$ it must be

$$0=2-3+\frac{3}{2^2}+\frac{5}{2^3\cdot3}+\frac{5}{2^6\cdot3}+\frac{7}{2^7\cdot3}+\frac{7}{2^9\cdot5}+\frac{9}{2^{10}\cdot7}+\frac{5\cdot9}{2^{14}\cdot7}+\text{etc.},$$

and this equation will immediately be seen to be true by actual calculation.

Scholium

§56 For this case also the solution found should be considered with more attention; recalling the results of § 36 because of

$$\triangle=\frac{(n-\omega)\pi\sin2n\pi}{2n\sin(n-\omega)\pi\cdot\sin(n-\omega)\pi}\quad\text{and}\quad y=\frac{f\sin\omega\pi}{\sin n\pi},$$

since it is

$$p = f, \quad q = f, \quad r = -f, \quad s = -f, \quad t = f, \quad u = f \quad \text{etc.},$$

this solution is contained in this equation

$$\frac{\pi \cos n\pi \cdot \sin \omega\pi}{\omega \sin(n-\omega)\pi \cdot \sin(n+\omega)\pi}$$

$$= \frac{1}{nn - \omega\omega} - \frac{1}{(1-n) - \omega^2} - \frac{1}{(1+n) - \omega^2} + \frac{1}{(2-n) - \omega^2} + \frac{1}{(2+n) - \omega^2} + \text{etc.};$$

this series deviates a lot from the one we just found. But I observe the following things on this series:

I. If ω vanishes, it will be

$$\frac{\pi\pi \cos n\pi}{(\sin n\pi)^2} = \frac{1}{nn} - \frac{1}{(1-n)^2} - \frac{1}{(1+n)^2} + \frac{1}{(2-n)^2} + \frac{1}{(2+n)^2} - \frac{1}{(3-n)^2} - \text{etc.};$$

but if additionally n vanishes, because of $\sin n\pi = n\pi$ following paradoxical equation results

$$\frac{1}{nn} = \frac{1}{nn} - \frac{2}{1} + \frac{2}{4} - \frac{2}{9} + \frac{2}{16} - \text{etc.}$$

But in order to explain this paradox, let us not consider the number n only as infinitely small, and since it is

$$\cos n\pi = 1 - \frac{1}{2} nn\pi\pi$$

and also

$$\sin n\pi = n\pi - \frac{1}{6} n^3\pi^3 = n\pi\left(1 - \frac{1}{6} nn\pi\pi\right),$$

it will be

$$\frac{\cos n\pi}{(\sin n\pi)^2} = \frac{1 - \frac{1}{2}nn\pi\pi}{nn\pi\pi(1 - \frac{1}{3}nn\pi\pi)} = \frac{1 - \frac{1}{6}nn\pi\pi}{nn\pi\pi};$$

hence this true equation is obtained

$$\frac{1}{nn} - \frac{1}{6}\pi\pi = \frac{1}{nn} - \frac{2}{1} + \frac{2}{4} - \frac{2}{9} + \frac{2}{16} - \frac{2}{25} + \text{etc.}$$

For, it is

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\text{etc.}=\frac{1}{12}\pi\pi.$$

II. Now let us put $n=0$ and we will have

$$-\frac{\pi}{\omega\sin\omega\pi}=-\frac{1}{\omega^2}-\frac{1}{1-\omega^2}-\frac{1}{1-\omega^2}+\frac{1}{4-\omega^2}+\frac{1}{4-\omega^2}-\frac{1}{9-\omega^2}-\frac{1}{9-\omega^2}+\text{etc.}$$

or

$$\frac{\pi}{\omega\sin\omega\pi}=\frac{1}{\omega^2}+\frac{2}{1-\omega^2}-\frac{2}{4-\omega^2}+\frac{2}{9-\omega^2}-\frac{2}{16-\omega^2}+\frac{2}{25-\omega^2}-\text{etc.};$$

hence we obtain this memorable summation

$$\frac{1}{1-\omega^2}-\frac{1}{4-\omega^2}+\frac{1}{9-\omega^2}-\frac{1}{16-\omega^2}+\text{etc.}=\frac{\pi}{2\omega\sin\omega\pi}-\frac{1}{2\omega\omega},$$

whose truth I demonstrated elsewhere[8]. But hence for an infinitely small ω because of

$$\sin\omega\pi=\omega\pi\left(1-\frac{1}{6}\omega^2\pi^2\right)$$

the sum of the series

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\text{etc.}$$

as before is calculated to be

$$\frac{1}{2\omega\omega(1-\frac{1}{6}\omega^2\pi^2)}-\frac{1}{2\omega\omega}=\frac{1}{12}\pi\pi.$$

III. If one sets $n=\frac{1}{2}$, because of $\cos n\pi=0$ also the series itself vanishes, while all terms indeed cancel each other. But for this to happen, if n differs infinitely less from $\frac{1}{2}$, differentiate with respect to the variable n, whence it is

$$-\frac{n\pi\sin n\pi\sin\omega\pi(1+\cos(n-\omega)\pi\cdot\cos(n+\omega)\pi)}{\omega(\sin(n-\omega)\pi\cdot\sin(n+\omega)\pi)^2}=-\frac{2n}{(nn-\omega\omega)^2}-\frac{2(1-n)}{((1-n)^2-\omega^2)^2}$$

[8]Euler refers to E130.

$$\frac{2(1+n)}{((1+n)^2-\omega^2)^2}+\frac{2(2-n)}{((2-n)^2-\omega^2)^2}-\frac{2(2+n)}{((2+n)^2-\omega^2)^2}-\text{etc.}$$

Therefore, now take $n=\frac{1}{2}$ and it will be

$$-\frac{\pi\pi\sin\omega\pi}{\omega(\cos\omega\pi)^2}=-\frac{16}{(1-4\omega^2)^2}-\frac{16}{(1-4\omega^2)^2}+\frac{3\cdot 16}{(9-4\omega^2)^2}+\frac{3\cdot 16}{(9-4\omega^2)^2}-\text{etc.}$$

or

$$\frac{\pi\pi\sin\omega\pi}{32\omega(\cos\omega\pi)^2}=\frac{1}{(1-4\omega^2)^2}-\frac{3}{(9-4\omega^2)^2}+\frac{5}{(25-4\omega^2)^2}-\frac{7}{(49-4\omega^2)^2}+\text{etc.},$$

where for $\omega=0$ it follows that it will be

$$\frac{\pi^3}{32}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.},$$

which is certainly known from elsewhere.

But the series found in the preceding problem seems to be a lot more difficult. Yes, even the case expanded in corollary 1, even though it is highly special case, deserves a more diligent expansion, which I will try to give in the following problem.

PROBLEM 6

§57 *If n is an arbitrary number, to find the sum of this series*

$$s=\frac{1}{n}+\frac{n}{1(1-n)}+\frac{n(1-n)}{1\cdot 2(1+n)}+\frac{n(1-n)(1+n)}{1\cdot 2\cdot 3(2-n)}+\frac{n(1-n)(1+n)(2-n)}{1\cdot 2\cdot 3\cdot 4(2+n)}+\text{etc.},$$

which we certainly found before [§ 53] to be

$$s=\frac{\pi}{\sin n\pi}.$$

SOLUTION

Since in this series the structure of the terms is irregular, it will be convenient to split it into two parts. Therefore, let us set

$$P=\frac{1}{n}+\frac{n(1-n)}{1\cdot 2(1+n)}$$

$$+\frac{n(1-n)(1+n)(2-n)}{1\cdot2\cdot3\cdot4(2+n)}+\frac{n(1-n)(1+n)(2-n)(2+n)(3-n)}{1\cdot2\cdot3\cdot4\cdot5\cdot6(3+n)}+\text{etc.},$$

$$Q=\frac{n}{1(1-n)}+\frac{n(1-n)(1+n)}{1\cdot2\cdot3(2-n)}+\frac{n(1-n)(1+n)(2-n)(2+n)}{1\cdot2\cdot3\cdot4\cdot5(3-n)}+\text{etc.},$$

so that it is

$$s=P+Q.$$

Now, investigating the sum of these series I recall the following series known from the theory of trigonometric functions

$$\frac{\cos\mu\varphi}{\cos\varphi}=1+\frac{(1-\mu)(1+\mu)}{1\cdot2}\sin^2\varphi+\frac{(1-\mu)(1+\mu)(3-\mu)(3+\mu)}{1\cdot2\cdot3\cdot4}\sin^4\varphi+\text{etc.},$$

$$\frac{\sin\nu}{\cos\varphi}=\nu\sin\varphi+\frac{\nu(2-\nu)(2+\nu)}{1\cdot2\cdot3}\sin^3\varphi+\frac{\nu(2-\nu)(2+\nu)(4-\nu)(4+\nu)}{1\cdot2\cdot3\cdot4\cdot5}\sin^5\varphi+\text{etc.}$$

and first I will apply it to the first form P. Therefore, since these fractions

$$\frac{(1-\mu)(1+\mu)}{n(1-n)},\quad\frac{(3-\mu)(3+\mu)}{(1+n)(2-n)},\quad\frac{(5-\mu)(5+\mu)}{(2+n)(3-n)}\quad\text{etc.}$$

must be equal, I conclude that one has to take $\mu=1-2n$, whence it will be

$$\frac{\cos(1-2n)\varphi}{\cos\varphi}=1+\frac{n(1-n)}{1\cdot2}\cdot2^2\sin^2\varphi+\frac{n(1-n)(1+n)(2-n)}{1\cdot2\cdot3\cdot4}\cdot2^4\sin^4\varphi+\text{etc.}$$

Let us multiply this equation by $d\varphi\sin^{2n-1}\varphi\cos\varphi$ and integrate, it will be

$$\int\mathrm{d}\varphi\sin^{2n-1}\varphi\cos(1-2n)\varphi=\frac{1}{2n}\cdot\sin^{2n}\varphi+\frac{n(1-n)}{1\cdot2(n+1)}\cdot2\sin^{2n+2}\varphi$$

$$+\frac{n(1-n)(1+n)(2-n)}{1\cdot2\cdot3\cdot4(n+2)}\cdot2^3\sin^{2n+4}\varphi+\text{etc.}$$

Now after the integration set $\sin\varphi=\frac{1}{2}$ or $\varphi=30^\circ$ and it will be

$$P=2^{2n+1}\int\mathrm{d}\varphi\sin^{2n-1}\varphi\cos(1-2n)\varphi;$$

the series Q on the other hand will easily be deduced from the other known series by taking $\nu=2n$, whence it is

$$\frac{\sin2n\varphi}{\cos\varphi}=n\cdot2\sin\varphi+\frac{n(1-n)(1+n)}{1\cdot2\cdot3}\cdot2^3\sin^3\varphi$$

$$+\frac{n(1-n)(1+n)(2-n)(2+n)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot 2^5\sin^5\varphi+\text{etc.}$$

Multiply this equation by $\mathrm{d}\varphi\sin^{-2n}\varphi\ \cos\varphi$ and integrate again; it will be

$$\int \mathrm{d}\varphi\sin^{-2n}\varphi\ \sin 2n\varphi=\frac{n}{1(1-n)}\cdot\sin^{2-2n}\varphi+\frac{n(1-n)(1+n)}{1\cdot 2\cdot 3(2-n)}\cdot 2^2\sin^{4-2n}\varphi+\text{etc.}$$

After the integration set $\sin\varphi=\frac{1}{2}$ or $\varphi=30^{\circ}$ and this expression will result

$$Q=2^{2-2n}\int \mathrm{d}\varphi\sin^{-2n}\varphi\ \sin 2n\varphi.$$

Therefore, the sum of the propounded series will expressed in such a way that it is

$$s=2^{2n+1}\int \mathrm{d}\varphi\sin^{2n-1}\varphi\cos(1-2n)\varphi+2^{2-2n}\int \mathrm{d}\varphi\sin^{-2n}\varphi\ \sin 2n\varphi,$$

and since this sum is already known from elsewhere, one will have

$$\frac{\pi}{\sin n\pi}=4\int \mathrm{d}\varphi\ \cos(1-2n)\varphi\ (2\sin\varphi)^{2n-1}+4\int \mathrm{d}\varphi\ \sin 2n\varphi\ (2\sin\varphi)^{-2n}.$$

COROLLARY 1

§58 If one puts $2n=\frac{1-\lambda}{2}$, it will be $1-2n=\frac{1+\lambda}{2}$; this way our equation becomes more convenient and it will be

$$\frac{\pi}{\sin\frac{1-\lambda}{4}\pi}=4\int\frac{\mathrm{d}\varphi\cos\frac{1+\lambda}{2}\varphi}{(2\sin\varphi)^{\frac{1+\lambda}{2}}}+4\int\frac{\mathrm{d}\varphi\sin\frac{1-\lambda}{2}\varphi}{(2\sin\varphi)^{\frac{1-\lambda}{2}}}=\frac{\pi\sqrt{2}}{\cos\frac{\lambda\pi}{4}-\sin\frac{\lambda\pi}{4}},$$

having put $\varphi=30^{\circ}$ after the integration.

COROLLARY 2

§59 In like manner having taken a negative λ it will be

$$\frac{\pi}{\sin\frac{1+\lambda}{4}\pi}=4\int\frac{\mathrm{d}\varphi\cos\frac{1-\lambda}{2}\varphi}{(2\sin\varphi)^{\frac{1-\lambda}{2}}}+4\int\frac{\mathrm{d}\varphi\sin\frac{1+\lambda}{2}\varphi}{(2\sin\varphi)^{\frac{1+\lambda}{2}}}=\frac{\pi\sqrt{2}}{\cos\frac{\lambda\pi}{4}+\sin\frac{\lambda\pi}{4}},$$

where it will be helpful to have noted that in all cases, which can be expanded, the same value of these integral formulas we exhibited here is actually found.