On the memorable number occurring in the summation of the natural harmonic progression *

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§1 After I had once treated the summation of the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\text{etc.},$$

I detected its indefinite sum expressed in the following way that having put

$$s=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x}$$

this sum is

$$C+\log x+\frac{1}{2x}-\frac{\mathfrak{A}}{2x^2}+\frac{\mathfrak{B}}{4x^4}-\frac{\mathfrak{C}}{6x^6}+\frac{\mathfrak{D}}{8x^8}-\text{etc.},$$

where the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. are those numbers called Bernoulli numbers, namely

$$\mathfrak{A}=\frac{1}{6},\quad \mathfrak{B}=\frac{1}{30},\quad \mathfrak{C}=\frac{1}{42},\quad \mathfrak{D}=\frac{1}{30},\quad \mathfrak{E}=\frac{5}{66},\quad \mathfrak{F}=\frac{691}{2730},\quad \mathfrak{G}=\frac{7}{6},$$

$$\mathfrak{H}=\frac{3617}{510},\quad \mathfrak{I}=\frac{43867}{798},\quad \mathfrak{K}=\frac{174611}{330},\quad \mathfrak{L}=\frac{854513}{138},\quad \mathfrak{M}=\frac{236364091}{2370},$$

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$$\mathfrak{R} = \frac{8553103}{6}, \quad \mathfrak{Q} = \frac{23749461029}{870}, \quad \mathfrak{P} = \frac{8615841276005}{14322} \quad \text{etc.},$$

but then $\log x$ denotes the hyperbolic logarithm of the number x, but the letter C, which entered through integration, is a certain defined number to be found from a particular case, which from the case $x = 10$ I found to be

$$C = 0,5772156649015325,$$

which number seems to be even more remarkable since it has still not been possible for me in any way to reduce it to a certain known measure.

§**2** Therefore, if the number x is taken infinitely large, then it will be

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x} = C + \log x,$$

whence it is natural to suspect that that number C is the hyperbolic logarithm of a certain notable number, which we want to put $= N$, such that

$$C = \log N$$

and the sum of that infinite series becomes equal to the logarithm of the number Nx; hence it will be worth one's while to inquire the value of this number N, which will certainly suffice to have defined it to five or six digits, since hence it will not be difficult to estimate, whether is agrees with a certain known number or not. Therefore, for this to be achieved more easily, let us find a certain simpler number whose logarithm hardly deviates from C; but such a number is detected as $\frac{3}{2} \cdot \frac{6}{5} = \frac{9}{5}$, whose logarithm is $= 0,58778$, slightly larger than C, whence we conclude that $N < \frac{9}{5}$. Therefore, let us set $N = \frac{9}{5} - \omega$, and since in general

$$\log(a - \omega) = \log a - \frac{\omega}{a} - \frac{\omega^2}{2a^2} - \frac{\omega^3}{3a^3} - \frac{\omega^4}{4a^4} - \text{etc.},$$

in this case it will be $a = \frac{9}{5}$ and hence $\frac{\omega}{a} = \frac{5\omega}{9}$, for which we want to write z that $\omega = \frac{9}{5}z$;

$$\log\left(\frac{9}{5} - \omega\right) = \log\frac{9}{5} - z - \frac{1}{2}zz - \frac{1}{3}z^3 - \frac{1}{4}z^4 - \text{etc.} = \log N = C.$$

Therefore, since $\log\frac{9}{5} = C = 0,01057$, it will be

$$z+\frac{1}{2}zz+\frac{1}{3}z^3+\frac{1}{4}z^4+\text{etc.}=0,01057.$$

Therefore, since $z<0,01057$, let us take $z=0,01000+y$; it will be $zz=0,0001+0,02\cdot y$, which is sufficient for our purpose. Having substituted these values it will result

$$0,01005+1,01000\cdot y=0,01057;$$

hence one deduces $y=0,00052$ and hence $z=0,01052$; therefore, hence $\omega=0,01894$, as a logical consequence the number in question $N=1,78106$. Therefore, the whole task reduces to this that it investigated whether that number N might has a certain assignable ratio to a known quantity.

§3 But since I deduced that value of the letter C from the series in which there is no clear structure, since the Bernoulli numbers proceed in a very strange pattern, it will be useful to inquire a more regular series, whose sum is equal to the number C and which converges rapidly, that its value can also defined from there, what seems even more necessary since the Bernoulli numbers increase a lot soon that they go over into a highly divergent series and hence justified doubt can be raised, whether the found value can be considered as sufficiently certain or not.

§4 Therefore, since the propounded number is indeed equal to this formula

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x}-\log x$$

while x denotes an infinite number, it is easily understood that hence the following series can be constructed

$$
\begin{aligned}
C = 1 + \frac{1}{2} &\quad - \log\frac{2}{1} \\
+ \frac{1}{3} &\quad - \log\frac{3}{2} \\
+ \frac{1}{4} &\quad - \log\frac{4}{3} \\
+ \frac{1}{5} &\quad - \log\frac{5}{4} \\
+ \frac{1}{6} &\quad - \log\frac{6}{5} \\
+ \frac{1}{7} &\quad - \log\frac{7}{6} \\
+ \text{etc.} &
\end{aligned}
$$

For, it is obvious that after having collected all these terms into one sum the formula

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x} - \log x$$

results. And so we have an infinite series equal to our number C, any arbitrary term of which will be

$$\frac{1}{n} - \log\frac{n}{n-1},$$

which formula will be the general of the found series.

§**5** Therefore, let us consider this formula $\frac{1}{n} - \log\frac{n}{n-1}$ more accurately, and since

$$-\log\frac{n}{n-1} = \log\frac{n-1}{n} = \log\left(1 - \frac{1}{n}\right),$$

by an infinite series it will be

$$-\log\frac{n}{n-1} = -\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} - \frac{1}{4n^4} - \frac{1}{5n^5} - \text{etc.}$$

and hence

$$\frac{1}{n} - \log\frac{n}{n-1} = -\frac{1}{2n^2} - \frac{1}{3n^3} - \frac{1}{4n^4} - \frac{1}{n^5} - \text{etc.},$$

whence for finding the number C one must expand the following series

$$\begin{aligned}
1 - C = \; & + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} + \frac{1}{5\cdot 2^5} + \text{etc.} \\
& + \frac{1}{2\cdot 3^2} + \frac{1}{3\cdot 3^3} + \frac{1}{4\cdot 3^4} + \frac{1}{5\cdot 3^5} + \text{etc.} \\
& + \frac{1}{2\cdot 4^2} + \frac{1}{3\cdot 4^3} + \frac{1}{4\cdot 4^4} + \frac{1}{5\cdot 4^5} + \text{etc.} \\
& + \frac{1}{2\cdot 5^2} + \frac{1}{3\cdot 5^3} + \frac{1}{4\cdot 5^4} + \frac{1}{5\cdot 5^5} + \text{etc.} \\
& \qquad\qquad \text{etc.}
\end{aligned}$$

§6 For the sake of brevity, let α denote the sum of the reciprocals of the squares

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.}$$

and in like manner β the sum of the series of the reciprocals of the cubes, γ the sum of the series of reciprocals of the fourth powers etc., and I already exhibited the numerical values of these values sufficiently accurately (see Inst. Calculi differentialis p. 456); therefore, hence it will be

$$1 - C = \frac{1}{2}(\alpha - 1) + \frac{1}{3}(\beta - 1) + \frac{1}{4}(\gamma - 1) + \frac{1}{5}(\delta - 1) + \frac{1}{6}(\varepsilon - 1) + \text{etc.}$$

$$
\begin{aligned}
\frac{1}{2}(\alpha-1) &= 0,3224670\\
\frac{1}{3}(\beta-1) &= 0,0673523\\
\frac{1}{4}(\gamma-1) &= 0,0205808\\
\frac{1}{5}(\delta-1) &= 0,0073856\\
\frac{1}{6}(\varepsilon-1) &= 0,0028905\\
\frac{1}{7}(\zeta-1) &= 0,0011928\\
\frac{1}{8}(\eta-1) &= 0,0005097\\
\frac{1}{9}(\theta-1) &= 0,0002232\\
\frac{1}{10}(\iota-1) &= 0,0000995\\
\frac{1}{11}(\varkappa-1) &= 0,0000449\\
\frac{1}{12}(\lambda-1) &= 0,0000205\\
\frac{1}{13}(\mu-1) &= 0,0000094\\
\frac{1}{14}(\nu-1) &= 0,0000044\\
\frac{1}{15}(\xi-1) &= 0,0000020\\
\frac{1}{16}(o-1) &= \underline{0,0000010}\\
1-C &= 0,4227836
\end{aligned}
$$

whence $C = 0,5772164$ would result, which value, because of the following terms of the series, must be diminished to $0,5772167$, where even in the last digit there is no error.

§7 But for the more accurate investigation of the same value a much more convergent series can be found. For, since

$$\log \frac{a+1}{a-1} = \frac{2}{a} + \frac{2}{3a^3} + \frac{2}{5a^5} + \frac{2}{7a^7} + \frac{2}{9a^9} + \text{etc.}$$

because of

$$\log \frac{n}{n-1} = \log \frac{2n}{2n-2}$$

take $a = 2n-1$ and it will be

$$\log \frac{n}{n-1} = \frac{2}{2n-1} + \frac{2}{3(2n-1)^3} + \frac{2}{5(2n-1)^5} + \frac{2}{7(2n-1)^7} + \text{etc;}$$

if from this the fraction $\frac{1}{n}$ is subtracted, the general term of our series will result as

$$\log \frac{n}{n-1} - \frac{1}{n} = \frac{1}{n(2n-1)} + \frac{2}{3(2n-1)^3} + \frac{2}{5(2n-1)^5} + \frac{2}{7(2n-1)^7} + \text{etc,}$$

Therefore, if one writes the numbers 2, 3, 4, 5 etc. instead of n successively, the following series will arise

$$\begin{aligned} 1 - C = \ & + \frac{1}{2\cdot 3} + \frac{2}{3\cdot 3^3} + \frac{2}{5\cdot 3^5} + \frac{2}{7\cdot 3^7} + \frac{2}{9\cdot 3^9} + \text{etc.} \\ & + \frac{1}{3\cdot 5} + \frac{2}{3\cdot 5^3} + \frac{2}{5\cdot 5^5} + \frac{2}{7\cdot 5^7} + \frac{2}{9\cdot 5^9} + \text{etc.} \\ & + \frac{1}{4\cdot 7} + \frac{2}{3\cdot 7^3} + \frac{2}{5\cdot 7^5} + \frac{2}{7\cdot 7^7} + \frac{2}{9\cdot 7^9} + \text{etc.} \\ & + \frac{1}{5\cdot 9} + \frac{2}{3\cdot 9^3} + \frac{2}{5\cdot 9^5} + \frac{2}{7\cdot 9^7} + \frac{2}{9\cdot 9^9} + \text{etc.} \\ & \qquad\qquad + \text{ etc.,} \end{aligned}$$

where the first vertical line, because of

$$\frac{1}{n(2n-1)} = \frac{2}{2n-1} - \frac{2}{2n},$$

is reduced to this series

$$\frac{2}{3}-\frac{2}{4}+\frac{2}{5}-\frac{2}{6}+\frac{2}{7}-\frac{2}{8}+\text{etc.},$$

whose sum obviously is $2\log 2-1$, having brought which sum to the other side it will now be

$$\begin{aligned}
2-2\log 2-C = {} & \frac{2}{3\cdot 3^3}+\frac{2}{5\cdot 3^5}+\frac{2}{7\cdot 3^7}+\frac{2}{9\cdot 3^9}+\text{etc.}\\
& +\frac{2}{3\cdot 5^3}+\frac{2}{5\cdot 5^5}+\frac{2}{7\cdot 5^7}+\frac{2}{9\cdot 5^9}+\text{etc.}\\
& +\frac{2}{3\cdot 7^3}+\frac{2}{5\cdot 7^5}+\frac{2}{7\cdot 7^7}+\frac{2}{9\cdot 7^9}+\text{etc.}\\
& +\frac{2}{3\cdot 9^3}+\frac{2}{5\cdot 9^5}+\frac{2}{7\cdot 9^7}+\frac{2}{9\cdot 9^9}+\text{etc.}\\
& + \quad \text{etc.}
\end{aligned}$$

This series converges so fast that its sum can easily be found to a lot more digits than before. But note here that the sum of the first series will be $\log 2-\frac{2}{3}$; and in like manner the sum of the second series will be $\log\frac{3}{2}-\frac{2}{5}$, but the sum of the third series $=\log\frac{4}{3}-\frac{2}{7}$, the sum of the following series $=\log\frac{5}{4}-\frac{2}{9}$, the following $=\log\frac{6}{5}-\frac{2}{11}$, which values should therefore be computed separately. Therefore, having collected these values, which are found in tables, it remains that the following series are gathered into one sum:

$$
\begin{aligned}
&+\ \frac{2}{3}\left(\frac{1}{13^3}+\frac{1}{15^3}+\frac{1}{17^3}+\frac{1}{19^3}+\text{etc.}\right)\\
&+\ \frac{2}{5}\left(\frac{1}{13^5}+\frac{1}{15^5}+\frac{1}{17^5}+\frac{1}{19^5}+\text{etc.}\right)\\
&+\ \frac{2}{7}\left(\frac{1}{13^7}+\frac{1}{15^7}+\frac{1}{17^7}+\frac{1}{19^7}+\text{etc.}\right)\\
&+\ \frac{2}{9}\left(\frac{1}{13^9}+\frac{1}{15^9}+\frac{1}{17^9}+\frac{1}{19^9}+\text{etc.}\right)\\
&\quad +\ \text{etc.},
\end{aligned}
$$

which will be easily achieved applying prescriptions I had once given for the summation of such series.

§**8** But since we are already certain about the true value of our number $C = 0,5772156649015325$ to 16 digits, it would be superfluous to perform this task again; hence let us check other more regular series, whose sum becomes equal to this number. And first the most simple series achieving this is deduced from the principal form

$$C = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x} - \log x$$

by this resolution:

$$
\begin{aligned}
C = 1 &\quad - \log 2 \\
+ \frac{1}{2} &\quad - \log \frac{3}{2} \\
+ \frac{1}{3} &\quad - \log \frac{4}{3} \\
+ \frac{1}{4} &\quad - \log \frac{5}{4} \\
+ \cdots &\quad - \cdots \\
+ \frac{1}{n} &\quad - \log \frac{n+1}{n}.
\end{aligned}
$$

For, having actually collected these up to $\frac{1}{x}$ this sum results

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x} - \log(1+x);$$

but since $\log x$ is supposed to be infinite, $\log(x+1)$ is not to be considered to differ from $\log x$.

§9 Therefore, since by an infinite series

$$\log \frac{n+1}{n} = \log\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \frac{1}{5n^5} - \text{etc.},$$

the general term of that series will be

$$= \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \frac{1}{5n^5} + \text{etc.},$$

whence our number C will be expressed by the following series:

$$
\begin{aligned}
C = {} & \frac{1}{2\cdot 1^2} - \frac{1}{3\cdot 1^3} + \frac{1}{4\cdot 1^4} - \frac{1}{5\cdot 1^5} + \text{etc.} \\
& + \frac{1}{2\cdot 2^2} - \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} - \frac{1}{5\cdot 2^5} + \text{etc.} \\
& + \frac{1}{2\cdot 3^2} - \frac{1}{3\cdot 3^3} + \frac{1}{4\cdot 3^4} - \frac{1}{5\cdot 3^5} + \text{etc.} \\
& + \frac{1}{2\cdot 4^2} - \frac{1}{3\cdot 4^3} + \frac{1}{4\cdot 4^4} - \frac{1}{5\cdot 4^5} + \text{etc.} \\
& + \frac{1}{2\cdot 5^2} - \frac{1}{3\cdot 5^3} + \frac{1}{4\cdot 5^4} - \frac{1}{5\cdot 5^5} + \text{etc.} \\
& \qquad\qquad + \ \text{etc.}
\end{aligned}
$$

§10 If now as above the letters α, β, γ, δ etc. denotes the sums of the series of the reciprocals of the squares, cubes, fourth powers and higher powers, by them our number C will be expressed this way

$$C = \frac{1}{2}\alpha - \frac{1}{3}\beta + \frac{1}{4}\gamma - \frac{1}{5}\delta + \frac{1}{6}\varepsilon - \frac{1}{7}\zeta + \text{etc.}$$

But above we already found this series involving the same letters

$$1 - C = \frac{1}{2}(\alpha - 1) + \frac{1}{3}(\beta - 1) + \frac{1}{4}(\gamma - 1) + \frac{1}{5}(\delta - 1) + \text{etc.};$$

these two series are even more remarkable since they exhibit the value of C differently in terms of the letters α, β, γ, δ etc.

§11 Therefore, these two series can be combined in various ways, whence it will be possible to derive conclusions worth one's complete attention. And first, these series added to each other will produce the following summation

$$1 = \alpha - \frac{1}{2} - \frac{1}{3}, \quad + \frac{1}{2}\gamma - \frac{1}{4} - \frac{1}{5}, \quad + \frac{1}{3}\varkappa - \frac{1}{6} - \frac{1}{7}, \quad \frac{1}{4}\eta - \frac{1}{8} - \frac{1}{9}, \quad \text{etc.},$$

where just the sum of the even powers occur, which, as I found, can be exhibited in terms of powers of the circumference of the circle π, since

$$\alpha = \frac{\pi^2}{6}, \quad \gamma = \frac{\pi^4}{90}, \quad \varepsilon = \frac{\pi^6}{945}, \quad \eta = \frac{\pi^8}{9450}, \quad \iota = \frac{\pi^{10}}{93555} \quad \text{etc.}$$

Hence, since the sum of this completely extraordinary series is $= 1$, it will be worth one's while to expand at least the first terms into decimal fractions. Therefore, it will be

$$\begin{aligned}
\alpha - \frac{1}{2} - \frac{1}{3} &= 0,8116007335, \\
\frac{1}{2}\gamma - \frac{1}{4} - \frac{1}{5} &= 0,0911616169, \\
\frac{1}{3}\varepsilon - \frac{1}{6} - \frac{1}{7} &= 0,0295905445, \\
\frac{1}{4}\eta - \frac{1}{8} - \frac{1}{9} &= 0,0149082279, \\
\frac{1}{5}\iota - \frac{1}{10} - \frac{1}{11} &= 0,0092898241,
\end{aligned}$$

the sum of which five terms already is

$$= 0,9565509469,$$

such that the sum of all remaining ones must produce

$$0,0434490531.$$

§12 This summation is even more remarkable since series of this kind have not been considered in analysis. It must be carefully noted that the terms of this series must be arranged the same way as they resulted from the combination of the two preceding series, such that each term consists of three parts. For, if for the sake of an example we wanted to bring all negative parts to the left side, this equation would result

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \text{etc.} = \alpha + \frac{1}{2}\gamma + \frac{1}{3}\varepsilon + \frac{1}{4}\eta + \frac{1}{5}\iota + \text{etc.},$$

whence nothing could be seen, since each of both sides would have an infinite magnitude; here, since the first terms of the series α, γ, ε etc. are one, from them alone for the right-hand side this series originates

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{etc.},$$

which series is also found on the left-hand side. And nevertheless hence it does not follow that remaining terms on the right-hand side become equal to zero, since obviously the series on the left contains twice as many terms as occur on the right-hand side.

§**13** Now let us also subtract the two series for C and $1-C$ given above in § 10 from each other and the following not less remarkable series will result

$$2C-1=+\frac{1}{2}+\frac{1}{3}-\frac{2}{3}\beta, \quad +\frac{1}{4}+\frac{1}{5}-\frac{2}{5}\delta, \quad +\frac{1}{6}+\frac{1}{7}-\frac{2}{7}\zeta, \quad +\frac{1}{8}+\frac{1}{9}-\frac{2}{9}\theta, \quad +\text{etc.},$$

where only the sums of the even powers occur. But we know that

$$2C-1=0,1544313298.$$

Therefore, let us see, which numerical values arise from the first five terms, and since

$$\begin{aligned}
\frac{1}{2}+\frac{1}{3}-\frac{2}{3}\beta &= 0,0319620646,\\
\frac{1}{4}+\frac{1}{5}-\frac{2}{5}\delta &= 0,0352288979,\\
\frac{1}{6}+\frac{1}{7}-\frac{2}{7}\zeta &= 0,0214240160,\\
\frac{1}{8}+\frac{1}{9}-\frac{2}{9}\theta &= 0,0134425794,\\
\frac{1}{10}+\frac{1}{11}-\frac{2}{11}\varkappa &= 0,0090010566,
\end{aligned}$$

since the sum of these five terms just is $0,1110586145$, this series is to be considered to be not suitable at all to explore the true value of C, if it would still be unknown.

§14 This way let us also expand the expression found in § 7 and let us, according to the vertical lines, represent it this way:

$$
\begin{aligned}
2-2\log 2 - C = &+ \quad \frac{2}{3}\left(\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{9^3}+\frac{1}{11^3}+\text{etc.}\right)\\
&+ \quad \frac{2}{5}\left(\frac{1}{3^5}+\frac{1}{5^5}+\frac{1}{7^5}+\frac{1}{9^5}+\frac{1}{11^5}+\text{etc.}\right)\\
&+ \quad \frac{2}{7}\left(\frac{1}{3^7}+\frac{1}{5^7}+\frac{1}{7^7}+\frac{1}{9^7}+\frac{1}{11^7}+\text{etc.}\right)\\
&+ \quad \frac{2}{9}\left(\frac{1}{3^9}+\frac{1}{5^9}+\frac{1}{7^9}+\frac{1}{9^9}+\frac{1}{11^9}+\text{etc.}\right)\\
&+ \text{etc.};
\end{aligned}
$$

since here only odd power of odd numbers occur, express these series in terms of the letters β, δ, ζ, θ etc. assumed above, since we know to be

$$
\begin{aligned}
1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{9^3}+\text{etc.} &= \frac{7}{8}\ \beta,\\
1+\frac{1}{3^5}+\frac{1}{5^5}+\frac{1}{7^5}+\frac{1}{9^5}+\text{etc.} &= \frac{31}{32}\ \delta,\\
1+\frac{1}{3^7}+\frac{1}{5^7}+\frac{1}{7^7}+\frac{1}{9^7}+\text{etc.} &= \frac{127}{128}\zeta,\\
1+\frac{1}{3^9}+\frac{1}{5^9}+\frac{1}{7^9}+\frac{1}{9^9}+\text{etc.} &= \frac{511}{512}\theta,\\
&\text{etc.}
\end{aligned}
$$

Therefore, having substituted these values, we will have this series

$$2-2\log 2-C=\frac{2}{3}\cdot\frac{7}{8}\beta-\frac{2}{3}+\frac{2}{5}\cdot\frac{31}{32}\delta-\frac{2}{5}+\frac{2}{7}\cdot\frac{127}{128}\zeta-\frac{2}{7}$$
$$+\frac{2}{9}\cdot\frac{511}{512}\theta-\frac{2}{9}+\text{etc.}$$

But just before we found that

$$2C-1=\frac{1}{2}+\frac{1}{3}-\frac{2}{3}\beta,\quad +\frac{1}{4}+\frac{1}{5}-\frac{2}{5}\delta,\quad +\frac{1}{6}+\frac{1}{7}-\frac{2}{7}\zeta,$$
$$+\frac{1}{8}+\frac{1}{9}-\frac{2}{9}\theta,\quad +\text{etc.},$$

which series added to the one just found yields

$$1-2\log 2+C=\frac{1}{2}-\frac{1}{3}-\frac{2}{3\cdot 2^3}\beta,\quad +\frac{1}{4}-\frac{1}{5}-\frac{2}{5\cdot 2^5}\delta,\quad +\frac{1}{6}-\frac{1}{7}-\frac{2}{7\cdot 2^7}\zeta,\quad \text{etc.},$$

where all absolute fractions constitute this series

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{8}-\text{etc.};$$

since its sum is finite and $=1-\log 2$, one can safely write this value for it and hence one will get to the following summation

$$\log 2-C=\frac{1}{3\cdot 2^2}\beta+\frac{1}{5\cdot 2^4}\delta+\frac{1}{7\cdot 2^6}\zeta+\frac{1}{9\cdot 2^8}\theta+\text{etc.},$$

which series converges so fast that from it one can easily find the value of our number C.

§**15** Therefore, it will be worth one's while to expand this series more accurately; that this can be done more easily, since all letters β, δ, ζ, θ etc. contain 1, these units taken separately yield

$$\frac{1}{3\cdot 2^2}+\frac{1}{5\cdot 2^4}+\frac{1}{7\cdot 2^6}+\frac{1}{9\cdot 2^8}+\text{etc.},$$

whose sum is $\log 3-1$, having introduced which value here it will be

$$1-\log\frac{3}{2}-C=\frac{1}{3\cdot 2^2}(\beta-1)+\frac{1}{5\cdot 2^4}(\delta-1)+\frac{1}{7\cdot 2^6}(\zeta-1)+\frac{1}{9\cdot 2^8}(\theta-1)+\text{etc.}$$

Here, for the left-hand side

$$1 - \log \frac{3}{2} = 0,5945348918918356,$$

but for the right-hand side the following values for the respective terms are found:

$$\frac{1}{3\cdot 4}(\beta-1)=0,0168380752632995,$$

$$\frac{1}{5\cdot 4^2}(\delta-1)=0,0004615969392921,$$

$$\frac{1}{7\cdot 4^3}(\zeta-1)=0,0000186367798704,$$

$$\frac{1}{9\cdot 4^4}(\theta-1)=0,0000008716982752,$$

$$\frac{1}{11\cdot 4^5}(\varkappa-1)=0,0000000438732781,$$

$$\frac{1}{13\cdot 4^6}(\mu-1)=0,0000000023045626,$$

$$\frac{1}{15\cdot 4^7}(\xi-1)=0,0000000001244639,$$

$$\frac{1}{17\cdot 4^8}(\pi-1)=0,0000000000068550,$$

$$\frac{1}{19\cdot 4^9}(\sigma-1)=0,0000000000003831,$$

$$\text{for the remaining}\quad \underline{0,0000000000000230,}$$

$$\text{sum}=0,0173192269903029,$$

$$1-\log\frac{3}{2}=\underline{0,5945348918918356}$$

$$C=0,5772156649015327,$$

which value is to be considered to agree with the one found once because of the inevitable errors.

§**16** But aside from these series we have given for the determination of the number C up to this point, one can find innumerable others. For, since one can put in general

$$C = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{x+p} - \log(x+q),$$

since the number x must be assumed infinitely large, hence always the same value will result, whatever numbers are assumed for p and q. If, for the sake of brevity, we put

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{p} = \Pi,$$

it will be

$$C = \Pi + \frac{1}{1+p} + \frac{1}{2+p} + \frac{1}{3+p} + \cdots + \frac{1}{x+p} - \log(x+q).$$

This form can be resolved into the following series

$$\begin{aligned} \Pi - \log q &+ \frac{1}{1+p} - \log\frac{q+1}{q} \\ &+ \frac{1}{2+p} - \log\frac{q+2}{q+1} \\ &+ \frac{1}{3+p} - \log\frac{q+3}{q+2} \\ &\qquad \vdots \qquad\quad \vdots \\ &+ \frac{1}{n+p} - \log\frac{q+n}{q+n-1}, \end{aligned}$$

whence therefore innumerable different infinite series can be exhibited, the sum of all of which is the same, $= C$, of course.

§**17** But for this series to converge faster, it will be convenient to assume such a relation among q and p, that, whenever $\log\frac{q+n}{q+n-1}$ is expanded into a series, its first term becomes equal to $\frac{1}{n+p}$. But one mainly has three ways of expanding this logarithm into a series; the first originates from the general series

$$\log(1+z) = z - \frac{1}{2}zz + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \text{etc.},$$

where having put

$$1+z = \frac{q+n}{q+n-1}$$

we have

$$z = \frac{1}{q+n-1};$$

therefore, in this case it will be convenient to take $p = q-1$. But if we want to use this resolution

$$\log\frac{1}{1-z} = z + \frac{1}{2}zz + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \text{etc.},$$

having put

$$\frac{1}{1-z} = \frac{q+n}{q+n-1},$$

it becomes

$$z = \frac{1}{q+n},$$

in which case one therefore will have to take $p = q$. The third way is taken from this resolution

$$\log\frac{1+z}{1-z} = 2z + \frac{2}{3}z^3 + \frac{2}{5}z^5 + \frac{2}{7}z^7 + \text{etc.};$$

therefore, having put

$$\frac{1+z}{1-z} = \frac{q+n}{q+n-1},$$

it will be

$$z = \frac{1}{2(q+n)-1},$$

in which case one must take $p = q - \frac{1}{2}$. Therefore, in this case for p to become an integer, for q one has to assume a fraction of the form $m + \frac{1}{2}$, for, then it will be $p = m$. But if q was an integer number, p will will obviously be the fraction $= q - \frac{1}{2}$; but since the value of Π is not not clear in this case, it must be investigated before everything else.

§18 To this end, let us consider the following series with its indices

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \\ 1, & 1+\frac{1}{2}, & 1+\frac{1}{2}+\frac{1}{3}, & 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, & 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, & \text{etc.,} \end{array}$$

where to the general index n this term will correspond

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n};$$

if this is said to be $= N$, the following terms corresponding to the indices $n+1$, $n+2$, $n+3$ etc. will be these

$$N + \frac{1}{n+1}, \quad N + \frac{1}{n+1} + \frac{1}{n+2}, \quad N + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3},$$

where therefore no difficulty occurs, if n was an integer number. Therefore, let us put the term corresponding to the index $\frac{1}{2}$ z, to the investigation of which the task is reduced, of course; if it was found, the following terms will proceed as follows

$$\begin{array}{cccc} \frac{1}{2}, & 1+\frac{1}{2}, & 2+\frac{1}{2}, & 3+\frac{1}{2} \\ z, & z+\frac{2}{3}, & z+\frac{2}{3}+\frac{2}{5}, & z+\frac{2}{3}+\frac{2}{5}+\frac{2}{7} \end{array}$$

and so in general to the index $n + \frac{1}{2}$ this term will correspond

$$z + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \cdots + \frac{2}{2n+1}.$$

Hence, if the number n is taken infinitely large, in which case the terms corresponding to the indices n and $n+1$ do not differ from each other anymore, the middle term, corresponding to the index $n+\frac{1}{2}$, must become equal to them and from this principle the following equation is constructed

$$z+\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\frac{2}{9}+\cdots+\frac{2}{2n+1}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n},$$

where the number of terms on each of both sides is the same; hence having brought each term for the left-hand side to the right-hand side and having interpolated accordingly it will result

$$z=1-\frac{2}{3}+\frac{1}{2}-\frac{2}{5}+\frac{1}{3}-\frac{2}{7}+\frac{1}{4}-\frac{2}{9}+\frac{1}{5}-\frac{2}{11}+\text{etc. to infinity}$$

and so the value of z is expressed by this infinite series, from which, since

$$\frac{1}{2}z=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\text{etc.},$$

it will obviously be

$$\frac{1}{2}z=1-\log 2 \quad \text{and hence} \quad z=2-2\log 2.$$

§19 Therefore, if p was a fraction of the form $n+\frac{1}{2}$, the corresponding values of the letter Π will look as follows:

p	Π
$\frac{1}{2}$	$2 - 2\log 2$
$1 + \frac{1}{2}$	$2 - 2\log 2 + \frac{2}{3}$
$2 + \frac{1}{2}$	$2 - 2\log 2 + \frac{2}{3} + \frac{2}{5}$
$3 + \frac{1}{2}$	$2 - 2\log 2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7}$
etc.	etc.
$m + \frac{1}{2}$	$2 - 2\log 2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \cdots + \frac{2}{2m+1}$

§**20** Let us use the second resolution of logarithms, where $p = q$ and

$$\log \frac{n+q}{n+q-1} = \frac{1}{n+q} + \frac{1}{2(n+q)^2} + \frac{1}{3(n+q)^3} + \frac{1}{4(n+q)^4} + \text{etc.},$$

whence we obtain this series for our number C

$$\begin{aligned}
\Pi - \log q &- \frac{1}{2(q+1)^2} - \frac{1}{3(q+1)^3} - \frac{1}{4(q+1)^4} - \frac{1}{5(q+1)^5} - \text{etc.} \\
&- \frac{1}{2(q+2)^2} - \frac{1}{3(q+2)^3} - \frac{1}{4(q+2)^4} - \frac{1}{5(q+2)^5} - \text{etc.} \\
&- \frac{1}{2(q+3)^2} - \frac{1}{3(q+3)^3} - \frac{1}{4(q+3)^4} - \frac{1}{5(q+3)^5} - \text{etc.} \\
&\qquad\qquad - \quad \text{etc.},
\end{aligned}$$

where

$$\Pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{q},$$

whence it is plain that the greater the number q is taken the more convergent this series will become.

§21 If anybody wanted to define the value of our number C more accurately, it will not even necessary to resolve the logarithms, which occur in each term, into series. Yes, it is not even necessary to constitute a certain relation among the two numbers p and q, but each one can be assumed arbitrarily, such that, after the value of the first term $\Pi - \log q$ had be dealt with, while

$$\Pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{p},$$

the whole task is reduced to the sum of this infinite series

$$\left(\frac{1}{p+1} - \log\frac{q+1}{q}\right) + \left(\frac{1}{p+2} - \log\frac{q+2}{q+1}\right) + \left(\frac{1}{p+3} - \log\frac{q+3}{q+2}\right) + \text{etc.},$$

instead of which we want to consider this general series which is to be summed

$$S = X + X' + X'' + X''' + \text{etc.},$$

where X is a function of x, but the following terms arise, if one writes the values $x+1$, $x+2$, $x+3$ etc. instead of x successively; to this end, let us write x instead of the letter q, and since the difference among p and q is given, set $p = x - a + 1$, such that

$$X = \frac{1}{x+a} - \log\frac{x+1}{x} \quad \text{and} \quad X' = \frac{1}{x+a+1} - \log\frac{x+2}{x+1}$$

and so forth. And so having taken any number for x the value of our number C will be

$$C = \Pi - \log q + S$$

or

$$C = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x+a-1} - \log x + S,$$

where both numbers a and x are arbitrary; since there is no obstruction, in the following expansion one can treat the number x as a variable.

§22 Therefore, now let us write $x+1$ instead of x and let the value of S go over into S' such that

$$S' = X' + X'' + X''' + \text{etc.},$$

and it will be $S' - S = -X$. But since S is a certain function of x, by a well-known reduction it will be

$$S' = S + \frac{dS}{dx} + \frac{ddS}{1 \cdot 2dx^2} + \frac{d^3S}{1 \cdot 2 \cdot 3dx^3} + \text{etc.},$$

whence this equation results

$$X + \frac{dS}{dx} + \frac{ddS}{1 \cdot 2x^2} + \frac{d^3S}{1 \cdot 2 \cdot 3dx^3} + \frac{d^4S}{1 \cdot 2 \cdot 3 \cdot 4dx^4} + \text{etc.} = 0,$$

from the first two terms of which, since we consider the following ones to decrease continuously, it is concluded that it will approximately be

$$dS = -Xdx$$

and hence

$$S = -\int Xdx,$$

which integral must be taken in such a way that $S = 0$ if $x = \infty$, since for $x = \infty$ it will be $C = \Pi - \log q$, of course.

§23 Therefore, $-\int Xdx$ will be the first term of our new series, by which we want to express the letter S, and from the form of the equation it is easily understood that one must set

$$S = -\int Xdx + \alpha X + \beta\frac{dX}{dx} + \gamma\frac{ddX}{dx^2} + \delta\frac{d^3X}{dx^3} + \varepsilon\frac{d^4X}{dx^4} + \text{etc.},$$

whence

$$\begin{array}{lllllllll}
\frac{dS}{dx} & = - & X & + & \frac{\alpha dX}{dx} & + \beta\frac{d^2X}{dx^2} & + \gamma\frac{d^3X}{dx^3} & + \frac{\delta d^4X}{dx^4} & + \text{etc.,} \\
\frac{ddS}{dx^2} & = - & \frac{dX}{dx} & + & \frac{\alpha ddX}{dx^2} & + \beta\frac{d^3X}{dx^3} & + \gamma\frac{d^4X}{dx^4} & + \frac{\delta d^5X}{dx^5} & + \text{etc.,} \\
\frac{d^3S}{dx^3} & = - & \frac{ddX}{dx^2} & + & \frac{\alpha d^3X}{dx^3} & + \beta\frac{d^4X}{dx^4} & + \gamma\frac{d^5X}{dx^5} & + \frac{\delta d^6X}{dx^6} & + \text{etc.,} \\
 & & & \text{etc.,} & & & & &
\end{array}$$

having substituted which values and having ordered them accordingly one will obtain the following equation

$$\begin{array}{rcccccccccc}
0 = X & + & \frac{\alpha dX}{dx} & + & \frac{\beta ddX}{dx^2} & + & \frac{\gamma d^3X}{dx^3} & + & \frac{\delta d^4X}{dx^4} & + \text{etc.,} \\
-X & - & \frac{1}{2} & + & \frac{\alpha}{2} & + & \frac{\beta}{2} & + & \frac{\gamma}{2} & \\
 & & & - & \frac{1}{6} & + & \frac{\alpha}{6} & + & \frac{\beta}{6} & \\
 & & & & & - & \frac{1}{24} & + & \frac{\alpha}{24} & \\
 & & & & & & & - & \frac{1}{120} &
\end{array}$$

whence the following determinations originate

$$\alpha = \frac{1}{2}, \quad \beta = -\frac{1}{2}\alpha + \frac{1}{6}, \quad \gamma = -\frac{1}{2}\beta - \frac{1}{6}\alpha + \frac{1}{24},$$
$$\delta = -\frac{1}{2}\gamma - \frac{1}{6}\beta - \frac{1}{24}\alpha$$

and hence

$$\alpha = \frac{1}{2}, \quad \beta = -\frac{1}{12}, \quad \gamma = 0, \quad \delta = \frac{1}{720} \quad \text{etc.}$$

§**24** But this way the determination of the letters α, β, γ, δ etc. becomes to cumbersome; hence, to simplify the work, let us consider the following series, where the same coefficients occur, and write its derivatives under them

$$\begin{array}{rlllllll}
v = & -1 + & \alpha z + & \beta z^2 + & \gamma z^3 + & \delta z^4 + & \varepsilon z^5 & + \text{etc.}, \\
+\ \frac{1}{2} vz = & & -\frac{1}{2} & +\frac{1}{2}\alpha & +\frac{1}{2}\beta & +\frac{1}{2}\gamma & +\frac{1}{2}\delta & + \text{etc.}, \\
+\ \frac{1}{6} vz^2 = & & & -\frac{1}{6} & +\frac{1}{6}\alpha & +\frac{1}{6}\beta & +\frac{1}{6}\gamma & + \text{etc.}, \\
+\ \frac{1}{24} vz^3 = & & & & -\frac{1}{24} & +\frac{1}{24}\alpha & +\frac{1}{24}\beta & + \text{etc.}, \\
+\ \frac{1}{120} vz^4 = & & & & & -\frac{1}{120}\alpha & +\frac{1}{120} & + \text{etc.}, \\
& & \text{etc.} & & & & &
\end{array}$$

Therefore, having collected these series into one sum, because of

$$\alpha - \frac{1}{2} = 0, \quad \beta + \frac{1}{2}\alpha - \frac{1}{6} = 0, \quad \gamma + \frac{1}{2}\beta + \frac{1}{6}\alpha - \frac{1}{24} = 0 \quad \text{etc.}$$

it will arise

$$v\left(1 + \frac{1}{2}z + \frac{1}{6}z^2 + \frac{1}{24}z^3 + \frac{1}{120}z^4 + \frac{1}{720}z^5 + \text{etc.}\right) = -1.$$

Therefore, since

$$e^z = 1 + z + \frac{1}{2}zz + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \text{etc.},$$

it is evident that the quantity v is multiplied by

$$\frac{e^z - 1}{z},$$

such that

$$\frac{v(e^z-1)}{z} = -1,$$

whence

$$v = \frac{-z}{e^z-1};$$

and it is just necessary that hence the value of v is expanded into a series, which must be identical to the assumed one, of course, and so the values of the letters α, β, γ etc. will manifest by themselves.

§**25** Therefore, to convert the value of v into a series from this in a convenient manner, since that equation gives us

$$e^z = \frac{v-z}{v},$$

let us set $v = u + \frac{1}{2}$ that

$$e^z = \frac{u-\frac{1}{2}z}{u+\frac{1}{2}z} = \frac{2u-z}{2u+z};$$

hence it will be

$$z = \log(2u-z) - \log(2u+z)$$

and by differentiating

$$dz = \frac{4(zdu - udz)}{4uu - zz},$$

which formula must therefore vice versa be integrated in such a way that for $z = 0$ one has $u = 1$. Now set $u = sz$ and for $z = 0$ it will be $s = \infty$; but then it will be

$$dz = \frac{4ds}{4ss-1},$$

from which equation one must now find a series of such a kind for s that for $z = 0$ $s = \infty$.

§**26** Therefore, since we hence have

$$4ss - 1 = \frac{4ds}{dz},$$

let us assume this series for s

$$2s = \frac{A}{z} + Bz + Cz^3 + Dz^5 + Ez^7 + \text{etc.},$$

whence

$$\frac{2ds}{dz} = -\frac{A}{zz} + B + 3Czz + 5Dz^4 + 7Ez^6 + \text{etc.},$$

but then

$$\begin{array}{rllll} 4ss = \dfrac{AA}{zz} & + 2AB & + 2ACzz & + 2ADz^4 & + 2AEz^5 + \text{etc.}, \\ & + \ BB & + \ 2BC & + 2BD & \\ & & & + \ CC & \end{array}$$

having substituted which series the equation

$$4ss - 1 - \frac{4ds}{dz} = 0$$

yields this expression

$$\begin{array}{llllllll} 2A\dfrac{1}{zz} & - 2B & - \ 6Czz & - \ 10Dz^4 & - \ 14Ez^6 & - \ 18Fz^8 & - \text{etc.} = 0, \\ AA & + 2AB & + 2AC & + 2AD & + 2AE & + 2AF & \\ & - 1 & + \ BB & + \ 2BC & + 2BD & + \ 2BE & \\ & & & & + \ CC & + 2CD & \end{array}$$

Therefore, hence from the first terms $A = -2$; from the following it is further deduced

$$
\begin{aligned}
2B &= 2AB - 1, \\
6C &= 2AC + BB, \\
10D &= 2AD + 2BC, \\
14E &= 2AE + 2BD + CC, \\
18F &= 2AF + 2BE + 2CD, \\
2GG &= 2AG + 2BF + 2CE + DD \\
&\text{etc.}
\end{aligned}
$$

§27 Therefore, since $A = -2$, the following determinations will be obtained

$$
\begin{aligned}
B &= -\frac{1}{6}, \qquad & E &= \frac{2BD + CC}{18}, \\
C &= \frac{BB}{10}, \qquad & F &= \frac{2BE + 2CD}{22}, \\
D &= \frac{2BC}{14}, \qquad & G &= \frac{2BF + 2CE + DD}{26} \\
&\text{etc.}
\end{aligned}
$$

Therefore, to simplify these equations, let us set

$$B = 2\mathfrak{A}, \quad C = 2\mathfrak{B}, \quad D = 2\mathfrak{C}, \quad E = 2\mathfrak{D} \quad \text{etc.};$$

for, then the found determinations will give:

$$\mathfrak{A} = -\frac{1}{12}, \qquad \mathfrak{E} = \frac{2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C}}{11},$$

$$\mathfrak{B} = \frac{\mathfrak{A}\mathfrak{A}}{5}, \qquad \mathfrak{F} = \frac{2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C}}{13},$$

$$\mathfrak{C} = \frac{2\mathfrak{A}\mathfrak{B}}{7}, \qquad \mathfrak{G} = \frac{2\mathfrak{A}\mathfrak{F} + 2\mathfrak{B}\mathfrak{C} + 2\mathfrak{C}\mathfrak{D}}{15},$$

$$\mathfrak{D} = \frac{2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B}}{9}, \qquad \mathfrak{H} = \frac{2\mathfrak{A}\mathfrak{G} + 2\mathfrak{B}\mathfrak{F} + 2\mathfrak{C}\mathfrak{E} + \mathfrak{D}\mathfrak{D}}{17}$$

etc.,

which values can be defined a lot more easily than the above ones α, β, γ, δ etc.

§**28** But having found the values of these letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. first it will be

$$s = -\frac{1}{z} + \mathfrak{A}z + \mathfrak{B}z^3 + \mathfrak{C}z^5 + \mathfrak{D}z^7 + \mathfrak{E}z^9 + \text{etc.},$$

but then it will hence be $u = sz$ and $v = u + \frac{1}{2}z$, whence we obtained this series for v

$$v = -1 + \frac{1}{2}z + \mathfrak{A}z^2 + \mathfrak{B}z^4 + \mathfrak{C}z^6 + \mathfrak{D}z^8 + \text{etc.}.$$

where it is seen that the odd powers, except the first, are missing here. Therefore, since we put

$$v = -1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \zeta z^6 + \text{etc.},$$

having made the comparison it will be

$$\alpha = \frac{1}{2}, \quad \beta = \mathfrak{A}, \quad \gamma = 0, \quad \delta = \mathfrak{B}, \quad \varepsilon = 0, \quad \zeta = \mathfrak{C}, \quad \eta = 0 \quad \text{etc.}$$

Therefore, having introduced these values the general sum S investigated above will become

$$S = -\int X dx + \frac{1}{2}X + \mathfrak{A}\frac{dX}{dx} + \mathfrak{B}\frac{d^3X}{dx^3} + \mathfrak{C}\frac{d^5X}{dx^5} + \mathfrak{D}\frac{d^7X}{dx^7} + \text{etc.}$$

§**29** Since here the value $\mathfrak{A}$ is negative and $= -\frac{1}{12}$, the following $\mathfrak{B}$ on the other becomes positive, the following $\mathfrak{C}$ again negative ans so forth alternately, to take this condition into account and at the same time reduce these numbers to those usually called Bernoulli numbers, let us put

$$2s = -\frac{2}{z} - Az + Bz^3 - Cz^5 + Dz^7 - Ez^9 + \text{etc.},$$

and after the expansion one finds

$$A = \frac{1}{6}, \qquad E = \frac{2AD + 2BC}{22},$$

$$B = \frac{AA}{10}, \qquad F = \frac{2AE + 2BD + CC}{26},$$

$$C = \frac{2AB}{14}, \qquad G = \frac{2AF + 2BE + 2CD}{30}$$

$$D = \frac{2AC + BB}{18}, \qquad \text{etc.};$$

but then it will be

$$v = -1 + \frac{1}{2}z - \frac{1}{2}Azz + \frac{1}{2}Bz^4 - \frac{1}{2}Cz^6 + \frac{1}{2}Dz^8 - \text{etc.},$$

if which series is compared to the assumed one, it will be

$$\alpha = \frac{1}{2}, \quad \beta = -\frac{1}{2}A, \quad \gamma = 0, \quad \delta = \frac{1}{2}B, \quad \varepsilon = 0, \quad \zeta = -\frac{1}{2}C, \quad \eta = 0 \quad \text{etc.}$$

Therefore, having introduced these letters A, B, C, D etc. the general sum will be

$$S = -\int Xdx + \frac{1}{2}X - \frac{1}{2}A\frac{dX}{dx} + \frac{1}{2}B\frac{d^3X}{dx^3} - \frac{1}{2}C\frac{d^5X}{dx^5} + \text{etc.}$$

If we further set $A = 2\mathfrak{A}$, $B = 2\mathfrak{B}$, $C = 2\mathfrak{C}$ etc., the relations between these letters will look as follows:

$$\mathfrak{A} = \frac{1}{12}, \qquad \mathfrak{E} = \frac{2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C}}{11},$$

$$\mathfrak{B} = \frac{\mathfrak{A}\mathfrak{A}}{5}, \qquad \mathfrak{F} = \frac{2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C}}{13},$$

$$\mathfrak{C} = \frac{2\mathfrak{A}\mathfrak{B}}{7}, \qquad \mathfrak{G} = \frac{2\mathfrak{A}\mathfrak{F} + 2\mathfrak{B}\mathfrak{E} + 2\mathfrak{C}\mathfrak{D}}{15}$$

$$\mathfrak{D} = \frac{2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B}}{9}, \qquad \text{etc.};$$

but then it will be

$$S = -\int X dx + \frac{1}{2}X - \mathfrak{A}\frac{dX}{dx} + \mathfrak{B}\frac{d^3X}{dx^3} - \mathfrak{C}\frac{d^5X}{dx^5} + \text{etc.}$$

From these formulas it is now understood that those numbers are connected to those, which enter into the even reciprocal powers; for, it will be

$$\mathfrak{A} = \frac{1}{2}\cdot\frac{1}{6}, \quad \mathfrak{B} = \frac{1}{2^3}\cdot\frac{1}{90}, \quad \mathfrak{C} = \frac{1}{2^5}\cdot\frac{1}{945}, \quad \mathfrak{D} = \frac{1}{2^7}\cdot\frac{1}{9450}, \quad \mathfrak{E} = \frac{1}{2^9}\cdot\frac{1}{93555} \quad \text{etc.},$$

which numbers I once already gave up to the thirtieth power.

§**30** Therefore, hence for our undertaking let us propound the following universal theorem.

THEOREM

If the following infinite series was propounded

$$S = X + X' + X'' + X''' + \text{etc.},$$

then its sum will be expressed in the following way

$$S = -\int X dx + \frac{1}{2}X - \frac{1}{2\cdot 6}\cdot\frac{dX}{dx} + \frac{1}{2^3\cdot 90} - \frac{1}{2^5\cdot 945}\cdot\frac{d^5X}{dx^5}$$
$$+\frac{1}{2^7\cdot 9450}\cdot\frac{d^7X}{dx^7} - \frac{1}{2^9\cdot 93555}\cdot\frac{d^9X}{dx^9} + \text{etc.},$$

where the following coefficients can be taken from my Introductio in Analysin Infinitorum p. 131. But note that here the integral $\int Xdx$ must be taken in such a way that it vanishes for $x = \infty$.

Application to our case in which $X = \frac{1}{x+a} - \log\frac{x+1}{x}$

§ **31** Therefore, since

$$X = \frac{1}{x+a} + \log x - \log(x+1),$$

first for the integral formula it will be

$$\int Xdx = \log(x+a) + x\log x - (x+1)\log(x+1) + C;$$

since this constant C must be taken in such a way that the integral vanishes for $x = \infty$, bring that integral into this form

$$\int Xdx = -x\log\frac{x+1}{x} + \log\frac{x+a}{x+1} + C,$$

which expression for $x = \infty$ becomes

$$\int Xdx = -\infty\log\left(\frac{\infty+1}{\infty}\right) + \log 1 + C = 0,$$

where, since

$$\log\frac{\infty+1}{\infty} = \log\left(1+\frac{1}{\infty}\right) = \frac{1}{\infty} - \frac{1}{2\infty^2} + \frac{1}{3\infty^3} - \text{etc.},$$

it will be

$$\infty\log\frac{\infty+1}{\infty} = 1$$

and hence that integral $C - 1 = 0$, therefore, the constant $C = 1$; for this reason, the first term of our expression is

$$\int Xdx = -x\log\frac{x+1}{x} + \log\frac{x+a}{x+1} + 1,$$

whence for the first two terms we will have

$$-\int Xdx + \frac{1}{2}X = \left(x - \frac{1}{2}\right)\log\frac{x+1}{x} - \log\frac{x+a}{x+1} + \frac{1}{2(x+a)} - 1.$$

§**32** The remaining terms of our expression do not cause any trouble, since they are found by iterated differentiation

$$\frac{dX}{dx} = -\frac{1}{(x+a)^2} + \frac{1}{x} - \frac{1}{x+1},$$

$$\frac{d^3X}{dx^3} = 1\cdot 2\left(-\frac{3}{(x+a)^4} + \frac{1}{x^3} - \frac{1}{(x+1)^3}\right),$$

$$\frac{d^5X}{dx^5} = 1\cdot 2\cdot 3\cdot 4\left(-\frac{5}{(x+a)^6} + \frac{1}{x^3} - \frac{1}{(x+1)^5}\right),$$

$$\frac{d^7X}{dx^7} = 1\cdot 2\cdots 6\left(-\frac{7}{(x+a)^8} + \frac{1}{x^7} - \frac{1}{(x+1)^7}\right),$$

$$\frac{d^9X}{dx^9} = 1\cdot 2\cdots 8\left(-\frac{9}{(x+a)^{10}} + \frac{1}{x^9} - \frac{1}{(x+1)^9}\right),$$

etc.

§33 Therefore, from these the sum of our series S is concluded to be

$$
\begin{aligned}
S = \left(x - \frac{1}{2}\right)\log\frac{x+1}{x} - \log\frac{x+a}{x+1} + \frac{1}{2(x+a)-1} \\
- \quad \frac{1}{1\cdot 2\cdot 3} \cdot \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+a)^2}\right) \\
+ \quad \frac{1}{3\cdot 4\cdot 5} \cdot \frac{1}{6}\left(\frac{1}{x^3} - \frac{1}{(x+1)^3} - \frac{3}{(x+a)^4}\right) \\
- \quad \frac{1}{5\cdot 6\cdot 7} \cdot \frac{1}{6}\left(\frac{1}{x^5} - \frac{1}{(x+1)^5} - \frac{5}{(x+a)^6}\right) \\
+ \quad \frac{1}{7\cdot 8\cdot 9} \cdot \frac{3}{10}\left(\frac{1}{x^5} - \frac{1}{(x+1)^5} - \frac{7}{(x+a)^8}\right) \\
- \quad \frac{1}{9\cdot 10\cdot 11} \cdot \frac{5}{6}\left(\frac{1}{x^9} - \frac{1}{(x+1)^9} - \frac{9}{(x+a)^{10}}\right) \\
+ \quad \frac{1}{11\cdot 12\cdot 13} \cdot \frac{691}{210}\left(\frac{1}{x^{11}} - \frac{1}{(x+1)^{11}} - \frac{11}{(x+a)^{12}}\right) \\
- \quad \frac{1}{13\cdot 14\cdot 15} \cdot \frac{35}{2}\left(\frac{1}{x^{13}} - \frac{1}{(x+1)^{13}} - \frac{13}{(x+a)^{14}}\right) \\
+ \quad \text{etc.},
\end{aligned}
$$

where the fractions at the second position

$$\frac{1}{2}, \quad \frac{1}{6}, \quad \frac{1}{6}, \quad \frac{3}{10}, \quad \frac{5}{6}, \quad \frac{691}{210} \quad \text{etc.}$$

are the numbers referred to as Bernoulli numbers, which proceed further this way

$$\frac{35}{2}, \quad \frac{3617}{30}, \quad \frac{43867}{42}, \quad \frac{1222277}{110}, \quad \frac{854513}{6}, \quad \frac{1181820455}{546}, \quad \frac{76977927}{2}, \quad \frac{23749461029}{30},$$
$$\frac{861584127005}{462}, \quad \frac{84802531453387}{170}, \quad \frac{90219075042845}{6} \quad \text{etc.}$$

§**34** Up to this point we have considered x as a variable, but now after the expansion both numbers x and a can be assumed arbitrarily and hence always the same value for our constant C will result, which will of course be

$$C = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x+a-1} - \log x + S,$$

and the series S will converge the faster the larger the numbers x and a are taken. But it will be convenient to chose a relation among the two numbers x and a of such a kind that the formula

$$\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+a)^2}$$

vanishes approximately. As if, e.g. $x = 10$, this formula

$$\frac{1}{110} - \frac{1}{(10+a)^2}$$

becomes very small, if either $a = 0$ or $a = 1$; but it will become the smallest for $a = \frac{1}{2}$, for, then that value will become $\frac{1}{110} - \frac{4}{441}$, whence it is plain that it always advisable to take $a = \frac{1}{2}$. Therefore, if x was an integer number, the first term will be

$$2 - 2\log 2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{9} + \cdots + \frac{2}{2x-1}.$$

Therefore, this way our expression will converge very fast, and as long as the number x is taken moderately large, a few terms will suffice to find the value of the number C sufficiently exactly.

§**35** Therefore, after we had expressed the value of our number C by infinite series, let us see, whether they can also be exhibited by finite integral formulas, whence it can more safely be deduced to which kind of quantities this number is to be referred. And first the indefinite series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

arises from the expansion of this integral formula

$$\int \frac{(1-x^n)dx}{1-x},$$

if after the integration one sets $x = 1$, where now the number n can be a fraction or an integer, and so we will have

$$C = \int \frac{(1-x^n)dx}{1-x} - \log n,$$

if the an infinite number is taken for n, of course. Therefore, let us also investigate an integral formula which for $x = 1$ gives us $\log n$. But since for $x = 1$

$$\frac{1-x^n}{1-x} = \log n,$$

it will be

$$\log \frac{1-x^n}{1-x} = \log n;$$

hence it is plain, if n denotes an infinitely large number, that then it will be

$$C = \int \frac{(1-x^n)dx}{1-x} - \log \frac{1-x^n}{1-x},$$

after one had set $x = 1$, of course.

§**36** But further

$$\log(1-x^n) = -\int \frac{x^{n-1}dx}{1-x^n}$$

and

$$\log(1-x) = -\int \frac{dx}{1-x},$$

having substituted which values in terms of mere integral formulas we will have

$$C = \int \frac{(1-x^n)dx}{1-x} + n\int \frac{x^{n-1}dx}{1-x^n} - \int \frac{dx}{1-x},$$

which is reduced to this form

$$C = -\int \frac{x^n dx}{1-x} + n\int \frac{x^{n-1}dx}{1-x^n}$$

such that C becomes equal to the difference of these two integral formulas, if one sets $x = 1$ after the integration, of course, but the exponent n is taken infinitely large; hence it is plain that this formula will exhibit the true value of the formula C the more closely the greater the number n is taken.

§**37** But it is possible to free these formulas from the infinitely large exponents by setting $x^n = z$, and since in the case $x = 1$ also $z = 1$, one has to take $z = 1$ in the integral formulas to arise from this. Hence, since from this $nx^{n-1}dx = dz$, but then $x = z^{\frac{1}{n}}$ and hence

$$dx = \frac{1}{n} z^{\frac{1}{n}-1} dz = \frac{z^{\frac{1}{n}} dz}{nz}$$

our formulas will go over into the following

$$C = -\frac{1}{n} \int \frac{z^{\frac{1}{n}} dz}{1 - z^{\frac{1}{n}}} + \int \frac{dz}{1-z}.$$

But it is known to be

$$\log z = n \left(z^{\frac{1}{n}} - 1 \right)$$

while $n = \infty$, having substituted which value in the first part

$$C = \int \frac{dz}{\log z} + \int \frac{dz}{1-z},$$

if just one puts $z = 1$ after the integration, such that this formula is now completely free from the infinite, which formula can be exhibited by one single integral sign this way

$$C = \int dz \left(\frac{1}{1-z} + \frac{1}{\log z} \right);$$

and so the whole question on our constant C is reduced to this that the value of this integral formula is investigated

$$\int dz \left(\frac{1}{1-z} + \frac{1}{\log z} \right)$$

extended from the limit $z = 0$ to $z = 1$, which formula seems to be worth one's complete attention. But it is evident that the first part of this formula

$$\int \frac{dz}{1-z} = -\log(1-z)$$

becomes $+\infty$ in this case. Further, I also showed that the other part $\int \frac{dz}{\log z}$ yields negative infinity; from this it is understood that both formulas combined can produce a finite value.

§38 Furthermore, since above we detected several extraordinary properties of the number C, it will be worth one's while to list them up here all together. Therefore, note that the letters α, β, γ, δ etc. denote the sums of the following series

$$\begin{aligned}
\alpha &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.},\\
\beta &= 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.},\\
\gamma &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.},\\
\delta &= 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.},\\
&\text{etc.};
\end{aligned}$$

having constituted these values the following theorems were found:

$$\text{I.} \qquad 1 - C = \frac{1}{2}(\alpha - 1) + \frac{1}{3}(\beta - 1) + \frac{1}{4}(\gamma - 1) + \frac{1}{5}(\delta - 1) + \text{etc.}$$

$$\text{II.} \qquad C = \frac{1}{2}\alpha - \frac{1}{3}\beta + \frac{1}{4}\gamma - \frac{1}{5}\delta + \frac{1}{6}\varepsilon - \frac{1}{7}\zeta + \frac{1}{8}\eta - \frac{1}{9}\theta + \text{etc.}$$

$$\text{III.} \qquad 1 = \left(\alpha - \frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{2}\gamma - \frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{3}\varepsilon - \frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{4}\eta - \frac{1}{8} - \frac{1}{9}\right) + \text{etc.}$$

$$\text{IV.} \qquad 2C - 1 = \left(\frac{1}{2} + \frac{1}{3} - \frac{2}{3}\beta\right) + \left(\frac{1}{4} + \frac{1}{5} - \frac{2}{5}\delta\right) + \left(\frac{1}{6} + \frac{1}{7} - \frac{2}{7}\zeta\right) + \left(\frac{1}{8} + \frac{1}{9} - \frac{2}{9}\theta\right) + \text{etc.}$$

$$\text{V.} \qquad 2 - 2\log 2 - C = \left(\frac{2}{3} \cdot \frac{7}{8}\beta - \frac{2}{3}\right) + \left(\frac{2}{5} \cdot \frac{31}{32}\delta - \frac{2}{5}\right) + \left(\frac{2}{7} \cdot \frac{127}{128}\zeta - \frac{2}{7}\right) + \left(\frac{2}{9} \cdot \frac{511}{512}\theta - \frac{2}{9}\right) + \text{etc.}$$

$$\text{VI.} \qquad 1 - 2\log 2 + C = \left(\frac{1}{2} - \frac{1}{3} - \frac{2}{3 \cdot 2^3}\beta\right) + \left(\frac{1}{4} - \frac{1}{5} - \frac{2}{5 \cdot 2^5}\delta\right) + \left(\frac{1}{6} - \frac{1}{7} - \frac{2}{7 \cdot 2^7}\zeta\right) + \text{etc.}$$

$$\text{VII.} \qquad \log 2 - C = \frac{1}{3 \cdot 2^2}\beta + \frac{1}{5 \cdot 2^4}\delta + \frac{1}{7 \cdot 2^6}\zeta + \frac{1}{9 \cdot 2^8}\theta + \frac{1}{11 \cdot 2^{10}}\varkappa + \text{etc.}$$

$$\text{VIII.} \qquad 1 - \log\frac{3}{2} - C = \frac{1}{3 \cdot 2^2}(\beta - 1) + \frac{1}{5 \cdot 2^4}(\delta - 1) + \frac{1}{7 \cdot 2^6}(\zeta - 1) + \frac{1}{9 \cdot 2^8}(\theta - 1) + \text{etc.}$$

Here, of course, everywhere

$$C = 0{,}5772156649015325$$

or

$$C = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \log n$$

for an infinite number n.