The universal Method to sum series further promoted *

Leonhard Euler

§1 The universal method to sum series I explained at the end of the last year[1], certainly extends very far, since using only the given general term of the series it exhibits a formula equal to the sum of the series.; nevertheless, it is difficult to apply it to series of such a kind, whose general terms cannot be expressed algebraically, but involve either exponential or even transcendental quantities. For, since having put the term corresponding to the index x X the sum of the series from the first term to X is equal to

$$\int Xdx + \frac{X}{1\cdot 2} + \frac{dX}{1\cdot 2\cdot 3\cdot 2dx} - \frac{d^3X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx^3} + \text{etc.},$$

it is easily seen, if X involves at least quantities of this kind a^x, that so the expression $\int Xdx$ as the differentials of X lead to logarithms, causing a huge difficulty to assign the sum in question even at least approximately.

§2 Furthermore, even if X was an algebraic quantity, nevertheless in many cases the differentials, which must be summed to obtain the sum, become so complicated that it is not only very difficult to exhibit them, but also yield a slowly converging series, as it is the case in the series

*Original title: " Methodus universalis Series summandi ulterius promota", first published in „*Commentarii academiae scientiarum Petropolitanae* 8 (1736), 1741, p. 147-158", reprinted in in „*Opera Omnia*: Series 1, Volume 14, pp. 124 - 137 ", Eneström-Number E55, translated by: Alexander Aycock for „Euler-Kreis Mainz"

[1] Euler refers to his papers "Methodus universalis serierum convergentium summas quam proxime inveniendi" and "Inventio summae cuiusque seriei ex dato termino generali". These are the papers E46 and E47 in the Eneström-Index, respectively.

$$\frac{1}{3}+\frac{1}{35}+\frac{1}{99}+\text{etc.}$$

containing the quadrature of the circle. The main reason for this difficulty is that I assumed the indices of the terms to grow by one; if I would have assumed them to grow by another number, a more tractable general term X might would have resulted. Finally, if the general term X can even not be exhibited at all, as it is the case in most series, then the given formula exhibiting the sum is completely useless.

§3 I have thought about this a long time how I could overcome these difficulties and observed that using the same principle, I applied to find that formula, also other formulas appropriate to sum series can be found; having exhibited these formulas for every series the most convenient formula is to be chosen. But it seems to be convenient that two formulas of each class of formulas of this kind are given, the one of which is appropriate to sum series from the first term up to a given term, of which kind the formula already mentioned was, but the other is appropriate to sum series from a given term to infinity. For, although this last summation follows from the first, it will nevertheless be useful to have given a peculiar formula for this case.

§4 Therefore, I will start from series, whose general term can be exhibited algebraically; for this the method given in the preceding paper[2] can be used; but I will assume the indices to proceed in an arbitrary arithmetic progression, so that the found formula extends further and will more often ensure a convenient calculation. Therefore, let the series to be summed from the beginning to a given term along with its indices be the following

$$\begin{array}{cccccc} a & a+b & a+2b & & x & \\ A & +B & +C & +\cdots & +X & =S, \end{array}$$

where the indices grow by the quantity b and the index of the first term A is a. Put the sum of this series $= S$; if in this expression $x-b$ is substituted for for x, it is perspicuous that it will exhibit the same sum without the term X or that it will be equal to $S-X$. But if $x-b$ is put instead of x in S, then it will result

[2]Euler refers to E47 again.

$$S - \frac{bdS}{1dx} + \frac{b^2ddS}{1\cdot 2dx^2} - \frac{b^3d^3S}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

whence one will have the following equation

$$X = \frac{bdS}{1dx} - \frac{b^2ddS}{1\cdot 2dx^2} + \frac{b^3d^3S}{1\cdot 2\cdot 3dx^3} - \frac{b^4d^4S}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

But from this equation one finds this formula

$$S = \int \frac{Xdx}{b} + \frac{X}{1\cdot 2} + \frac{bdX}{1\cdot 2\cdot 3\cdot 2dx} - \frac{b^3d^3X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx^3} + \frac{b^5d^5X}{1\cdot 2\cdot 3\cdots 7\cdot 6dx^5}$$
$$- \frac{3b^7d^7X}{1\cdot 2\cdot 3\cdot 9\cdot 10dx^7} + \frac{5b^9d^9X}{1\cdot 2\cdot 3\cdots 11\cdot 6dx^9} - \frac{691b^{11}d^{11}X}{1\cdot 2\cdot 3\cdots 12\cdot 210dx^{11}}$$
$$+ \frac{35b^{13}d^{13}X}{1\cdot 2\cdot 3\cdots 15\cdot 2dx^{13}} - \frac{3617b^{15}d^{15}X}{1\cdot 2\cdot 3\cdots 17\cdots 30dx^{15}} + \frac{43867b^{17}d^{17}X}{1\cdot 2\cdot 3\cdots 19\cdot 42dx^{17}} - \text{etc.};$$

to this expression such a large constant is to be added that for $x = a$ it is $S = A$ or having put $x = a - b$ it is $S = 0$.

§5 If one puts $X = x^n$ or if the sum of this series is to be found

$$a^n + (a+b)^n + (a+2b)^n + \cdots + x^n,$$

it will be

$$\int Xdx = \frac{x^{n+1}}{n+1}, \quad \frac{dX}{dx} = nx^{n-1}, \quad \frac{d^3X}{dx^3} = n(n-1)(n-2)x^{n-3} \quad \text{etc.}$$

Hence the sum in question will be

$$S = \frac{x^{n+1}}{(n+1)b} + \frac{x^n}{1\cdot 2} + \frac{nbx^{n-1}}{1\cdot 2\cdot 3\cdot 2} - \frac{n(n-1)(n-2)b^3x^{n-3}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)b^5x^{n-5}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 6} - \text{etc.} - \frac{a^{n+1}}{(n+1)b} + \frac{a^n}{1\cdot 2} - \frac{nba^{n-1}}{1\cdot 2\cdot 3\cdot 2}$$
$$+ \frac{n(n-1)(n-2)b^3a^{n-3}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{n(n-1)(n-2)(n-3)(n-4)b^5a^{n-5}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 6} + \text{etc.}$$

having added the corresponding constant. Therefore, the sum of the series

$$a+(a+b)+(a+2b)+\cdots+x$$

will be

$$=\frac{x^2}{2b}+\frac{x}{2}+\frac{b}{12}-\frac{a^2}{2b}+\frac{a}{2}-\frac{b}{12}=\frac{x^2-a^2+bx+ab}{2b}$$

and the sum of the series

$$a^2+(a+b)^2+(a+2b)^3+\cdots+x^2=\frac{x^3}{3b}+\frac{x^2}{2}+\frac{bx}{6}-\frac{a^3}{3b}+\frac{a^2}{2}-\frac{ab}{6}$$
$$=\frac{2x^3-2a^3+3bx^2+3a^2b+b^2x-ab^2}{6b}.$$

These expressions are similar to those I gave for the sum of powers of natural numbers in the upper dissertation[3], and using these expressions they are also easily formed.

§6 Now to find the other formula of this class let a series to be summed from the first term X, whose index we want to put x, to infinity, namely this one

$$\begin{array}{llll} x & x+b & x+2b & \\ X & +Y & +Z & +\text{ etc. to infinity } = S. \end{array}$$

Therefore, if one writes $x+b$ for x in the sum S, $S-X$ will result; hence it will be

$$X=-\frac{bdS}{1dx}-\frac{b^2ddS}{1\cdot 2dx^2}-\frac{b^3d^3S}{1\cdot 2\cdot 3dx^3}-\text{etc.},$$

whence as above it will be found

$$S=-\int\frac{Xdx}{b}+\frac{X}{1\cdot 2}+\frac{bdX}{1\cdot 2\cdot 3\cdot 2dx}-\frac{b^3d^3X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx^3}-\frac{b^5d^5X}{1\cdot 2\cdot 3\cdots 7\cdot 6dx^5}$$
$$+\frac{3b^7d^7X}{1\cdot 2\cdot 3\cdot 9\cdot 10dx^7}-\frac{5b^9d^9X}{1\cdot 2\cdot 3\cdots 11\cdot 6dx^9}+\frac{691b^{11}d^{11}X}{1\cdot 2\cdot 3\cdots 12\cdot 210dx^{11}}$$
$$-\frac{35b^{13}d^{13}X}{1\cdot 2\cdot 3\cdots 15\cdot 2dx^{13}}+\frac{3617b^{15}d^{15}X}{1\cdot 2\cdot 3\cdots 17\cdots 30dx^{15}}-\frac{43867b^{17}d^{17}X}{1\cdot 2\cdot 3\cdots 19\cdot 42dx^{17}}+\text{etc.}$$

[3]Euler refers to E47 again.

To this formula such a large constant is to be added that it is $S=0$, if one puts $x=\infty$; for, if X already was the infinitesimal term or the last term of the series, the sum must vanish, if the series has a finite sum, of course, what we will always assume here.

§7 In order to understand the application of this formula, let $X=\frac{1}{x^2}$ or this series

$$\frac{1}{x^2}+\frac{1}{(x+b)^2}+\frac{1}{(x+2b)^2}+\text{etc.}$$

to be summed to infinity; therefore, because of

$$\int Xdx=-\frac{1}{x},\quad \frac{dX}{dx}=\frac{-2}{x^3},\quad \frac{dX}{dx^3}=\frac{-2\cdot 3\cdot 4}{x^5}\quad \text{etc.}$$

it will be

$$S=\frac{1}{bx}+\frac{1}{2x^2}+\frac{b}{6x^3}-\frac{b^3}{3030x^5}+\frac{b^5}{42x^7}-\frac{b^7}{30x^9}+\frac{5b^9}{66x^{11}}-\frac{691b^{11}}{2730x^{13}}$$
$$+\frac{7b^{13}}{6x^{15}}-\frac{3617b^{15}}{510x^{17}}+\frac{43867b^{17}}{35910x^{19}}-\text{etc.},$$

which expression does not require a constant, since it vanishes per se having put $x=\infty$. But the series will converge the more the greater x was with respect to b. Hence, if several initial terms of the given series are actually added, the sum of the remaining ones found by this method added to that aggregate will give the sum of the propounded series continued to infinity.

§8 But having mentioned all these things, which only render the first rule more convenient, I proceed to the summation of series for which that formula does not suffice. For, let a series to be summed in which the signs of the terms alternate as, e.g.,

$$\begin{array}{cccccccc} a & a+b & a+2b & a+3b & & x & x+b & \\ A & +B & +C & -D & +\cdots+ & X & -Y & =S; \end{array}$$

since in this series Y is such a function of $x+b$ as X is one of x, it will be

$$Y=X+\frac{bdX}{1dx}+\frac{b^2ddX}{1\cdot 2dx^2}+\frac{b^3d^3X}{1\cdot 2\cdot 3dx^3}+\text{etc.}$$

Further, if one puts $x-2b$ instead of x in S, $S-X+Y$ will result; therefore, it will be

$$-Y+X=\frac{2bdS}{1dx}-\frac{4b^2ddS}{1\cdot 2dx^2}+\frac{8b^3d^3S}{1\cdot 2\cdot 3dx^3}-\frac{16b^4d^4S}{1\cdot 2\cdot 3\cdot 4dx^4}+\frac{32b^5d^5S}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^5}-\text{etc.}$$
$$=-\frac{bdX}{1dx}-\frac{b^2d^2X}{1\cdot 2dx^2}-\frac{b^3d^3X}{1\cdot 2\cdot 3dx^3}-\frac{b^4d^4X}{1\cdot 2\cdot 3\cdot 4dx^4}-\text{etc.}$$

Put

$$\frac{dS}{dx}=\frac{\alpha dX}{dx}+\frac{\beta d^2X}{dx^2}+\frac{\gamma d^3X}{dx^3}+\frac{\delta d^4X}{dx^4}+\text{etc.}$$

and by comparing the homologous terms[4] it will result

$$S=C-\frac{X}{2}-\frac{3bdX}{4dx}-\frac{b^2ddX}{2dx^2}-\frac{3b^3d^3X}{16dx^3}-\frac{b^4d^4X}{24dx^4}-\frac{9b^5d^5X}{1440dx^5}-\text{etc.}$$

and by introducing Y it will be

$$S=C-Y+\frac{X}{2}+\frac{bdX}{1\cdot 2\cdot 2dx}-\frac{b^3d^3X}{1\cdot 2\cdot 3\cdot 4\cdot 2dx^3}+\frac{3b^5d^5X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2dx^5}-\text{etc.}$$

And hence it will be

$$A-B+C-D+E-F+\cdots+X$$
$$=\text{Const.}+\frac{X}{2}+\frac{bdX}{1\cdot 2\cdot 2dx}-\frac{b^3d^3X}{1\cdot 2\cdot 3\cdot 4\cdot 2dx^3}+\frac{3b^5d^5X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 2dx^5}-\text{etc.,}$$

where the constant, as before, must be of such a nature that this sum becomes $=A$ having put $x=a$.

§9 But having continued the terms of this formula further and having put the sum of this series

$$\begin{array}{cccccccc} a & a+b & a+2b & a+3b & & & x & \\ A & -B & +C & -D & +\cdots & + & X & =S; \end{array}$$

[4]By this Euler means the terms containing the same order of differentials.

it will result

$$S = \text{Const.} + \frac{X}{1 \cdot 2} + \frac{bdX}{1 \cdot 2 \cdot 2dx} - \frac{b^3d^3X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2dx^3} + \frac{3b^5d^5X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 2dx^5}$$

$$-\frac{17b^7d^7X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 1dx^7} + \frac{155b^9d^9X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 2dx^9} - \frac{2073b^{11}d^{11}X}{1 \cdot 2 \cdot 3 \cdots 12 \cdot 2dx^{11}}$$

$$+\frac{38227b^{13}d^{13}X}{1 \cdot 2 \cdot 3 \cdots 14 \cdot 2dx^{13}} - \text{etc.}$$

Therefore, if the value of this progression is to be found

$$a^2 - (a+b)^2 + (a+2b)^2 - (a+3b)^2 + \cdots + x^2,$$

which we want to put S, because of

$$\frac{dX}{dx} = 2x$$

it will be

$$S = \text{Const.} + \frac{x^2}{2} + \frac{bx}{2}.$$

But the constant C on the other will be found by putting $x = a$ and it will be

$$S = \frac{a^2 - ab + x^2 + bx}{2}$$

For the sake of an example, it will be

$$1 - 4 + 9 - 16 + 25 - \cdots + 121 = 66.$$

§10 Now let us consider infinite series of this kind, namely

$$\begin{array}{cccccc} & & x & x+b & x+2b & \\ S & = & X & -Y & +Z & -\text{ etc. to infinity} \end{array}$$

Therefore, it will be

$$S + \frac{2bdS}{1dx} + \frac{4b^2ddS}{1 \cdot 2dx^2} + \frac{8b^3d^3S}{1 \cdot 2 \cdot 3dx^3} + \text{etc.} = S - X + Y$$

or

$$X - Y = -\frac{2bdS}{1dx} - \frac{4b^2ddS}{1 \cdot 2dx^2} - \text{etc.}$$

whence, because it is

$$Y = X + \frac{bdX}{1dx} + \frac{b^2ddX}{1 \cdot 2dx^2} + \frac{b^3d^3X}{1 \cdot 2 \cdot 3dx^3} + \text{etc.}$$

it will be found

$$S = \frac{X}{1 \cdot 2} - \frac{bdX}{1 \cdot 2 \cdot 2dx} + \frac{b^3d^3X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2dx^3} - \frac{3b^5d^5X}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 2dx^5} + \frac{17b^7d^7X}{1 \cdot 2 \cdot 3 \cdots 8 \cdot 2dx^7}$$
$$- \frac{155b^9d^9X}{1 \cdot 2 \cdot 3 \cdots 10 \cdot 2dx^9} + \frac{2073b^{11}d^{11}X}{1 \cdot 2 \cdot 3 \cdots 12 \cdot 2dx^{11}} - \frac{38227b^{13}d^{13}X}{1 \cdot 2 \cdot 3 \cdots 14 \cdot 2dx^{13}} + \text{etc.} + \text{Const.}$$

This constant must be of such a nature that it is $S = 0$ for $x = \infty$. Therefore, by means of this formula many other slowly converging series, in which the sings of the terms alternate, can be summed approximately.

§11 Let $X = \frac{1}{x}$ such that the sum of this series must be found

$$\frac{1}{x} - \frac{1}{x+b} + \frac{1}{x+2b} - \frac{1}{x+3b} + \text{etc. to infinity}$$

Therefore, because it is $X = \frac{1}{x}$, it will be

$$\frac{dX}{dx} = \frac{-1}{x^2}, \quad \frac{d^3X}{dx^3} = \frac{-2 \cdot 3}{x^4}, \quad \frac{d^5X}{dx^5} = \frac{-2 \cdot 3 \cdot 4 \cdot 5}{x^6} \quad \text{etc.}$$

and hence it will be

$$S = \frac{1}{2x} + \frac{b}{2 \cdot 2x^2} - \frac{b^3}{4 \cdot 2x^4} + \frac{3b^5}{6 \cdot 2x^6} - \frac{17b^7}{8 \cdot 2x^8} + \frac{155b^9}{10 \cdot 2x^{10}} - \frac{203b^{11}}{12 \cdot 2x^{12}} + \frac{38227b^{13}}{14 \cdot 2x^{14}} - \text{etc.},$$

where the addition of the constant is not necessary. For the sake of an example, let $b = 2$ and $x = 25$; the sum of the series

$$\frac{1}{25} - \frac{1}{27} + \frac{1}{29} - \frac{1}{31} + \frac{1}{33} - \text{etc.}$$

will be

$$= \frac{2}{100} + \frac{8}{100^2} - \frac{256}{100^4} + \frac{8 \cdot 4^6}{100^6} - \frac{17 \cdot 8 \cdot 4^8}{100^8} + \frac{31 \cdot 128 \cdot 4^{10}}{100^{10}} - \frac{691 \cdot 256 \cdot 4^{12}}{100^{12}}$$
$$+ \frac{5461 \cdot 2048 \cdot 4^{14}}{100^{14}} - \text{etc.} = 0.020797471915 \quad \text{approximately;}$$

but if the found sum is actually added to the sum of the preceding terms $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{23}$, the fourth part of the circumference of this circle, whose diameter is $= 1$, will result.

§12 Now, that the calculation can be done without any difficulty, the formulas given up to this point require that X is an algebraic function of x, whose differentials of each order can be exhibited conveniently. For, these formulas could hardy be applied or not even used at all, if an exponential quantity of this kind n^x would be contained in the general term of the progression. Therefore, it will be convenient to find peculiar summation formulas for progressions contained in this general term Xn^x, where X, as before, denotes an algebraic function of x. Therefore, let this series

$$\begin{array}{ccccccccc} a & & a+b & & a+2b & & & & x \\ An^a & + & Bn^{a+b} & + & Cn^{a+2b} & + \cdots + & Xn^x & = & S; \end{array}$$

be propounded to be summed and let the sum be $= Sn^x$. But this formula, having put $x - b$ instead of x, will go over into this one

$$n^{x-b}\left(S - \frac{bdS}{1dx} + \frac{b^2d^2S}{1 \cdot 2dx^2} - \frac{b^3d^3S}{1 \cdot 2 \cdot 3dx^3} + \text{etc.}\right),$$

which must be equal to the first sum Sn^x without the last term Xn^x. Therefore, one will have this equation

$$Sn^b - Xn^b = S - \frac{bdS}{1dx} + \frac{b^2d^2S}{1 \cdot 2dx^2} - \frac{b^3d^3S}{1 \cdot 2 \cdot 3dx^3} + \frac{b^4d^4S}{1 \cdot 2 \cdot 3 \cdot 4dx^4} - \text{etc.}$$

From this equation the value of S must be found.

§13 Therefore, put $n^b = m$ and it will be

$$S = \frac{mX}{m-1} - \frac{\alpha b dX}{1(m-1)^2 dx} + \frac{\beta b^2 ddX}{1\cdot 2(m-1)^3 dx^3} - \frac{\gamma b^3 d^3 X}{1\cdot 2\cdot 3(m-1)^4 dx^3}$$
$$+\frac{\delta b^4 d^4 X}{1\cdot 2\cdot 3\cdot 4(m-1)^5 dx^4} - \frac{\varepsilon b^5 d^5 X}{1\cdot 2\cdot 3\cdot 4\cdot 5(m-1)^6 dx^5} + \text{etc.}$$

Hence by comparing the homologous terms having, for the sake of brevity, put $m-1=p$ it will be as follows

$$\begin{aligned}
\alpha &= m,\\
\beta &= 2\alpha + mp,\\
\gamma &= 3\beta + 3\alpha p + mp^2,\\
\delta &= 4\gamma + 6\beta p + 4\alpha p^2 + mp^3,\\
\varepsilon &= 5\delta + 10\gamma p + 10\beta p^2 + 5\alpha p^3 + mp^4
\end{aligned}$$

whence for the letters α, β, γ, δ etc. the following values are obtained

$$\begin{aligned}
\alpha &= m,\\
\beta &= 2m + mp,\\
\gamma &= 6m + 6mp + mp^2,\\
\delta &= 24m + 36mp + 14mp^2 + mp^3,\\
\varepsilon &= 120m + 240mp + 150mp^2 + 30mp^3 + mp^4
\end{aligned}$$

etc.

Here, each number is a multiple of the one written above it combined with the preceding one, as

$$\begin{aligned}
30 &= 2(1+11),\\
150 &= 3(14+36),\\
240 &= 4(36+24),\\
120 &= 5(24+0),
\end{aligned}$$

or having substituted $m-1$ for p again it is as follows

$$\begin{array}{ll}
1. & \alpha = m, \\
2. & \beta = m + m^2, \\
3. & \gamma = m + 4m^2 + m^3, \\
4. & \delta = m + 11m^2 + 11m^3 + m^4, \\
5. & \varepsilon = m + 26m^2 + 66m^3 + 26m^4 + m^5, \\
6. & \zeta = m + 57m^2 + 302m^3 + 302m^4 + 57m^5 + m^6,
\end{array}$$

etc.,

which values proceed in such a way that the coefficient ψ, whose index is k, is

$$= m + \left(2^k - \frac{k+1}{1}\right) m^2 + \left(3^k - \frac{k+1}{1}2^k + \frac{(k+1)k}{1 \cdot 2}\right) m^3$$
$$+ \left(4^k - \frac{k+1}{1}3^k + \frac{(k+1)k}{1 \cdot 2}2^k - \frac{(k+1)k(k-1)}{1 \cdot 2 \cdot 3}\right) m^4 + \cdots + m^k.$$

§14 Therefore, from these the sum of the propounded series

$$\begin{array}{ccccccccc} a & & a+b & & a+2b & & & & x \\ An^a & + & Bn^{a+b} & + & Cn^{a+2b} & + & \cdots & + & Xn^x \end{array}$$

is calculated to be

$$= n^x \left\{ \begin{array}{c} \dfrac{n^b X}{n^b - 1} - \dfrac{n^b b dX}{1(n^b-1)dx} + \dfrac{(n^{2b}+n^b)^2 b^2 ddX}{1 \cdot 2(n^b-1)^3 dx^2} - \dfrac{(n^{3b}+4n^{2b}+n^b)b^3 d^3 X}{1 \cdot 2 \cdot 3(n^b-1)^4 dx^3} \\ + \dfrac{(n^{4b}+11n^{3b}+11n^{2b}+n^b)b^4 d^4 X}{1 \cdot 2 \cdot 3 \cdot 4(n^b-1)^5 dx^4} - \text{etc.} + \text{Const.}, \end{array} \right\} + \text{Const.},$$

which constant must be of such a nature that having put $x = a$ the sum is $= An^a$. If one puts $n^b = -1$, the series will go over into a purely algebraic one, in which the signs alternate, and hence by putting -1 instead of n^b the same formula we found for the same case already above in § 9 results. But the sum of the series running to infinity

$$\begin{array}{cccccc} x & & x+b & & x+2b & \\ An^x & + & Yn^{x+b} & + & Zn^{x+2b} & + \text{etc. to infinity} \end{array}$$

is

$$= n^x \left\{ \begin{array}{c} \dfrac{-X}{n^b-1} + \dfrac{n^b b dX}{1(n^b-1)^2 dx} - \dfrac{(n^{2b}+n^b)b^2 ddX}{1\cdot 2(n^b-1)^3 dx^2} + \dfrac{(n^{3b}+4n^{2b}+n^b)b^3 d^3X}{1\cdot 2\cdot 3(n^b-1)^4 dx^3} \\ -\dfrac{(n^{4b}+11n^{3b}+11n^{2b}+n^b)b^4 d^4X}{1\cdot 2\cdot 3\cdot 4(n^b-1)^5 dx^4} + \text{etc.} + \text{Const.,} \end{array} \right\} + \text{Const.,}$$

which constant must be of such a nature that having put $x = \infty$ the sum becomes $= 0$; this always happens per se such that a constant is not necessary, if the series converges and has a finite sum, of course.

§15 From the formula for summing series of this kind up to a given term it is understood, if X was such an algebraic function of x that its higher differentials finally vanish, then the summatory term can indeed be exhibited. Therefore, if the series, whose general term is X, was summable, then also the series, whose general term is Xn^x, will be summable. So having propounded the series

$$a^2n^a + (a+b)n^{a+b} + (a+2b)n^{a+2b} + \cdots + x^2n^x$$

its sum will be

$$= n^x \left(\frac{n^b x^2}{n^b-1} - \frac{2n^b x}{(n^b-1)^2} + \frac{(n^{2b}+n^b)b^2}{(n^b-1)^3} \right) + \text{Const.,}$$

which constant having put $x = a$ and the sum $= a^2n^a$ will result to be

$$n^a \left(a^2 - \frac{n^b a^2}{n^b-1} + \frac{2n^b ab}{(n^b-1)^2} - \frac{(n^{2b}+n^b)b^2}{(n^b-1)^3} \right).$$

§16 Furthermore, the other formula has an immense use for the summation of infinite series; in order to illustrate this, let this series be propounded

$$\frac{n^x}{x} + \frac{n^{x+2}}{x+2} + \frac{n^{x+4}}{x+4} + \frac{n^{x+6}}{x+6} + \text{etc.,}$$

whose sum we want to put $= S$. Therefore, it will be $b = 2$ and $X = \frac{1}{x}$, whence the sum will become

$$S = n^x \left\{ \begin{array}{c} \dfrac{-1}{(n^2-1)x} - \frac{2n^2}{(n^2-1)^2x^2} - \dfrac{4(n^4+n^2)}{(n^2-1)^3x^3} - \dfrac{8(n^6+4n^4+n^2)}{(n^2-1)^4x^4} \\ - \dfrac{16(n^8+11n^6+11n^4+n^2)}{(n^2-1)^5x^5} - \text{etc.} \end{array} \right\}.$$

Now let $x = 25$ and $n^2 = -\frac{1}{3}$ or $n = \frac{1}{\sqrt{-3}}$; it will be

$$\begin{aligned} &\frac{1}{\sqrt{-3}} \left(\frac{1}{25 \cdot 3^{12}} - \frac{1}{27 \cdot 3^{13}} + \frac{1}{29 \cdot 3^{14}} - \frac{1}{31 \cdot 3^{15}} + \text{etc.} \right) \\ &= \frac{1}{3^{12}\sqrt{-3}} \left(\frac{3}{4 \cdot 25} + \frac{3}{8 \cdot 25^2} - \frac{3}{8 \cdot 25^3} - \frac{3}{16 \cdot 25^4} + \frac{15}{8 \cdot 25^5} + \text{etc.} \right). \end{aligned}$$

Therefore, because in the circle with radius $= 1$ the arc of thirty degree is

$$= \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \frac{1}{9 \cdot 3^4} - \text{etc.} \right)$$

if 12 of these terms are actually added, the following remaining ones will be

$$\begin{aligned} &\frac{1}{\sqrt{3}} \left(\frac{1}{25 \cdot 3^{12}} - \frac{1}{27 \cdot 3^{15}} + \frac{1}{29 \cdot 3^{14}} - \text{etc.} \right) \\ &= \frac{\sqrt{3}}{3^{12}} \left(\frac{1}{4 \cdot 25} + \frac{1}{8 \cdot 25^2} - \frac{1}{8 \cdot 25^3} - \frac{1}{16 \cdot 25^4} + \frac{55}{8 \cdot 25^5} + \text{etc.} \right). \end{aligned}$$

§17 If the terms of the series to be summed were composed of factors such that the series has a form of this kind

$$\begin{array}{ccccccc} a & & a+b & & a+2b & & & & x \\ A & + & AB & + & ABC & + & \cdots & + & ABC \cdots VX, \end{array}$$

put the sum

$$= S \cdot ABC \cdots VX.$$

Therefore, this sum having put $x - b$ instead of x will go over into this one

$$ABC \cdots V \left(S - \frac{bdS}{1dx} + \frac{b^2ddS}{1 \cdot 2dx^2} - \frac{b^3d^3S}{1 \cdot 2 \cdot 3dx^3} + \text{etc.} \right),$$

which must be equal to the first expression without the last term, this means the quantity

$$AB \cdot V(SX - X).$$

Therefore, one will have this equation

$$SX - X = S - \frac{bdS}{1dx} + \frac{b^2 ddS}{1 \cdot 2dx^2} - \frac{b^3 d^3 S}{1 \cdot 2 \cdot 3dx^3} + \frac{b^4 d^4 S}{1 \cdot 2 \cdot 3 \cdot 4dx^4} - \text{etc.}$$

From this equation the found value of S will be this

$$S = \frac{X}{X-1} + E + F + G + \text{etc.},$$

which terms proceed in such a way that having put

$$\frac{X}{X-1} = D$$

it is

$$\begin{aligned}
E &= -\frac{bdD}{(X-1)dx}, \\
F &= -\frac{bdE}{(X-1)dx} && + \frac{b^2 ddD}{1 \cdot 2(X-1)dx^2}, \\
G &= -\frac{bdF}{(X-1)dx} + \frac{b^2 ddE}{1 \cdot 2(X-1)dx^2} && - \frac{b^3 d^3 D}{1 \cdot 2 \cdot 3(X-1)dx^3}
\end{aligned}$$

and so forth, that hence the sum of the propounded series is

$$= ABC \cdots VX(D + E + F + G + \text{etc.}).$$

But the sum of the series continued to infinity

$$\overset{x}{ABC \cdots VX} + \overset{x+b}{ABCD \cdots VXY} + \text{etc.}$$

will be

$$ABC \cdots VX(1 - D - E - F - G - \text{etc.}) + \text{Const.}$$

If, e.g, the sum of this series is in question

$$\frac{1}{1\cdot 2\cdot 3\cdots x}+\frac{1}{1\cdot 2\cdot 3\cdots x(x+1)}+\text{etc. to infinity,}$$

it will be

$$b=1,\quad X=\frac{1}{x}\quad\text{and}\quad D=\frac{1}{1-x},$$

but then

$$E=\frac{-x}{(1-x)^3},\quad F=\frac{x(2+x)}{(1-x)^5}\quad\text{etc.;}$$

therefore, the sum of the propounded series, because of the vanishing constant, will be

$$=\frac{1}{1\cdot 2\cdot 3\cdots x}\left(1+\frac{1}{x-1}-\frac{x}{(x-1)^3}+\frac{x(x+2)}{(x-1)^5}-\text{etc.}\right).$$

From these things explained here it is indeed easily understood, a form of what kind of the sum must be assumed in each case that the sum is found by minimal work.